Universal features of BPS strings in six-dimensional SCFTs

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Abstract: In theories with extended supersymmetry the protected observables of UV superconformal fixed points are found in a number of contexts to be encoded in the BPS solitons along an IR Coulomb-like phase. For six-dimensional SCFTs such a role is played by the BPS strings on the tensorial Coulomb branch. In this paper we develop a uniform description of the worldsheet theories of a BPS string for rank-one 6d SCFTs. These strings are the basic constituents of the BPS string spectrum of arbitrary rank six-dimensional models, which they generate by forming bound states. Motivated by geometric engineering in F-theory, we describe the worldsheet theories of the BPS strings in terms of topologically twisted 4d $\mathcal{N} = 2$ theories in the presence of 1/2-BPS 2d $(0,4)$ defects. As the superconformal point of a 6d theory with gauge group G is approached, the resulting worldsheet theory flows to an $\mathcal{N} = (0,4)$ NLSM with target the moduli space of one G instanton, together with a nontrivial left moving bundle characterized by the matter content of the six-dimensional model. We compute the anomaly polynomial and central charges of the NLSM, and argue that the 6d flavor symmetry F is realized as a current algebra on the string, whose level we compute. We find evidence that for generic theories the G dependence is captured at the level of the elliptic genus by characters of an affine Kac-Moody algebra at negative level, which we interpret as a subsector of the chiral algebra of the BPS string worldsheet theory. We also find evidence for a spectral flow relating the R-R and NS-R elliptic genera. These properties of the string CFTs lead to constraints on their spectra, which in combination with modularity allow us to determine the elliptic genera of a vast number of string CFTs, leading also to novel results for 6d and 5d instanton partition functions.

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Contents

1 Introduction and summary

Recently much progress has been made in understanding six-dimensional $\mathcal{N} = (1, 0)$ supersymmetric theories and their compactifications thanks to the development of novel techniques relating to their holographic description, to their geometric engineering in F-theory, and to 6d field theory itself [1–60]. Six-dimensional theories have tensorial Coulomb-like phases in which the tensor dynamics abelianizes and can be more easily understood. In particular, a feature of this 'tensor branch' of the moduli space is that a spectrum of BPS strings is generated (see e.g. [61]) which possess 2d $(0, 4)$ supersymmetry on their worldsheet; approaching the origin of the tensor branch corresponds to a renormalization group flow on the strings' worldsheet to an IR (0,4) CFT. Considerable advances have been made

in characterizing these BPS strings and in understanding their elliptic genera [62–94],[1] the main motivation for such studies being the conjecture that the Ω-background partition function for six-dimensional theories localizes on contributions from the BPS strings, which can also be exploited to reconstruct the corresponding superconformal index and other partition functions [63, 64, 108–111].[2] The currently available methods, which often involve UV realizations of the strings' worldsheet theories in which RG flow invariant quantities can more easily be computed, are extremely powerful and can frequently be employed to compute at once the elliptic genera of arbitrary bound states of BPS strings. A downside of relying on UV techniques however is that they tend to be tailored to specific classes of 6d SCFTs, and as a consequence certain features which are common to the BPS strings of all 6d (1,0) theories tend to be somewhat obscured. Motivated by this, in this paper we take a different route and seek to reformulate the various known features of the BPS strings directly in the language of 2d conformal field theory. The payoff of this approach is that we will find a very natural and uniform picture for how the global symmetries of the string are realized at the level of the CFT, which also turns out to be a quite powerful asset in computing the strings' elliptic genera. In particular, this will allow us to determine the elliptic genera for the one string sectors of essentially all rank one 6d SCFTs, which also leads to several new results concerning 6d and 5d Nekrasov partition functions. Along the way, we develop a description of the BPS string CFTs in terms of twisted compactifications of 4d $\mathcal{N} = 2$ theories on $\mathbb{P}^1$ in the presence of 2d (0,4) surface defects.

In this paper we focus on models that admit a geometric engineering in F-theory without frozen singularities [129], and therefore whenever referring to an F-theory geometry we implicitly make such an assumption.[3] The relevant six-dimensional F-theory geometric backgrounds have been classified [2, 5, 10]; in this geometric setup the BPS strings are realized as stacks of D3-branes wrapped on combinations of intersecting rational curves in the F-theory base. For almost all 6d SCFTs these rational curves also support wrapped seven-branes which lead to nontrivial gauge groups in the SCFT. The Hilbert spaces of these models along the tensor branch are divided in superselection sectors labeled by the BPS string charges. In particular, the one-string subsector of the Hilbert space can always be identified with the one-string subsector of specific rank-one theories.[4] For this reason, in this work we choose to focus on the spectrum of BPS strings of rank-one 6d SCFTs. A review of the F-theory geometries and of various properties of rank-one theories can be found in section 2.

The tensor branches of rank-one 6d SCFTs are realized geometrically in terms of an elliptically fibered Calabi-Yau such that the base of the fibration has local model given by the total space of a line bundle $\mathcal{O}(-n) \to \mathbb{P}^1$ [2]. A specific model is characterized by two

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[1]This type of analysis has also been carried out in the related contexts of 5d supersymmetric field theories [95–97], little string theories [97–104], and 6d (1,0) supergravity theories [105–107].

[2]In analogy with the case of 4d $\mathcal{N} = 2$ SCFTs [112–128], one might ask whether it is possible to recover some, if not all, BPS properties of the 6d SCFTs from the knowledge of the corresponding BPS string CFTs.

[3]At the time of this writing progress is being made towards understanding the frozen phase of F-theory; a very nice account of these recent advances can be found in the recent talks given by Alessandro Tomasiello at the Banff 2018 and at Madrid 2018 F-theory conferences, which are available online.

[4]We define the rank of a 6d SCFT as the dimension of its tensor branch, in analogy with the definition of the rank of a 4d $\mathcal{N} = 2$ SCFT.

pieces of data: an integer n between 1 and 12 specifying the degree of the line bundle, and a Lie group G. The former is interpreted in the SCFT as the Dirac pairing of an elementary BPS string with itself; the latter, which is determined geometrically from the structure of the elliptic fiber along the base $\mathbb{P}^1$, specifies the 6d gauge symmetry. The BPS string of the 6d SCFT can be viewed as an instanton for G, and indeed the Green-Schwarz term in the 6d tensor branch Lagrangian identifies the BPS string charge with the instanton charge for the gauge group G. From anomaly cancellation it follows that not all pairs n and G are allowed; in particular, given a pair (n, G) the corresponding matter content and flavor symmetry F is often uniquely determined (with only a few exceptions in which more than one choice of matter is allowed [130]). We adopt the notation $\mathbf{n}_G$ for the corresponding six-dimensional SCFT. It is well known that D3-brane probes of seven-branes in F-theory give rise to four-dimensional theories with $\mathcal{N} = 2$ supersymmetry [131–134]. Therefore, exploiting an adiabatic approximation, we can view the worldsheet theories for the 6d BPS strings as twisted compactifications of such 4d $\mathcal{N} = 2$ theories on the base $\mathbb{P}^1$. This strategy has been adopted in [83] to describe the BPS string instantons for the minimal 6d SCFTs without matter. In presence of matter, the geometry is modified by introducing transverse seven-branes that wrap noncompact curves intersecting the base $\mathbb{P}^1$ at points. These extra seven-branes are interpreted as surface defects preserving 2d $(0, 4)$ supersymmetry for the corresponding 4d $\mathcal{N} = 2$ theories on the base $\mathbb{P}^1$, which generalize the chiral defects studied by Martucci in [135] in the context of duality-twisted compactifications of 4d $\mathcal{N} = 4$ SYM. The geometric engineering setup allows us to deduce several properties of these generalized chiral defects as well as of the corresponding BPS string CFTs, as we discuss in section 3.

The BPS string and its bound states are interesting probes into the physics of the six-dimensional SCFT. This is true first and foremost at the level of the Nekrasov partition function on the tensorial Coulomb branch, which, specializing to theories with one tensor multiplet, is given by

$$Z_{\text{Nekrasov}} = Z_{\text{pert}} \cdot Z_{\text{inst}}, \qquad Z_{\text{inst}} = \sum_{k=0}^{\infty} Q^k Z_{k \text{ inst}}. \tag{1.1}$$

The 6d Nekrasov partition function is an elliptic generalization of the 4d and 5d Nekrasov partition functions [136], which are given respectively in terms of rational and trigonometric functions of various fugacities. The Z_{pert} factor includes contributions to the partition function from the BPS particles, which do not carry instanton charge, whereas Z_{inst} is the contribution of the instantons, which are identified with BPS strings wrapped on T^2 in the $T^2 \times \mathbb{R}^4$ omega background. Indeed, $Z_{k \text{ inst}}$ is given by the Ramond-Ramond elliptic genus of the worldsheet theory of a bound state of k strings [63], with periodic boundary conditions on the left movers. This interpretation holds for all 6d SCFTs including the E- and M- string SCFTs which may be viewed respectively as 6d SCFTs with an Sp(0) or SU(1) gauge group. We review elliptic genera of the BPS strings and their relation to the 6d Nekrasov partition function in section 4.

In the following sections we also encounter other ways in which the CFT of a string carries nontrivial information about the 6d SCFT. For example, the $\mathbf{2}_{\text{SU(2)}}$ SCFT has SO(8) flavor symmetry on the tensor branch, but only SO(7) flavor symmetry at the origin [20].

This is reflected in the spectrum of the string, which as we will see in section 9 can be organized in terms of $SO(8)_1$ affine characters (with a certain specialization of fugacities), but only has an $SO(7)$ adjoint representation worth of chiral currents rather than a full $SO(8)$ adjoint representation. Moreover we will find evidence that some features of 6d SCFTs that emerge in their F-theory classification have a natural explanation from the perspective of the string worldsheet CFT. For example, F-theory predicts that 6d SCFTs with unpaired tensors (i.e. with a tensor multiplet that is not paired up to a gauge algebra via the Green-Schwarz mechanism) only exist for $n = 1$ and 2. From the perspective of the string worldsheet CFT, this is a consequence of unitarity: the requirement that the left moving central charge

$$c_L = 6\,h_G^\vee - 6\,n + 12 \geq 0 \tag{1.2}$$

for $n \geq 3$ is only possible if $h_G^\vee > 0$.

Motivated by these observations, in sections 5 and 6 we seek a uniform description for the BPS string CFTs; though several aspects of our discussion apply to bound states of arbitrary number of strings, for brevity we choose to restrict our attention to the CFTs of a single string. We find that these CFTs share a number of universal features which strongly constrain their spectrum and the form of their elliptic genera. We choose to take an empirical approach, relying on known results for various BPS string CFTs (such as computation of the elliptic genera by localization or by other methods) and inferring general features of the CFTs from the existing results. We first focus in section 5 on the BPS strings of the $\mathbf{1}_{Sp(n)}$ SCFTs, which are essentially free theories (since the reduced moduli space of one $Sp(N)$ instanton is just a $\mathbb{Z}_2$ orbifold of $\mathbb{C}^{2N}$), and then rephrase our findings in full generality in section 6.

The CFT describing a BPS string consists of two components: a center of mass piece, and an interacting piece on which we focus, which we denote by $\mathbf{h}_n^G$ for the string of the $\mathbf{n}_G$ 6d SCFT. The interacting piece can be understood as a $\mathcal{N} = (0,4)$ nonlinear sigma model with target space the reduced moduli space of one G instanton, $\widetilde{\mathcal{M}}_{G,1}$. The information on the 6d gauge symmetry is carried by scalar (bosonic) superfields, whereas the 6d flavor symmetry F which acts on the matter content is coupled to spinor (fermionic) superfields, which are sections of a chiral vector bundle on $\widetilde{\mathcal{M}}_{G,1}$. We find that the NLSM consists of the following numbers of bosonic and fermionic components (where we further distinguish between L (chiral) and R (anti-chiral) degrees of freedom):

	L	R
# noncompact bosons	$4(h_G^\vee - 1)$	$4(h_G^\vee - 1)$
# fermions	$4h_G^\vee - 12(n-2)$	$4(h_G^\vee - 1)$

From this data one recovers the central charges for $\mathbf{h}_n^G$

$$c_L = 6h_G^\vee - 6n + 8; \tag{1.3}$$
$$c_R = 6h_G^\vee - 6. \tag{1.4}$$

Since the flavor symmetry F couples chirally to the fermionic superfields, we are immediately led to the conclusion that it is realized in the CFT as a chiral current algebra

(that is, as the chiral half of a WZW model). This is of course already a very well known fact for the E-string CFT, whose flavor symmetry is captured by the level 1 E_8 current algebra [137]; in this paper we will see very concretely how the statement generalizes to all $\mathbf{h}_n^G$ CFTs. The level can be read off from the anomaly polynomial as the coefficient of the corresponding 't Hooft anomaly. In turn, the string's anomaly polynomial can be read off from the anomaly polynomial of the $\mathbf{n}_G$ SCFT by anomaly inflow, following the approach of [31, 82]. Anticipating our discussion from section 6.5, in table 1 we list the current algebras realizing the flavor symmetry of all BPS string CFTs. We only encounter subtleties for certain theories with $G = \mathrm{SO}(11)$ or $\mathrm{SO}(12)$, which we address in section 6.5.

At the level of the spectrum of the CFT, for $F = \prod_{i=1}^{n_F} F_i$ a product of simple and abelian factors, this implies that the Hilbert space factorizes as

$$
H_n^G = \bigoplus_{\vec{\lambda}} \left(\bigotimes_{i=1}^{n_F} H_{\lambda_i}^{WZW_{F_i}} \right) \otimes H_{\vec{\lambda}}^{\text{residual}},
\tag{1.5}
$$

where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_{n_F})$ labels highest weights corresponding to integrable highest weight representations of the F_i WZW models. The residual factor of the Hilbert space includes both chiral and anti-chiral degrees of freedom that depend on G and v. Correspondingly, the dependence of the elliptic genus also factorizes:

$$
\mathbb{E}_n^G(\mathbf{m}_G, \mathbf{m}_F, v, q) = \sum_{\vec{\lambda}} \left(\prod_{i=1}^{n_F} \widehat{\chi}_{\lambda_i}^{F_i}(\mathbf{m}_{F_i}, q) \right) \xi_{\vec{\lambda}}^{n,G}(\mathbf{m}_G, v, q),
\tag{1.6}
$$

where $\widehat{\chi}_{\lambda_i}^{F_i}(\mathbf{m}_{F_i}, q)$ are WZW_{F_i} characters, and $\xi_{\vec{\lambda}}^{n,G}(\mathbf{m}_G, v, q)$ are the holomorphic contributions of $H_{\vec{\lambda}}^{\text{residual}}$ to the elliptic genus. In these expressions, $\mathbf{m}_F$ and $\mathbf{m}_G$ are respectively exponentiated fugacities for F and G, while $v = e^{2\pi i \epsilon_+}$ is a fugacity coupling to $J_R^3 + J_I^3$, where J_R^3 and J_I^3 are respectively Cartan generators for the $\mathrm{SU}(2)_R$ subgroup of the $\mathrm{SO}(4) \sim \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ isometry group of $\mathbb{R}^4$ and for the $\mathrm{SU}(2)_I$ R-symmetry group of the 6d SCFT, which is also identified with the superconformal R-symmetry group of the 2d (0,4) CFT (consistent with the fact that the latter cannot act on the non-compact target space [138]).

Rewriting the elliptic genus as in equation (1.6) reveals a number of interesting features which we ultimately interpret as properties of the chiral algebra of the $(0, 4)$ CFT. First of all, we find that for $\mathbf{h}_n^G$ theories with $n \neq h_G^\vee$ the functions $\xi_{\vec{\lambda}}^{n,G}$ admit the following expansion:

$$
\xi_{\vec{\lambda}}^{n,G}(\mathbf{m}_G, v, q) =
$$
$$
= \sum_{\nu \in \mathrm{Rep}(G)} \sum_{\ell=-2|n-h_G^\vee|+1}^{0} \sum_{m \in \mathbb{Z}} n_{\nu,\ell,m}^{\vec{\lambda}} \times \frac{q^{-\frac{c_G}{24}+h_\nu^G} \chi_\nu^G(\mathbf{m}_G)}{\widetilde{\Delta}_G(\mathbf{m}_G, q)} \times \frac{q^{-\frac{c_v}{24}+h_{\ell,m}^v} v^{\ell+2(n-h_G^\vee)m}}{\prod_{j=1}^\infty (1-q^j)}, \tag{1.7}
$$

where the $n_{\nu,\ell,m}^{\vec{\lambda}}$ are integer coefficients, and c_G, h_ν^G (resp. $c_v = 1$, and $h_{\ell,m}^v$) are the central charges and conformal dimensions of operators of the level $-n$ G Kac-Moody algebra (resp. $\mathrm{U}(1)$ Kac-Moody algebra at $R^2 \sim n - h_G^\vee$). By analyzing the elliptic genus in

n	G	F
12	E_8	–
8	E_7	–
7	E_7	–
6	E_6	–
6	E_7	$SO(2)_{12}$
5	F_4	–
5	E_6	$U(1)_6$
5	E_7	$SO(3)_{12}$
4	$SO(N), N \geq 8$	$Sp(N-8)_1$
4	F_4	$Sp(1)_3$
4	E_6	$SU(2)_6 \times U(1)_{12}$
4	E_7	$SO(4)_{12}$
3	$SU(3)$	–
3	$SO(7)$	$Sp(2)_1$
3	$SO(8)$	$Sp(1)_1^3$
3	$SO(9)$	$Sp(2)_1 \times Sp(1)_2$
3	$SO(10)$	$Sp(3)_1 \times (SU(1)_4 \times U(1)_4)$
3	$SO(11)$	$Sp(4)_1 \times \mathrm{Ising}$
3	$SO(12)$	$Sp(5)_1$
3	G_2	$Sp(1)_1$
3	F_4	$Sp(2)_3$
3	E_6	$SU(3)_6 \times U(1)_{18}$
3	E_7	$SO(5)_{12}$
2	$SU(1)$	$SU(2)_1$
2	$SU(2)$	$SO(8)_1 \to SO(7)_1 \times \mathrm{Ising}$
2	$SU(N), N > 2$	$SU(2N)_1$
2	$SO(7)$	$Sp(1)_1 \times Sp(4)_1$

n	G	F
2	$SO(8)$	$Sp(2)_1^3$
2	$SO(9)$	$Sp(3)_1 \times Sp(2)_2$
2	$SO(10)$	$Sp(4)_1 \times (SU(2)_4 \times U(1)_8)$
2	$SO(11)$	$Sp(5)_1 \times ???$
2	$SO(12)_a$	$Sp(6)_1 \times SO(2)_8$
2	$SO(12)_b$	$Sp(6)_1 \times \mathrm{Ising} \times \mathrm{Ising}$
2	$SO(13)$	$Sp(7)_1$
2	G_2	$Sp(4)_1$
2	F_4	$Sp(3)_3$
2	E_6	$SU(4)_6 \times U(1)_{24}$
2	E_7	$SO(6)_{12}$
1	$Sp(0)$	$(E_8)_1$
1	$Sp(N), N \geq 1$	$SO(4N+16)_1$
1	$SU(3)$	$SU(12)_1$
1	$SU(4)$	$SU(12)_1 \times SU(2)_1$
1	$SU(N), N \geq 4$	$SU(N+8)_1 \times U(1)_{2N(N-1)(N+8)}$
1	$SU(6)_*$	$SU(15)_1$
1	$SO(7)$	$Sp(2)_1 \times Sp(6)_1$
1	$SO(8)$	$Sp(3)_1^3$
1	$SO(9)$	$Sp(4)_1 \times Sp(3)_2$
1	$SO(10)$	$Sp(5)_1 \times (SU(3)_4 \times U(1)_{12})$
1	$SO(11)$	$Sp(6)_1 \times ???$
1	$SO(12)_a$	$Sp(7)_1 \times SO(3)_8$
1	$SO(12)_b$	$Sp(7)_1 \times ???$
1	G_2	$Sp(7)_1$
1	F_4	$Sp(4)_3$
1	E_6	$SU(5)_6 \times U(1)_{30}$
1	E_7	$SO(7)_{12}$

Table 1. Current algebra associated to the flavor symmetry F of the $\mathbf{n}_G$ 6d SCFTs. The ??? indicate cases for which we do not have a good understanding of the worldsheet realization of the flavor symmetry. The notation $SO(12)_{a,b}$ and $SU(6)_*$ is explained in section 2.4.

specific examples, we find evidence that the G dependence indeed organizes itself in terms of irreducible characters of the $-n$ G Kac-Moody algebra, where the level $k_G = -n$ is in agreement with the expectation from the anomaly polynomial of the string.

The central charge of the left-moving part of the CFT can be written as follows:

$$c_L = c_F + c_G + c_v - 24 \frac{(h_G^\vee - 1)^2}{4(n - h_G^\vee)}, \tag{1.8}$$

where c_F is the total Sugawara central charge of the WZW sector capturing the 6d flavor symmetry F. The last term arises because the G- and F-neutral vacuum in the chiral

algebra carries nonzero $U(1)_v$ charge $h_G^\vee - 1$ and therefore sits in a Verma module of $U(1)_v$ which is distinct from the vacuum Verma module.

The diagonal subgroup $SU(2)_v$ of $SU(2)_R \times SU(2)_I$, to which the chemical potential v couples, does not act chirally on the full spectrum of the CFT; however, it can be thought of as a chiral symmetry once we restrict to the chiral algebra underlying the elliptic genus; we find that shifting its fugacity $v \to q^{1/2}/v$ implements a spectral flow which leads to a relation between the Ramond-Ramond elliptic genus $\mathbb{E}_n^G$ and the Neveu-Schwarz-Ramond elliptic genus $\mathcal{E}_n^G$:

$$\mathcal{E}_n^G(\mathbf{m}_G, \mathbf{m}_F, v, q) = q^{\frac{n-h_G^\vee}{4}} v^{-(n-h_G^\vee)} \mathbb{E}_n^G(\mathbf{m}_G, \mathbf{m}_F, q^{1/2}/v, q). \tag{1.9}$$

This fact turns out to be quite convenient: whereas the low energy spectrum that contributes to the Ramond-Ramond elliptic genus is complicated due to the presence of fermionic zero modes on top of the bosonic zero modes, the low energy spectrum contributing to the Neveu-Schwarz-Ramond elliptic genus is much simpler. At the zero energy level only the bosonic generators of the moduli space of one G instanton contribute; their contribution is given by:

$$\mathcal{E}_n^G(\mathbf{m}_G, \mathbf{m}_F, v, q)\Big|_{q^{-\frac{c_L}{24}}} = v^{h_G^\vee - 1} \sum_{k=0}^\infty v^{2k} \chi_{k \cdot \theta_G}^G(\mathbf{m}_G), \tag{1.10}$$

where up to the overall factor of $v^{h_G^\vee - 1}$ the right hand side coincides with the Hilbert series of $\widetilde{\mathcal{M}}_{G,1}$ [139].

At the first excited level, if the 6d matter fields transform in a direct sum of representations

$$\bigoplus_{i=1}^r (R_i^G, R_i^F), \tag{1.11}$$

we find the following set of contributions:

$$\mathcal{E}_n^G(\mathbf{m}_G, \mathbf{m}_F, v, q)\Big|_{q^{-\frac{c_L}{24}+\frac{1}{2}}} = -v^{h_G^\vee - 1} \sum_{k=0}^\infty v^{2k+1} \sum_{i=1}^r \chi_{\lambda_i^G + k \cdot \theta_G}^G(\mathbf{m}_G) \chi_{\lambda_i^F}^F(\mathbf{m}_F), \tag{1.12}$$

where λ_i^G and λ_i^F are respectively the highest weights of the representations R_i^G and R_i^F, θ_G is the highest weight of the adjoint representation of G, and χ_λ^G, χ_λ^F are Lie algebra characters. Using the spectral flow (1.9), equations (1.10) and (1.12) determine an infinite number of coefficients in the Ramond-Ramond elliptic genus as well.

In section 7 we discuss in some detail the five-dimensional limit of the 6d Nekrasov partition function (1.1). Geometric considerations suggest three different behaviors according to whether $n \geq 3$, $n = 2$, or $n = 1$. For 6d SCFTs with $n \geq 3$ the five-dimensional limit we consider is a 5d $\mathcal{N} = 1$ theory with the gauge group and matter content obtained by the naive dimensional reduction of the 6d field content; for $n = 2$ one in addition finds free decoupled states (see also [140]); finally for $n = 1$ one simply obtains the 5d $\mathcal{N} = 1$ theory of one free hypermultiplet. This distinction between the three cases is reflected in the $\tau \to i\infty$ behavior of the Ramond-Ramond elliptic genus: for $n = 1$ at lowest energy one finds a

single state, while for $n \geq 2$ one obtains the one instanton piece $Z_{1\,\text{inst}}$ of the 5d Nekrasov partition function with same gauge symmetry and matter content as the 6d theory, plus additional extra states for $n = 2$. The $n \geq 2$ case is of particular interest, since for many of the 6d SCFTs that we study the 5d Nekrasov partition function is not known, and the elliptic genus of the BPS string can be used to obtain new information about $Z_{1\,\text{inst}}$ in 5d.

In section 8 we translate these features of the $\mathbf{h}_n^G$ CFTs into a series of constraints on their Ramond-Ramond elliptic genera. For convenience, we switch off the F and G fugacities $\mathbf{m}_F, \mathbf{m}_G$; then the elliptic genus can be expressed as the following meromorphic Jacobi form:

$$\mathbb{E}_n^G(v, q) = \frac{\mathcal{N}_n^G(v, q)}{\eta(q)^{12(n-2) - 4 + 24\,\delta_{n,1}} \varphi_{-2,1}(v^2, q)^{h_G^\vee - 1}}. \tag{1.13}$$

The numerator $\mathcal{N}_n^G(v, q)$ is now a holomorphic Jacobi form of even weight

$$m_n^G = 6n - 2h_G^\vee - 12 + 12\,\delta_{n,1} \quad \in \quad 2\,\mathbb{Z} \tag{1.14}$$

and index

$$k_n^G = n + 3\,h_G^\vee - 4 \quad \in \quad \mathbb{Z}_+ \tag{1.15}$$

with respect to ϵ_+. According to a structure theorem for the bi-graded ring of even weight holomorphic Jacobi forms, the vector space of forms of given weight and index is finite-dimensional and has a known basis, given by products of the standard Jacobi and modular forms $\varphi_{0,1}(v, q)$, $\varphi_{-2,1}(v, q)$, $E_4(q)$, and $E_6(q)$. We find that the constraints from section 6 are strong enough to uniquely determine the elliptic genus for 59 out of the 72 CFTs for which $\text{rank}(G) \leq 7$. In particular, we are able to compute the elliptic genus for a number of CFTs for which the elliptic genus was not previously known.

In section 9 we discuss two BPS string CFTs which fall outside of the general discussion of section 6: the theories $\mathbf{h}_2^{\text{SU}(2)}$ and $\mathbf{h}_3^{\text{SU}(3)}$, for which $n = h_G^\vee$ (the only other CFT belonging to this class is the E-string CFT $\mathbf{h}_1^{\text{Sp}(0)}$ which however is trivially given by the E_8 current algebra at level 1). For these theories the expansion (1.7) is not valid, but we are able to find simple alternative series expansions in v. Moreover, for these two CFTs the level of the Kac-Moody algebra implied from the anomaly polynomial is the critical level $k = -h_G^\vee$. The irreducible highest-weight modules of Kac-Moody algebras at critical level have markedly different structure from the noncritical case. Interestingly, for these two CFTs the G dependence of the elliptic genus does not seem to be captured in terms of the corresponding irreducible characters.

In section 10 we present our conclusions and formulate a number of questions which we leave to future research. The appendices are organized as follows: in appendix A we briefly review simple and affine Lie algebras and set up our notation. In section B we review the properties of modular and Jacobi forms we make use of in the main text. In appendix C we collect a list of references to other works where the elliptic genera of various BPS string CFTs are presented in a form which may be readily compared to our results. In appendix D we compute the elliptic genera of the strings of the $\mathbf{4}_{\text{SO}(2M+1)}$ SCFTs (with $M \geq 4$), which have not previously appeared in the literature. In appendix E we recall basic properties of

chiral WZW models that we make use of in the text. In appendix F we give a detailed description of the $\xi_\lambda^{n,G}$ functions that appear in equation (1.6) for a specific choice of n, G, and we also show how the form of these functions leads to constraints on the elliptic genus. In appendix G we provide extensive tables of coefficients of the series expansions of the elliptic genera which we determined by exploiting modularity in section 8. Finally, in appendix H we discuss the computation of the one-instanton component of 5d Nekrasov partition functions starting from the elliptic genera of the $\mathbf{h}_n^G$ theories, and provide our results for a number of theories with $n \geq 2$, several of which have not previously appeared in the literature.

2 Review of 6d rank-one SCFTs

In this section we review the geometric engineering of 6d SCFTs in the context of F-theory, highlighting a number of aspects that will be relevant to describing their BPS strings. We begin in section 2.1 with a broad overview of the geometric engineering of 6d SCFTs. In section 2.2 we discuss in more detail the example of 6d D-type conformal matter, whose BPS strings we will study in some detail in section 5. In section 2.3 we review how 6d anomaly polynomials are encoded in the F-theory geometry. Finally in section 2.4 we discuss in more detail the 6d SCFTs of rank 1 which are the focus of this paper.

2.1 Six-dimensional SCFTs from F-theory

Consider an F-theory compactification to six dimensions, defined by a Calabi-Yau threefold X that is elliptically fibered over a base B, which is a complex (Kähler) surface [141–145]. As we are interested in the geometric engineering of SCFTs decoupled from gravity, we consider a local models such that B has infinite volume. Moreover, any geometry corresponding to an SCFT has no scales in it, which leaves us with bases that typically have the form

$$B \simeq \mathbb{C}^2/\Gamma \,, \tag{2.1}$$

where Γ is a discrete subgroup of U(2) under whose action the only fixed point is the origin $(0,0) \in \mathbb{C}^2$. In case the elliptic fibration is trivial

$$X \simeq B \times T^2 \,, \tag{2.2}$$

the F-theory background has a perturbative type IIB interpretation, and the CY condition on X forces $\Gamma \subset$ SU(2). This background preserves a higher amount of supersymmetry, and the corresponding 6d SCFTs are the ADE $(2,0)$ theories. Otherwise, one obtains backgrounds preserving $(1,0)$ supersymmetry. In this case, the elliptic fibration can degenerate along a codimension-one locus in the base B, along a curve which is called the discriminant locus. The complex structure parameter τ of the elliptic fiber of X is interpreted in IIB as the axio-dilaton field, and a nontrivial discriminant signals that τ undergoes monodromies which are sourced by seven-branes; hence, the discriminant locus is interpreted in F-theory backgrounds as a curve of coalesced stacks of wrapped seven-branes.[5] Requiring a geometry that has no scales in it forces the possible components of the discriminant to be

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[5]In this paper we consider only singularities that are not frozen. Frozen singularities correspond to bound states of seven-branes that involve an $O7^+$ [129], see also the remark in footnote 3.

noncompact curves through the origin, which gives rise to (generalized) flavor symmetries for the SCFT. The noncompact flavor curves, being of codimension one in the F-theory base, support degenerate elliptic fibers that obey the Kodaira classification; this has to be contrasted with the behavior of the elliptic fiber at the origin, which is a codimension-two locus: there, the elliptic fiber can degenerate into a non-Kodaira type fiber.

To determine the tensor branch geometry, one resolves the origin of the base B by successive blow-ups until no further codimension-two components in the discriminant support singularities of non-Kodaira type. Indeed, a non-Kodaira singularity in codimension-two signals the presence of tensionless strings in the geometry, corresponding to a partially unresolved tensor branch, which can be resolved by further blow-ups. By this process of repeatedly blowing up one obtains a curve Σ which can have several compact irreducible rational components intersecting transversally; we denote these components by Σ_I, with $I = 1, \ldots, R$, where R is the rank of the corresponding tensor branch. This can be understood by considering the corresponding IIB reduction. Let ω_I be harmonic forms such that

$$\int_{\Sigma_J} \omega_I = \delta_{IJ}. \tag{2.3}$$

In the decomposition of the IIB RR potential

$$C_4^+ = \sum_{I=1}^{R} B^I \wedge \omega_I \tag{2.4}$$

one obtains R anti-self-dual two-forms B^I that are part of the 6d $\mathcal{N} = (1,0)$ tensor multiplets. The scalar components ϕ^I of the tensor multiplets correspond to the periods of the Kähler form of $\widehat{B}$:

$$J = \sum_{I=1}^{R} \phi^I \omega_I. \tag{2.5}$$

Hence the vacuum expectation values of the scalar fields ϕ^I correspond to the Kähler moduli given by the volumes of the rational curves Σ_I. In particular, the superconformal fixed point at the origin of the tensor branch is attained by setting all such volumes to zero, shrinking the curve Σ to a point. This is possible iff $\Sigma \cdot \Sigma < 0$ by the Artin-Grauert criterion [146, 147]. Whenever the curve has multiple irreducible components, this implies that the matrix

$$A_{IJ} \equiv -\Sigma_I \cdot \Sigma_J \tag{2.6}$$

is positive definite. This is not a surprise because the matrix in equation (2.6) gives the kinetic terms for the effective action of the scalars along the tensor branch $A_{IJ} \partial_\mu \phi^I \partial^\mu \phi^J$.

Along the tensor branch, we can use the standard dictionary relating Kodaira singularities to coalesced seven-brane stacks. To make this dictionary explicit in the discussion that follows, we are going to assume that the elliptic fibration of the resolved CY $\widehat{X}$ has a section.[6] In this case, the elliptic fibration can be described by a local Weierstrass model

$$y^2 = x^3 + fx + g, \tag{2.7}$$

---

[6]This is not strictly necessary for general F-theory engineerings of six-dimensional models, see [148]. For some recent application to six-dimensional models decoupled from gravity see e.g. [22, 55, 59, 149].

ord(f)	ord(g)	ord(Δ)	type	singularity	non-abelian algebra
≥ 0	≥ 0	0	I_0	none	none
0	0	1	I_1	none	none
0	0	$n \geq 2$	I_n	A_{n-1}	su_n or $sp_{[n/2]}$
≥ 1	1	2	II	none	none
1	≥ 2	3	III	A_1	su_2
≥ 2	2	4	IV	A_2	su_3 or su_2
≥ 2	≥ 3	6	I_0^*	D_4	so_8 or so_7 or g_2
2	3	$n \geq 7$	I_{n-6}^*	D_{n-2}	so_{2n-4} or so_{2n-5}
≥ 3	4	8	IV*	E_6	e_6 or f_4
3	≥ 5	9	III*	E_7	e_7
≥ 4	5	10	II*	E_8	e_8
≥ 4	≥ 6	≥ 12	non-Kodaira	-	-

Type	Monodromy Cover equation
I_m, $m \geq 3$	$\psi^2 + (9g/2f)\vert_{z=0}$
IV	$\psi^2 - (g/z^2)\vert_{z=0}$
I_0^*	$\psi^3 + (f/z^2)\vert_{z=0} \cdot \psi + (g/z^3)\vert_{z=0}$
I_{2n-5}^*, $n \geq 3$	$\psi^2 + \frac{1}{4}(\Delta/z^{2n+1})(2zf/9g)^3\vert_{z=0}$
I_{2n-4}^*, $n \geq 3$	$\psi^2 + (\Delta/z^{2n+2})(2zf/9g)^2\vert_{z=0}$
IV*	$\psi^2 - (g/z^4)\vert_{z=0}$

Table 2. *Top:* summary of Kodaira singularities and corresponding non-abelian gauge algebras for F-theory seven-branes. *Bottom:* monodromy covers for Σ using adapted coordinates in which Σ is the locus $\{z = 0\}$ in the F-theory base.

where f and g are local functions on $\widehat{B}$ that globally are sections respectively of $-4K$ and $-6K$, K being the canonical bundle of $\widehat{B}$. The discriminant of the elliptic fibration is the local function

$$\Delta = 4f^3 + 27g^2 \tag{2.8}$$

which is a section of $-12K$. By an abuse of notation, we also denote by Δ the discriminant locus, which is the curve $\Delta = 0$. If the discriminant Δ has several irreducible components Δ_α, such that the order of vanishing of Δ along such irreducible components is

$$\mathrm{ord}(\Delta\big|_{\Delta_\alpha}) = N_\alpha > 0\,, \tag{2.9}$$

this signals that a configuration of seven-branes with RR charge N_α is wrapped on the curve Δ_α. Since the base $\widehat{B}$ is noncompact there are two possibilites: the curve Δ_α itself can be either compact or noncompact. In the latter case the divisor is a flavor divisor, corresponding to a global symmetry. In the former case, the corresponding stack of seven-branes contributes a gauge sector to the model with coupling

$$1/g_\alpha^2 \sim \mathrm{vol}(\Delta_\alpha). \tag{2.10}$$

The gauge and global symmetries corresponding to the various components of Δ_α are determined by the structure of the corresponding Kodaira singularity (see the top part of table 2). In particular, the Calabi-Yau condition on $\widehat{X}$ is satisfied provided that

$$c_1(B) = -\frac{1}{12} \sum_\alpha N_\alpha \, \delta(\Delta_\alpha)\,, \tag{2.11}$$

in obvious notation. The irreducible compact components Σ_I of Σ that are part of the discriminant correspond to gauge theory subsectors of the model; we denote by $\mathfrak{g}_{\Sigma_I}$ the Lie algebra of the corresponding gauge group. This is computed in terms of two pieces of data: the first is the order of vanishing of (f, g, Δ) along Σ_I, and the second is the monodromy of the fibration, which determines which gauge algebra occurs for a given singularity type, according to Tate's algorithm. Tate's algorithm assigns to each curve Σ_I a monodromy cover which is captured by an equation in an auxiliary variable ψ, valued in a line bundle over Σ_I. Explicit equations for these monodromy covers are given in table 2: *Bottom*, where we have adapted locally the base coordinates so that Σ_I is identified with locus $z = 0$. For all cases except I_0^*, the equation of the monodromy cover takes the form

$$\psi^2 - P(f,g,z) = 0\,, \tag{2.12}$$

where $P(f, g, z)$ is a Laurent polynomial in f, g and z. The cover splits (leading to no monodromy) if P is a perfect square. In the I_0^* case, the monodromy cover equation defines a degree 3 cover of Σ_I, and one must analyze this system further to determine whether the cover is irreducible ($\mathfrak{g}_\Sigma = g_2$), splits into two components ($\mathfrak{g}_\Sigma = so(7)$), or splits into three components ($\mathfrak{g}_\Sigma = so(8)$).[7]

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[7]We are being very explicit here since later on we will be interpreting these geometric structures in terms of surface defects of the theory describing D3-branes wrapped on two-cycles in $\widehat{B}$.
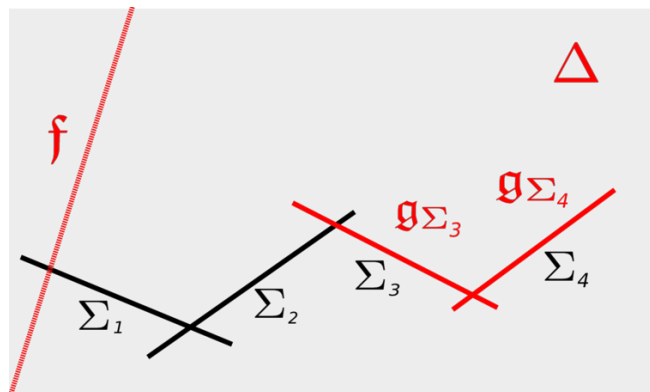
Figure 1. Schematic description of the geometric engineering of the tensor branch of a 6d SCFT in F-theory. The rational curves Σ_I are in one-to-one correspondence with tensor multiplets, whose scalars' vacuum expectation values coincide with vol(Σ_I). In red we have depicted the discriminant locus, corresponding to the divisor Δ of the F-theory base. In this example, we have a flavor divisor intersecting the curve Σ_1, carrying flavor symmetry $\mathfrak{f}$, while the compact curves Σ_3 and Σ_4 both support non-abelian gauge groups.

A schematic description of the typical geometries for the 6d tensor branches can be found in figure 1. The Calabi-Yau condition on $\widehat{X}$ gives several restrictions both at the level of the base geometries and at the level of the corresponding elliptic fibrations. In particular, the base geometries have to satisfy [2]

1.) For all $I = 1, \ldots, R$ the self-intersection numbers are bounded:

$$1 \leq -\Sigma_I \cdot \Sigma_I \leq 12 \,; \tag{2.13}$$

2.) Whenever for some $I \in \{1, \ldots, R\}$

$$-\Sigma_I \cdot \Sigma_I \geq 3 \,, \tag{2.14}$$

the rational curve Σ_I is forced to be part of the discriminant by the CY condition;

3.) Whenever

$$\Sigma_I \cdot \Sigma_J \neq 0, \tag{2.15}$$

the self intersections of Σ_I and Σ_J are constrained; for instance, compact rational curves with self-intersection smaller than 3 never intersect in $\widehat{B}$.[8]

By property 2.), the elliptic fibration has to degenerate along the curves with self-intersections smaller than -3. For each such rational curve the minimal degeneration gives the gauge algebras in table 3, that can be further enhanced by tuning the Weierstrass model [150]. In particular, for a given base geometry, characterized by a collection of compact rational curves with intersection matrix A_{IJ}, we have a plethora of possible compatible F-theory models, each characterized by a different structure of the discriminant.

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[8]The precise rules for composition of the bases can be found in [2, 10]. We do not dwell on these details here because our focus in this work are the rank-one theories.

$-\Sigma_I \cdot \Sigma_I$	3	4	5	6	7 − 8	9 − 12
Kodaira type	IV	I_0^*	IV_{ns}^*	IV^*	III^*	II^*
gauge symmetry	$\mathfrak{su}_3$	$\mathfrak{so}_8$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Table 3. Minimal gauge algebras for rational curves with $-\Sigma_I \cdot \Sigma_I \geq 3$. For the rational curve with $-\Sigma_I \cdot \Sigma_I = 7$ the theory also has matter in the $\frac{1}{2}\mathbf{56}$, while for the rational curves with $-\Sigma_I \cdot \Sigma_I = 12 - k$ and $k = 1, 2, 3$ one has k additional E-string subsectors.
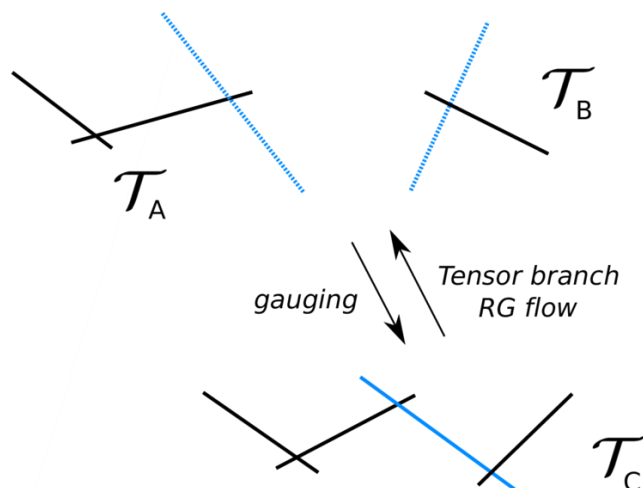
Figure 2. Schematic description of the gauging of a (generalized) flavor symmetry in 6d SCFTs. Two models $\mathcal{T}_A$ and $\mathcal{T}_B$ that share the same flavor divisor can give rise to a consistent theory $\mathcal{T}_C$ by making the corresponding noncompact flavor curve (dotted blue line in the figure) into a compact one.

These models have to satisfy further consistency conditions dictated by the geometry of $\widehat{X}$. The resulting geometries are related to each other by Higgs branch RG flows, which amount to deformations of the complex structure of $\widehat{X}$ (that can be translated into deformations of the complex structure of X). The effect of such Higgs branch RG flows is to make the singularities of the elliptic fiber less and less severe. In IIB language, the Higgs branch deformations correspond to separating the stacks of coalesced seven-branes and sending some at infinity, thus lowering their effective number. At the bottom of a Higgs deformation chain sits a minimal model that is characterized solely by the intersection matrix A_{IJ}, which automatically determines the corresponding minimal gauge algebras and matter content as in table 3. At the intersection of the curves that are part of the discriminant the corresponding singularity enhances in codimension two, giving rise to matter charged under the gauge groups supported by the curves in the base geometry [151].

Another salient aspect of the base geometries is that they give a natural geometric mechanism for gauging a 6d flavor symmetry: gauging simply amounts to making a flavor divisor compact, as illustrated in figure 2. The precise restrictions imposed by the geometries on this type of mechanism are implicit in the classification result. At any

rate, the lesson one learns from this mechanism is that all higher rank 6d SCFTs obtained from F-theory (without frozen singularities) can be obtained by coupling together rank-one building blocks. Below we will see the counterpart of this phenomenon at the level of the BPS strings of these models.

2.2 Example: conformal matter of D-type

Let us discuss now a simple example corresponding to the rank-one conformal matter of D-type. In this case the base B is smooth, and we have a Weierstrass model for X that in a neighborhood of the origin is given by

$$f(s,t) = -3\alpha^2 s^2 t^2 \qquad g(s,t) = (2\alpha^3 + \beta s^{k-6} t^{k-6}) s^3 t^3 \qquad k \geq 6 \,, \tag{2.16}$$

where $\alpha, \beta \in \mathbb{C}$. The corresponding discriminant is given by

$$\Delta = -432 \, \beta s^{k-6} t^{k-6} (4\alpha^3 s^6 t^6 + \beta s^k t^k). \tag{2.17}$$

We clearly see that we have a pair of noncompact flavor curves given by $D_s \equiv \{t = 0\}$ and $D_t \equiv \{s = 0\}$ that intersect at the origin. These are part of the discriminant locus and support singularities of type I^*_{k-6} (where $k \geq 6$). This can be easily seen because the order of vanishing

$$\mathrm{ord}(f, g, \Delta)\big|_{D_s} = (2, 3, k-6) = \mathrm{ord}(f, g, \Delta)\big|_{D_t} \tag{2.18}$$

away from the origin, and in each case the monodromy cover of table 2 factorizes, which corresponds to a singularity of split type. The origin $D_s \cdot D_t$, however, has a non-Kodaira type singularity, where

$$\mathrm{ord}(f, g, \Delta)\big|_{D_s \cdot D_t} = (4, 6, 2k) \,. \tag{2.19}$$

By a single blow-up, we obtain an exceptional rational curve Σ with self intersection $-\Sigma \cdot \Sigma = 1$, along which the order of vanishing of (f, g, Δ) is $(0, 0, 2k - 12)$. The corresponding monodromy cover in this case is non-split, and we obtain an I^{ns}_{2k-12} fiber, corresponding to the gauge algebra $\mathfrak{sp}_{k-6}$. Higgsing in this case corresponds to deforming the polynomial for $g(s,t)$ in equation (2.16) in such a way that $k \mapsto k-1$. The endpoint for this chain of Higgsings is the E-string theory, with trivial gauge algebra $\mathfrak{sp}_0$.

Now consider taking two of these models and compactifying a common flavor divisor with gauge algebra $\mathfrak{so}_{8+k}$ corresponding to the I^*_{k-6} fiber. The corresponding intersection pairing is easily determined from the matter content of these models by requiring gauge anomaly cancellation (see discussion below). We obtain

$$A_{IJ} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix} \tag{2.20}$$

where

$$\mathfrak{g}_{\Sigma_1} = \mathfrak{g}_{\Sigma_3} = \mathfrak{sp}_k \qquad \mathfrak{g}_{\Sigma_2} = \mathfrak{so}_{8+k}. \tag{2.21}$$

Blowing such rational curves down, one obtains a Weierstrass model over a singular base. In this case, the base is no longer $\mathbb{C}^2$, but rather $\mathbb{C}^2/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on (s, t) as the cyclic

discrete subgroup of SU(2). Proceeding similarly, considering N rank one D-type theories one obtains a Weierstrass model over $\mathbb{C}^2/\mathbb{Z}_N$, where $\mathbb{Z}_N$ acts as

$$(s,t) \to (\omega s, \omega^{-1} t) \qquad \omega^N = 1. \tag{2.22}$$

One straightforwardly sees that the Weierstrass model in equation (2.16) is compatible with this action. To proceed, one first has to blow-up the base, giving rise to a collection of -2 curves intersecting transverally according to an A-type Dynkin graph; at each intersection, one finds a local model of the type discussed above, which can then be resolved along similar lines [5].

2.3 Anomaly polynomials

It is well known that there is a deep interplay between the intersection theory of the F-theory base and anomaly cancellation in 6d [130, 152, 153]. The invariant field strengths for the tensor fields B^I in the theory are given by

$$H^I = dB^I + L_3^I \tag{2.23}$$

where L_3^I is a Chern-Simons three-form associated to the 4-form

$$dL_3^I = X_4^I = \frac{1}{4}\alpha^I p_1(M_6) + \sum_\ell \beta_\ell^I c_2(\mathfrak{g}_\ell) + \frac{1}{2}\sum_{ij} \beta_{ij}^I c_1(\mathrm{U}(1)^i)c_1(\mathrm{U}(1)^j); \tag{2.24}$$

L_3^I can be interpreted as an effective degree-3 abelian gauge field with field strength X_4^I, coupled to the tensor multiplets through the Green-Schwarz terms. Moreover, $p_1(M_6)$ is the first Pontryagin class of the tangent bundle of the 6d worldvolume M_6, while $c_1(\mathrm{U}(1)^i)$ and $c_2(\mathfrak{g}_\ell)$ are respectively the first Chern class of the i-th U(1) bundle and the second Chern class associated to $\mathfrak{g}_\ell$, i.e. a degree-4 characteristic form associated to the quadratic Casimir of $\mathfrak{g}_\ell$. The vectors $\mathbf{a}$, $\mathbf{b}_\ell$, $\mathbf{b}_{ij}$ with components respectively α^I, β_ℓ^I and β_{ij}^I are the anomaly coefficients of the theory; $\mathbf{a}$ is a gravitational anomaly coefficient, while $\mathbf{b}_\ell$ and $\mathbf{b}_{ij}$ are respectively non-abelian and abelian anomaly coefficients. It is convenient to group the X_4^I into a vector of four-forms $\mathbf{X}_4$.

Let us first consider the case of a compact elliptic threefold. In order for the local gravitational, gauge and mixed anomalies of the theory to be canceled via the Green-Schwarz mechanism, the following anomaly cancellation condition must be satisfied [153–155]: the anomaly polynomial $\mathcal{A}_8$ has to factor:

$$\mathcal{A}_8 = \frac{1}{2}\mathbf{X}_4 \cdot \mathbf{X}_4 = \frac{1}{2}A_{IJ} X_4^I \wedge X_4^J, \tag{2.25}$$

where A_{IJ} is the intersection pairing. If this is the case, the gauge and gravitational anomalies can be cured with local counterterms of the form $B_I \wedge X_4^I$ in the six-dimensional effective action. The cancellation conditions that do not involve the abelian anomaly coefficients are as follows. The cancellation of the gravitational anomaly requires

$$0 = H - V + 29T - 273 \qquad \mathbf{a} \cdot \mathbf{a} = 9 - T, \tag{2.26}$$

$\mathfrak{g}$	R	$\mathrm{tr}_R F^2$	$\mathrm{tr}_R F^4$
$\mathfrak{su}_N$	Adj	$2N\,\mathrm{tr}F^2$	$(6+N)(\,\mathrm{tr}F^2)^2$
$N=2,3$	Λ	$\mathrm{tr}F^2$	$\frac{1}{2}(\,\mathrm{tr}F^2)^2$
$\mathfrak{su}_N$	Adj	$2N\,\mathrm{tr}F^2$	$6(\,\mathrm{tr}F^2)^2 + 2N\,\mathrm{tr}F^4$
$N\geq 4$	Λ	$\mathrm{tr}F^2$	$0(\,\mathrm{tr}F^2)^2 + \,\mathrm{tr}F^4$
	Λ^2	$(N-2)\,\mathrm{tr}F^2$	$3(\,\mathrm{tr}F^2)^2 + (N-8)\,\mathrm{tr}F^4$
$\mathfrak{su}_6$	Λ^3	$6\,\mathrm{tr}F^2$	$6(\,\mathrm{tr}F^2)^2 - 6\,\mathrm{tr}F^4$
$\mathfrak{so}_8$	Adj	$12\,\mathrm{tr}F^2$	$4\,\mathrm{tr}_V F^4 + 4\,\mathrm{tr}_{S_+} F^4 + 4\,\mathrm{tr}_{S_-} F^4$
	$\Lambda = V, S_+, S_-$	$2\ \mathrm{tr}_\Lambda F^2$	$\sum_{\Lambda=V,S_+,S_-}\,\mathrm{tr}_\Lambda F^4 = 3(\,\mathrm{tr}F^2)^2$
$\mathfrak{so}_N$	Adj	$(2N-4)\,\mathrm{tr}F^2$	$12(\,\mathrm{tr}F^2)^2 + (2N-16)\,\mathrm{tr}F^4$
$N\geq 7$	V	$2\,\mathrm{tr}F^2$	$0(\,\mathrm{tr}F^2)^2 + 2\,\mathrm{tr}F^4$
$N\neq 8$	S_*	$\dim(S_*)(\frac{1}{4}\,\mathrm{tr}F^2)$	$\dim(S_*)(\frac{3}{16}(\,\mathrm{tr}F^2)^2 - \frac{1}{8}\,\mathrm{tr}F^4)$
$\mathfrak{sp}_N$	Adj	$(2N+2)\,\mathrm{tr}F^2$	$3(\,\mathrm{tr}F^2)^2 + (2N+8)\,\mathrm{tr}F^4$
$N\geq 2$	Λ	$\mathrm{tr}F^2$	$0(\,\mathrm{tr}F^2)^2 + \,\mathrm{tr}F^4$
	Λ^2_{irr}	$(2N-2)\,\mathrm{tr}F^2$	$3(\,\mathrm{tr}F^2)^2 + (2N-8)\,\mathrm{tr}F^4$
$\mathfrak{g}_2$	Adj	$8\,\mathrm{tr}F^2$	$10(\,\mathrm{tr}F^2)^2$
	$\mathbf{7}$	$2\,\mathrm{tr}F^2$	$(\,\mathrm{tr}F^2)^2$
$\mathfrak{f}_4$	Adj	$18\,\mathrm{tr}F^2$	$15(\,\mathrm{tr}F^2)^2$
	$\mathbf{26}$	$6\,\mathrm{tr}F^2$	$3(\,\mathrm{tr}F^2)^2$
$\mathfrak{e}_6$	Adj	$24\,\mathrm{tr}F^2$	$18(\,\mathrm{tr}F^2)^2$
	$\mathbf{27}$	$6\,\mathrm{tr}F^2$	$3(\,\mathrm{tr}F^2)^2$
$\mathfrak{e}_7$	Adj	$36\,\mathrm{tr}F^2$	$24(\,\mathrm{tr}F^2)^2$
	$\mathbf{56}$	$12\,\mathrm{tr}F^2$	$6(\,\mathrm{tr}F^2)^2$
$\mathfrak{e}_8$	Adj	$60\,\mathrm{tr}F^2$	$36(\,\mathrm{tr}F^2)^2$

Table 4. Anomaly coefficients in our normalization [130, 152].

where H is the total number of hypermultiplets in the theory, V is the total number of vector multiplets, and T is the total number of tensor multiplets. The mixed gauge-gravitational anomaly gives:[9]

$$\mathbf{a}\cdot\mathbf{b}_\ell = \frac{1}{6}\left(-A_{\mathrm{Adj}_\ell} + \sum_{R_\ell} n^\ell_{R_\ell} A_{R_\ell}\right). \qquad (2.27)$$

---

[9]We are adopting the same conventions as section 9 of [152] with $\mathbf{a}|_{\mathrm{here}} = -K_B|_{\mathrm{there}}$ and $\mathbf{b}|_{\mathrm{here}} = \Sigma|_{\mathrm{there}}$.

The cancellation of non-abelian gauge anomalies requires the following constraint on the matter representations

$$0 = B_{\mathrm{Adj}_\ell} - \sum_{R_\ell} n_{R_\ell}^\ell B_{R_\ell}, \tag{2.28}$$

and moreover

$$\mathbf{b}_\ell \cdot \mathbf{b}_\ell = \frac{1}{3}\left(-C_{\mathrm{Adj}_\ell} + \sum_{R_\ell} x_{R_\ell}^\ell C_{R_\ell}\right) \quad \text{and} \quad \mathbf{b}_\ell \cdot \mathbf{b}_{\ell'} = \sum_{R_\ell, R_{\ell'}} n_{R_\ell, R'_\ell}^{\ell,\ell'} A_{R_\ell} A_{R_{\ell'}} \ (\ell \neq \ell') \tag{2.29}$$

need to be satisfied. The anomaly cancellation conditions that depend on the abelian anomaly coefficients are the following:

$$\mathbf{a} \cdot \mathbf{b}_{ij} = \frac{1}{6}\sum_{R_\ell} n_{q_i,q_j}^{i,j} q_i q_j \tag{2.30}$$

$$0 = \sum_{R_\ell, q_i} n_{R_\ell, q_i}^{\ell, i} q_i E_{R_\ell}, \tag{2.31}$$

$$\mathbf{b}_\ell \cdot \mathbf{b}_{ij} = \sum_{R_\ell q_i, q_j} n_{R_\ell, q_i, q_j}^{\ell, i, j} q_i q_j A_{R_\ell}, \tag{2.32}$$

$$\mathbf{b}_{ij} \cdot \mathbf{b}_{kp} + \mathbf{b}_{ip} \cdot \mathbf{b}_{kj} + \mathbf{b}_{ik} \cdot \mathbf{b}_{jp} = \sum_{q_i,q_j,q_k,q_p} n_{q_i,q_j,q_k,q_p}^{i,j,k,p} q_i q_j q_k q_p. \tag{2.33}$$

The Lie algebra coefficients A, B, C and E are defined by

$$\mathrm{tr}_{R_\ell} F^2 = A_{R_\ell} \mathrm{tr} F^2, \quad \mathrm{tr}_{R_\ell} F^4 = B_{R_\ell} \mathrm{tr} F^4 + C_{R_\ell} (\mathrm{tr} F^2)^2, \quad \mathrm{tr}_{R_\ell} F^3 = E_{R_\ell} \mathrm{tr} F^3 \tag{2.34}$$

for $F \in \mathfrak{g}_i$. For $\mathfrak{g}_\ell$ with rank ≤ 2, B_{R_i} is defined to vanish for any R_i. The coefficients $n_{R_{\ell_1}, \cdots, R_{\ell_k}, q_{i_1}, \cdots, q_{i_K}}^{\ell_1, \cdots, \ell_k, i_1, \cdots, i_K}$ are defined to be the number of hypermultiplets simultaneously in representation R_{ℓ_1} of $\mathfrak{g}_{\ell_1}$, ..., R_{ℓ_k} of $\mathfrak{g}_{\ell_k}$, q_{i_1} of $u(1)_{i_1}$, ... and q_{i_K} of $u(1)_{i_K}$. In table 4 we give the conventions we use to compute all possible traces needed for this paper [130, 152, 153].

The link to the algebraic geometry of the F-theory base is as follows [130, 152, 153]: the $\mathbf{a}$ coefficient can be identified with the class $-K$, while the $\mathbf{b}_\ell$ anomaly coefficients can be identified with the curve classes Σ_I.[10] On the other hand, the U(1) anomaly coefficients can be identified with appropriate curve classes in the F-theory base (see e.g. [59] for a detailed review).

Decoupling gravity amounts to taking the volume of the F-theory base to infinity, and focusing on a subset of curves which remains compact, see e.g. the discussions in [9, 22, 59]. This results in some curves becoming noncompact; the noncompact curves which intersect the compact ones are interpreted as flavor divisors. Note that there is no known six-dimensional theory with a U(1) gauge symmetry, but there are several examples that have U(1) global symmetries. Along the way all of the above conditions can be dropped except for the subset of equations (2.28) and (2.29) that corresponds to gauge anomaly cancellation. Nevertheless the anomaly coefficients $\mathbf{a}$, as well as the ones corresponding to flavor divisors, give rise to 't Hooft anomalies for our models, which can be determined by using the second equation in (2.29) as well as equation (2.32).

---

[10]Here we are assuming that all the curves Σ_I belong to the discriminant, which is the case for all the rank-one models that appear in this paper except for the E-string theory.

$HJ_{n,1}$	$HJ_{1,1}$	$HJ_{2,1}$	$HJ_{3,1}$	$HJ_{4,1}$	$HJ_{5,1}$	$HJ_{6,1}$	$HJ_{7,1}$	$HJ_{8,1}$	$HJ_{12,1}$
fiber	I_0	I_1	IV	I_0^*	IV_{ns}^*	IV^*	III^*	III^*	II^*
$\mathfrak{g}_{\min}$	none	none	$\mathfrak{su}_3$	$\mathfrak{so}_8$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7 \oplus \frac{1}{2}\mathbf{56}$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Table 5. $HJ_{p,q}$ singularities and corresponding minimal gauge groups for rank one 6d SCFTs.

2.4 Rank one 6d SCFTs and their Higgsing trees

There are only nine F-theory bases that can give rise to rank-one SCFTs. These are given by Hirzebruch-Jung singularities, denoted by $HJ_{p,q}$, that can be resolved by blowing up the origin leading to a single rational curve $\Sigma \subset \widehat{B}$. Recall that a singularity of type $HJ_{p,q}$ is the orbifold of $\mathbb{C}^2$

$$HJ_{p,q} : \qquad (z_1, z_2) \to (\omega z_1, \omega^q z_2) \qquad \omega^p = 1. \qquad (2.35)$$

The rank one theories correspond to Hirzebruch-Jung orbifold singularities of type

$$(p,q) = (n,1) \quad \text{with} \quad n = 1, 2, 3, 4, 5, 6, 7, 8, 12. \qquad (2.36)$$

Performing a single blow up in the base results in a single rational curve Σ of self-intersection

$$\Sigma \cdot \Sigma = -n. \qquad (2.37)$$

The resolved base is

$$\widehat{B} = \text{Tot}\left(\mathcal{O}(-n) \to \mathbb{P}^1\right), \qquad (2.38)$$

where the Kähler class of the base $\mathbb{P}^1$ corresponds to the vev of the tensor multiplet scalar parametrizing the 6d tensorial Coulomb branch. For $n > 2$, all these curves are forced to be part of the discriminant for $\widehat{X}$ and have minimal singularities associated to them which we list in table 5. For a generic F-theory model, the curves with $n = 1, 2$ in the base are not necessarily part of the discriminant. For 6d SCFTs with $(1,0)$ supersymmetry, the situation is slightly more constrained. Indeed, the minimal singularity associated to $n = 2$ corresponds to the M-string theory is a type I_1 fiber. On the other hand, in the minimal case for $n = 1$ the generic fiber on the $\mathbb{P}^1$ is nondegenerate, corresponding to the E-string CFT. By tuning the corresponding Weierstrass models, one can obtain higher Kodaira singularities for each of these bases, giving rise to higher rank gauge symmetries at the price of introducing flavor divisors. At the locus where a flavor divisor meets a gauge divisor, the singularity of the fiber enhances and one obtains localized matter.

We introduce the following special notation for rank one models:

$$\mathbf{n}_G^F \qquad (2.39)$$

where $\mathbf{n}$ refers to the Hirzebruch-Jung singularity type for the F-theory base, G labels the gauge group corresponding to the seven-brane stack wrapping the resolved base curve $\Sigma \subset \widehat{B}$, whose self intersection is n, and F is the flavor symmetry for the corresponding model. In particular, in our notation the M-string corresponds to the model $\mathbf{2}_{\mathrm{SU}(1)}^{\mathrm{SU}(2)}$, while the E-string

n	G_Σ	R_Σ	F_Σ
1,2	SU(2)	$(32 - 12n)\frac{1}{2}\Lambda$	SO(32 − 12n)
1,2,3	SU(3)	$(18 - 6n)\Lambda$	SU(18 − 6n)
1,2	SU(4)	$(16 - 4n)\Lambda \oplus (2 - n)\Lambda^2$	SU(16 − 4n) ⊗ Sp(2 − n)
1,2	SU(5)	$(16 - 3n)\Lambda \oplus (2 - n)\Lambda^2$	SU(16 − 3n) ⊗ SU(2 − n) ⊗ U(1)$^{2-n}$
1,2	SU(6)	$(16 - 2n)\Lambda \oplus (2 - n)\Lambda^2$	SU(16 − 2n) ⊗ SU(2 − n) ⊗ U(1)$^{2-n}$
1	SU(6)$_*$	$(16 - n)\Lambda \oplus \frac{1}{2}(2 - n)\Lambda^3$	SU(16 − n) ⊗ SO(2 − n)
1,2	SU(N), $N \geq 7$	$(16 + (N - 8)n)\Lambda \oplus (2 - n)\Lambda^2$	SU(16+(N − 8)n)⊗SU(2−n)⊗U(1)$^{2-n}$
1	Sp(N), $N \geq 2$	$(16 + 4N)\frac{1}{2}\Lambda$	SO(16 + 4N)
1,2,3	SO(7)	$(3 - n)V \oplus 2(4 - n)S$	Sp(3 − n) ⊗ Sp(8 − 2n)
1,2,3,4	SO(8)	$(4 - n)(V \oplus S_+ \oplus S_-)$	Sp(4 − n) ⊗ Sp(4 − n) ⊗ Sp(4 − n)
1,2,3,4	SO(9)	$(5 - n)V \oplus (4 - n)S$	Sp(5 − n) ⊗ Sp(4 − n)
1,2,3,4	SO(10)	$(6 - n)V \oplus (4 - n)S_+$	Sp(6 − n) ⊗ SU(4 − n) ⊗ U(1)
1,2,3,4	SO(11)	$(7 - n)V \oplus (4 - n)\frac{1}{2}S$	Sp(7 − n) ⊗ SO(4 − n)
1,2,3,4	SO(12)	$(8 - n)V \oplus (4 - n)\frac{1}{2}S_\pm$	Sp(8 − n) ⊗ SO(4 − n)
2,4	SO(13)	$(9 - n)V \oplus (2 - \frac{1}{2}n)\frac{1}{2}S$	Sp(9 − n) ⊗ SO(2 − $\frac{1}{2}$n)
4	SO(N), $N \geq 14$	$(n - 8)V$	Sp(n − 8)
1,2,3	G_2	$(10 - 3n)\mathbf{7}$	Sp(10 − 3n)
1…5	F_4	$(5 - n)\mathbf{26}$	Sp(5 − n)
1…6	E_6	$(6 - n)\mathbf{27}$	SU(6 − n) ⊗ U(1)
1…8	E_7	$(8 - n)\frac{1}{2}\mathbf{56}$	SO(8 − n)
12	E_8	none	none

Table 6. Global symmetries and matter of rank one 6D SCFTs. For SU(N) and Sp(N), Λ is the fundamental representation, and Λ^k its exterior powers. For SO(N), V denotes the vector representation; for N odd, S denotes the spinor representation, while for N even we denote respectively by S_+ and S_- the spinor and conjugate spinor representations. Recall that k fundamentals in the SU(N) case contribute a factor of SU(k) to the flavor symmetry group, instead of U(k), see e.g. [73, 156]. There are two theories with $G = $ SU(6); we use the terminology $G = $ SU(6)$_*$ to distinguish the one with $F_\Sigma = $ SU(15) from the other.

is the model $\mathbf{1}^{E_8}_{\mathrm{Sp}(0)}$. The flavor symmetry for a given SCFT is encoded by the matter content of the corresponding model, which is in turn uniquely determined (with a few exceptions) by the requirement of anomaly cancellation once n and G are specified. Therefore we often drop the flavor symmetry label. We refer the reader to table 6 for a summary of the rank one models and their flavor symmetries [157]. In consulting the table, it is useful to recall that N hypermultiplets in a complex representation of the gauge group support an U(N) symmetry group, which is enhanced to SO($2N$) for quaternionic (i.e., pseudoreal) representations, and to Sp(N) for real representations. In the quaternionic case, the underlying complex representation is a representation by half-hypermultiplets, so that N is allowed to be a half-integer. The fundamental (resp. anti-symmetric) representation of SU(N) is quaternionic

for $N = 2$ and is complex for $N > 2$ (resp. real for $N = 4$ and complex for $N > 4$); the fundamental representation of Sp(N) is always quaternionic; the spinor representations of SO(N) are real for $N = 0, 1, 7 \mod 8$, complex for $N = 2, 6 \mod 8$, and quaternionic for $N = 3, 4, 5 \mod 8$. It is interesting to remark that table 6 respects the exceptional isomorphisms of Lie algebras [157]. Extending the SO(N) formula to the representation of SO(6), one obtains $(2 - n)V + 4(4 - n)S_{\pm}$, which agrees with the result for SU(4) as $(V, S_{\pm}) \simeq (\Lambda^2, \Lambda)$. Similarly, for SO(5), one obtains $(1 - n)V + 8(4 - n)\frac{1}{2}S$, which implies that $n = 1$ and V does not occur; this agrees with the Sp(2) case as $S \simeq \Lambda$.

For the theories $\mathbf{n}_{\mathrm{SO}(12)}$ with $\mathbf{n} = 1, 2$ additional remarks are in order. In these cases the matter sector includes respectively 3 or 2 half-spinors. Recall that spinors of different chirality are Casimir equivalent, so one may entertain making different choices of spinorial matter. Consider the case $\mathbf{n} = 1$ first; choosing the spinorial matter to be in the $\frac{3}{2}S_+$ or $\frac{3}{2}S_-$ representation gives rise to a flavor symmetry SO(3), while the choices $\frac{1}{2}S_+ \oplus S_-$ and $S_+ \oplus \frac{1}{2}S_-$ give SO(1) × SO(2) $\simeq$ U(1) flavor symmetry; these choices lead naively to different 6d SCFTs, as the flavor symmetry groups are different. In later sections we also use the terminology SO(12)$_a$ and SO(12)$_b$ to distinguish between the $G = $ SO(12) theories with $F = $ SO(3) and $F = $ SO(1) × SO(2). Similarly for $\mathbf{n} = 2$ one can obtain a flavor symmetry SO(2) $\simeq$ U(1) or SO(1) × SO(1) associated different choices of chirality for the spinorial matter. We again use the terminology SO(12)$_a$ and SO(12)$_b$ to distinguish between these two possibilities. Notice that these possibilities do not arise for the models $\mathbf{n}_{\mathrm{SO}(10)}$ because the representations S_+ and S_- are each other's complex conjugate. Naively one would think that a similar phenomenon might arise also for SO(8) as the representations $V = \mathbf{8}_v$, $S_+ = \mathbf{8}_s$ and $S_- = \mathbf{8}_c$ are rotated by triality, but this is not the case, as these representations are not Casimir equivalent — see appendix C of [130] for a proof.[11]

In the remainder of this section we review the classification of rank-one SCFTs [158, 159] from the F-theory perspective [10]. A convenient organizing principle is to distinguish nine different classes of theories based on the value of n; in each class, one can arrange the theories in 'Higgsing trees' where each node of the tree corresponds to a specific rank 1 SCFT, and arrow connecting two theories indicates that it is possible to Higgs the first theory to obtain the second. In drawing the Higgsing trees, since n is fixed, we denote the Higgsable SCFTs by the alternative notation $G \oplus R_G$ emphasizing the representation of G in which the matter transforms.

2.4.1 Models with finite-length Higgsing trees

The rank-one models corresponding to F-theory bases with

$$n = 3, 5, 6, 7, 8, 12 \tag{2.40}$$

all have finite-length Higgsing trees. The rank one theories

$$\mathbf{12}_{E_8}, \qquad \mathbf{8}_{E_7}, \qquad \mathbf{7}_{E_7} \tag{2.41}$$

are isolated, meaning that these theories are not part of any rank-one Higgsing tree that includes higher rank gauge groups. They can nevertheless be obtained by Higgsing higher

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[11] We thank D. R. Morrison for pointing this out to us.

rank 6d SCFTs, obtained by coupling the $\mathbf{12}_{E_8}$ with several copies of $\mathbf{1}^{E_8}_{\mathrm{Sp}(0)}$ theories, gauging the diagonal E_8 flavor symmetry of these models [82].

The rank one theory $\mathbf{6}_{E_6}$ has a Higgsing tree of length one:

$$E_7 \oplus \mathbf{56} \longrightarrow \mathbf{6}_{E_6}, \tag{2.42}$$

where the theory $\mathbf{6}^{\mathrm{U}(1)}_{E_7}$ has matter charged with respect to E_7 transforming as a hypermultiplet in the $\mathbf{56}$ representation. The theory $\mathbf{5}_{F_4}$ has a Higgsing tree of length two,

$$E_7 \oplus \frac{3}{2}\mathbf{56} \longrightarrow E_6 \oplus \mathbf{27} \longrightarrow \mathbf{5}_{F_4}. \tag{2.43}$$

The richest finite-length Higgsing tree is the one for the theory $\mathbf{3}_{\mathrm{SU}(3)}$, given by

$$\tag{2.44}$$

$$
\begin{array}{ccc}
\mathbf{3}_{\mathrm{SU}(3)} & & \\
\uparrow & & \\
G_2 \oplus \mathbf{7} & & \\
\uparrow & & \\
\mathrm{SO}(7) \oplus S^{\oplus 2} & & \\
\uparrow & & \\
\mathrm{SO}(8) \oplus V \oplus S_+ \oplus S_- & \nwarrow & \\
\uparrow & & \\
F_4 \oplus \mathbf{26}^{\oplus 2} & & \mathrm{SO}(9) \oplus V^{\oplus 2} \oplus S \\
\uparrow & & \uparrow \\
E_6 \oplus \mathbf{27}^{\oplus 3} & & \mathrm{SO}(10) \oplus V^{\oplus 3} \oplus S_+ \\
\uparrow & & \uparrow \\
E_7 \oplus (\tfrac{1}{2}\mathbf{56})^{\oplus 5} & & \mathrm{SO}(11) \oplus V^{\oplus 4} \oplus \tfrac{1}{2}S \\
& & \uparrow \\
& & \mathrm{SO}(12) \oplus V^{\oplus 5} \oplus \tfrac{1}{2}S_{\pm}
\end{array}
$$

2.4.2 Models with infinite-length Higgsing trees

The remaining bases with

$$n = 1, 2, 4 \tag{2.45}$$

all have infinitely long Higgsing trees, at the bottom of which one finds the theories

$$\mathbf{1}^{E_8}_{\mathrm{Sp}(0)} \qquad \mathbf{2}_{\mathrm{SU}(1)} \qquad \mathbf{4}_{\mathrm{SO}(8)} \tag{2.46}$$

We draw the corresponding Higgsing trees in figures 3, 4, and 5.

The M-strings and E-strings deserve special attention. The first is in fact technically a Higgsable theory. In the F-theory setup this is related to the fact that the discriminant
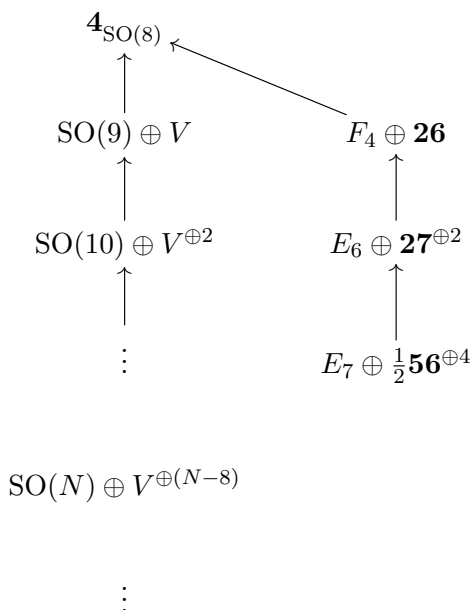
Figure 3. The Higgsing tree for $\mathbf{4}_{\mathrm{SO}(8)}$.

has an I_1 singular fiber along the base curve Σ, this explains our notation $\mathbf{2}_{\mathrm{SU}(1)}$ for this model. This can be deformed to a smooth configuration with fiber I_0 and no monodromy. In this case we have an enhancement of supersymmetry and we obtain the A_1 (2,0) theory. This can also be readily understood from the perspective of type IIA string theory, where the M-string is realized by the following brane configuration:

	0	1	2	3	4	5	6	7	8	9
NS5$_1$	X	X	X	X	X	X	$\{p_0\}$	–	–	–
NS5$_2$	X	X	X	X	X	X	$\{p_1\}$	–	–	–
D6	X	X	X	X	X	X	X	–	–	–

$$(2.47)$$

In this IIA setup, Higgsing amounts to initiating an RG flow by moving the D6 brane away from the NS5 branes along a direction transverse to them. In the IR limit, the D6 brane is moved infinitely far away from the NS5 branes, and the 6d theory reduces to the A_1 $\mathcal{N} = (2,0)$ SCFT.

A similar picture holds for the E-string SCFT [156]. One can consider the following brane configuration

	0	1	2	3	4	5	6	7	8	9
O8$^-$+8 D8	X	X	X	X	X	X	$\{p_0\}$	X	X	X
$\frac{1}{2}$NS5$_1$	X	X	X	X	X	X	$\{p_0\}$	–	–	–
NS5$_2$	X	X	X	X	X	X	$\{p_1\}$	–	–	–
D6	X	X	X	X	X	X	X	–	–	–

$$(2.48)$$

$$A_1(2,0)$$

$$\mathbf{2}_{\mathrm{SU}(1)}$$

$$\mathrm{SU}(2) \oplus \Lambda^{\oplus 4}$$

$$\mathrm{SU}(3) \oplus \Lambda^{\oplus 6}$$

$$\mathrm{SU}(4) \oplus \Lambda^{\oplus 8} \qquad G_2 \oplus \mathbf{7}^{\oplus 4}$$

$$\vdots \qquad \mathrm{SO}(7) \oplus V \oplus S^{\oplus 4}$$

$$\mathrm{SU}(N) \oplus \Lambda^{\oplus 2N} \qquad \mathrm{SO}(8) \oplus V^{\oplus 2} \oplus S_+^{\oplus 2} \oplus S_-^{\oplus 2}$$

$$\vdots \qquad F_4 \oplus \mathbf{26}^{\oplus 3} \qquad \mathrm{SO}(9) \oplus V^{\oplus 3} \oplus S^{\oplus 2}$$

$$E_6 \oplus \mathbf{27}^{\oplus 4} \qquad \mathrm{SO}(10) \oplus V^{\oplus 4} \oplus S_+^{\oplus 2}$$

$$E_7 \oplus \tfrac{1}{2}\mathbf{56}^{\oplus 6} \qquad \mathrm{SO}(11) \oplus V^{\oplus 5} \oplus \tfrac{1}{2}S^{\oplus 2}$$

$$\mathrm{SO}(12) \oplus V^{\oplus 6} \oplus \tfrac{1}{2}S_\pm^{\oplus 2} \qquad \mathrm{SO}(12) \oplus V^{\oplus 6} \oplus \tfrac{1}{2}S_+ \oplus \tfrac{1}{2}S_-$$

$$\mathrm{SO}(13) \oplus V^{\oplus 7} \oplus \tfrac{1}{2}S$$
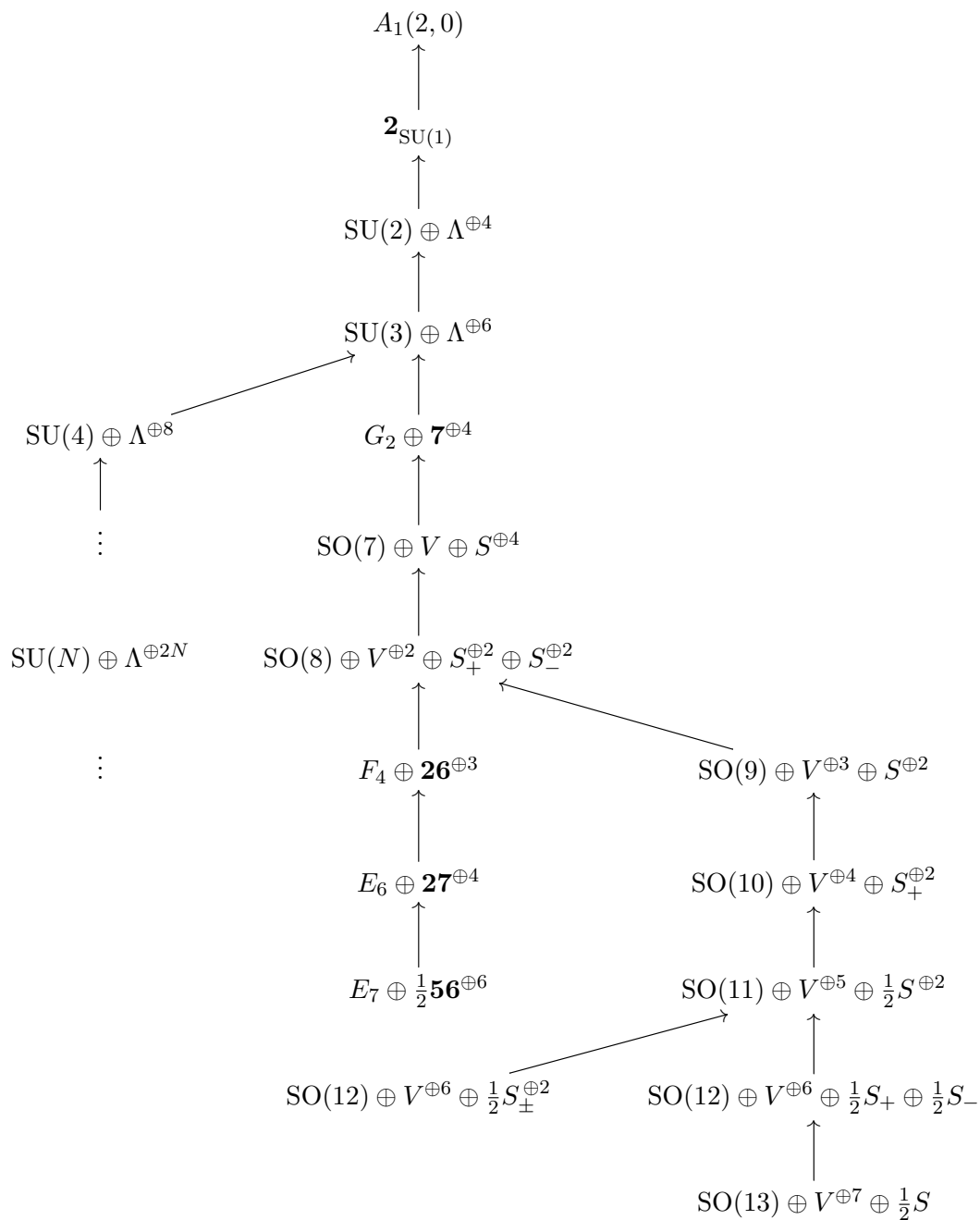
Figure 4. The M-string Higgsing tree.

$$\mathbf{1}^{E_8}_{\mathrm{Sp}(0)}$$

$$\mathrm{Sp}(1) \oplus \Lambda^{\oplus 10}$$

$$\mathrm{SU}(3) \oplus \Lambda^{\oplus 12}$$

$$\mathrm{Sp}(2) \oplus \Lambda^{\oplus 14} \qquad \mathrm{SU}(4) \oplus \Lambda^{\oplus 12} \oplus \Lambda^2 \qquad G_2 \oplus \mathbf{7}^{\oplus 7}$$

$$\mathrm{Sp}(3) \oplus \Lambda^{\oplus 16} \qquad \mathrm{SU}(5) \oplus \Lambda^{\oplus 13} \oplus \Lambda^2 \qquad \mathrm{SO}(7) \oplus V^{\oplus 2} \oplus S^{\oplus 6}$$

$$\mathrm{SU}(6) \oplus \Lambda^{\oplus 14} \oplus \Lambda^2 \qquad \mathrm{SU}(6) \oplus \Lambda^{\oplus 15} \oplus \tfrac{1}{2}\Lambda^3 \qquad \mathrm{SO}(8) \oplus V^{\oplus 3} \oplus S_+^{\oplus 3} \oplus S_-^{\oplus 3}$$

$$\mathrm{Sp}(N) \oplus \Lambda^{\oplus 8+2N} \qquad \mathrm{SO}(9) \oplus V^{\oplus 4} \oplus S^{\oplus 3} \qquad F_4 \oplus \mathbf{26}^{\oplus 4}$$

$$\mathrm{SU}(N) \oplus \Lambda^{\oplus 8+N} \oplus \Lambda^2 \qquad \mathrm{SO}(10) \oplus V^{\oplus 5} \oplus S_+^{\oplus 3} \qquad E_6 \oplus \mathbf{27}^{\oplus 5}$$

$$\mathrm{SO}(11) \oplus V^{\oplus 6} \oplus \tfrac{1}{2}S^{\oplus 3} \qquad E_7 \oplus \tfrac{1}{2}\mathbf{56}^{\oplus 7}$$

$$\mathrm{SO}(12) \oplus V^{\oplus 7} \oplus \tfrac{1}{2}S_\pm^{\oplus 3} \qquad \mathrm{SO}(12) \oplus V^{\oplus 7} \oplus \tfrac{1}{2}S_\pm \oplus \tfrac{1}{2}(S_\mp)^{\oplus 2}$$

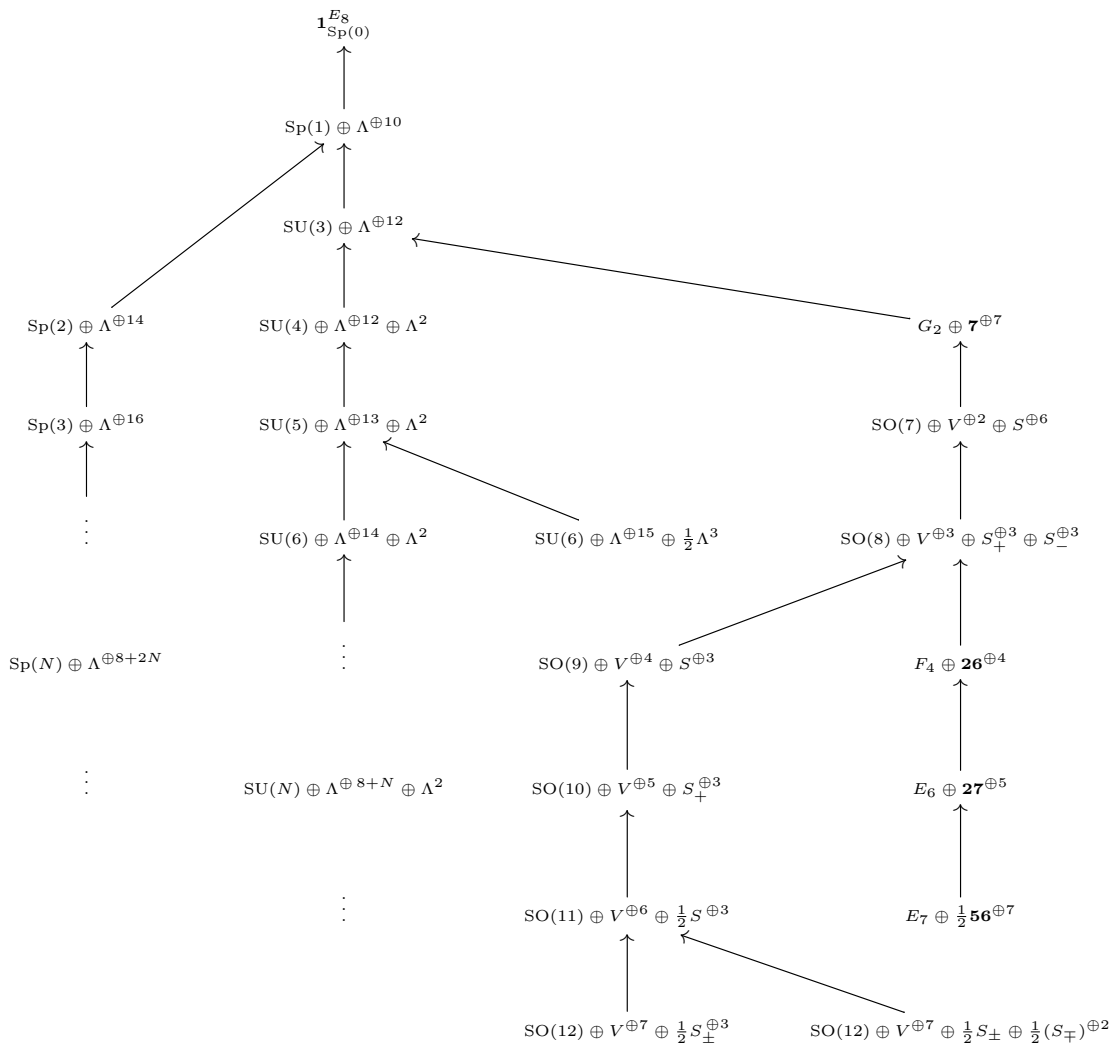**Figure 5.** The E-string Higgsing tree.

which gives a brane realization of the $G = \text{`SU}(1)\text{'}$ theory, sitting at the end of the $\mathrm{SU}(n)$ Higgsing chain. Alternatively, we can consider the following brane configuration:

	0	1	2	3	4	5	6	7	8	9
O8$^-$+8 D8	X	X	X	X	X	X	$\{p_0\}$	X	X	X
NS5	X	X	X	X	X	X	$\{p_1\}$	–	–	–

(2.49)

obtained by removing the D6 brane and also moving away the 1/2-NS5 brane along $x_{6,7,8,9}$. This provides a realization of the $G = \text{`Sp}(0)\text{'}$ theory. The only difference between the two theories is that the latter describes the E-string theory, while the former describes the E-string theory plus a decoupled free (1,0) hypermultiplet. Going from the former to the latter can be viewed as a quite trivial Higgs branch RG flow. As the E-string's 2d CFT is not sensi-

tive to the presence of the extra decoupled hypermultiplet [73], in the remainder of the paper we do not distinguish between the two theories and only consider the $G = \text{`Sp}(0)\text{'}$ theory.

3 BPS strings and wrapped D3 branes

In this section we discuss the geometric engineering of the BPS strings of rank-one 6d SCFTs. In section 3.1 we point out that these sectors are the basic elements of the BPS spectrum of 6d SCFTs of arbitrary rank. In section 3.2 we discuss some of the features of the (0,4) surface defects that appear in our construction. In section 3.3 we use anomaly inflow from 6d to determine the anomaly polynomial of the 2d worldsheet theories of the strings. In section 3.4 we discuss about an analogy between the structure of the singular elliptic fibers and UV curves for wrapped M5 branes.

3.1 Universality of rank-one BPS strings

Along the tensor branch, six-dimensional theories have BPS strings that are engineered in F-theory by D3 branes wrapping the rational curves Σ_I. By an abuse of notation, we denote the classes of the curves Σ_I with the same symbol used for the curves themselves. The integer lattice

$$\Gamma \equiv \sum_{I=1}^{R} \mathbb{Z}\,\Sigma_I \subseteq H_{2,c}(B,\mathbb{Z}) \tag{3.1}$$

is identified with the string charge lattice of the model. A BPS string configuration with charge

$$\mathbf{S} = N_1\,\Sigma_1 + \cdots + N_R\,\Sigma_R\,, \tag{3.2}$$

where $N_i \geq 0$, is engineered as a bound state of N_1 D3 branes wrapping the curve Σ_1, N_2 D3 branes wrapping the curve Σ_2, etcetera. The configuration is irreducible only if the support of the N_I is on curves that mutually intersect. We denote the cone of positive charges corresponding to the Σ_I generators as Γ_+. All the resulting worldsheet theories preserve $(0,4)$ supersymmetry.

Let us restrict momentarily to 6d SCFT that do not possess abelian global symmetries. For a given charge vector $\mathbf{S}$, let us denote by $\mathcal{I}(\mathbf{S}) \subset \{1,\ldots,R\}$ the support of the charge vector, e.g. the $I \in \{1,\ldots,R\}$ for which $N_I \neq 0$. Let us also denote by $\mathcal{I}_G(\mathbf{S}) \subset \mathcal{I}(\mathbf{S})$ the index set of those curves in the support of the charge vector that are also part of the discriminant. Let us further denote by $\mathcal{I}_F(\mathbf{S})$ an index set running over those compact and noncompact irreducible components of the discriminant that are not in the support of the charge vector but intersecting the compact curves Σ_I such that $I \in \mathcal{I}(\mathbf{S})$. For all curves $\{\Sigma_I\}_{I \in \mathcal{I}_G(\mathbf{S})}$ (resp. $\{\Delta_\alpha\}_{\alpha \in \mathcal{I}_F(\mathbf{S})}$) we denote by G_I (resp. F_α) the gauge group associated to the corresponding seven-brane stack. The $(0,4)$ model corresponding to a BPS string of charge $\mathbf{S} \in \Gamma_+$ has global symmetry

$$\text{SU}(2)_L \times \text{SU}(2)_R \times \text{SU}(2)_I \times \prod_{I \in \mathcal{I}_G(\mathbf{S})} G_I \times \prod_{\alpha \in \mathcal{I}_F(\mathbf{S})} F_\alpha. \tag{3.3}$$

The subgroup $\text{SU}(2)_L \times \text{SU}(2)_R$ arises from the $\text{SO}(4)$ symmetry rotating the $\mathbb{R}^4$ plane transverse to the D3 branes within $\mathbb{R}^{1,5}$; $\text{SU}(2)_I$ is the R-symmetry of the small $(0,4)$

superconformal algebra governing the IR fixed point, which is also identified with the Sp(1) R-symmetry of the 6d SCFT. Of course, for 6d SCFTs that have global U(1) symmetries one can modify equation (3.3) by adding a factor $\prod_a \mathrm{U}(1)_a$ to the global symmetry of the model.[12] Both gauge and global symmetries of the 6d SCFT give rise to global symmetries for the 2d $(0, 4)$ worldsheet SCFTs, but as we shall see below they are realized differently in the BPS string worldsheet theory: the 6d flavor symmetry is realized chirally, while the 6d gauge symmetry is not.

From this description of the BPS string sector of a given 6d SCFT, it follows that the subsectors with unit total BPS string charge

$$\mathbf{S} = \delta^{IJ} \Sigma_J \tag{3.4}$$

are universal, and can always be described as strings of rank-one models. To characterize these universal subsectors, therefore, it is enough to focus on the rank-one 6d SCFTs, whose salient features we have described above. The main topic of this paper is to probe the physics of rank-one 6d SCFTs exploiting their one-string BPS subsectors. Moreover, understanding the structure of these subsectors is a necessary first step towards the a general understanding of multi-string BPS sectors (though various results have already been obtained for various specific instances of higher rank SCFTs [63–65, 73, 76, 82, 86, 93, 94]).

3.2 BPS strings of rank-one models and surface defects

In this subsection we discuss the BPS string worldsheet theories from the perspective of geometric engineering in F-theory. We find that the BPS strings are obtained by reduction on $\mathbb{P}^1$ of the 4d $\mathcal{N} = 2$ worldvolume theory of an instanton of an 8d SYM, with the insertion of appropriate classes of surface defects. These defects are generalizations of the chiral defects studied by Martucci [135]. More precisely, in eight dimensions for any simple, simply laced gauge group there exists a corresponding type of seven-brane. The instantons on the seven-brane worldvolume are engineered by parallel stacks of D3-branes; for A_n and D_n gauge groups the worldvolume theories of the D3 branes are determined by the corresponding ADHM quivers, while for exceptional gauge groups they are given by the Minahan-Nemeschansky SCFTs [160, 161] and their higher rank generalizations [133, 134, 162]. While the chiral defects studied in [135] are defined for $\mathcal{N} = 4$ SYM which is a Lagrangian theory, our chiral defects arise in $\mathcal{N} = 2$ theories that generally do not admit a Lagrangian description, which happens for exceptional gauge groups. In these cases the complex structure of the elliptic fiber is frozen, hence these are not strictly speaking duality defects in the sense of [135]. Nevertheless, these support chiral fermionic degrees of freedom, which are responsible for shifting the central charges of the IR CFT on the BPS string, as we will see below. In addition to these chiral defects, in order to realize the BPS strings of 6d theories with non-simply laced gauge group one is also led to consider a

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[12]Since U(1) symmetries in F-theory are read off from the MW group of the elliptic fibration, they are a global property of the base; it has been suggested to one of us by J.J. Heckman that as a consequence they cannot give rise to gauge symmetries, but only to global ones. While this paper was in preparation [59] appeared which contains a proof of this conjecture for models that can be coupled to gravity.

seven-brane	D3 probe	Δ	H_G^1
I_n	U(1) with n charge-one hypers	1	$H_{\mathrm{SU}(n)}^1$
I_n^*	SU(2) with $n+4$ fundamentals	2	$H_{\mathrm{SO}(2n+8)}^1$
II	(A_1, A_2) AD	6/5	$H_\varnothing^1$
III	(A_1, A_3) AD	4/3	$H_{\mathrm{SU}(2)}^1$
IV	(A_1, D_4) AD	3/2	$H_{\mathrm{SU}(3)}^1$
IV^*	E_6 MN	3	$H_{E_6}^1$
III^*	E_7 MN	4	$H_{E_7}^1$
II^*	E_8 MN	6	$H_{E_8}^1$

Table 7. Rank-one 4d $\mathcal{N}=2$ theories of D3-brane probes of F-theory seven-branes. Each entry of this table must be supplemented with a free center of mass hypermultiplet.

different class of 'folding' defects, which implement the projection from a suitable simply-laced gauge group to the desired non-simply laced one. In this section we only discuss basic properties of these classes of defects which we infer from geometric engineering, while a more detailed study is left to future work [163].

For rank-one models, the D3 brane configurations considered in section 3.1 simplify, as the BPS string charge lattice is one dimensional. We denote the corresponding base curve Σ. The global symmetry for the worldvolume theory of a stack of Q D3 branes in a generic 6d (1,0) SCFT is

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(2)_I \times G \times \prod_{\alpha \in I_F} F_\alpha \times \prod_a \mathrm{U}(1)_a. \tag{3.5}$$

The worldsheet theory describing a bound state of BPS strings with charge $\mathbf{S} = Q\Sigma$ for a rank-one 6d SCFT arises from of a stack of Q D3-branes wrapping the rational curve Σ. When Σ is not part of the discriminant, one can think of the corresponding string worldsheet theory as a twisted compactification of the 4D $\mathcal{N}=4$ U(Q) SYM theory that lives on the D3-branes worldvolume [105, 106, 135]. However, this is not the situation that one encounters for the generic rank-one 6d SCFT; rather, for a typical member of this class of theories Σ is part of the discriminant, the sole exception being the E-string SCFT $\mathbf{1}_{\mathrm{Sp}(0)}^{E_8}$.

More precisely, the twisted compactification of the D3 branes' worldvolume theory leads to a 2d UV description of the strings with $\mathcal{N}=(0,4)$ supersymmetry whose properties
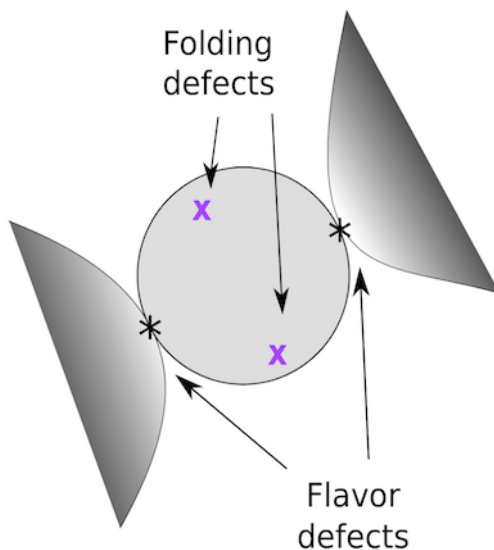
Figure 6. Generic geometry for a rank-one F-theory base.

we study in this section. This UV theory flows in the IR to the $\mathcal{N} = (0,4)$ CFT which is the subject of later sections.

Consider a collection of F-theory seven-branes in flat space. The worldvolume theories of D3 branes that are placed on top of the seven-branes enjoy $\mathcal{N} = 2$ supersymmetry. For seven-brane stacks corresponding to Kodaira singularities of type I_N and I_N^* the world-volume theories governing the low energy behavior of a stack of parallel Q D3-branes are ADHM theories: for I_N singularities we have the 4d $\mathcal{N} = 2$ theory

$$H_{\mathrm{SU}(N)}^Q \equiv \mathrm{U}(Q) \oplus \Lambda^{\oplus N} \tag{3.6}$$

where Λ is the fundamental representation, while for I_N^* singularities with $N \geq 1$ we have the 4d $\mathcal{N} = 2$ theory

$$H_{\mathrm{SO}(N+8)}^Q \equiv \mathrm{Sp}(Q) \oplus \Lambda^{\oplus N+8} \oplus \Lambda_{\mathrm{irr}}^2 \tag{3.7}$$

where Λ_{irr}^2 denotes the irreducible component of the antisymmetric representation of the $\mathrm{Sp}(Q)$ gauge group. Notice that with the exception of the singularity I_0^* the corresponding 4d $\mathcal{N} = 2$ theories are not conformal and often they are not asymptotically free either - this is the case for instance for a single D3-brane probe of the I_N seven-brane stack, which is a U(1) theory. For the exotic seven-branes of type II, III, IV and II^*, III^*, IV^* the corresponding $\mathcal{N} = 2$ models are strongly interacting, non-Lagrangian SCFTs, that we denote H_G^Q, which have flavor symmetry $\mathrm{SU}(2)_L \times G$ where $G = \varnothing, \mathrm{SU}(2), \mathrm{SU}(3), E_6, E_7, E_8 \equiv E_{n=0,1,2,6,7,8}$ respectively.

In considering six-dimensional F-theory backgrounds, by invoking an adiabatic approximation we can identify the worldsheet theory on the BPS strings with the twisted compactification on Σ of precisely the 4D models governing the worldvolumes of D3-brane stacks, along the lines of [164]. If Σ belongs to the discriminant, the resulting worldvolume

theory is the twisted compactification of one of the 4D $\mathcal{N} = 2$ theories we introduced above. We refer the reader to table 7 for the list of the corresponding $\mathcal{N} = 2$ theories for the case of a single D3-brane, which is the case we address in detail in this paper. This strategy was exploited in [83] for the theories $\mathbf{3}_{\mathrm{SU}(3)}$, $\mathbf{4}_{\mathrm{SO}(8)}$, $\mathbf{6}_{E_6}$, $\mathbf{8}_{E_7}$, and $\mathbf{12}_{E_8}$, building upon the results of [165]. The relevant twisted compactification in these cases is the β-twist on $\mathbb{R}^{1,1} \times \mathbb{P}^1$ by the $\mathrm{U}(1)_R$ R-symmetry of the corresponding 4d $\mathcal{N} = 2$ SCFTs [166]. In the language of [167–170], one couples the 4d $\mathcal{N} = 2$ theories to a rigid supersymmetry background of the form $\mathbb{R}^{1,1} \times S^2$ or $T^2 \times S^2$ by switching on a unit monopole flux on the S^2 for the R-symmetry background gauge field [72, 171–174]. Notice that this is defined for theories with a $\mathrm{U}(1)_R$ R-symmetry such that the corresponding R-charges are quantized over the integers. For a review, see e.g. [83] and references therein. Most of the theories in table 7 are not superconformal and typically have an anomalous $\mathrm{U}(1)_R$ symmetry. Nevertheless, in the geometric engineering picture one sees that each time such a non-conformal system of branes arises the compactification on Σ requires the insertion of additional flavor divisors intersecting the D3 worldvolume along the directions transverse to Σ. On the one hand, these defects are responsible for compensating the $\mathrm{U}(1)_R$ anomaly, thus rendering the background consistent; on the other hand, they introduce extra degrees of freedom which make the corresponding 2d $(0,4)$ models consistent. The models $\mathbf{3}_{\mathrm{SU}(3)}$, $\mathbf{4}_{\mathrm{SO}(8)}$, $\mathbf{6}_{E_6}$, $\mathbf{8}_{E_7}$, and $\mathbf{12}_{E_8}$ are special in that the 6d gauge anomaly polynomial coefficient multiplying $\mathrm{tr}\, F^4$ cancels automatically, with no need for extra matter; correspondingly, these theories have no flavor divisors. For all other theories, in addition to the gauge seven-branes wrapped on the base curve Σ the background involves extra components of the discriminant that are noncompact and intersect $\Sigma \simeq S^2$, corresponding to flavor seven-branes. Schematically, the corresponding geometry is the following:

IIB background	$\mathbb{R}^{1,5}$						$\widehat{B}$			
	$\mathbb{C}_\parallel^2$				$\mathbb{R}^{1,1}$		Σ			
	0	1	2	3	4	5	6	7	8	9
seven-branes	X	X	X	X	X	X	X	X	-	-
flavor seven-branes	X	X	X	X	X	X	-	-	X	X
D3 branes	-	-	-	-	X	X	X	X	-	-

$$(3.8)$$

From the adiabatic perspective of the theory on the D3-brane, it is manifest that these additional flavor seven-branes behave like surface defects which preserve $(0,4)$ supersymmetry. The insertion of these defects alters the S^2 metric on Σ by introducing conical defects, which lead to an effective shift of the $\mathrm{U}(1)_R$ symmetry of the model. This is a key feature that allows for a twisted compactification even for the 4D $\mathcal{N} = 2$ theories in table 7 that do not have an R-symmetry to begin with. We denote the corresponding systems as

$$\left(H_G^Q \, , \, \mathfrak{D}_F^{R_\mathfrak{D}} \right)_{S^2} \, , \tag{3.9}$$

where H_G^Q is the theory on the worldvolume of Q D3-branes probing a stack of seven-branes of type G, while $\mathfrak{D}_F^{R_\mathfrak{D}}$ is a defect preserving 2d $(0,4)$ supersymmetry, which supports a flavor
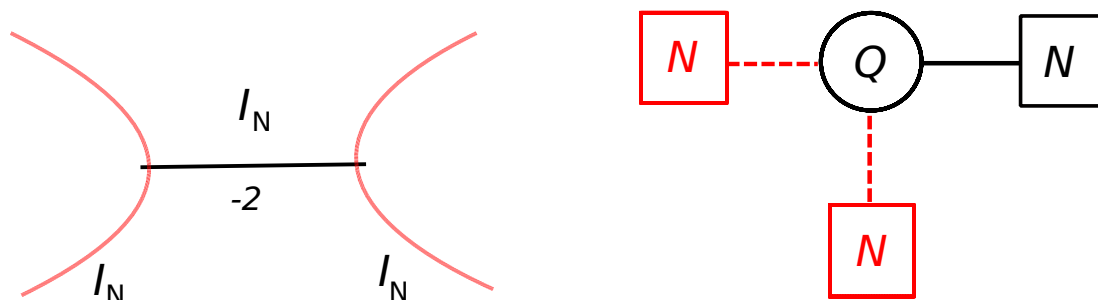
Figure 7. *Left:* schematic description of the geometry of the F-theory model correspoding to the $\mathbf{2}_{\mathrm{SU}(N)}$ theory. *Right:* 2d quiver theory for a bound state of Q BPS strings [64].

symmetry F and contributes an effective shift to the R-charge by $R_{\mathfrak{D}}$ units. The subscript S^2 on the right hand side of equation (3.9) is a reminder that these surface defects are inserted at points on the S^2 used for the compactification. In the absence of defects, the direction normal to the rational curve Σ has the structure of the fiber of an $\mathcal{O}(-n)$ bundle, where

$$n = R_G = 2\Delta_G, \tag{3.10}$$

where R_G is the R-charge of the Coulomb branch operator for the H_G^1 theories. Whenever the corresponding generalized chiral defects have a nonzero effective $R_{\mathfrak{D}}$ charge, this formula gets modified to

$$n = 2\Delta_G - R_{\mathfrak{D}}. \tag{3.11}$$

This allows one to determine the effective R-charge shift $R_{\mathfrak{D}}$ associated to a defect by inspection of the geometric engineering setup.

As an example, let us consider the six-dimensional models $\mathbf{2}_{\mathrm{SU}(N)}$ which involve seven-branes of I_N type. The corresponding geometry is shown in figure 7. Proceeding naively, reducing the 4d $\mathcal{N} = 2$ theories $H_{\mathrm{SU}(N)}^Q$ on S^2 gives rise to 2d $(0,4)$ theories with field content analogous to that of the 4d theory, since the S^2 reduction gives a 2d $(0,4)$ vector multiplet (resp. hypermultiplet) for each 4d $\mathcal{N} = 2$ vector multiplet (resp. hypermultiplet). This is of course not the complete description of the worldsheet degrees of freedom. On the left of the figure one can see the F-theory geometry for the corresponding rank-one 6d tensor branch, which involves two extra noncompact flavor seven-branes of I_N type, corresponding to generalized chiral defects that make the corresponding compactification consistent by contributing extra degrees of freedom. In other words, we have a compactification of the $H_{\mathrm{SU}(N)}^Q$ theory with two defects:

$$\left(H_{\mathrm{SU}(N)}^Q, \mathfrak{D}_{\mathrm{SU}(N)}^0 \oplus \mathfrak{D}_{\mathrm{SU}(N)}^0 \right)_{S^2} \approx \left(H_{\mathrm{SU}(N)}^Q, \mathfrak{D}_{\mathrm{SU}(2N)}^0 \right)_{S^2} \tag{3.12}$$

where on the right hand side we have emphasized the fact that the two defects can be brought on top of each other and merged, leading to a single defect supporting the larger flavor symmetry SU(2N). Notice that for these models the effective R-charge shift $R_{\mathfrak{D}}$ is null, where to determine the effective R-charge shift $R_{\mathfrak{D}}$ we read off the dimension of

$\Delta_{\mathrm{SU}(N)} = 1$ from table 7, which implies that

$$n = 2 = 2\Delta_G, \tag{3.13}$$

so that $R_{\mathfrak{D}} = 0$ in equation (3.11). This has a natural explanation: in this case, the effective shift is used up to compensate for the $\mathrm{U}(1)_R$ anomaly. The resulting two-dimensional $(0,4)$ theory admits a quiver description, which is shown on the right hand side of figure 7. In the figure the degrees of freedom arising from the 4d $\mathcal{N} = 2$ theory $H^Q_{\mathrm{SU}(N)}$ on S^2 are shown in black, while the degrees of freedom corresponding to the defect insertions are shown in red. Each $\mathfrak{D}^0_{\mathrm{SU}(N)}$ defect supports a $(0,4)$ bifundamental Fermi multiplet and therefore contributes chirally to the quiver theory.

As an example without a Lagrangian description, consider the models with gauge group E_7 and matter hypermultiplets in the $\frac{1}{2}\mathbf{56}$. We claim that the following configurations for the $H^Q_{E_7}$ theory involving defects,

$$\left(H^Q_{E_7}, \underbrace{\mathfrak{D}^1_{\mathrm{SO}(1)} \oplus \cdots \oplus \mathfrak{D}^1_{\mathrm{SO}(1)}}_{m \text{ times}}\right)_{S^2} \approx \left(H^Q_{E_7}, \mathfrak{D}^m_{\mathrm{SO}(m)}\right)_{S^2}, \tag{3.14}$$

give rise to the BPS strings for the entire class of 6d SCFTs

$$(\mathbf{8} - \mathbf{m})^{\mathrm{SO}(\ell)}_{E_7} \qquad m = 0, 1, \ldots, 7 \tag{3.15}$$

whose gauge symmetry on the tensor branch is E_7.

If an effective tensor branch gauge group is not simply laced one must include additional $(0,4)$ defects that implement the relevant outer automorphism twist on the flavor symmetry (see figure 9), which we will refer to as folding defects. We denote backgrounds with folding defects as

$$\left(H^Q_G, \mathfrak{F}^{R_{\mathfrak{F}}}_{\mathbb{G}}\right)_{S^2} \tag{3.16}$$

where $\mathbb{G}$ is the outer automorphism that the folding defect implements on the flavor symmetry of the model H^Q_G and $R_{\mathfrak{F}}$ is the shift to the $\mathrm{U}(1)_R$ charge induced by the defect. Crucially, the folding defects do not act as orbifold projections on the degrees of freedom of the BPS string, which would correspond to gauging the corresponding discrete symmetry. Rather, their role is to restrict the algebra of operators of the theory H^Q_G on S^2 to a smaller closed consistent subalgebra in which only the non-simply laced flavor symmetry $G/\mathbb{G}$ is manifest.

A simple example of folding occurs for the BPS strings of the $\mathbf{1}^{\mathrm{SO}(16+4N)}_{\mathrm{Sp}(N)}$ theories. In that context, we need to insert two folding defects to get from $\mathrm{SU}(2N)$ to $\mathrm{Sp}(N)$. Each folding defect produces an effective shift of the R-charge by $1/2$. The resulting models are given by

$$\begin{aligned} \mathbf{h}^{\mathrm{Sp}(N)}_1 &= \left(H^Q_{\mathrm{SU}(2N)}, \mathfrak{F}^{1/2}_{\mathbb{Z}_2} \oplus \mathfrak{F}^{1/2}_{\mathbb{Z}_2} \oplus \mathfrak{D}^0_{\mathrm{SO}(2N+8)} \oplus \mathfrak{D}^0_{\mathrm{SO}(2N+8)}\right)_{S^2} \\ &= \left(H^Q_{\mathrm{SU}(2N)}, \mathfrak{F}^{1/2}_{\mathbb{Z}_2} \oplus \mathfrak{F}^{1/2}_{\mathbb{Z}_2} \oplus \mathfrak{D}^0_{\mathrm{SO}(4N+16)}\right)_{S^2}. \end{aligned} \tag{3.17}$$

We draw the corresponding quiver theory in figure 14(b). The insertion of the folding defects projects the $\mathrm{SU}(2N)$ ADHM theory to the $\mathrm{Sp}(N)$ one, which has a gauge group
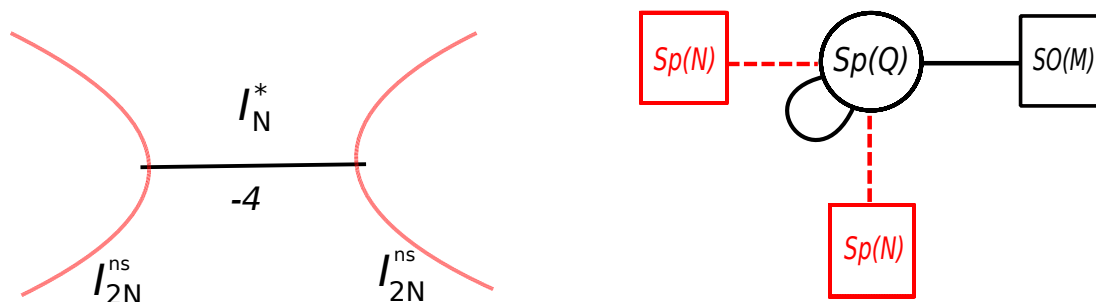
Figure 8. *Left:* schematic description of the geometry of the F-theory model correspoding to the $\mathbf{4}_{\mathrm{SO}(2N+8)}$ theory. *Right:* 2d quiver theory for a bound state of Q BPS strings, here $M = 2N+8$ [73].

$O(Q)$ with an anti-symmetric multiplet as well as a bifundamental hypermultiplet carrying $\mathrm{Sp}(N)$ flavor symmetry. The $\mathfrak{D}^0_{\mathrm{SO}(4N+16)}$ flavor defect on the other hand supports an ortho-symplectic bifundamental $(0,4)$ Fermi multiplet, which couples to the $\mathrm{Sp}(N)$ gauge symmetry.

One can similarly interpret the strings for the models $\mathbf{4}_{\mathrm{SO}(8+2N)}$ as arising from compactifications of the theory $H^Q_{\mathrm{SO}(8+2N)}$ on S^2 with $(0,4)$-defect insertions

$$\mathbf{h}_4^{\mathrm{SO}(8+2N)} = \left(H^Q_{\mathrm{SO}(8+2N)}, \, \mathfrak{D}^0_{\mathrm{Sp}(N)} \oplus \mathfrak{D}^0_{\mathrm{Sp}(N)} \right)_{S^2}. \tag{3.18}$$

We draw the resulting quiver in figure 8. Again here we see that the 2d $(0,4)$ defects support Lagrangian degrees of freedom (bifundamental ortho-symplectic half-hypermultiplets). The $\mathrm{SO}(8+2N)$ ADHM quiver of the theory without defects is clearly visible in black in the figure.

Next, we provide some additional examples that further clarify the nature of folding and generalized chiral defects.

3.2.1 Folding defects: a detailed example

The theory $\mathbf{5}_{F_4}$ is the only example of a 6d SCFT of rank one with non-simply laced group and no matter, and therefore involves folding defects but no generalized chiral defects. The BPS strings for this theory are realized starting from an S^2 compactification of the theory $H^Q_{E_6}$ with the insertion of folding defects corresponding to the $\mathbb{Z}_2$ outer automorphism

$$\mathbb{Z}_2 \colon E_6 \to F_4. \tag{3.19}$$

We claim that the model

$$\left(H^Q_{E_6}, \, \mathfrak{F}^{1/2}_{\mathbb{Z}_2} \oplus \mathfrak{F}^{1/2}_{\mathbb{Z}_2} \right)_{S^2} \tag{3.20}$$

corresponds to the worldsheet theory of Q BPS strings of the $\mathbf{5}_{F_4}$ theory. The fact that two insertions are needed is manifest in the geometry (see figure 10), which has been studied in detail in [41], (see also [175]) and is due to the compactness of the gauge divisor in the fully resolved CY geometry. The generic fiber along the -5 curve in the F-theory base is indeed
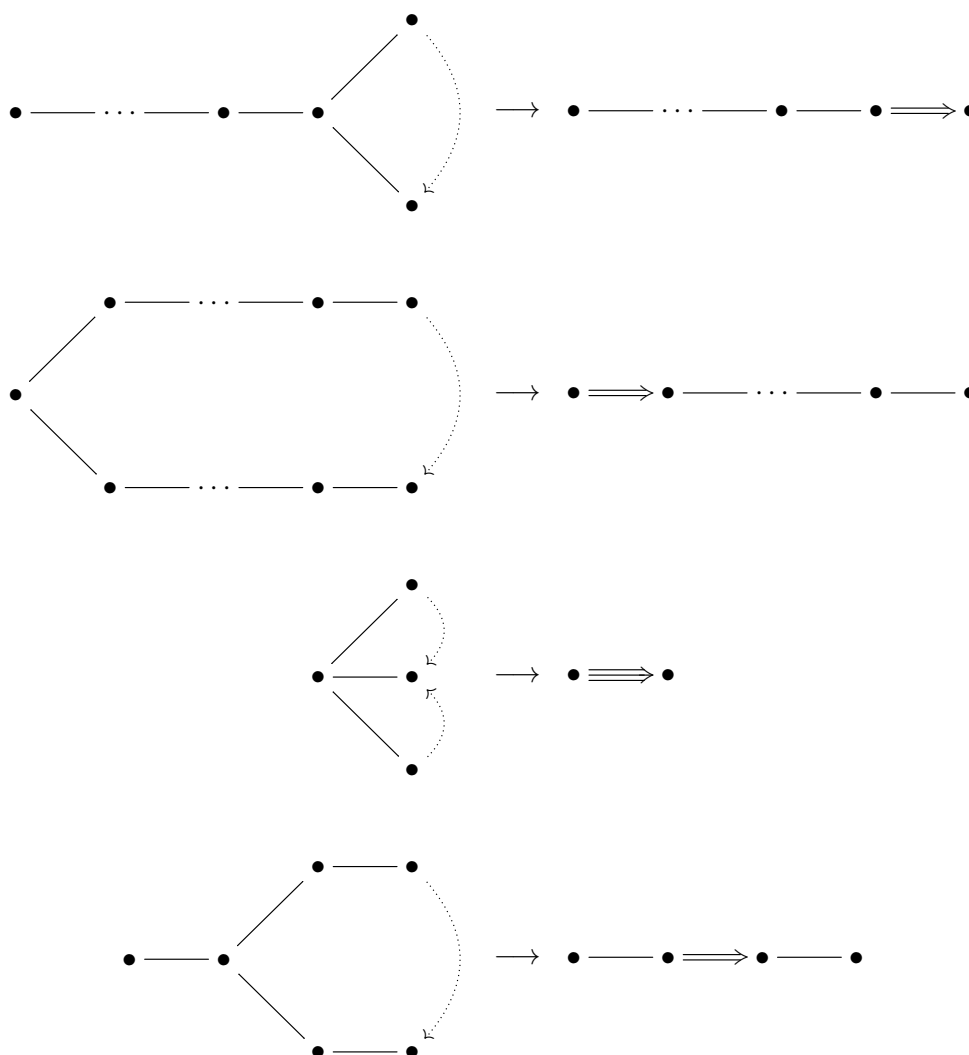
Figure 9. Dynkin diagram foldings.

a fiber of type IV^*, which correspond to an exotic E_6 seven-brane stack. A stack of Q D3 branes probing the seven-branes supports a generalized rank Q MN theory of E_6 type, denoted $H_{E_6}^Q$, which has flavor symmetry $\mathrm{SU}(2)_L \times E_6$. There are two points over the base $\mathbb{P}^1$ where two legs of the affine E_6 Dynkin diagram 'merge', which corresponds to the 'folding' of the surfaces in red in figure 10. From the perspective of the $H_{E_6}^Q$ theory supported on the stack of D3-branes, these two points on the S^2 are the locations of the folding defects. The R-charge of the folding defects is determined by the order of the monodromy cover equation in table 2. From the table we see that there are only two options: a monodromy cover of order two implies an R-charge of $\frac{1}{2}$, while a monodromy cover of order three (which occurs only in the G_2 case) implies an R-charge of $\frac{1}{3}$. These considerations result in the 4d-2d system of equation (3.20) giving rise to the BPS strings of the $\mathbf{5}_{F_4}$ theory.

Figure 10. Geometry of the $\mathbf{5}_{F_4}$ theory [41].

We remark that an alternative characterization for the $\mathbf{5}_{F_4}$ BPS strings worldsheet theories can be achieved by first adding generalized chiral defects to the theory

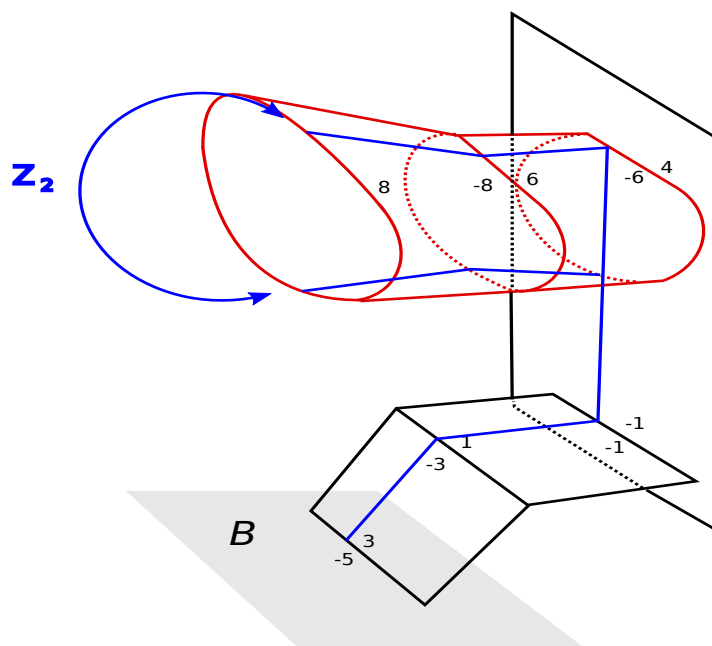$$\left(H_{E_6}^Q \,, \varnothing\right)_{S^2} \tag{3.21}$$

and then Higgsing to the $\mathbf{5}_G^F$ theory along the Higgsing three of equation (2.43). We summarize this in figure 11. From this perspective it is clear that the role of the folding defects is to project out degrees of freedom from the worldsheet 2d (0,4) SCFT.

The authors of [176] have discussed in an example how one may directly obtain the Hilbert series of the moduli spaces of instantons for a non-simply laced gauge group $G/\mathbb{G}$ by projecting out states from the Hilbert series of $\mathcal{M}_{G,1}$. It would be interesting to see whether such an explicit procedure for folding can also be implemented at the level of the elliptic genera.

3.2.2 Global anomalies and surface defects

Six-dimensional theories may suffer from global anomalies that are the analogue of the famous Witten anomaly constraining the number of half-hypers in SU(2) theories in four-dimensions [177]. These Bershadsky-Vafa anomalies occur for groups SU(2), SU(3) and G_2 only [178]. In this section we are going to study the interplay of these anomalies with the structure of chiral defects in the G_2 case.

The geometries of the G_2 models involve both flavor divisors and non-split singularities, and therefore the BPS strings are described by compactifications of D3 brane configurations
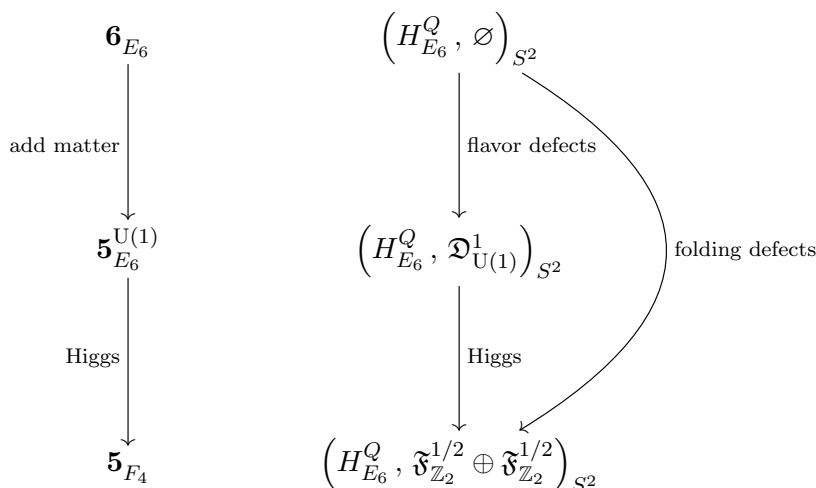
$$\mathbf{6}_{E_6}$$

$$\left(H_{E_6}^Q \,, \varnothing\right)_{S^2}$$

add matter

flavor defects

folding defects

$$\mathbf{5}_{E_6}^{\mathrm{U}(1)}$$

$$\left(H_{E_6}^Q \,, \mathfrak{D}_{\mathrm{U}(1)}^1\right)_{S^2}$$

Higgs

Higgs

$$\mathbf{5}_{F_4}$$

$$\left(H_{E_6}^Q \,, \mathfrak{F}_{\mathbb{Z}_2}^{1/2} \oplus \mathfrak{F}_{\mathbb{Z}_2}^{1/2}\right)_{S^2}$$

Figure 11. Higgsing tree and folding defects: the case of $\mathbf{5}_{F_4}$.

that involve both flavor and folding surface defects. We claim that the models

$$\left(H_{\mathrm{SO}(8)}^Q \,, \mathfrak{F}_{\mathbb{Z}_3}^{1/3} \oplus \mathfrak{F}_{\mathbb{Z}_3}^{1/3} \oplus \mathfrak{D}_{\mathrm{Sp}(1)}^{1/3} \oplus \underbrace{\mathfrak{D}_{\mathrm{Sp}(1)}^{1/3} \oplus \cdots \oplus \mathfrak{D}_{\mathrm{Sp}(1)}^{1/3}}_{3m \text{ times}}\right)_{S^2} \qquad m = 0, 1, 2 \qquad (3.22)$$

give rise to the strings of the family of six-dimensional (1,0) theories

$$(\mathbf{3} - \mathbf{m})_{G_2}^{\mathrm{Sp}(1+3m)} \qquad m = 0, 1, 2. \qquad (3.23)$$

In 6d the Bershadsky-Vafa global anomaly requires the matter to consist of either $1, 4$, or 7 fundamental hypermultiplets. From the point of view of the D3 brane probe, this corresponds to the fact that the R-charges must be quantized over the integers in order for the compactification to 2d to be consistent. The folding defects $\mathfrak{F}_{\mathbb{Z}_3}^{1/3}$ have fractional $R_{\mathfrak{F}}$ effective charges which are dictated by the order of the corresponding monodromy cover; this in turn forces the charges of the defects to be multiples of $1/3$ and their overall number to be equal to 1 mod 3. In the M-theory picture, the G_2 symmetry is realized in terms of a non-split Kodaira fiber, see figure 12 (details about G_2 fibers can also be found in [179]); the surface defects are located at the trivalent points in the fibration of the D_4 singularity along the base $\mathbb{P}^1$. Only in the G_2 case one encounters a third order non-split fiber; in all other instances where the 6d gauge algebra is non simply-laced the order of the monodromy cover is 2, and correspondingly the effective R-charges are half-integers.

3.3 BPS string anomaly inflow

The aim of this section is to determine certain coefficients in the anomaly polynomial of the worldsheet theories of the BPS strings, which will be used in sections 5 and 6 to make various statements about the structure of their IR CFTs. In particular, we first focus
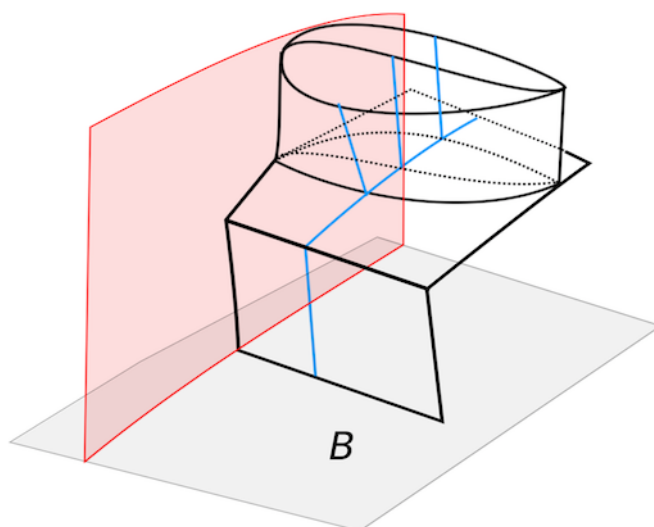
Figure 12. Geometry corresponding to a non-split singularity for the model $\mathbf{3}_{G_2}^{\mathrm{Sp}(1)}$. We interpret the folding from D_4 to G_2 in this geometry as monodromy defects for the D3 branes; in addition to this, non compact flavor divisors give rise to matter in the $\mathbf{7}$ of G_2 localized at the intersection with the compact divisor.

on the anomalies associated to both abelian and non-abelian global symmetries, and then compute gravitational and R-symmetry anomaly coefficients in term of which the central charges of the IR CFT of the string are determined.

In the presence of BPS strings the Bianchi identities for the six-dimensional anti-self-dual three-forms[13] which are the curvatures of the two-form fields in the corresponding tensor multiplets get modified [180]. It is convenient to group the three-form curvatures into a single vector, which we denote by

$$\mathbf{H} = (H^1, \ldots, H^R), \tag{3.24}$$

where $H^I = dB^I$ and R is the rank of the six-dimensional SCFT. The modified Bianchi identities read as follows:

$$d\,\mathbf{H} = \mathbf{X}_4 + \mathbf{J}_s \tag{3.25}$$

where $\mathbf{J}_s$ denotes the BPS string current. Setting $\mathbf{J}_s = 0$ above, the GSSW mechanism [154, 155] entails that whenever the anomaly polynomial for the 6d theory factors as

$$\mathcal{A}_8 = \frac{1}{2}\mathbf{X}_4 \cdot \mathbf{X}_4, \tag{3.26}$$

the gauge anomalies can be canceled. However, in case $\mathbf{J}_s \neq 0$ the contribution to the anomaly due to BPS strings needs to be canceled via an anomaly inflow mechanism. This

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[13]In the F-theory setup the tensor fields of the 6d theory arise from the reduction of C_4^+ and it is natural to adopt conventions in which they are anti-self-dual. From the perspective of the 6d field theory the opposite convention is most often chosen and the BPS strings are thus commonly referred to as self-dual.

was studied in [31, 82] (see also [181–183]) whose results we adapt to our discussion in what follows. The anomaly inflow mechanism implies that the anomalies of the 2d worldsheet theories of the BPS strings are given by

$$\mathcal{A}_4 = -\int_{N_4} \left(\mathbf{J}_s \cdot \mathbf{X}_4 + \frac{1}{2} \mathbf{J}_s \cdot \mathbf{J}_s \right), \tag{3.27}$$

where N_4 is a manifold transverse to the worldsheet of a BPS string within the worldvolume of the 6d SCFT. In particular, for a configuration with BPS string charge $\mathbf{S}$ one obtains that

$$\int_{N_4} \mathbf{J}_s \cdot \mathbf{X}_4 = \mathbf{S} \cdot \mathbf{X}_4 \qquad \text{and} \qquad \int_{N_4} \mathbf{J}_s \cdot \mathbf{J}_s = \mathbf{S} \cdot \mathbf{S} \, \chi_4(N_4). \tag{3.28}$$

This implies that

$$\mathcal{A}_4 = -\frac{1}{2} \mathbf{S} \cdot \left(2 \, \mathbf{X}_4 + \mathbf{S} \left(c_2(\mathrm{SU}(2)_L) - c_2(\mathrm{SU}(2)_R) \right) \right) \tag{3.29}$$

where we have used the identity

$$\chi_4(N_4) = c_2(\mathrm{SU}(2)_L) - c_2(\mathrm{SU}(2)_R) \tag{3.30}$$

for the Euler class of the $\mathrm{SO}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ bundle TN_4. At this point, consider a bound state of BPS strings of charge

$$\mathbf{S} = \sum_{I=1}^{R} Q^I \Sigma_I. \tag{3.31}$$

The anomaly polynomial of the corresponding 2d $(0, 4)$ BPS string worldsheet theory is completely determined by inflow from the anomaly of the 6d $(1, 0)$ theory. In particular, we have a contribution from

$$\mathbf{S} \cdot \mathbf{X}_4 = \sum_{I=1}^{R} Q^I \Sigma_I \cdot \left(\frac{1}{4} \mathbf{a} \, p_1(M_6) + \sum_\ell \mathbf{b}_\ell c_2(G^\ell) + \frac{1}{2} \sum_{i\,j} \mathbf{b}_{ij} c_1(\mathrm{U}(1)^i) c_1(\mathrm{U}(1)^j) \right), \tag{3.32}$$

where we refer to section 2.3 for the notation. These intersections can be evaluated exploiting the equations in section 2.3 and table 4. For a rank-one model with gauge symmetry G such as the ones we consider,

$$\mathbf{S} = Q \, \Sigma = Q \, \mathbf{b}_G, \tag{3.33}$$

and these equations simplify. One finds that the coefficients associated to the 't Hooft anomaly of $\mathrm{U}(1)$ flavor symmetry factors are given by

$$\mathbf{S} \cdot \mathbf{b}_{ij} = Q \, \Sigma \cdot \mathbf{b}_{ij} = Q \sum_R n_{R,ij} A_R q^i q^j \tag{3.34}$$

while the coefficients for the non-abelian flavor symmetry 't Hooft anomalies are given by

$$\mathbf{S} \cdot \mathbf{b}_\alpha = Q \, \Sigma \cdot \mathbf{b}_\alpha = Q \sum_{R,R_\alpha} n_{R,R_\alpha} A_R A_{R_\alpha}. \tag{3.35}$$

Let us illustrate this more concretely by looking at some examples; we begin with the 6d theories involving abelian flavor symmetries. The models $(\mathbf{6} - \mathbf{k})_{E_6}$ always have a U(1) factor in the flavor symmetry for $k \geq 1$. The U(1) charges can be determined by exploiting the Higgsing

$$(\mathbf{6} - \mathbf{k})_{E_7} \to (\mathbf{6} - \mathbf{k})_{E_6} \tag{3.36}$$

where E_7 is broken to E_6 leaving a global U(1). The U(1) charge for the $\mathbf{27}$ of E_6 is determined by the branching rules

$$\mathbf{133} \to \mathbf{78}_0 \oplus \mathbf{27}_2 \oplus \mathbf{27}_{-2} \oplus \mathbf{1}_0 \qquad \mathbf{56} \to \mathbf{27}_1 \oplus \overline{\mathbf{27}}_{-1} \tag{3.37}$$

which gives a fundamental hyper in the $\mathbf{27}$ with U(1) charge 1 for each $\mathbf{56}$ which survives the Higgsing. This allows us to compute

$$\Sigma \cdot \mathbf{b}_{\mathrm{U}(1)} = k \times 6 \times 1 \times 1 = 6k \quad \text{for} \quad (\mathbf{6} - \mathbf{k})_{E_6}, \quad k \geq 1. \tag{3.38}$$

where we have used $A_{\mathbf{27}} = 6$ from table 4. This is twice the coefficient which multiplies the term $c_1(\mathrm{U}(1))^2$ in the anomaly polynomial of the 2d string.

Similarly, for the $(\mathbf{4} - \mathbf{k})_{\mathrm{SO}(10)}$ theories we determine the U(1) charges by exploiting the Higgsing

$$(\mathbf{4} - \mathbf{k})_{\mathrm{SO}(12)} \to (\mathbf{4} - \mathbf{k})_{\mathrm{SO}(10)} \tag{3.39}$$

where SO(12) is broken to SO(10) leaving a global U(1). From the branching rules

$$\mathbf{66} \to \mathbf{45}_0 \oplus \mathbf{10}_2 \oplus \mathbf{10}_{-2} \oplus \mathbf{1}_0 \qquad \mathbf{12} \to \mathbf{10}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2} \qquad \mathbf{32} \to \mathbf{16}_{-1} \oplus \overline{\mathbf{16}}_1, \tag{3.40}$$

we obtain

$$\Sigma \cdot \mathbf{b}_{\mathrm{U}(1)} = k \times 4 \times 1 \times 1 = 4k \quad \text{for} \quad (\mathbf{4} - \mathbf{k})_{\mathrm{SO}(10)} \quad k \geq 1, \tag{3.41}$$

using also the fact that the vectors of SO(10) are neutral with respect to the U(1).

For the model $\mathbf{1}_{\mathrm{SU}(N)}$, as remarked by [76] only a mixed U(1) survives from the flavor symmetry arising from the $\mathrm{U}(1) \times \mathrm{U}(1)$ factor which rotates the $N + 8$ fundamental multiplets and the anti-symmetric multiplet. With respect to this mixed U(1), we claim that the charges of the fundamental and anti-symmetric multiplets are given respectively by $4 - N$ and $N + 8$, which gives

$$\begin{aligned}
\Sigma \cdot \mathbf{b}_{\mathrm{U}(1)} &= (N + 8) \times 1 \times (4 - N)^2 + 1 \times (N - 2) \times (N + 8)^2 \\
&= 2N(N - 1)(N + 8),
\end{aligned} \tag{3.42}$$

where we have used that $A_{\Lambda^2} = N - 2$ from table 4. We find that this assignment of charges is required in order for the CFT describing the BPS strings to be consistent with the results of section 6 below.

To illustrate the calculation of anomaly coefficients for non-abelian flavor symmetries, consider the theories $(\mathbf{8} - \mathbf{k})_{E_7}^{\mathrm{SO}(k)}$ with $k \geq 3$. The SO(k) anomaly coefficient receives contributions from the bifundamental $(\frac{1}{2}\mathbf{56}, \mathbf{k})$, leading to the coefficient

$$\Sigma \cdot \mathbf{b}_{\mathrm{SO}(k)} = \frac{1}{2} \times 12 \times 2 = 12. \tag{3.43}$$

Similarly, for $(\mathbf{5} - \mathbf{k})_{F_4}^{\mathrm{Sp}(k)}$ the bifundamental matter $(\mathbf{26}, \frac{1}{2}\mathbf{2k})$ leads to the anomaly coefficient

$$\Sigma \cdot \mathbf{b}_{\mathrm{SO}(k)} = 6 \times \frac{1}{2} = 3. \tag{3.44}$$

The calculation for other 6d SCFTs proceeds analogously.

In order to compute the anomalies of the BPS strings' worldsheet theory under the remaining global symmetries $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ one can exploit the following identity:

$$p_1(M_6) = p_1(N_4) + p_1(M_2) \tag{3.45}$$

where we have split the 6d $p_1(M_6)$ characteristic class in its component along the 2d worldsheet M_2 and its component along the four normal directions N_4. Then the following identity can be used to recover the coefficients of the global anomalies for the factors $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ of the global symmetry:

$$p_1(N_4) = -2c_2(\mathrm{SU}(2)_L) - 2c_2(\mathrm{SU}(2)_R). \tag{3.46}$$

Specializing the anomaly polynomial to rank one theories one obtains [31, 82]

$$\begin{aligned}
\mathcal{A}_4 &= -\frac{1}{2}\mathbf{S} \cdot \left(2\,\mathbf{X}_4 + \mathbf{S}\,(c_2(\mathrm{SU}(2)_L) - c_2(\mathrm{SU}(2)_R)) \right) \\
&= -\frac{1}{2}\mathbf{S} \cdot \mathbf{S} \left(c_2(\mathrm{SU}(2)_L) - c_2(\mathrm{SU}(2)_R) \right) \\
&\quad - \frac{1}{4}\mathbf{S} \cdot \mathbf{a} \left(p_1(M_2) - 2c_2(\mathrm{SU}(2)_L) - 2c_2(\mathrm{SU}(2)_R) \right) \\
&\quad - \mathbf{S} \cdot \mathbf{b}_{\mathrm{SU}(2)_I}\, c_2(I) - \mathbf{S} \cdot \mathbf{b}_G\, c_2(G) \\
&\quad - \mathbf{S} \cdot \mathbf{b}_{F_\alpha}\, c_2(F_\alpha) - \frac{1}{2} \sum_{ij} \mathbf{S} \cdot \mathbf{b}_{ij}\, c_1(\mathrm{U}(1)^i) c_1(\mathrm{U}(1)^j)
\end{aligned} \tag{3.47}$$

As we have reviewed in section 2.3, the F-theory geometric engineering requires to identify $\mathbf{a}$ with the class of $-K$ in $H_2(B, \mathbb{Z})$. In this case a straightforward application of the Riemann-Roch formula gives

$$-\Sigma \cdot (\Sigma + K) = 2 \quad \Rightarrow \quad \Sigma \cdot \mathbf{a} = 2 + \Sigma \cdot \Sigma = 2 - n \quad \Rightarrow \quad \mathbf{S} \cdot \mathbf{a} = Q\,(2 - n). \tag{3.48}$$

A conjecture of [7] determines the $\mathrm{SU}(2)_I$ anomaly coefficient as:

$$-\mathbf{S} \cdot \mathbf{b}_{\mathrm{SU}(2)_I} = Q h_G^\vee. \tag{3.49}$$

Finally, the gravitational anomaly k of these models, which is a measure of the difference among the central charges c_L and c_R, can be read off from the coefficient of $-\frac{1}{24}\,p_1(M_2)$. In our normalization, we obtain:

$$k = c_R - c_L = 6\,Q\,(2 - n), \tag{3.50}$$

while the coefficient of $c_2(\mathrm{SU}(2)_I)$ is given by the 2d $\mathrm{SU}(2)_I$ R-symmetry anomaly coefficient k_R, which is related to c_R by a $(0, 4)$ Ward identity, namely

$$c_R = 6k_R. \tag{3.51}$$

This determines the central charges of the 2d $\mathcal{N} = (0, 4)$ infrared CFT describing a bound state of Q BPS strings of a rank-one 6d SCFT to be the following:

$$c_R = 6Q\, h_G^\vee \quad c_L = c_R + 6Q(2 - n)$$
$$= 6Q(h_G^\vee - n + 2). \tag{3.52}$$

Notice that the unitarity of the left moving sector requires, in particular, that

$$h_G^\vee + 2 \geq n. \tag{3.53}$$

For $n > 2$, this implies $h_G^\vee > 0$. This is a manifestation from the BPS string perspective of the fact that such theories necessarily support a nontrivial gauge algebra [150]. Of course, equation (3.52) relies on the validity of the conjectural expression for the 6d (1,0) anomaly polynomial.

3.4 An analogy with 4d UV curves

At low energies the worldvolume theory of a stack of N D3-branes is described by $\mathrm{U}(N)$ $\mathcal{N} = 4$ SYM theory. If we view F-theory as the limit of M-theory on an elliptically fibered Calabi-Yau where the volume of the elliptic fiber is sent to zero, the D3-branes wrapping a divisor in the base of the Calabi-Yau arise from M5-branes wrapping a divisor of the Calabi-Yau X which also includes the elliptic fiber. Consistent with this, the coupling τ of the theory governing the D3 branes is identified with the complex structure τ_E of the elliptic fiber. This was discussed in the context of the E-string e.g. in references [184, 185], where it was found that the E-string arises as a compactification of the $\mathcal{N} = 4$ SYM in presence of a network of chiral defects, encoded in the geometry of the elliptic fibration of a del Pezzo dP_9 surface, which can be realized as a $\mathbb{P}^2$ surface blown up at 9 points.

If on the other hand a stack of M5 branes is compactified on a curve $\mathcal{C}$ with appropriate boundary conditions at punctures on the curve, it is well known that one obtains a $\mathcal{N} = 2$ SQFT whose properties are encoded by the topology of $\mathcal{C}$ and by the boundary conditions at the punctures [186, 187]. Moreover, the punctures can be viewed as originating from collections of transverse M5 branes intersecting the $\mathcal{C}$. In the IR Coulomb phase, the stack of M5 branes wrapping the curve $\mathcal{C}$ recombines to a single M5 wrapping the Seiberg-Witten curve $\Sigma \subset T^*\mathcal{C}$.

For the rank-one 6d (1,0) theories without matter $\mathbf{4}_{\mathrm{SO}(8)}$, $\mathbf{6}_{E_6}$, $\mathbf{8}_{E_7}$, and $\mathbf{12}_{E_8}$, we find it worthwhile to point out that the D3 branes are described by $\mathcal{N} = 2$ SCFTs whose UV curve $\mathcal{C}$ resembles a limit of the exceptional Kodaira fiber associated to the two-cycle on which the D3 branes are wrapped, as illustrated in figure 13.

Let us see how this works out explicitly building upon the analysis in [27] for the various 6d (1,0) SCFTs without matter. In the $\mathbf{4}_{\mathrm{SO}(8)}$ case, the I_0^* fiber can be mapped to a sphere with 4 A_1 singularities; correspondingly, the theory on a wrapped D3 brane is the 4d $\mathcal{N} = 2$ CFT with SU(2) gauge group and $N_f = 4$, which has a Gaiotto curve of class S$[A_1]$ given by a sphere with 4 punctures, each carrying an SU(2) global symmetry (cf. figure 13 (bottom)). In the $\mathbf{6}_{E_6}$ case, the Kodaira type IV^* fiber depicted in figure 13 (top) degenerates to a $\mathbb{P}^1$ with three A_2 singularities, consistent with the fact that the D3
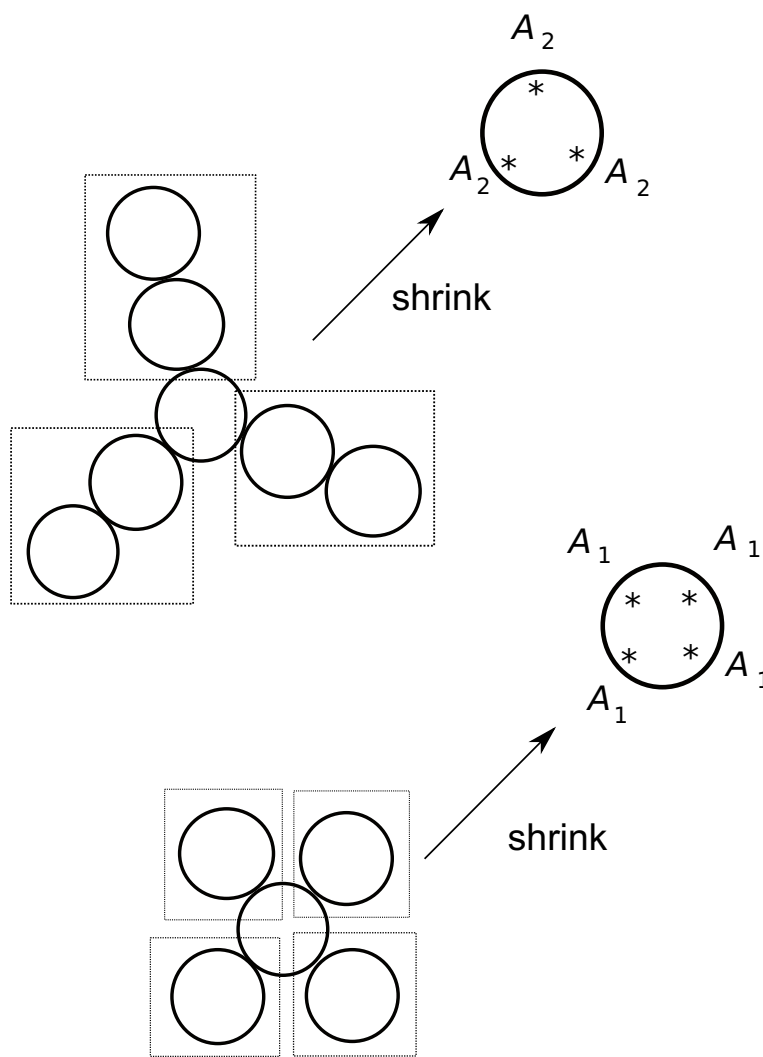
Figure 13. Analogy with Gaiotto curves. In the top part of the figure, a Kodaira fiber of type IV^* is shrunk to a sphere with three punctures reminiscent of the Gaiotto curve of the E_6 Minahan-Nemeschansky theory. In the bottom part, the analogous procedure is shown for a fiber of type I_0^*, which shrinks to the four-punctured sphere reminiscent of the Gaiotto curve for the $\mathcal{N} = 2$ SCFT with $G = \mathrm{SU}(2)$ and $N_f = 4$.

brane is described by the E_6 Minahan-Nemeschansky theory, which is the class $\mathrm{S}[A_2]$ theory associated to a sphere with 3 maximal punctures each carrying a flavor symmetry of type $\mathrm{SU}(3)$. In the $\mathbf{8}_{E_7}$ case, the Kodaira type III^* fiber degenerates to a $\mathbb{P}^1$ with singularities of type A_3, A_3 and A_1, while the worldvolume theory of the D3 brane is the E_7 MN theory, which is identified with a Sicilian theory of class $\mathrm{S}[A_4]$ that has two maximal puncturs with symmetry $\mathrm{SU}(4)$ and one puncture of a different kind supporting $\mathrm{SU}(2)$ global symmetry. Finally, in the $\mathbf{12}_{E_8}$ case the Kodaira type II^* fiber degenerates to a $\mathbb{P}^1$ with singularities of type A_5, A_2 and A_1, and correspondingly the D3 brane worldvolume theory is the E_8

MN theory, which can be realized as a Sicilian theory of class S[A_5] with three punctures with global symmetries $SU(2), SU(3)$ and $SU(5)$ respectively.

4 Elliptic genera and 6d $T^2 \times \mathbb{R}^4$ partition functions

By subjecting a (1,0) SCFT to the 6d Ω-background $T^2 \times \mathbb{R}^4_{\epsilon_1,\epsilon_2}$ [188, 189] one can compute a partition function that receives two distinct kinds of contributions arising from:

1. Towers of BPS particles arising from the 6d tensor, vector, and hypermultiplets, with KK momentum along the 6d circle, and

2. BPS strings wrapped around the 6d circle, which couple to the various tensor multiplets of the 6d SCFT.

These two classes of contributions organize themselves as follows

$$Z^{6d}_{T^2 \times \mathbb{R}^4}(\vec{\varphi}, \mathbf{m}, x, v, q) = Z_{\text{pert}}(\mathbf{m}, x, v, q) \times \left(\sum_{\mathbf{S} \in \Gamma^+} e^{-\vec{\Phi} \cdot \mathbf{S}} \, \mathbb{E}_{\mathbf{S}}(\mathbf{m}, x, v, q) \right). \tag{4.1}$$

The BPS particles contribute to the factor Z_{pert}, while the strings contribute via their elliptic genera to the remaining factor [63]. More precisely, $\mathbb{E}_{\mathbf{S}}$ is the elliptic genus of the 2d $\mathcal{N} = (0, 4)$ theory that describes a bound state of strings with charge $\mathbf{S} \in \Gamma^+$. In particular,

$$\mathbb{E}_{\mathbf{0}} = 1. \tag{4.2}$$

Let us define

$$\vec{\varphi} = \vec{\phi} + i \int_{T^2} \vec{B}, \tag{4.3}$$

where $\vec{\phi}$ are the vevs of the scalar fields in the tensor multiplets, while the imaginary components are fluxes on the torus of the two-form fields belonging to the tensor multiplets. If the I-th tensor multiplet is coupled to a gauge multiplet transforming under the gauge algebra G_I, the Yang-Mills coupling for G_I is identified with ϕ_I as follows:

$$\phi_I = \frac{8\pi^2}{g^2_{G_I}}. \tag{4.4}$$

The chemical potentials $\vec{\Phi}$ that appear on the right hand side of equation (4.1) and couple to the BPS string charges are identified with $\vec{\varphi}$, up to shifts by linear combinations of the parameters τ and $\mathbf{m}$ (see e.g. [70, 85]).

The partition function depends on several further parameters: on the fugacities $\mathbf{m}$ for the flavor and gauge symmetries of the 6d SCFT, on $q = e^{2\pi i \tau}$, where τ is the complex modulus of T^2, and on the Ω-background parameters $x = e^{2\pi i \epsilon_-}, v = e^{2\pi i \epsilon_+}$, which couple respectively to the Cartan generator of $SU(2)_L$ and to the linear combination $J^3_R + J^3_I$ of the Cartan generators J^3_R and J^3_I of $SU(2)_R$ and $SU(2)_I$, where $SU(2)_L \times SU(2)_R = SO(4)$ is the rotation group of $\mathbb{R}^4$ and $SU(2)_I$ is the R-symmetry group of the 6d (1,0) SCFT.

We focus on the single-string sector of the BPS spectrum of any 6d SCFT, for which $\mathbf{S} = (0, \ldots, 1, \ldots, 0)$. The theory that describes a single string coupled to the I-th tensor

multiplet depends on the following data: the Dirac self-pairing of the string n_I, the gauge group G_I (which couples via a Green-Schwarz term to the I-th tensor multiplet), and the effective flavor symmetry group for the string, which was discussed in section 3.1. The catalogue of theories of a single BPS string mirrors the list of rank-one 6d SCFTs of section 2.4, and Higgsing the 6d (1,0) SCFT corresponds from the perspective of its BPS string to introducing a deformation that triggers an RG flow to the CFT of the BPS string of the Higgsed 6d SCFT. As discussed in section 2.4, most of the allowed combinations of n_I and G_I allow for a single choice of flavor group F, the exceptions being

$$(n, G) = (1, \mathrm{SU}(6)) \tag{4.5}$$

for which F is either $\mathrm{SU}(14) \times \mathrm{U}(1)$ or $\mathrm{SU}(15)$,

$$(n, G) = (2, \mathrm{SO}(12)) \tag{4.6}$$

for which F is either $\mathrm{SO}(2)$ or $\mathrm{SO}(1) \times \mathrm{SO}(1)$, and

$$(n, G) = (1, \mathrm{SO}(12)) \tag{4.7}$$

for which F is either $\mathrm{SO}(3)$ or possibly $\mathrm{SO}(2) \times \mathrm{SO}(1)$; see section 6.5 for further details on these theories.

For a single string, we can split the fugacities $\mathbf{m}$ into two sets: a first set, $\mathbf{m}_{G_I}$, which couples to the Cartan of the 6d gauge algebra, and a second set, $\mathbf{m}_{F_I}$, which couples to the Cartan of the flavor symmetry group.[14] The Ramond-Ramond elliptic genus of the string associated to the I-th tensor multiplet is defined as the following trace over the Hilbert space of the theory:

$$\mathbb{E}_{\Sigma_I}(\mathbf{m}, x, v, q) = \tag{4.8}$$
$$= Tr_{RR}(-1)^{F_L + F_R} e^{2\pi i \tau H_L} e^{2\pi i \bar{\tau} H_R} x^{J_L^3} v^{J_R^3 + J_I^3} \prod_{j=1}^{\mathrm{rank}(F_I)} (m_{F_I}^j)^{K_{F_I}^j} \prod_{j=1}^{\mathrm{rank}(G_I)} (m_{G_I}^j)^{K_{G_I}^j}.$$

where $K_{F_I}^j, K_{G_I}^j$ are respectively Cartan generators for the Lie algebras F_I and G_I, and for a Lie group G the chemical potential μ_G, which takes values in the complexification of the dual of the Cartan of G, is related to the exponentiated fugacity $\mathbf{m}_G$ as follows:

$$\mathbf{m}_G = e^{2\pi i \mu_G}. \tag{4.9}$$

In explicit computations, we find it convenient to pick a specific basis in which to expand these chemical potentials. The choices we make for various groups are explained in section A.1.

The elliptic genus of the theories under consideration consists of two factors:

$$\mathbb{E}_{\Sigma_I}(\mathbf{m}, x, v, q) = \mathbb{E}_{c.m.}(x, v, q) \mathbb{E}_{n_I}^{G_I}(\mathbf{m}_{G_I}, \mathbf{m}_{F_I}, v, q). \tag{4.10}$$

---

[14]We collect basic facts of Lie algebras and their representations, as well as our notational conventions, in appendix A.1.

The first factor captures the degrees of freedom of the string associated to its motion in $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ and is given by[15]

$$\mathbb{E}_{c.m.}(x,v,q) = \frac{\eta(q)^2}{\theta_1(v\,x,q)\theta_1(v/x,q)}.$$ (4.11)

The second factor in the elliptic genus captures the 'internal' dynamics of the string that describe its propagation on the reduced moduli space $\widetilde{\mathcal{M}}_{G_I,1}$ of one G_I instanton. For a single string, this component turns out to be independent of x but does depend on v.

The Ramond boundary conditions on the left movers are the relevant ones for computing the 6d Nekrasov partition function, but we will also find it useful to consider the elliptic genus with Neveu-Schwarz boundary conditions on the left-movers; we will denote the latter elliptic genus by

$$\mathcal{E}^{G_I}_{n_I}(\mathbf{m}_{G_I}, \mathbf{m}_{F_I}, v, q).$$ (4.12)

The theory of a single BPS string flows in the infrared to an interacting fixed point described by a $\mathcal{N} = (0,4)$ CFT which we denote by $\mathbf{h}^{G_I}_{n_I}$. At the time of writing of this paper, the elliptic genera for a number of BPS string worldsheet theories have been computed, but a sizable number of cases remains for which the elliptic genera are not known. Our approach in the next sections will be to analyze the elliptic genera of the BPS string worldsheet theories from the perspective of the infrared CFT and to identify features shared by the various CFTs. In appendix C we provide a summary of the results available in the literature on which we rely in the coming sections.

5   A motivating example: strings of D-type conformal matter SCFTs

In this section we study in some detail the $\mathbf{h}^{\mathrm{Sp}(N)}_1$ CFTs describing the strings of the $\mathbf{1}_{\mathrm{Sp}(N)}$ 6d SCFTs (which correspond to conformal matter of (D_{4+N}, D_{4+N}) type). The $\mathbf{h}^{\mathrm{Sp}(N)}_1$ CFTs are particularly simple, which allows for a very detailed analysis. This brings to light several interesting properties of this class of CFTs, which in section 6 we will argue are also shared by other $\mathbf{h}^G_n$ theories. We begin in section 5.1 by reviewing the brane construction of the $\mathbf{1}_{\mathrm{Sp}(N)}$ SCFTs and give explicit formulas for elliptic genera of the $\mathbf{h}^{\mathrm{Sp}(N)}_1$ CFTs; in section 5.2 we argue that the 6d $\mathrm{SO}(16 + 4N)$ flavor symmetry is realized in the CFT as a level 1 current algebra; in section 5.3 we discuss how the 6d gauge symmetry G is realized in the string CFT, and find evidence that its contribution to the elliptic genus is captured by irreducible characters of the affine Kac-Moody algebra for $\mathrm{Sp}(N)$ at level -1; finally, in section 5.4 we discuss the NS-R elliptic genus of these theories and argue for the existence of a spectral flow that relates it to the R-R elliptic genus; we also discuss the spectrum of operators of low conformal weight counted by the NS-R elliptic genus.

5.1   Brane engineering and elliptic genus

Recall that six-dimensional (1,0) SCFTs with orthogonal or symplectic groups can be engineered by means of massive Type IIA brane constructions involving orientifold planes [156,

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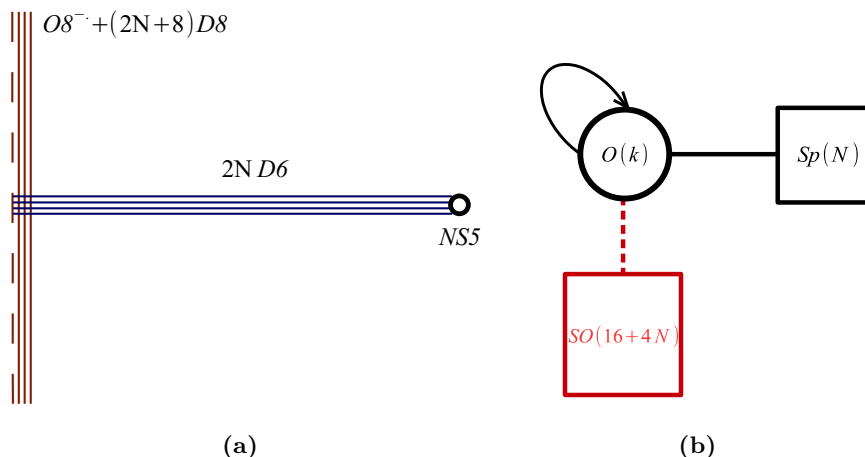[15]We refer to appendix B for our notation for modular and Jacobi forms.

Figure 14. (a): type IIA brane engineering of the $\mathbf{1}_{\mathrm{Sp}(N)}$ 6d SCFTs, and (b): 2d $(0,4)$ quiver theory for k BPS strings of the SCFT. The dashed and continuous lines symbolize respectively bifundamental Fermi and hypermultiplets; the gauge node supports an $O(k)$ adjoint vector multiplet and a hypermultiplet in the symmetric representation of $O(k)$ represented by the arrow.

190]. In particular, the SCFT of interest to us can be engineered from the brane configuration of figure 14(a), which involves one $O8^-$ orientifold plane and $(8+2N)$ D8 branes, as well as a NS5 brane and N D6 branes stretching between the NS5 brane and its mirror under the orientifold. This brane engineering is closely related to the ones considered in [68, 79]. The strings are realized in terms of D2 branes suspended between the NS5 brane and its mirror image (not shown in the figure).

The theories for k BPS strings with $G = \mathrm{Sp}(N)$ admit a UV Lagrangian description that can be read off from the brane construction presented above; the Lagrangian theory involves $(0,4)$ multiplets coupled to a $O(k)$ gauge group [68, 79]. This 2d gauge theory is summarized by the quiver of figure 14(b). The theory $\mathbf{h}_1^{\mathrm{Sp}(N)}$ of a single string in particular is extremely simple: the gauge group is just $O(1) = \mathbb{Z}_2$, which does not have an adjustable gauge coupling, so the Lagrangian realization describes directly the BPS string CFT. The spectrum of the $\mathbf{h}_1^{\mathrm{Sp}(N)}$ theory, expressed in terms of $\mathcal{N} = (0,4)$ consists of $4+N$ Fermi multiplets, transforming under the vector representation of $\mathrm{SO}(16+4N)$, and N hypermultiplets transforming under the $2N$-dimensional representation of $\mathrm{Sp}(N)$. All multiplets are charged under the $O(1)$ gauge group. From the CFT point of view, the $O(1)$ gauge symmetry can be viewed as implementing a GSO projection.

The elliptic genus of the $\mathbf{h}_1^{\mathrm{Sp}(N)}$ theory is given by [79]:

$$
\mathbb{E}_1^{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, \mathbf{m}_{\mathrm{SO}(16+4N)}, v, q) = \frac{1}{2} \sum_{a \in \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}} \left(\prod_{i=1}^{8+2N} \frac{\theta_1(e^{2\pi i a} m_{\mathrm{SO}(16+4N)}^i, q)}{\eta(q)} \right) \quad (5.1)
$$
$$
\times \left(\prod_{i=1}^{N} \frac{\eta(q)^2}{\theta_1(e^{2\pi i a} v\, m_{\mathrm{Sp}(N)}^i, q) \theta_1(e^{2\pi i a} v/m_{\mathrm{Sp}(N)}^i, q)} \right),
$$

where the chemical potentials for $\mathrm{Sp}(N)$ and $\mathrm{SO}(16+4N)$ that appear as arguments in the right hand side given by equations (A.15) and (A.15). The four sectors labeled by different

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values of a correspond to the four possible choices of $O(1)$ holonomy around the two circles of the T^2. We can alternatively use the Jacobi theta functions $\theta_\ell(z,q)$, $\ell = 1,\ldots,4$ (see appendix B for our conventions) to rewrite the elliptic genus as

$$\mathbb{E}_1^{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, \mathbf{m}_{\mathrm{SO}(16+4N)}, v, q) = \frac{1}{2} \sum_{\ell=1}^{4} \left(\prod_{i=1}^{8+2N} \frac{\theta_\ell(m_{\mathrm{SO}(16+4N)}^i, q)}{\eta(q)} \right) \qquad (5.2)$$
$$\times \left(\prod_{i=1}^{N} \frac{\eta(q)^2}{\theta_\ell(v\, m_{\mathrm{Sp}(N)}^i, q)\theta_\ell(v/m_{\mathrm{Sp}(N)}^i, q)} \right).$$

The simplicity of this class of CFTs reflects the fact that the moduli space of one $\mathrm{Sp}(N)$ instanton is just

$$\mathcal{M}_{\mathrm{Sp}(N),1} = \mathbb{H}^N/\mathbb{Z}_2 = \mathbb{C}^{2N}/\mathbb{Z}_2, \qquad (5.3)$$

where the $\mathbb{Z}_2$ acts on the $\mathbb{C}^{2N}$ coordinates $z_1,\ldots,z_{2N}$ simultaneously as:

$$z_n \to -z_n, \qquad n = 1,\ldots,2N. \qquad (5.4)$$

The bosonic degrees of freedom that capture the target space of the BPS string CFT consist simply of $2N$ free complex bosons and give rise to the second term on the right hand side of equation (5.2), with the orbifolding by $\mathbb{Z}_2$ being reflected by the sum over $\ell = 1,\ldots,4$.

Incidentally, it is straightforward to see the effect of Higgsing the $G = \mathrm{Sp}(N)$ 6d theory to the $G = \mathrm{Sp}(N-1)$ theory at the level of the elliptic genus: the former reduces to the latter when one sets[16]

$$m_{\mathrm{SO}(4N+16)}^{2N+8} = v\, m_{\mathrm{Sp}(N)}^N \qquad m_{\mathrm{SO}(4N+16)}^{2N+7} = v/m_{\mathrm{Sp}(N)}^N, \qquad (5.5)$$

which leads to cancellations between the contributions to the elliptic genus from one Fermi and one hypermultiplet. For this specialization of values, it is possible to simultaneously give a mass to the Fermi multiplet and to the hypermultiplet and remove them from the spectrum of the CFT. In the brane construction, the Higgsing is realized by removing a pair of D6 branes from the stack of $2N$ D6 branes as in figure 15.

By successively removing all D6 branes, the 6d gauge group can be completely Higgsed, and one is left with the E-string (or 'Sp(0)') theory which consists of four (0,4) Fermi multiplets (or, in other words, 16 real fermions). The elliptic genus is given by:

$$\mathbb{E}_1^{\mathrm{E\text{-}string}}(\mathbf{m}_{\mathrm{SO}(16)}, v, q) = \frac{1}{2} \sum_{\ell=1}^{4} \left(\prod_{i=1}^{8} \frac{\theta_\ell(m_{\mathrm{SO}(16)}^i, q)}{\eta(q)} \right). \qquad (5.6)$$

5.2 Current algebra realization of $F = \mathrm{SO}(16 + 4N)$

In this section we show that the flavor symmetry F of the $\mathbf{h}_1^{\mathrm{Sp}(N)}$ theories is realized in terms of a Kac-Moody algebra, a statement which we will eventually extend to all $\mathbf{h}_n^G$ theories in section 6.2.

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[16]Notice the appearance of a shift of the flavor symmetry fugacities by $v = e^{2\pi i \epsilon_+}$. See [191] for a related discussion.
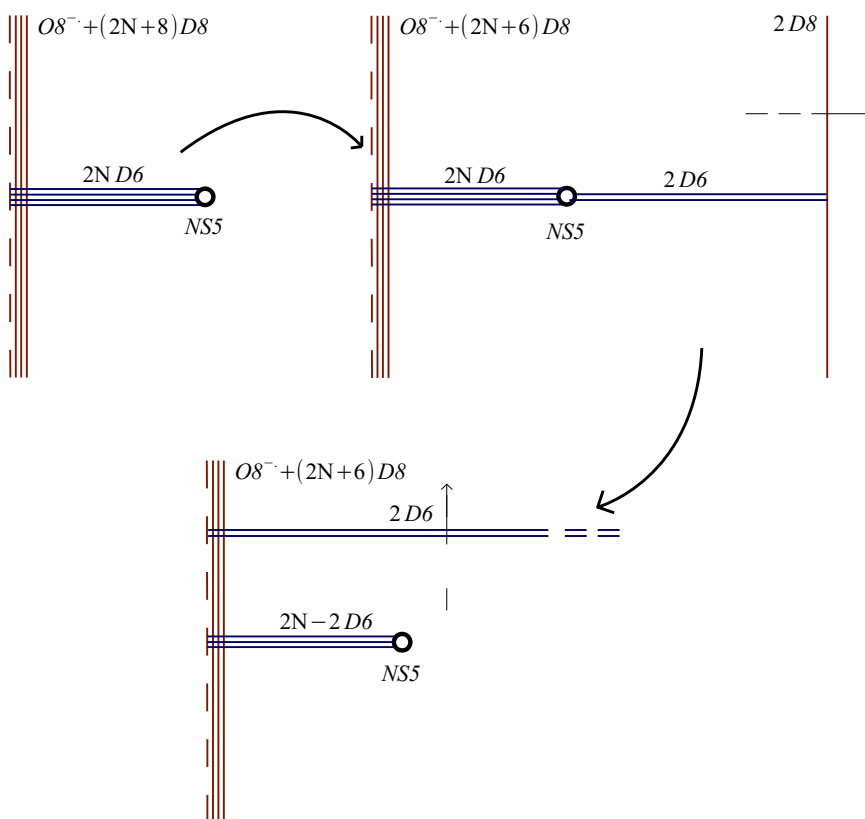
Figure 15. Higgsing of the $\mathbf{1}_{\mathrm{Sp}(N)}$ theory to the $\mathbf{1}_{\mathrm{Sp}(N-1)}$ theory. Higgsing can be performed by moving a pair of D8 branes to infinity past the NS5 brane, which creates two additional D6 branes via a Hanany-Witten transition [192]. The new D6 branes combine with two of the already existent D6 branes and are also moved off to infinity, lowering the overall number of D6 branes to $2(N-1)$.

Recall that the CFT of one E-string, equation (5.6), is given by the E_8 Kac-Moody algebra at level 1. Indeed, the elliptic genus of one E-string coincides with the character of the unique level 1 character of E_8,[17]

$$\mathbb{E}_1^{\text{E-string}} = \widehat{\chi}_{\mathbf{1}}^{E_8}(\mathbf{m}_{E_8}, q),\tag{5.7}$$

where the regular embedding of SO(16) into E_8 fixes the precise relation between $\mathbf{m}_{E_8}$ and $\mathbf{m}_{\mathrm{SO}(16)}$ fugacities. The appearance of the E_8 level 1 chiral algebra is understood since the early work of [137]: the E-string can be realized in M-theory as a M2 brane suspended between an M5 brane and an M9 brane; the latter carries E_8 gauge degrees of freedom to which the M2 brane couples chirally, giving rise to a current algebra. Generalizing to multiple E-strings, it was found in [185] that the theory of k E-strings involves a E_8 at level k sector, describing the M2-M9 boundary degrees of freedom, coupled to a residual piece of the CFT describing propagation of the bound state of strings in $\mathbb{R}^4$.

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[17]See appendices A.2 and E for our conventions on affine algebras and their characters.

In the Type IIA brane construction at weak coupling only the SO(16) maximal subgroup of E_8 is visible [193]. The elliptic genus for k strings may also be expressed in terms of characters of level k modules of the Kac-Moody algebra associated to this SO(16) subgroup. For example, for one string the elliptic genus receives contributions from the level 1 SO(16) WZW primaries associated to the vacuum and spinor representations:

$$\mathbb{E}_1^{\text{E-string}}(\mathbf{m}_{\text{SO(16)}}, v, q) = \widehat{\chi}_{\mathbf{1}}^{\text{SO(16)}}(\mathbf{m}_{\text{SO(16)}}, q) + \widehat{\chi}_{\mathbf{s}}^{\text{SO(16)}}(\mathbf{m}_{\text{SO(16)}}, q), \qquad (5.8)$$

where[18]

$$\widehat{\chi}_{\mathbf{1}}^{\text{SO(16)}}(\mathbf{m}_{\text{SO(16)}}, q) = \frac{1}{2}\left(\prod_{i=1}^{8} \frac{\theta_3(m_{\text{SO(16)}}^i, q)}{\eta(q)} + \prod_{i=1}^{8} \frac{\theta_4(m_{\text{SO(16)}}^i, q)}{\eta(q)}\right), \qquad (5.9)$$

$$\widehat{\chi}_{\mathbf{s}}^{\text{SO(16)}}(\mathbf{m}_{\text{SO(16)}}, q) = \frac{1}{2}\left(\prod_{i=1}^{8} \frac{\theta_2(m_{\text{SO(16)}}^i, q)}{\eta(q)} + \prod_{i=1}^{8} \frac{\theta_1(m_{\text{SO(16)}}^i, q)}{\eta(q)}\right). \qquad (5.10)$$

The fact that the elliptic genus can also be interpreted in terms of a SO(16) Kac-Moody algebra is of course hardly surprising: the theory of figure 14 with $N = 0$ consists simply of 16 free chiral fermions with identical boundary conditions, transforming in the vector representation of the flavor symmetry SO(16); this free CFT indeed provides a free field realization of the SO(16) current algebra at level 1, which explains the appearance of the affine flavor symmetry in the elliptic genus.

The situation for the $\mathbf{h}_1^{\text{Sp}(N)}$ theories is only slightly more complicated: the chiral half of the $\mathbf{h}_1^{\text{Sp}(N)}$ CFT again consists of a theory of free fermions in the vector representation of $F = \text{SO}(16 + 4N)$, all with identical periodicities, now coupled to a CFT of noncompact free bosons charged under $\text{SU}(2)_R \times \text{Sp}(N)$. The only coupling between the fermionic and bosonic components of the CFT arises through the assigment of identical periodicities to the two sectors. So in analogy with the E-string case one also expects the $\mathbf{h}_1^{\text{Sp}(N)}$ CFT to involve a $\text{SO}(16 + 4N)$ current algebra at level one realized in terms of the free chiral fermions, which arise from the D2-D8 strings in the brane setup of figure 14; only, for $N > 0$ we expect this sector to be coupled nontrivially to an additional sector of the CFT describing the dependence on the 6d gauge algebra.

Recall that for a WZW model the level is proportional to $\frac{1}{2}$ of the anomaly coefficient for the term in the anomaly polynomial that is associated to the corresponding 't Hooft anomaly. In our case this is precisely the coefficient of the term $c_2(F)$ within $\Sigma \cdot \mathbf{X}_4$. For the models $\mathbf{1}_{\text{Sp}(N)}$ the matter transforms as a bifundamental half hypermulti-

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[18]For $G = \text{SO}(2N)$ we adopt the shorthand notation $\mathbf{1}, \mathbf{v}, \mathbf{s}, \mathbf{c}$ to indicate respectively the level 1 WZW primaries corresponding to the trivial, vector, spinor, and conjugate spinor representations of $\text{SO}(2N)$; for a WZW model of specified level k we label the WZW primaries in terms of the Dynkin label λ of the corresponding finite representation. Characters of the WZW model are denoted by a hat (e.g. $\widehat{\chi}_{\lambda}^G(\mathbf{m}_G, q)$), as opposed to the characters of the simple Lie group G (e.g. $\chi_{\lambda}^G(\mathbf{m}_G)$). We sometimes omit the arguments of these functions when they are clear from the context. We refer to appendices A and E for our conventions for Lie algebras and WZW models.

plet $\frac{1}{2}(\mathbf{2N}, \mathbf{16+4N})$, and so we obtain

$$\Sigma \cdot \mathbf{b}_{\mathrm{SO}(16+4N)} = \frac{1}{2} \times A_{\mathrm{Sp}(N),\mathbf{2N}} \times A_{\mathrm{SO}(4N+16),\mathbf{16+4N}}$$
$$= \frac{1}{2} \times 1 \times 2 = 1. \tag{5.11}$$

The occurrence of the level 1 $\mathrm{SO}(16+4N)$ current algebra coupled to a gauge sector is easily seen at the level of the elliptic genus, which can be rewritten in terms of the four characters

$$\widehat{\chi}_{\mathbf{1}}^{\mathrm{SO}(16+4N)}, \quad \widehat{\chi}_{\mathbf{v}}^{\mathrm{SO}(16+4N)}, \quad \widehat{\chi}_{\mathbf{s}}^{\mathrm{SO}(16+4N)}, \quad \text{and} \quad \widehat{\chi}_{\mathbf{c}}^{\mathrm{SO}(16+4N)} \tag{5.12}$$

of the current algebra as follows:

$$\mathbb{E}_1^{\mathrm{Sp}(N)} = \sum_{\lambda=\mathbf{1},\mathbf{v},\mathbf{s},\mathbf{c}} \widehat{\chi}_\lambda^{\mathrm{SO}(16+4N)}(\mathbf{m}_{\mathrm{SO}(16+4N)}, q)\, \xi_\lambda^{1,\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, v, q) \tag{5.13}$$

where the functions

$$\xi_{\mathbf{1}}^{1,\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, v, q) =$$
$$= \frac{1}{2}\left(\prod_{i=1}^{N} \frac{\eta(q)}{\theta_3(v\, m_{\mathrm{Sp}(N)}^i, q)} \frac{\eta(q)}{\theta_3(v/m_{\mathrm{Sp}(N)}^i, q)} + \prod_{i=1}^{N} \frac{\eta(q)}{\theta_4(v\, m_{\mathrm{Sp}(N)}^i, q)} \frac{\eta(q)}{\theta_4(v/m_{\mathrm{Sp}(N)}^i, q)} \right),$$

$$\xi_{\mathbf{v}}^{1,\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, v, q) =$$
$$= \frac{1}{2}\left(\prod_{i=1}^{N} \frac{\eta(q)}{\theta_3(v\, m_{\mathrm{Sp}(N)}^i, q)} \frac{\eta(q)}{\theta_3(v/m_{\mathrm{Sp}(N)}^i, q)} - \prod_{i=1}^{N} \frac{\eta(q)}{\theta_4(v\, m_{\mathrm{Sp}(N)}^i, q)} \frac{\eta(q)}{\theta_4(v/m_{\mathrm{Sp}(N)}^i, q)} \right),$$

$$\xi_{\mathbf{s}}^{1,\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, v, q) =$$
$$= \frac{1}{2}\left(\prod_{i=1}^{N} \frac{\eta(q)}{\theta_2(v\, m_{\mathrm{Sp}(N)}^i, q)} \frac{\eta(q)}{\theta_2(v/m_{\mathrm{Sp}(N)}^i, q)} + (-1)^N \prod_{i=1}^{N} \frac{\eta(q)}{\theta_1(v\, m_{\mathrm{Sp}(N)}^i, q)} \frac{\eta(q)}{\theta_1(v/m_{\mathrm{Sp}(N)}^i, q)} \right),$$

$$\xi_{\mathbf{c}}^{1,\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, v, q) =$$
$$= \frac{1}{2}\left(\prod_{i=1}^{N} \frac{\eta(q)}{\theta_2(v\, m_{\mathrm{Sp}(N)}^i, q)} \frac{\eta(q)}{\theta_2(v/m_{\mathrm{Sp}(N)}^i, q)} - (-1)^N \prod_{i=1}^{N} \frac{\eta(q)}{\theta_1(v\, m_{\mathrm{Sp}(N)}^i, q)} \frac{\eta(q)}{\theta_1(v/m_{\mathrm{Sp}(N)}^i, q)} \right),$$

capture the contribution to the elliptic genus of $2N$ free noncompact complex bosons with both periodic and anti-periodic boundary conditions, which reflects the fact that the moduli space of one $\mathrm{Sp}(N)$ instanton $\mathcal{M}_{\mathrm{Sp}(N),1}$ is the orbifold space $\mathbb{C}^{2N}/\mathbb{Z}_2$.

In the $N=0$ case, corresponding to the E-string, one obtains $\xi_{\mathbf{1}}^{1,\mathrm{Sp}(0)} = \xi_{\mathbf{s}}^{1,\mathrm{Sp}(0)} = 1$ and $\xi_{\mathbf{v}}^{1,\mathrm{Sp}(0)} = \xi_{\mathbf{c}}^{1,\mathrm{Sp}(0)} = 0$, which neatly recovers expression (5.8).

At the level of central charges, the current algebra of $F = \mathrm{SO}(16+4N)$ contributes

$$c_L^F = 8 + 2N \qquad \text{and} \qquad c_R^F = 0 \tag{5.14}$$

while the residual piece of the CFT contributes

$$c_L^{\mathrm{residual}} = 4N \qquad \text{and} \qquad c_R^{\mathrm{residual}} = 6N, \tag{5.15}$$

so that the total central charge of the $\mathbf{h}_1^{\mathrm{Sp}(N)}$ CFT adds up to

$$c_L = 8 + 6N \qquad \text{and} \qquad c_R = 6N. \qquad (5.16)$$

Finally, adding the contributions $(c_L, c_R) = (4, 6)$ from the decoupled center of mass hypermultiplet which we have suppressed in this section, one obtains

$$c_L \to 6(N + 2) \qquad \text{and} \qquad c_R \to 6(N + 1), \qquad (5.17)$$

consistent with the values (3.52) computed from the anomaly polynomial.

5.3 Worldsheet realization of $G = \mathrm{Sp}(N)$

In this section we turn to a discussion of the 6d gauge symmetry $G = \mathrm{Sp}(N)$ from the perspective of the $\mathbf{h}_1^{\mathrm{Sp}(N)}$ CFT. The analysis of the G-dependent sector of the CFT is complicated by the non-compactness of the target space, on which both G and $\mathrm{SU}(2)_R$ act as isometries in a nonchiral fashion. Nonetheless, we find that the elliptic genus is captured in terms of a $\mathrm{Sp}(N)$ Kac-Moody algebra at level -1, coupled to a $c = 1$ theory that encodes the dependence on v. Eventually, we will provide a rationale for this structure as being part of the chiral algebra of the theory, but we begin by taking an agnostic approach and studying the G dependence of the series expansion of the $\xi_\lambda^{1,\mathrm{Sp}(N)}$ functions.

In order to obtain a clearer understanding of the G-dependence of the $\xi_\lambda^{1,\mathrm{Sp}(N)}$ functions, it turns out to be helpful to first rescale them by a factor of

$$\eta(q) \cdot \widetilde{\Delta}_{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, q), \qquad (5.18)$$

where

$$\widetilde{\Delta}_G(\mathbf{m}_G, q) = \prod_{j=1}^{\infty} (1 - q^j)^{\mathrm{rank}(G)} \prod_{\alpha \in \Delta_+^G} (1 - q^j m_\alpha)(1 - q^j m_\alpha^{-1}) \qquad (5.19)$$

is closely related to the Weyl-Kac determinant for the affine Lie algebra G, equation (A.37). Here, Δ_+^G denotes the set of positive roots of the Lie algebra of G, and to any root α we associate a fugacity m_α as in equation (A.9). We then expand the rescaled $\xi_\lambda^{1,\mathrm{Sp}(N)}$ functions in terms of q and v; this requires care, as the dependence on these variables is meromorphic. We choose to expand in the region $0 < q < v$, and furthermore require that $v < m_\alpha$ and $v < m_\alpha^{-1}$ for any positive root α; this allows us to first expand the $\xi_\lambda^{1,\mathrm{Sp}(N)}$ in powers of q and next perform a series expansion in v, leading to the following expressions:

$$\xi_{\mathbf{1}}^{1,\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, v, q) = \frac{q^{N/12}}{\eta(q)\widetilde{\Delta}_{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, q)} \sum_{j,k=0}^{\infty} a_{jk}^{(1)}(\mathbf{m}_{\mathrm{Sp}(N)})q^j v^k, \qquad (5.20)$$

$$\xi_{\mathbf{v}}^{1,\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, v, q) = \frac{q^{N/12+1/2}}{\eta(q)\widetilde{\Delta}_{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, q)} \sum_{j,k=0}^{\infty} a_{jk}^{(2)}(\mathbf{m}_{\mathrm{Sp}(N)})q^j v^k, \qquad (5.21)$$

$$\xi_{\mathbf{s}}^{1,\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, v, q) = \frac{q^{-N/6}}{\eta(q)\widetilde{\Delta}_{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, q)} \sum_{j,k=0}^{\infty} a_{jk}^{(3)}(\mathbf{m}_{\mathrm{Sp}(N)})q^j v^k, \qquad (5.22)$$

$$\xi_{\mathbf{c}}^{1,\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, v, q) = \frac{q^{-N/6}}{\eta(q)\widetilde{\Delta}_{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, q)} \sum_{j,k=0}^{\infty} a_{jk}^{(4)}(\mathbf{m}_{\mathrm{Sp}(N)})q^j v^k. \qquad (5.23)$$

$j \backslash k$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
0									$\chi_{(0)}$								
1							$\chi_{(2)}$				$\chi_{(2)}$						
2					$\chi_{(4)}$				$\chi_{(2)}$				$\chi_{(4)}$				
3			$\chi_{(6)}$												$\chi_{(6)}$		
4	$\chi_{(8)}$																$\chi_{(8)}$
5							$\chi_{(4)}$				$\chi_{(4)}$						

Table 8. Nonvanishing coefficients $a_{j,k}^{(1)}(\mathbf{m}_{\mathrm{Sp}(1)})$ in the series expansion of $\xi_{\mathbf{1}}^{1,\mathrm{Sp}(1)}$, written in terms of the Sp(1) characters $\chi_{(n)} = \chi_{(n)}^{\mathrm{Sp}(1)}(\mathbf{m}_{\mathrm{Sp}(1)})$.

$j \backslash k$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
0								$\text{-}\chi_{(1)}$		$\text{-}\chi_{(1)}$							
1						$\text{-}\chi_{(3)}$						$\text{-}\chi_{(3)}$					
2				$\text{-}\chi_{(5)}$										$\text{-}\chi_{(5)}$			
3		$\text{-}\chi_{(7)}$						$\text{-}\chi_{(3)}$		$\text{-}\chi_{(3)}$						$\text{-}\chi_{(7)}$	
4																	
5																	

Table 9. Nonvanishing coefficients $a_{j,k}^{(2)}(\mathbf{m}_{\mathrm{Sp}(1)})$ in the series expansion of $\xi_{\mathbf{v}}^{1,\mathrm{Sp}(1)}$, written in terms of the Sp(1) characters $\chi_{(n)} = \chi_{(n)}^{\mathrm{Sp}(1)}(\mathbf{m}_{\mathrm{Sp}(1)})$.

This expansion turns out to be quite natural, as is indicated by the fact that many of the coefficients $a_{jk}^{(i)}$ turn out to vanish, and the nonvanishing coefficients are simple linear combinations of characters $\chi_{\lambda}^{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)})$ of irreducible representations λ of Sp(N). We illustrate this by looking at some explicit examples. For $G = \mathrm{Sp}(1)$ we display the first several nonvanishing coefficients of $\xi_{\mathbf{1}}^{1,\mathrm{Sp}(1)}, \xi_{\mathbf{v}}^{1,\mathrm{Sp}(1)}, \xi_{\mathbf{s}}^{1,\mathrm{Sp}(1)}$, and $\xi_{\mathbf{c}}^{1,\mathrm{Sp}(1)}$ in tables 8–11; in tables 12 and 13 we display a similar set of coefficients for the $G = \mathrm{Sp}(2)$ functions $\xi_{\mathbf{1}}^{1,\mathrm{Sp}(2)}$ and $\xi_{\mathbf{v}}^{1,\mathrm{Sp}(2)}$ (from which tables of coefficients for $\xi_{\mathbf{s}}^{1,\mathrm{Sp}(2)}, \xi_{\mathbf{c}}^{1,\mathrm{Sp}(2)}$ may easily be obtained by spectral flow, as discussed in the next section). For $\mathrm{Sp}(N), N > 2$ one obtains similar tables of coefficients, though increasingly higher numbers of representations of Sp(N) appear.

We find that we can fully account for the Sp(N) dependence of the elliptic genus if we assume that the functions $\xi_{\lambda}^{1,\mathrm{Sp}(N)}$ take the following form:

$$\xi_{\lambda}^{1,\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, \mathbf{m}_{\mathrm{SO}(16+4N)}, v, q) = \tag{5.24}$$
$$= \sum_{\nu \in \mathrm{Rep}(\mathrm{Sp}(N))} \sum_{\ell=-2N+1}^{0} \sum_{m \in \mathbb{Z}} n_{\nu,\ell,m}^{\lambda} \times \frac{q^{-\frac{c_{\mathrm{Sp}(N)}}{24}+h_{\nu}^{\mathrm{Sp}(N)}} \chi_{\nu}^{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)})}{\widetilde{\Delta}_{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, q)} \times \frac{q^{-\frac{c_v}{24}+h_{\ell,m}^v} v^{\ell-2Nm}}{\prod_{j=1}^{\infty}(1-q^j)},$$

$j\backslash k$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
0										$\chi_{(0)}$		$\chi_{(2)}$		$\chi_{(4)}$		$\chi_{(6)}$	
1																	
2							$\chi_{(2)}$		$\chi_{(2)}$								
3																	
4					$\chi_{(4)}$					$\chi_{(4)}$							
5																	

Table 10. Nonvanishing coefficients $a_{j,k}^{(3)}(\mathbf{m}_{\mathrm{Sp}(1)})$ in the series expansion of $\xi_{\mathbf{s}}^{1,\mathrm{Sp}(1)}$, written in terms of the Sp(1) characters $\chi_{(n)} = \chi_{(n)}^{\mathrm{Sp}(1)}(\mathbf{m}_{\mathrm{Sp}(1)})$.

$j\backslash k$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
0											$\text{-}\chi_{(1)}$		$\text{-}\chi_{(3)}$		$\text{-}\chi_{(5)}$		$\text{-}\chi_{(7)}$
1								$\text{-}\chi_{(1)}$									
2																	
3						$\text{-}\chi_{(3)}$					$\text{-}\chi_{(3)}$						
4								$\text{-}\chi_{(3)}$									
5				$\text{-}\chi_{(5)}$									$\text{-}\chi_{(5)}$				

Table 11. Nonvanishing coefficients $a_{j,k}^{(4)}(\mathbf{m}_{\mathrm{Sp}(1)})$ in the series expansion of $\xi_{\mathbf{c}}^{1,\mathrm{Sp}(1)}$, written in terms of the Sp(1) characters $\chi_{(n)} = \chi_{(n)}^{\mathrm{Sp}(1)}(\mathbf{m}_{\mathrm{Sp}(1)})$.

where $\chi_{\nu}^{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)})$ is the character of the irreducible representation ν of $\mathrm{Sp}(N)$, and the quantities

$$c_{\mathrm{Sp}(N)} = -(2N+1) \qquad \text{and} \qquad h_{\nu}^{\mathrm{Sp}(N)} = \frac{\langle \nu, \nu + 2\rho_{\mathrm{Sp}(N)}\rangle}{2N} \qquad (5.25)$$

are respectively the central charge of the (chiral half of the) $\mathrm{Sp}(N)$ WZW model at level $k = -1$, and the conformal dimension of a WZW primary of this theory. On the other hand, the v-depedence of the elliptic genus is suggestive of a CFT at $c = 1$ (consistent with the fact that in the q, v expansion of the elliptic genus only a $\mathrm{U}(1)_v$ generated by the Cartan of the $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_I$ global symmetry is visible), where $h_{\ell,m}^v$ is the conformal dimension of a $\mathrm{U}(1)_{-N}$ WZW primary of charge $\ell - 2\,N\,m$:

$$h_{\ell,m}^v = -\frac{(\ell - 2Nm)^2}{4N}. \qquad (5.26)$$

The only data that is not fixed in equation (5.24) are the coefficients $n_{\nu,\ell,m}^\lambda$, which by inspection all turn out to be either 1, 0, or -1.[19] For fixed λ, ν, ℓ, only a finite number

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[19]Though for the theories considered in section 6 other values also occur.

$j\backslash k$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0								$\chi_{(00)}$							
1						$\chi_{(20)}$		$\chi_{(01)}$		$\chi_{(20)}$					
2				$\chi_{(40)}$		$-\chi_{(02)}$				$-\chi_{(02)}$		$\chi_{(40)}$			
3		$\chi_{(60)}$		$-\chi_{(22)}$			$-\chi_{(21)}$					$-\chi_{(22)}$		$\chi_{(60)}$	

Table 12. Nonvanishing coefficients $a_{j,k}^{(1)}(\mathbf{m}_{\mathrm{Sp}(2)})$ in the series expansion of $\xi_{\mathbf{1}}^{1,\mathrm{Sp}(2)}$, written in terms of the Sp(2) characters $\chi_{(n_1 n_2)} = \chi_{(n_1 n_2)}^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)})$.

$j\backslash k$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0							$-\chi_{(10)}$		$-\chi_{(10)}$						
1					$-\chi_{(30)}$						$-\chi_{(30)}$				
2			$-\chi_{(50)}$		$\chi_{(12)}$						$\chi_{(12)}$		$-\chi_{(50)}$		
3	$-\chi_{(70)}$		$\chi_{(32)}$				$\chi_{(12)}$		$\chi_{(12)}$				$\chi_{(32)}$		$-\chi_{(70)}$

Table 13. Nonvanishing coefficients $a_{j,k}^{(2)}(\mathbf{m}_{\mathrm{Sp}(2)})$ in the series expansion of $\xi_{\mathbf{v}}^{1,\mathrm{Sp}(2)}$, written in terms of the Sp(2) characters $\chi_{(n_1 n_2)} = \chi_{(n_1 n_2)}^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)})$.

of coefficients are nonvanishing due to the requirement that H_L be bounded from below. Moreover, the level-matching condition for the spectrum of the CFT on the torus implies that $n_{\nu,\ell,m}^{\lambda} = 0$ for the choices of λ, ν, ℓ, m for which $H_L + \frac{c_L - c_R}{12}$ is not an integer.

Equation (5.24) turns out to be quite constraining, as one sees by looking at an example. We consider the case $G = \mathrm{Sp}(2)$ and for definiteness focus on the function $\xi_{\mathbf{1}}^{1,\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)}, v, q)$. We compute the first several coefficients in equation (5.24), only keeping terms consistent with H_L being a positive integer, and expand the resulting expression as in equation (5.20). The nonvanishing coefficients are displayed in table 14 (we do not display the ones with $k > 0$ since they are specular to the ones with $k < 0$). Comparing with table 12, one finds that of the nine undetermined coefficients $n_{\nu,\ell,m}^{\mathbf{0}}$, five turn out to be equal to 1:

$$n_{(00),0,0}^{\mathbf{0}} = n_{(20),-2,0}^{\mathbf{0}} = n_{(40),0,1}^{\mathbf{0}} = n_{(60),-2,1}^{\mathbf{0}} = n_{(01),0,0}^{\mathbf{0}} = 1; \qquad (5.27)$$

four turn out to be equal to -1:

$$n_{(02),-2,0}^{\mathbf{0}} = n_{(22),0,1}^{\mathbf{0}} = n_{(21),0,0}^{\mathbf{0}} = -1, \qquad (5.28)$$

and one vanishes:

$$n_{(23),-2,1}^{\mathbf{0}} = 0. \qquad (5.29)$$

$j\backslash k$	-7	-6	-5	-4	-3	-2	-1	0
0								$n_{(00),0,0}^{\mathbf{0}}\chi_{(00)}$
1						$n_{(20),-2,0}^{\mathbf{0}}\chi_{(20)}$		$n_{(01),0,0}^{\mathbf{0}}\chi_{(01)}$
2				$n_{(40),0,1}^{\mathbf{0}}\chi_{(40)}$	$n_{(02),-2,0}^{\mathbf{0}}\chi_{(02)}$			
3		$n_{(60),-2,1}^{\mathbf{0}}\chi_{(60)} + n_{(23),-2,1}^{\mathbf{0}}\chi_{(23)}$	$n_{(22),0,1}^{\mathbf{0}}\chi_{(22)}$					$n_{(21),0,0}^{\mathbf{0}}\chi_{(21)}$

Table 14. Potential nonvanishing coefficients $a_{j,k}^{(1)}(\mathbf{m}_{\mathrm{Sp}(2)})$ in the series expansion of $\xi_{\mathbf{1}}^{1,\mathrm{Sp}(2)}$, obtained from equation (5.24). The coefficients are written in terms of the Sp(2) characters $\chi_{(n_1 n_2)} = \chi_{(n_1 n_2)}^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)})$.

For $G = \mathrm{Sp}(N)$ with other values of N we have similarly performed extensive checks that (5.24) is consistent with the known expressions for the elliptic genus.

While $F = \mathrm{SO}(16 + 4N)$ is truly a chiral symmetry of the $\mathbf{h}_1^{\mathrm{Sp}(N)}$ CFT, the flavor symmetry $G = \mathrm{Sp}(N)$ is not. Nevertheless, one can argue on quite general grounds that flavor symmetries of 2d theories with $(0,2)$ supersymmetry are realized as affine Kac-Moody subalgebras of the chiral algebra [194]. In particular, flavor symmetries that act on noncompact degrees of freedom lead to current algebras at negative level.

This is of course consistent with the string worldsheet anomaly. Indeed, by the same argument that fixed the level of the 6d flavor symmetry contribution $\mathrm{SO}(16 + 4N)_1$, one can see that the coefficient of $c_2(G)$ in the string worldsheet anomaly is given by

$$\Sigma \cdot \mathbf{b}_{\mathrm{Sp}(N)} = \Sigma \cdot \Sigma = -1 \tag{5.30}$$

because of the identification of the anomaly coefficient $\mathbf{b}_{\mathrm{Sp}(N)}$ with the unit string charge vector of the corresponding BPS string instanton Σ.

As a consequence, we expect the G dependence of the elliptic genus to be given in terms of irreducible characters of the level $k = -1$ affine $\mathrm{Sp}(N)$ algebra. In the remainder of the section we provide empirical evidence that this is indeed the case, while a more thorough study of the chiral algebra will be undertaken in a future work [195].

The Kazhdan-Lusztig conjecture, which has been proven for affine Kac-Moody algebras at level $k > h_G^\vee$ in [196, 197], provides a prescription to determine the irreducible modules of the Kac-Moody algebra and their characters by evaluating a certain collection of Kazhdan-Lusztig polynomials [198]; see e.g. appendix C of [199] for an explanation of this relation. We have implemented the computation of the characters with the aid of the `Mathematica` and `Sage` platforms and the `Coxeter` program [200]. We illustrate our results for the two cases $G = \mathrm{Sp}(1)$ and $\mathrm{Sp}(2)$. The first case turns out to be trivial: the irreducible $\mathrm{Sp}(1)_{-1}$ characters turn out to be given simply by

$$\widehat{\chi}_{(n)}^{\mathrm{Sp}(1)}(\mathbf{m}_{\mathrm{Sp}(1)}, q) = q^{-\frac{c_{\mathrm{Sp}(1)}}{24} + h_\nu^{\mathrm{Sp}(1)}} \frac{\chi_{(n)}^{\mathrm{Sp}(1)}(\mathbf{m}_{\mathrm{Sp}(1)})}{\widetilde{\Delta}_{\mathrm{Sp}(1)}(\mathbf{m}_{\mathrm{Sp}(1)}, q)}, \tag{5.31}$$

ν	$\widehat{\chi}_\nu^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)},q) \cdot \widetilde{\Delta}_{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)},q)$
(00)	$q^{\frac{5}{24}}\left[\chi_{(00)}^{\mathrm{Sp}(2)} - \chi_{(22)}^{\mathrm{Sp}(2)}q^5 + \chi_{(41)}^{\mathrm{Sp}(2)}q^6 + \chi_{(04)}^{\mathrm{Sp}(2)}q^7 - \chi_{(61)}^{\mathrm{Sp}(2)}q^{10} + \chi_{(80)}^{\mathrm{Sp}(2)}q^{12} + \mathcal{O}(q^{16})\right]$
(10)	$q^{\frac{5}{24}+\frac{1}{2}}\left[\chi_{(10)}^{\mathrm{Sp}(2)} - \chi_{(12)}^{\mathrm{Sp}(2)}q^3 + \chi_{(50)}^{\mathrm{Sp}(2)}q^5 + \chi_{(14)}^{\mathrm{Sp}(2)}q^8 - \chi_{(52)}^{\mathrm{Sp}(2)}q^{10} \right.$ $\left. + \chi_{(90)}^{\mathrm{Sp}(2)}q^{14} - \chi_{(16)}^{\mathrm{Sp}(2)}q^{15} + \mathcal{O}(q^{16})\right]$
(01)	$q^{\frac{5}{24}+1}\left[\chi_{(01)}^{\mathrm{Sp}(2)} - \chi_{(21)}^{\mathrm{Sp}(2)}q^2 + \chi_{(40)}^{\mathrm{Sp}(2)}q^3 + \chi_{(05)}^{\mathrm{Sp}(2)}q^9 - \chi_{(62)}^{\mathrm{Sp}(2)}q^{12} - \chi_{(25)}^{\mathrm{Sp}(2)}q^{13} \right.$ $\left. + (\chi_{(44)}^{\mathrm{Sp}(2)} + \chi_{(81)}^{\mathrm{Sp}(2)})q^{14} + \mathcal{O}(q^{15})\right]$
(20)	$q^{\frac{5}{24}+1}\left[\chi_{(20)}^{\mathrm{Sp}(2)} - \chi_{(02)}^{\mathrm{Sp}(2)}q - \chi_{(03)}^{\mathrm{Sp}(2)}q^3 + (\chi_{(60)}^{\mathrm{Sp}(2)} + \chi_{(23)}^{\mathrm{Sp}(2)})q^6 - \chi_{(42)}^{\mathrm{Sp}(2)}q^7 + \chi_{(24)}^{\mathrm{Sp}(2)}q^9 \right.$ $\left. - \chi_{(43)}^{\mathrm{Sp}(2)}q^{10} - \chi_{(06)}^{\mathrm{Sp}(2)}q^{12} + \mathcal{O}(q^{15})\right]$
(30)	$q^{\frac{5}{24}+\frac{3}{2}}\left[\chi_{(30)}^{\mathrm{Sp}(2)} - \chi_{(12)}^{\mathrm{Sp}(2)}q + \chi_{(14)}^{\mathrm{Sp}(2)}q^6 + \chi_{(70)}^{\mathrm{Sp}(2)}q^7 - \chi_{(52)}^{\mathrm{Sp}(2)}q^8 - \chi_{(16)}^{\mathrm{Sp}(2)}q^{13} + \mathcal{O}(q^{15})\right]$
(40)	$q^{\frac{5}{24}+2}\left[\chi_{(40)}^{\mathrm{Sp}(2)} - \chi_{(22)}^{\mathrm{Sp}(2)}q + \chi_{(04)}^{\mathrm{Sp}(2)}q^3 + \chi_{(05)}^{\mathrm{Sp}(2)}q^6 + \chi_{(80)}^{\mathrm{Sp}(2)}q^8 - \chi_{(62)}^{\mathrm{Sp}(2)}q^9 - \chi_{(25)}^{\mathrm{Sp}(2)}q^{10} \right.$ $\left. + \chi_{(44)}^{\mathrm{Sp}(2)}q^{11} + \mathcal{O}(q^{14})\right]$
(50)	$q^{\frac{5}{24}+\frac{5}{2}}\left[\chi_{(50)}^{\mathrm{Sp}(2)} - \chi_{(32)}^{\mathrm{Sp}(2)}q + \chi_{(14)}^{\mathrm{Sp}(2)}q^3 + \chi_{(90)}^{\mathrm{Sp}(2)}q^9 - (\chi_{(72)}^{\mathrm{Sp}(2)} + \chi_{(16)}^{\mathrm{Sp}(2)})q^{10} \right.$ $\left. + \chi_{(54)}^{\mathrm{Sp}(2)}q^{12} + \mathcal{O}(q^{14})\right]$
(60)	$q^{\frac{5}{24}+3}\left[\chi_{(60)}^{\mathrm{Sp}(2)} - \chi_{(42)}^{\mathrm{Sp}(2)}q + \chi_{(24)}^{\mathrm{Sp}(2)}q^3 - \chi_{(06)}^{\mathrm{Sp}(2)}q^6 + (\chi_{(10,0)}^{\mathrm{Sp}(2)} - \chi_{(07)}^{\mathrm{Sp}(2)})q^{10} \right.$ $\left. - \chi_{(82)}^{\mathrm{Sp}(2)}q^{11} + \mathcal{O}(q^{13})\right]$
(70)	$q^{\frac{5}{24}+\frac{7}{2}}\left[\chi_{(70)}^{\mathrm{Sp}(2)} - \chi_{(52)}^{\mathrm{Sp}(2)}q + \chi_{(34)}^{\mathrm{Sp}(2)}q^3 - \chi_{(16)}^{\mathrm{Sp}(2)}q^6 + \chi_{(11,0)}^{\mathrm{Sp}(2)}q^{11} - \chi_{(92)}^{\mathrm{Sp}(2)}q^{12} + \mathcal{O}(q^{13})\right]$
(80)	$q^{\frac{5}{24}+4}\left[\chi_{(80)}^{\mathrm{Sp}(2)} - \chi_{(62)}^{\mathrm{Sp}(2)}q + \chi_{(44)}^{\mathrm{Sp}(2)}q^3 - \chi_{(26)}^{\mathrm{Sp}(2)}q^6 + \chi_{(08)}^{\mathrm{Sp}(2)}q^{10} + \mathcal{O}(q^{12})\right]$
(90)	$q^{\frac{5}{24}+\frac{9}{2}}\left[\chi_{(90)}^{\mathrm{Sp}(2)} - \chi_{(72)}^{\mathrm{Sp}(2)}q + \chi_{(54)}^{\mathrm{Sp}(2)}q^3 - \chi_{(36)}^{\mathrm{Sp}(2)}q^6 + \chi_{(18)}^{\mathrm{Sp}(2)}q^{10} + \mathcal{O}(q^{12})\right]$
(10,0)	$q^{\frac{5}{24}+5}\left[\chi_{(10,0)}^{\mathrm{Sp}(2)} - \chi_{(82)}^{\mathrm{Sp}(2)}q + \chi_{(64)}^{\mathrm{Sp}(2)}q^3 - \chi_{(46)}^{\mathrm{Sp}(2)}q^6 + \chi_{(28)}^{\mathrm{Sp}(2)}q^{10} + \mathcal{O}(q^{11})\right]$
(11,0)	$q^{\frac{5}{24}+\frac{11}{2}}\left[\chi_{(11,0)}^{\mathrm{Sp}(2)} - \chi_{(92)}^{\mathrm{Sp}(2)}q + \chi_{(74)}^{\mathrm{Sp}(2)}q^3 - \chi_{(56)}^{\mathrm{Sp}(2)}q^6 + \chi_{(38)}^{\mathrm{Sp}(2)}q^{10} + \mathcal{O}(q^{11})\right]$
...	...
(n,0)	$q^{\frac{5}{24}+\frac{n}{2}}\left[\chi_{(n,0)}^{\mathrm{Sp}(2)} - \chi_{(n-2,2)}^{\mathrm{Sp}(2)}q + \chi_{(n-4,4)}^{\mathrm{Sp}(2)}q^3 - \chi_{(n-6,6)}^{\mathrm{Sp}(2)}q^6 + \mathcal{O}(q^{10})\right] \qquad (n>12)$

Table 15. Irreducible characters of Sp(2) at level -1 (rescaled by an overall factor of $\widetilde{\Delta}_{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)},q)$), computed up to $\mathcal{O}(q^{\frac{5}{24}+16})$. For n sufficiently high, the leading order terms in the $\nu=(n,0)$ representation behave regularly, as indicated in the last entry; we have checked the validity of this formula up to $\mathcal{O}(q^{\frac{5}{24}+16})$ for $n \leq 31$.

which are exactly the quantities appearing in equation (5.24), and the $\xi_\lambda^{1,\mathrm{Sp}(N)}$ functions can be written in terms of them as:

$$\xi_{\mathbf{1}}^{1,\mathrm{Sp}(1)}(\mathbf{m}_{\mathrm{Sp}(1)},v,q) = \sum_{n=0}^{\infty}\widehat{\chi}_{(2n)}^{\mathrm{Sp}(1)}(\mathbf{m}_{\mathrm{Sp}(1)},q)\frac{\sum_{m=-n}^{n}q^{-m^2}v^{2m}}{\eta(q)}, \qquad (5.32)$$

$$\xi_{\mathbf{v}}^{1,\mathrm{Sp}(1)}(\mathbf{m}_{\mathrm{Sp}(1)},v,q) = -\sum_{n=0}^{\infty}\widehat{\chi}_{(2n+1)}^{\mathrm{Sp}(1)}(\mathbf{m}_{\mathrm{Sp}(1)},q)\frac{\sum_{m=-n-1}^{n}q^{-\frac{1}{4}(-1-2m)^2}v^{-1-2m}}{\eta(q)}, \qquad (5.33)$$

and

$$\xi_{\mathbf{s}}^{1,\mathrm{Sp}(1)}(\mathbf{m}_{\mathrm{Sp}(1)},v,q) = \xi_{\mathbf{1}}^{1,\mathrm{Sp}(1)}(\mathbf{m}_{\mathrm{Sp}(1)},q^{1/2}/v,q), \tag{5.34}$$

$$\xi_{\mathbf{c}}^{1,\mathrm{Sp}(1)}(\mathbf{m}_{\mathrm{Sp}(1)},v,q) = \xi_{\mathbf{v}}^{1,\mathrm{Sp}(1)}(\mathbf{m}_{\mathrm{Sp}(1)},q^{1/2}/v,q). \tag{5.35}$$

For $G = \mathrm{Sp}(2)$, on the other hand, we find:

$$\xi_{\mathbf{1}}^{1,\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)},v,q) = \left(\widehat{\chi}_{(0,0)}^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)},q) + \widehat{\chi}_{(0,1)}^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)},q)\right)\frac{1}{\eta(q)}$$
$$+ \sum_{n=1}^{\infty} \widehat{\chi}_{(2n,0)}^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)},q)\frac{q^{-n^2}(v^{-2n}+v^{2n})}{\eta(q)}, \tag{5.36}$$

$$\xi_{\mathbf{v}}^{1,\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)},v,q) = -\sum_{n=0}^{\infty} \widehat{\chi}_{(2n+1,0)}^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)},q)\frac{q^{-\frac{1}{4}(2n+1)^2}(v^{-2n-1}+v^{2n+1})}{\eta(q)}. \tag{5.37}$$

We have verified these expressions up to $\mathcal{O}(q^{\frac{1}{6}+15})$ and $\mathcal{O}(q^{\frac{2}{3}+15})$ respectively. The relevant level -1 $\mathrm{Sp}(2)$ characters, computed to sufficiently high order, are reported in table 15. One also finds that

$$\xi_{\mathbf{s}}^{1,\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)},v,q) = \xi_{\mathbf{1}}^{1,\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)},q^{1/2}/v,q), \tag{5.38}$$

$$\xi_{\mathbf{c}}^{1,\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)},v,q) = \xi_{\mathbf{v}}^{1,\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)},q^{1/2}/v,q). \tag{5.39}$$

In section 6.3 we will discuss the generalization of the results of this section to arbitrary $\mathbf{h}_n^G$ theories.

5.4 NS-R elliptic genus, spectral flow, and low energy spectrum

In this section we compute the NS-R elliptic genus[20] of the $\mathbf{h}_1^{\mathrm{Sp}(N)}$ theories and analyze the low energy states that contribute to it, remarking on their interpretation from the point of view of the parent 6d (1,0) SCFT. We also comment on the existence of a spectral flow that relates the NS-R and R-R elliptic genera.

The NS-R elliptic genus is obtained by imposing anti-periodic boundary conditions on the chiral fermions:

$$\mathcal{E}_1^{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)},\mathbf{m}_{\mathrm{SO}(16+4N)},v,q) = \frac{1}{2}\sum_{a\in\{0,\frac{1}{2},\frac{\tau}{2},\frac{1+\tau}{2}\}}\left(\prod_{i=1}^{8+2N}\frac{\theta_1(e^{2\pi ia}q^{1/2}m_{\mathrm{SO}(16+4N)}^i,q)}{\eta(q)}\right)$$
$$\times\left(\prod_{i=1}^{N}\frac{\eta(q)^2}{\theta_1(e^{2\pi ia}vm_{\mathrm{Sp}(N)}^i,q)\theta_1(e^{2\pi ia}v/m_{\mathrm{Sp}(N)}^i,q)}\right). \tag{5.40}$$

The NS boundary conditions on the fermions are reflected in the factor of $q^{1/2} = e^{\pi i\tau}$ in the argument of the theta functions in the first factor of the elliptic genus. As the same time, the sum over $O(1)$ holonomies also instructs us to sum over different boundary conditions. For

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[20]Note that for this particular class of 2d CFTs the terminology 'NS-R elliptic genus' is somewhat of a misnomer, since the sum over $O(1) = \mathbb{Z}_2$ holonomies leads automatically to summing over different fermion and boson periodicities.

the $N = 0$ case of the E-string, for example, the choice of NS versus R boundary conditions is immaterial since the shift by $q^{1/2}$ can be readsorbed by shifting $a \to a - \tau/2$ in the sum. On the other hand, for $N \geq 1$ the choice of boundary conditions does matter, since the sum over holonomies also affects the hypermultiplets in the second term on the right hand side of equation (5.40). One can again absorb the factor of $q^{1/2}$ by a shift in the holonomy a, but this must be compensated by simultaneously shifting $v \to v\, q^{1/2}$.[21] This leads to the following relation between NS-R and R-R elliptic genera (valid up to an overall minus sign):

$$\mathcal{E}_1^{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, \mathbf{m}_{\mathrm{SO}(16+4N)}, v, q) = q^{\frac{k_{\epsilon_+}}{4}} v^{-k_{\epsilon_+}} \mathbb{E}_1^{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, \mathbf{m}_{\mathrm{SO}(16+4N)}, q^{1/2}/v, q),$$
(5.41)

where $k_{\epsilon_+} = 1 - h_{\mathrm{Sp}(N)}^\vee$ is the ϵ_+-index of the elliptic genus, which can be read off from the ϵ_+^2 coefficient in the anomaly polynomial (3.47) [83, 91], upon setting $c_2(R) = c_2(I) = -\epsilon_+^2$ and subtracting the contributions to the anomaly polynomial from the center of mass hypermultiplet. Decomposing the elliptic genus as in equation (5.13), one finds that the effect of the transformation $v \to q^{1/2}/v$ is simply to exchange

$$\xi_{\mathbf{1}}^{1,\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, \mathbf{m}_{\mathrm{SO}(16+4\,N)}, v, q) \leftrightarrow \xi_{\mathbf{s}}^{1,\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, \mathbf{m}_{\mathrm{SO}(16+4\,N)}, v, q),$$
(5.42)

$$\xi_{\mathbf{v}}^{1,\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, \mathbf{m}_{\mathrm{SO}(16+4\,N)}, v, q) \leftrightarrow \xi_{\mathbf{c}}^{1,\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, \mathbf{m}_{\mathrm{SO}(16+4\,N)}, v, q),$$
(5.43)

which is evident from looking at e.g. tables 8–11 for $G = \mathrm{Sp}(1)$. We will discuss this spectral flow in more detail in section 6.4, where we will find it to be a property of all $\mathbf{h}_n^G$ theories.

We next turn to a discussion of the low energy spectrum in the NS-R sector. This sector has a unique fermionic vacuum tensored with a degenerate set of bosonic vacua which arise from the zero energy modes of the scalars $q_i, \widetilde{q}_i$, $1 = 1, \ldots, N$ in the hypermultiplets. In the Ramond sector one finds a set of fermionic zero modes, which originate from the Fermi multiplets charged under the 6d flavor symmetry group $\mathrm{SO}(16 + 4N)$; on the other hand, since in the Neveu-Schwarz sector the fermionic vacuum is unique, the zero energy sector is captured simply by the ADHM quantum mechanics of one pure $\mathrm{Sp}(N)$ instanton, represented by the following 1d $\mathcal{N} = 2$ quiver:
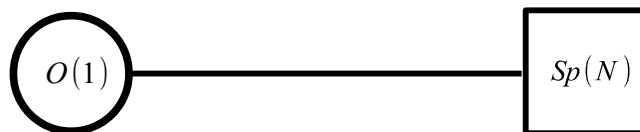
Accordingly, we find that the lowest energy component of the NS-R elliptic genus is given by

$$\mathcal{E}_1^{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, \mathbf{m}_{\mathrm{SO}(16+4N)}, v, q)\Big|_{q^{-\frac{1}{3}-\frac{N}{4}}} = \frac{1}{2} \sum_{s=\pm 1} \prod_{j=1}^{N} \frac{v}{(1 - s\, v\, m_{\mathrm{Sp}(N)}^j)(1 - s\, v/m_{\mathrm{Sp}(N)}^j)}.$$
(5.44)

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[21]In fact, we find it more convenient to combine this shift with the transformation $v \to 1/v$, which only changes the elliptic genus by an overall phase, so we express the spectral flow as $v \to q^{1/2}/v$.

This may be expanded in a power series in v as follows:[22]

$$\mathcal{E}_1^{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, \mathbf{m}_{\mathrm{SO}(16+4N)}, v, q)\Big|_{q^{-\frac{1}{3}-\frac{N}{4}}} = v^N \sum_{k=0}^{\infty} v^{2k} \chi_{k \cdot \theta_{\mathrm{Sp}(N)}}^{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}), \qquad (5.45)$$

which coincides with the Hilbert series of the moduli space of one $\mathrm{Sp}(N)$ instanton [139] up to an overall factor of v^N; in this expression, $\theta_{\mathrm{Sp}(N)}$ is the highest weight of the adjoint representation of $\mathrm{Sp}(N)$,

$$\theta_{\mathrm{Sp}(N)} = (20\ldots0). \qquad (5.46)$$

The contribution at $k = 0$ indicates that the $\mathrm{Sp}(N)$-neutral vacuum is assigned $\mathrm{U}(1)_v$ charge $+N = h_{\mathrm{Sp}(N)}^{\vee} - 1$ (see sections 6.4 and 7.2.1 for further remarks). The $k = 1$ terms correspond to mesonic operators constructed out $O(1)$ gauge-invariant quadratic combinations of the zero modes $q_i^{(0)}, \widetilde{q}_i^{(0)}$ of the hypermultiplet scalars; these terms appear with an accompanying power of v^2, consistent with the assignment of $\mathrm{U}(1)_v$ charge $+1$ to the fields $q_i^{(0)}, \widetilde{q}_i^{(0)}$. The $k > 1$ terms count operators constructed out of combinations of higher numbers the $q_i^{(0)}, \widetilde{q}_i^{(0)}$, taking into account the relations arising from F-terms in the quantum mechanics.

We can also easily construct a set of operators that appear at the first excited level $H_L = -c_L/24 + 1/2$ in the elliptic genus: these involve gauge-invariant combinations of the scalar zero modes $q_i^{(0)}, \widetilde{q}_i^{(0)}$ as well as the first excited component of a fermionic operator, $\psi_i^{(1/2)}$, which contributes $L_0 = 1/2$ to the energy. The simplest such combination consists of the gauge-invariant operators

$$q_i^{(0)} \psi_j \qquad \text{and} \qquad \widetilde{q}_i^{(0)} \psi_j, \qquad (5.47)$$

which transform together in the $(\mathbf{2N}, \mathbf{16 + 4N})$ bifundamental representation of $G \times F = \mathrm{Sp}(N) \times \mathrm{SO}(16 + 4N)$, which coincides with the representation in which the matter content of the 6d (1,0) SCFT on the tensor branch transforms. The fermion ψ^i is neutral under $\mathrm{U}(1)_v$, and therefore these fields contribute to the $q^{-c_L/24 + 1/2} v^{N+1}$ term in the v-expansion of the elliptic genus.

At the same energy level one can form further states by considering combinations of multiple of $q_i^{(0)}$ and $\widetilde{q}_i^{(0)}$ with a single $\psi_i^{(1/2)}$. Such fields will still transform in the $16 + 4N$-dimensional vector representation of F, but the precise representation of G will be affected by the existence of relations. By examining the explicit expressions for the elliptic genus, we find the first excited level terms take the following form:

$$\mathcal{E}_1^{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, \mathbf{m}_{\mathrm{SO}(16+4N)}, v, q)\Big|_{q^{-\frac{1}{3}-\frac{N}{4}+\frac{1}{2}}} = \qquad (5.48)$$

$$= v^N \sum_{k=0}^{\infty} v^{2k+1} \chi_{\omega_1}^{\mathrm{SO}(16+4N)}(\mathbf{m}_{\mathrm{SO}(16+4N)}) \chi_{\omega_1 + k \cdot \theta_{\mathrm{Sp}(N)}}^{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}),$$

---

[22]Note that only the $\mathrm{U}(1)_v$ Cartan subgroup of the diagonal subgroup $\mathrm{SU}(2)_v$ of $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_I$ is visible in this expansion.

where for both $SO(16 + 4N)$ and $Sp(N)$ ω_1 indicates the highest weight $(10\ldots0)$ of the fundamental representation, which is respectively $16 + 4N$ and $2N$-dimensional.

In other words, the ground state and first excited level contributions to the elliptic genus take a very simple form which is determined in terms of basic data of the 6d (1,0) SCFT: its gauge symmetry and matter content. As we will discuss in section 6.4 this feature generalizes to all BPS string CFTs $\mathbf{h}_n^G$.

On the other hand, at the lowest energy level of the Ramond elliptic genus we find a single state:

$$\mathbb{E}_1^{Sp(N)}(\mathbf{m}_{Sp(N)}, \mathbf{m}_{SO(16+4N)}, v, q) = \Big|_{q^{-1/3}} = 1. \tag{5.49}$$

This is a peculiarity of the theories $\mathbf{h}_n^G$ with $n = 1$; the theories with $n > 1$, on the other hand, have a rich set of lowest-energy states in the Ramond sector. We will give a more detailed analysis of the low energy states in the Ramond sector in section 7.

6 Universal features of BPS string CFTs

In this section we extend our discussion in the previous section of the $\mathbf{h}_1^{Sp(N)}$ theories to arbitrary $\mathbf{h}_n^G$ theories. We find several features which are common to all theories of BPS strings. This section is organized as follows: in section 6.1 we discuss the interpretation of the $\mathbf{h}_n^G$ CFTs as nonlinear sigma models on the moduli space of one G-instanton, and use this realization to compute their central charges. In section 6.2 we argue that the 6d flavor symmetry F is realized on the string as a current algebra whose level we determine. In section 6.3 we discuss the realization of the 6d gauge symmetry in the $\mathbf{h}_n^G$ theory, and find evidence that it contributes to the elliptic genus in terms of irreducible characters of the affine G Kac-Moody algebra at level $-n$. Finally, in section 6.4 we comment on the existence of a spectral flow relating the states that contribute to the NS-R and R-R elliptic genera.

6.1 BPS string CFTs as $\mathcal{N} = (0, 4)$ NLSMs

An $\mathcal{N} = (0, 4)$ NLSM can be described concisely in $(0,1)$ superspace $\Sigma^{(0,1)}$, which consists of two bosonic coordinates $z, \bar{z}$ and one fermionic coordinate θ^-, as follows [201, 202]: the field content for a generic $\mathcal{N} = (0, 1)$ nonlinear sigma model consists of D bosonic superfields $\phi^i(z, \bar{z}, \theta^-)$, $i = 1, \ldots, D$, defining a map

$$\phi : \Sigma^{(0,1)} \to \mathcal{M} \tag{6.1}$$

into a Riemaniann manifold $\mathcal{M}$,[23] and n fermionic superfields $\psi_+^a(z, \bar{z}, \theta^-)$ defining a section ψ of the bundle $S_+ \otimes \phi^* E$, where S_+ is the spinor bundle over $\Sigma^{(0,1)}$ and E is an n-dimensional vector bundle. The sigma model depends on the metric g_{ij} and B-field B_{ij} on $\mathcal{M}$, as well a connection Ω and a covariantly constant metric h_{ab} on E. The action

$$S = \int dz d\bar{z} \, d\theta^- \left[D_- \phi^i \partial_+ \phi^j (g_{ij} + B_{ij}) + i \psi_+^a \nabla_- \psi_+^b h_{ab} \right], \tag{6.2}$$

---

[23]Equivalently, the bosonic component of ϕ defines a map from $\mathbb{R}^2$ to $\mathcal{M}$ and the fermionic component defines a section of $S_- \otimes \phi^*(T\mathcal{M})$.

where $D_- = i\partial_{\theta^-} + \theta^- \partial_{\bar{z}}$ and $\nabla_- \psi_+^b = (D_- \psi_-^b + D_- \phi^i \Omega_{ic}^b \psi_+^c)$, defines a nonlinear sigma model with $\mathcal{N} = (0,1)$ supersymmetry.

The requirement that the NLSM display $\mathcal{N} = (0,4)$ supersymmetry imposes additional constraints on the data of the model. First of all, the target manifold $\mathcal{M}$ must be a hyperkähler manifold of quaternionic dimension $K = D/4$. The metric g_{ij} must be Hermitian with respect to each of the three complex structures. Furthermore, the holonomy of the connection

$$\Gamma_{ij}^{(+)K} = \hat{\Gamma}_{ij}^K + H_{ij}^K, \tag{6.3}$$

where $\hat{\Gamma}_{ij}^K$ is the Christoffel connection on $\mathcal{M}$ and $H_{ijk} = \frac{3}{2}\partial_{[i}B_{jk]}$, must be a subgroup of $\mathrm{Sp}(K)$. Finally, the bundle $E \otimes \mathbb{C}$ must be holomorphic with respect to all three complex structures.

The NLSMs describing the $\mathbf{h}_n^G$ theories consist of two separate components. The first component describes the propagation of the string in $\mathbb{R}^4$ and is given by a $\mathcal{N} = (0,4)$ NLSM with $\mathcal{M} = \mathbb{R}^4$ and trivial chiral fermion bundle; this is just the theory of one free $(0,4)$ hypermultiplet and has $c_L = 4, c_R = 6$. The second component describes the 'internal' degrees of freedom of the string and has as target space the (reduced) moduli space of one G-instanton, $\widetilde{\mathcal{M}}_{G,1}$. The information about the 6d matter content is reflected by the choice of chiral spinor bundle (for instance, for the $\mathbf{h}_1^{\mathrm{Sp}(N)}$ theories the $F = \mathrm{SO}(16 + 4N)$ flavor symmetry that rotates the 6d hypermultiplets is carried by $16 + 4N$ fermionic superfields).

Two issues afflict the latter component of the nonlinear sigma model. First, the target space is noncompact, which leads to a continuum of bosonic vacua. This is cured by turning on chemical potentials for the $G \times \mathrm{SU}(2)_R$ isometries of $\widetilde{\mathcal{M}}_{G,1}$, where $\mathrm{SU}(2)_R$ acts as hyperkähler rotations on the three complex structures of $\widetilde{\mathcal{M}}_{G,1}$. This leads to a still infinitely degenerate, but discrete, set of vacua. Second, the moduli space of one G-instanton has a singularity corresponding to the instanton shrinking to zero size. Lifting the ADHM quantum mechanics to 2d suggests a UV description flowing to the NLSM in the IR which is reliable at finite instanton size, but is potentially ill-behaved at the small-instanton singularity [203]; however, alternative UV completions have been found whose low energy behavior is identical to the ADHM model everywhere except at the small-instanton singularity, where one finds additional localized degrees of freedom [76] leading to a consistent UV description of the NLSM.

Regardless of these complications, we will be able to infer several nontrivial properties of the BPS strings and their elliptic genera from basic properties of the sigma model. We begin in this section by re-deriving the $\mathbf{h}_n^G$ central charges (3.52). It is easiest to begin by considering the maximally Higgsed theories for any n. First of all, recall that for $n = 3, 4, 5, 6, 8, 12$ the strings are instantons in theories without matter. The chiral fermion bundle is therefore trivial, and the only contributions come from the $4(h_G^\vee - 1)$ noncompact bosons describing the target space $\widetilde{\mathcal{M}}_{G,1}$ and an equal number of anti-chiral spinors [83]. Furthermore, using the relation $h_G^\vee = 3(n-2)$ which holds for these theories, we obtain

$$c_L = 4(3n - 7) \qquad \text{and} \qquad c_R = 6(3n - 7). \tag{6.4}$$

	L	R
# noncompact bosons	$4(h_G^\vee - 1)$	$4(h_G^\vee - 1)$
# fermions	$4h_G^\vee - 12(n-2)$	$4(h_G^\vee - 1)$

Table 16. Degrees of freedom of the $\mathbf{h}_n^G$ NLSM.

The M-string and E-string cases are different: as explained in section 2.4 they can respectively be thought of as theories with $G = \mathrm{SU}(1)$ and $\mathrm{Sp}(0)$, for which $h_G^\vee = 1$ and the moduli space of instantons is trivial. Nevertheless, both theories support a bundle of chiral fermions. In the M-string case this consists of four fermions [63], leading to $c_L = 2$ and $c_R = 0$ (i.e. $c_L = c_R = 6$ upon including the center of mass hypermultiplet contribution). The E-string CFT on the other hand consists of 16 chiral fermions, leading to $c_L = 8$ and $c_R = 0$.

The central charges for Higgsable theories can be computed from the ones of the maximally Higgsed theories as follows: the relevant deformation of the (0,4) CFT of the Higgsable theory that initiates the Higgs RG flow involves giving mass to an equal number of chiral fermionic superfields ψ_+^a and of bosonic superfields ϕ^i (in particular, this is consistent with the fact that the gravitational anomaly is invariant under the RG flow). In particular, the difference between the rank of the chiral fermion bundle and the dimension of the target space the same before and after Higgsing. The dimension of the target space is $4(h_G^\vee - 1)$, and therefore upon Higgsing from the $\mathbf{h}_n^G$ theory to the $\mathbf{h}_n^{G_0}$ theory one has

$$c_L \to c_L - 6(h_G^\vee - h_{G_0}^\vee), \qquad c_R \to c_R - 6(h_G^\vee - h_{G_0}^\vee). \tag{6.5}$$

This fixes the number of bosonic and fermionic degrees of freedom of the BPS string CFTs; the results are summarized in table 16. The only loophole in the argument above is for the $\mathbf{h}_7^{E_7}$ theory, which is maximally Higgsed but has nontrivial matter content. However, the number of chiral fermions given in table 16 depends linearly on n (or equivalently on the number of defects in the D3 brane worldvolume from the perspective of section 3); each defect contributes a fixed number of chiral fermions; this in particular also holds for $G = E_7$ for $n \neq 7$, so by extension this shows that table 16 also applies to the $\mathbf{h}_7^{E_7}$ theory.

From the data of table 16 one recovers immediately the correct central charges for the nonlinear sigma models:

$$c_L = 6(h_G^\vee - n + 2) - 4, \tag{6.6}$$

$$c_R = 6h_G^\vee - 6, \tag{6.7}$$

consistent with equation (3.52) (with $Q = 1$). For a generic number Q of strings, the same line of reasoning leads to the following prediction for the numbers of bosonic and fermionic degrees of freedom in the NLSM:

From table 16 one can also read off the ground state energies in the Ramond-Ramond sector of the $\mathbf{h}_n^G$ NLSM:[24]

$$E_{L,R}^{\mathrm{Ramond}} = -\frac{1}{24}(\#\mathrm{bosons}_{L,R} - \#\mathrm{fermions}_{L,R}); \tag{6.8}$$

---

[24]See also the recent paper [94] for a similar discussion.

	L	R
# noncompact bosons	$4(Qh_G^\vee - 1)$	$4(Qh_G^\vee - 1)$
# fermions	$4Qh_G^\vee - 12Q(n-2)$	$4(Qh_G^\vee - 1)$

Table 17. Degrees of freedom of the NLSM describing a bound states of Q strings for the 6d $(1,0)$ SCFT with gauge algebra G and string Dirac pairing $-n$.

this gives of course $E_R^{\mathrm{Ramond}} = 0$ consistent with supersymmetry, whereas on the chiral side one finds

$$E_L^{\mathrm{Ramond}} = \frac{1}{6} - \frac{n-2}{2}. \tag{6.9}$$

This determines the leading order power of q appearing in the Ramond elliptic genus, with the single exception of the $\mathbf{h}_1^G$ theories, as we will discuss in section 7. The general statement will turn out to be that for an arbitrary $\mathbf{h}_n^G$ theory

$$\mathbb{E}_n^G(\mathbf{m}_G, \mathbf{m}_F, v, q) = q^{E_L^{\mathrm{Ramond}}} \left(\delta_{n,1} q^{-1} + f_n^G(\mathbf{m}_G, \mathbf{m}_F, v) q^0 + \mathcal{O}(q) \right), \tag{6.10}$$

where $f_n^G(\mathbf{m}_G, \mathbf{m}_F, v)$ includes a combination of Ramond ground state contributions and, for $n = 1$, excitations above the $L_0 = -1$ singlet state (see section 7).

Similarly, for Q strings one finds from table 17 that the ground state energy is given by

$$E_R^{\mathrm{Ramond}} = \frac{1}{6} - Q\frac{n-2}{2}. \tag{6.11}$$

6.2 Current algebra realization of F (I)

In this section we discuss the realization of the flavor symmetry of the 6d SCFT in terms of a current algebra in the $\mathbf{h}_n^G$ CFTs. Indeed, the bosonic superfields describe the propagation on the one G-instanton moduli space, and are not charged under the flavor symmetry F; the only fields that are charged under it are the fermionic superfields ψ_+^a, which contribute chirally to the CFT. This is also consistent with the F-theory picture where the flavor symmetry is carried by the $(0,4)$ defects on the D3 worldvolume, which arise from seven-branes intersecting the D3 brane along a codimension 1 locus and therefore contribute chirally to the topologically twisted theory on the D3 brane.

In particular, this suggests that the 6d flavor symmetry F, which is also a flavor symmetry of the $\mathbf{h}_n^G$ theories, is realized in terms of chiral current algebras. Let us write F as a product of n_F simple or abelian factors, $F = \prod_{i=1}^{n_F} F_i$. Then, the currents $j_{F_i}^a(z)$, $a = 1, \ldots, \mathrm{rank}(F_i)$ obey the OPE

$$j_{F_i}^a(z) j_{F_i}^b(w) \sim \frac{k_i \delta^{ab}}{(z-w)^2} + \sum_c i f_{abc}^{F_i} \frac{j_{F_i}^c(w)}{z-w}, \tag{6.12}$$

where $f_{abc}^{F_i}$ are the structure constants of F_i, and the coefficient of the most singular term determines the level k.[25] The level can be read off from the anomaly polynomial, and it is given by

$$\Sigma \cdot \mathbf{b}_F \qquad \text{or} \qquad \Sigma \cdot \mathbf{b}_{\mathrm{U}(1)}. \tag{6.13}$$

---

[25] For abelian factors, which correspond to a boson of compact radius R, k_i is replaced by R^2.

The WZW model corresponding to each simple factor has central charge

$$c_{F_i} = \frac{\dim(F_i)\, k}{h_{F_i}^\vee + k},$$ (6.14)

and a set of WZW primaries which are in one-to-one correspondence with the integrable level k representations of F; abelian factors have $c_{U(1)} = 1$ and a set of WZW primaries detailed in section E. The total central charge associated to F is given by:

$$c_F = \sum_{i|\, F_i \text{ simple}} \frac{\dim(F_i) k_i}{h_{F_i}^\vee + k_i} + \sum_{i|\, F_i \text{ abelian}} 1.$$ (6.15)

This proposal implies that the Hilbert space of the $\mathbf{h}_n^G$ CFT decomposes as:

$$H_n^G = \bigoplus_{\vec\lambda} \left(\bigotimes_{i=1}^{n_F} H_{\lambda_i}^{WZW_{F_i}} \right) \otimes H_{\vec\lambda}^{\text{residual}},$$ (6.16)

where the i-th entry of $\vec\lambda = (\lambda_1, \ldots, \lambda_{n_F})$ runs over the primaries of the WZW_{F_i} model. While the WZW sector of the CFT is chiral, the spectrum of the residual part of the CFT includes both chiral and anti-chiral components and couples to $\text{SU}(2)_R$ and G.

From (6.16) it follows moreover that the R-R elliptic genus for any BPS string CFT decomposes in terms of characters $\widehat\chi_{\lambda_i}^{F_i}(\mathbf{m}_{F_i}, q)$ of the WZW model for F_i in the following way:

$$\mathbb{E}_n^G(\mathbf{m}_G, \mathbf{m}_F, v, q) = \sum_{\vec\lambda} \left(\prod_{i=1}^{n_F} \widehat\chi_{\lambda_i}^{F_i}(\mathbf{m}_{F_i}, q) \right) \xi_{\vec\lambda}^{n,G}(\mathbf{m}_G, v, q).$$ (6.17)

The $\widehat\chi_\lambda^{F_i}(\mathbf{m}_{F_i}, q)$ are vector-valued Jacobi forms with modular parameter τ and (exponentiated) elliptic parameters $\mathbf{m}_{F_i}$, and therefore the functions $\xi_{\vec\lambda}^{n,G}(\mathbf{m}_G, v, q)$ must also transform as a vector-valued Jacobi form with modular parameter τ and (exponentiated) elliptic parameters $\mathbf{m}_G$ and v. The modular S-matrix for this Jacobi form can be easily determined by requiring that the elliptic genus transform as a scalar Jacobi form with the correct modular anomaly.

We now turn to a discussion of the $G \times \text{SU}(2)_R$-dependent part of the CFT, and will return to a more in-depth discussion of the flavor symmetry F in section 6.5.

6.3 Worldsheet realization of G

In section 5.3 we found that the 6d gauge symmetry G contributes to the elliptic genus according to equation (5.24), which we furthermore refined by showing that the $\text{Sp}(N)$ representations that appear organize themselves in terms of irreducible characters of the affine $\text{Sp}(N)$ Kac-Moody algebra at level -1. In this section we discuss the generalization to other $\mathbf{h}_n^G$ theories. For the theories for which $n \neq h_G^\vee$ we discuss a realization of the 6d gauge symmetry in terms of an affine G Kac-Moody algebra at level $-n$, in agreement with the expectation from the coefficient of $c_2(G)$ in the anomaly polynomial of the $\mathbf{h}_n^G$ theory, equation (3.47).

The representation theory of affine Kac-Moody algebras is different for the three cases $k < -h_G^\vee$, $k = -h_G^\vee$, and $k > -h_G^\vee$. A survey of the $\mathbf{h}_n^G$ theories shows that the case $k < -h_G^\vee$ occurs only for the M-string CFT $\mathbf{h}_2^{\mathrm{SU}(1)}$, the critical case $k = -h_G^\vee$ occurs for the three theories $\mathbf{h}_3^{\mathrm{SU}(3)}$, $\mathbf{h}_2^{\mathrm{SU}(2)}$, and $\mathbf{h}_1^{\mathrm{Sp}(0)}$, while all other theories belong to the case $k > -h_G^\vee$. The critical case is subtle and requires a separate discussion, which we postpone to section 9. On the other hand, for $k \neq -h_G^\vee$ we find the following generalization of equation (5.24):

$$
\xi_{\vec\lambda}^{n,G}(\mathbf{m}_G, v, q) =
$$
$$
= \sum_{\nu \in \mathrm{Rep}(G)} \sum_{\ell = -2|n-h_G^\vee|+1}^{0} \sum_{m \in \mathbb{Z}} n_{\nu,\ell,m}^{\vec\lambda} \times \frac{q^{-\frac{c_G}{24}+h_\nu^G}\chi_\nu^G(\mathbf{m}_G)}{\widetilde\Delta_G(\mathbf{m}_G,q)} \times \frac{q^{-\frac{c_v}{24}+h_{\ell,m}^v}v^{\ell+2(n-h_G^\vee)m}}{\prod_{j=1}^{\infty}(1-q^j)}, \quad (6.18)
$$

which we have verified by comparing against known results for the elliptic genera of a large number of theories. The sum runs over all representations of G, and

$$
c_G = \frac{\dim(G)(-n)}{h_G^\vee - n}, \qquad\qquad h_\nu^G = \frac{\langle \nu, \nu + 2\rho_G\rangle}{2(h_G^\vee - n)},
$$
$$
c_v = 1, \qquad\qquad h_{\ell,m}^v = \frac{(\ell + 2(n-h_G^\vee)m)^2}{4(n-h_G^\vee)}, \quad (6.19)
$$

are the central charges and conformal dimensions for the G_{-n} and $\mathrm{U}(1)_{n-h_G^\vee}$ WZW models (see appendix E), suggestive that the 6d gauge symmetry G is again realized in terms of an affine Kac-Moody algebra at level $k = -n$, and the $\mathrm{U}(1)_v$ symmetry generated by $J_R^3 + J_I^3$ is realized in terms of a $\mathrm{U}(1)_{n-h_G^\vee}$ algebra; we will return to this shortly.

Analogously to the $n = 1, G = \mathrm{Sp}(N)$ case, the $n_{\nu,\ell,m}^{\vec\lambda}$ coefficients must obey various constraints:

- The $n_{\nu,\ell,m}^{\vec\lambda}$ must be integers, which follows immediately from the integrality of the coefficients of the elliptic genus.

- A look at equations (6.18) and (6.19) reveals that for $n < h_G^\vee$ only a finite number of terms in the sum over the integer m would correspond to states of energy above the ground state energy, since, $n < h_G^\vee$ implies that $h_{\ell,m}^v$ is negative and grows quadratically with m. As the elliptic genus describes a unitary theory whose energy is bounded from below, for fixed λ, ν, and ℓ all but a finite number of coefficients $n_{\nu,\ell,m}^{\vec\lambda}$ can be nonvanishing.

- The level-matching condition for the CFT on the torus forces the energy of any state that contributes to the elliptic genus to differ from the Ramond ground state energy $-\frac{(c_R-c_L)}{12}$ by an integer. For many choices of ν, ℓ, and m, the coefficient of $n_{\nu,\ell,m}^{\vec\lambda}$ in the elliptic genus must therefore vanish, since the corresponding term in equation (6.18) would violate the level-matching condition.

We next turn to a few examples. First of all, the M-string CFT $\mathbf{h}_2^{\mathrm{SU}(1)}$ is the only case for which $h_G^\vee < n$, and it is interesting to see how equation (6.18) captures the elliptic genus. As the group $G = \mathrm{SU}(1)$ is trivial, only the $\mathbf{1}$ representation contributes. Moreover since the

ν	$\widehat{\chi}_\nu^{\mathrm{SO}(8)}(\mathbf{m}_{\mathrm{SO}(8)},q)\cdot\widetilde{\Delta}_{\mathrm{SO}(8)}(\mathbf{m}_{\mathrm{SO}(8)},q)$
(0000)	$q^{\frac{56}{24}}\Big[\chi_{(0000)}^{\mathrm{SO}(8)}-2\chi_{(0100)}^{\mathrm{SO}(8)}q^3+(\chi_{(2000)}^{\mathrm{SO}(8)}+\chi_{(0020)}^{\mathrm{SO}(8)}+\chi_{(0002)}^{\mathrm{SO}(8)})q^4+3\chi_{(0200)}^{\mathrm{SO}(8)}q^7$ $-2(\chi_{(2100)}^{\mathrm{SO}(8)}+\chi_{(0120)}^{\mathrm{SO}(8)}+\chi_{(0102)}^{\mathrm{SO}(8)})q^8+(\chi_{(2020)}^{\mathrm{SO}(8)}+\chi_{(2002)}^{\mathrm{SO}(8)}+\chi_{(0022)}^{\mathrm{SO}(8)})q^9$ $+(\chi_{(4000)}^{\mathrm{SO}(8)}+\chi_{(0040)}^{\mathrm{SO}(8)}+\chi_{(0004)}^{\mathrm{SO}(8)})q^{10}+\mathcal{O}(q^{12})\Big]$
(0100)	$q^{\frac{56}{24}+3}\Big[\chi_{(0100)}^{\mathrm{SO}(8)}-(\chi_{(2000)}^{\mathrm{SO}(8)}+\chi_{(0020)}^{\mathrm{SO}(8)}+\chi_{(0002)}^{\mathrm{SO}(8)})q+\chi_{(1011)}^{\mathrm{SO}(8)}q^3-3\chi_{(0200)}^{\mathrm{SO}(8)}q^4$ $+2(\chi_{(2100)}^{\mathrm{SO}(8)}+\chi_{(0120)}^{\mathrm{SO}(8)}+\chi_{(0102)}^{\mathrm{SO}(8)})q^5-2(\chi_{(2020)}^{\mathrm{SO}(8)}+\chi_{(2002)}^{\mathrm{SO}(8)}+\chi_{(0022)}^{\mathrm{SO}(8)})q^6$ $-(\chi_{(4000)}^{\mathrm{SO}(8)}+\chi_{(0040)}^{\mathrm{SO}(8)}+\chi_{(0004)}^{\mathrm{SO}(8)})q^7+\mathcal{O}(q^9)\Big]$
(0200)	$q^{\frac{56}{24}+7}\Big[\chi_{(0200)}^{\mathrm{SO}(8)}-(\chi_{(2100)}^{\mathrm{SO}(8)}+\chi_{(0120)}^{\mathrm{SO}(8)}+\chi_{(0102)}^{\mathrm{SO}(8)})q+(\chi_{(2020)}^{\mathrm{SO}(8)}+\chi_{(2002)}^{\mathrm{SO}(8)}+\chi_{(0022)}^{\mathrm{SO}(8)})q^2$ $+(\chi_{(4000)}^{\mathrm{SO}(8)}+\chi_{(0040)}^{\mathrm{SO}(8)}+\chi_{(0004)}^{\mathrm{SO}(8)})q^3+\mathcal{O}(q^5)\Big]$

Table 18. Irreducible characters of SO(8) at level -4 (rescaled by an overall factor of $\widetilde{\Delta}_{\mathrm{SO}(8)}(\mathbf{m}_{\mathrm{SO}(8)},q)$), for a set of highest-weight representations that contribute in the elliptic genus.

dependence on v is holomorphic one expects the $\xi_\lambda^{2,\mathrm{SU}(1)}$ functions to be expressed in terms of affine characters for $\mathrm{SU}(2)_R$ at level $2-h_{\mathrm{SU}(1)}^\vee=1$. Indeed, the M-string elliptic genus

$$\mathbb{E}_2^{\mathrm{SU}(2)}(\mathbf{m}_{\mathrm{SU}(2)},v,q)=\frac{\theta_1(v\,m_{\mathrm{SU}(2)}^1,q)\theta_1(v/m_{\mathrm{SU}(2)}^1,q)}{\eta(q)^2},\tag{6.20}$$

can be written in terms of level 1 characters of $F=\mathrm{SU}(2)$ and of $\mathrm{SU}(2)_v$ as follows:[26]

$$\mathbb{E}_2^{\mathrm{SU}(2)}(\mathbf{m}_{\mathrm{SU}(2)},v,q)=\widehat{\chi}_{\mathbf{1}}^F(\mathbf{m}_F,q)\widehat{\chi}_{\mathbf{2}}^{\mathrm{SU}(2)_R}(v,q)-\widehat{\chi}_{\mathbf{2}}^F(\mathbf{m}_F,q)\widehat{\chi}_{\mathbf{1}}^{\mathrm{SU}(2)_R}(v,q).\tag{6.21}$$

On the other hand equation (6.18) reads:

$$\xi_\lambda^{2,\mathrm{SU}(1)}(\mathbf{m}_G,v,q)=\sum_{\ell=-1}^{0}\sum_{m\in\mathbb{Z}}n_{\mathbf{1},\ell,m}^\lambda\frac{q^{(\ell/2+m)^2}v^{\ell+2\,m}}{\eta(q)}.\tag{6.22}$$

Comparing this with equations (E.14) and (E.15) for the SU(2) level 1 characters, one finds that

$$n_{\mathbf{1},1,m}^{\mathbf{1}}=-n_{\mathbf{1},0,m}^{\mathbf{2}}=1\qquad\text{and}\qquad n_{\mathbf{1},0,m}^{\mathbf{1}}=n_{\mathbf{1},1,m}^{\mathbf{2}}=0\qquad\forall\,m\in\mathbb{Z}.\tag{6.23}$$

Next, we consider the $\mathbf{h}_4^{\mathrm{SO}(8)}$ theory, for which $n>-h_G^\vee$ and we expect the affine SO(8) Kac-Moody algebra at level -4 to appear. As F is trivial, the elliptic genus is given by the function $\xi_{\mathbf{1}}^{4,\mathrm{SO}(8)}$; we find that this function can be written in terms of $\mathrm{SO}(8)_{-4}$ irreducible representations as follows:

$$\begin{aligned}\xi_{\mathbf{1}}^{4,\mathrm{SO}(8)}(\mathbf{m}_{\mathrm{SO}(8)},v,q)=&\sum_{n=0}^{\infty}\widehat{\chi}_{(0n00)}^{\mathrm{SO}(8)}(\mathbf{m}_{\mathrm{SO}(8)},q)\\ &\times\sum_{m=0}^{2}a_m\left[\frac{v^{-b_{m,n}+1}q^{-\frac{(b_{m,n}-1)^2}{8}}}{\eta(q)}-\frac{v^{b_{m,n}+1}q^{-\frac{(b_{m,n}+1)^2)}{8}}}{\eta(q)}\right],\end{aligned}\tag{6.24}$$

---

[26]Since the elliptic genus is holomorphic in v rather than meromorphic, it can be expressed in terms of representations of the $\mathrm{SU}(2)_v$ diagonal subgroup of $\mathrm{SU}(2)_R\times\mathrm{SU}(2)_I$, and not solely of its Cartan subgroup $\mathrm{U}(1)_v$.

ν	$\widehat{\chi}_\nu^{G_2}(\mathbf{m}_{G_2},q)\cdot\widetilde{\Delta}_{G_2}(\mathbf{m}_{G_2},q)$
(00)	$q^{\frac{7}{4}}\left[\chi_{(00)}^{G_2}-\chi_{(11)}^{G_2}q^7+\chi_{(30)}^{G_2}q^8+\mathcal{O}(q^{18})\right]$
(01)	$q^{\frac{7}{4}+4}\left[\chi_{(01)}^{G_2}-\chi_{(11)}^{G_2}q^3+\chi_{(02)}^{G_2}q^6-\chi_{(12)}^{G_2}q^{10}+\chi_{(31)}^{G_2}q^{11}+\mathcal{O}(q^{14})\right]$
(02)	$q^{\frac{7}{4}+10}\left[\chi_{(02)}^{G_2}-\chi_{(40)}^{G_2}q^2-\chi_{(12)}^{G_2}q^4+\chi_{(31)}^{G_2}q^5+\chi_{(03)}^{G_2}q^8+\mathcal{O}(q^{13})\right]$
(03)	$q^{\frac{7}{4}+18}\left[\chi_{(03)}^{G_2}-\chi_{(41)}^{G_2}q^2+\chi_{(60)}^{G_2}q^4-\chi_{(13)}^{G_2}q^5+\chi_{(32)}^{G_2}q^6+\chi_{(04)}^{G_2}q^{10}-\chi_{(42)}^{G_2}q^{12}+\mathcal{O}(q^{16})\right]$
(04)	$q^{\frac{7}{4}+28}\left[\chi_{(04)}^{G_2}-\chi_{(42)}^{G_2}q^2+\chi_{(61)}^{G_2}q^4-\chi_{(14)}^{G_2}q^6+\chi_{(33)}^{G_2}q^7-\chi_{(71)}^{G_2}q^{11}+\chi_{(05)}^{G_2}q^{12}+\mathcal{O}(q^{14})\right]$
(05)	$q^{\frac{7}{4}+40}\left[\chi_{(05)}^{G_2}-\chi_{(43)}^{G_2}q^2+\chi_{(62)}^{G_2}q^4-\chi_{(15)}^{G_2}q^7+\chi_{(34)}^{G_2}q^8-\chi_{(10,0)}^{G_2}q^{10}-\chi_{(72)}^{G_2}q^{12}+\mathcal{O}(q^{14})\right]$
(06)	$q^{\frac{7}{4}+54}\left[\chi_{(06)}^{G_2}-\chi_{(44)}^{G_2}q^2+\chi_{(63)}^{G_2}q^4-\chi_{(16)}^{G_2}q^8+\chi_{(35)}^{G_2}q^9-\chi_{(10,1)}^{G_2}q^{10}+\mathcal{O}(q^{13})\right]$
(07)	$q^{\frac{7}{4}+70}\left[\chi_{(07)}^{G_2}-\chi_{(45)}^{G_2}q^2+\chi_{(64)}^{G_2}q^4-\chi_{(17)}^{G_2}q^9+(\chi_{(36)}^{G_2}-\chi_{(10,2)}^{G_2})q^{10}+\mathcal{O}(q^{14})\right]$
(08)	$q^{\frac{7}{4}+88}\left[\chi_{(08)}^{G_2}-\chi_{(46)}^{G_2}q^2+\chi_{(65)}^{G_2}q^4-(\chi_{(18)}^{G_2}+\chi_{(10,3)}^{G_2})q^{10}+\chi_{(37)}^{G_2}q^{11}+\mathcal{O}(q^{14})\right]$
(09)	$q^{\frac{7}{4}+108}\left[\chi_{(09)}^{G_2}-\chi_{(47)}^{G_2}q^2+\chi_{(66)}^{G_2}q^4-\chi_{(10,4)}^{G_2}q^{10}-\chi_{(19)}^{G_2}q^{11}+\chi_{(3,8)}^{G_2}q^{12}+\mathcal{O}(q^{14})\right]$
$(0,10)$	$q^{\frac{7}{4}+130}\left[\chi_{(0,10)}^{G_2}+\mathcal{O}(q^2)\right]$
$(0,11)$	$q^{\frac{7}{4}+154}\left[\chi_{(0,11)}^{G_2}+\mathcal{O}(q^2)\right]$
(10)	$q^{\frac{7}{4}+2}\left[\chi_{(10)}^{G_2}-\chi_{(01)}^{G_2}q^2-\chi_{(02)}^{G_2}q^8+\chi_{(40)}^{G_2}q^{10}+\chi_{(12)}^{G_2}q^{12}+\mathcal{O}(q^{13})\right]$
(11)	$q^{\frac{7}{4}+7}\left[\chi_{(11)}^{G_2}-\chi_{(30)}^{G_2}q-\chi_{(02)}^{G_2}q^3+\chi_{(12)}^{G_2}q^7-2\chi_{(03)}^{G_2}q^{11}+\mathcal{O}(q^{13})\right]$
(12)	$q^{\frac{7}{4}+14}\left[\chi_{(12)}^{G_2}-\chi_{(31)}^{G_2}q-\chi_{(03)}^{G_2}q^4+\chi_{(41)}^{G_2}q^6+\chi_{(13)}^{G_2}q^9-\chi_{(32)}^{G_2}q^{10}+\mathcal{O}(q^{14})\right]$
(13)	$q^{\frac{7}{4}+23}\left[\chi_{(13)}^{G_2}-\chi_{(32)}^{G_2}q+(\chi_{(70)}^{G_2}-\chi_{(04)}^{G_2})q^5+\chi_{(42)}^{G_2}q^7-\chi_{(61)}^{G_2}q^9+\chi_{(14)}^{G_2}q^{11}-\chi_{(33)}^{G_2}q^{12}+\mathcal{O}(q^{17})\right]$
(14)	$q^{\frac{7}{4}+34}\left[\chi_{(14)}^{G_2}-\chi_{(33)}^{G_2}q+\chi_{(71)}^{G_2}q^5-\chi_{(05)}^{G_2}q^6+(\chi_{(43)}^{G_2}-\chi_{(90)}^{G_2})q^8-\chi_{(62)}^{G_2}q^{10}+\mathcal{O}(q^{13})\right]$
(15)	$q^{\frac{7}{4}+47}\left[\chi_{(15)}^{G_2}-\chi_{(34)}^{G_2}q+\chi_{(72)}^{G_2}q^5-\chi_{(06)}^{G_2}q^7-\chi_{(91)}^{G_2}q^8+\chi_{(44)}^{G_2}q^9-\chi_{(63)}^{G_2}q^{11}+\mathcal{O}(q^{15})\right]$
(16)	$q^{\frac{7}{4}+62}\left[\chi_{(16)}^{G_2}-\chi_{(35)}^{G_2}q+\chi_{(73)}^{G_2}q^5-(\chi_{(92)}^{G_2}+\chi_{(07)}^{G_2})q^8+\chi_{(45)}^{G_2}q^{10}-\chi_{(64)}^{G_2}q^{12}+\mathcal{O}(q^{16})\right]$
(17)	$q^{\frac{7}{4}+79}\left[\chi_{(17)}^{G_2}-\chi_{(36)}^{G_2}q+\chi_{(74)}^{G_2}q^5-\chi_{(93)}^{G_2}q^8-\chi_{(08)}^{G_2}q^9+\chi_{(46)}^{G_2}q^{11}+\mathcal{O}(q^{13})\right]$
(18)	$q^{\frac{7}{4}+98}\left[\chi_{(18)}^{G_2}-\chi_{(37)}^{G_2}q+\chi_{(75)}^{G_2}q^5-\chi_{(94)}^{G_2}q^8-\chi_{(09)}^{G_2}q^{10}+\chi_{(47)}^{G_2}q^{12}+\mathcal{O}(q^{14})\right]$
(19)	$q^{\frac{7}{4}+119}\left[\chi_{(19)}^{G_2}-\chi_{(38)}^{G_2}q+\mathcal{O}(q^5)\right]$
$(1,10)$	$q^{\frac{7}{4}+142}\left[\chi_{(1,10)}^{G_2}+\mathcal{O}(q)\right]$

Table 19. Irreducible level -3 characters of G_2 (rescaled by an overall factor of $\widetilde{\Delta}_{G_2}(\mathbf{m}_{G_2},q)$), for a set of highest-weight representations that contribute to the elliptic genus. The expansion is given to an order sufficient to compute the elliptic genus to $\mathcal{O}(q^{13})$ and $\mathcal{O}(v^{22})$.

where $a_0=a_2=1$, $a_1=2$, and $b_{m,n}=2m+2n$. This expression is indeed also consistent with equation (6.18). In table 18 we list the first few terms in the q expansion of the irreducible characters of the relevant representations of $\mathrm{SO}(8)_{-4}$, which we obtained in terms of Kazhdan-Lusztig polynomials as outlined in section 5.3.

Finally, let us examine the $\mathbf{h}_3^{G_2}$ theory, for which $F=\mathrm{Sp}(1)$ at level 1. In this case we expect the affine G_2 Kac-Moody algebra at level -3 to appear. The elliptic genus can be

written as

$$
\begin{aligned}
\mathbb{E}_3^{G_2}(\mathbf{m}_{G_2}, \mathbf{m}_{\mathrm{Sp}(1)}, v, q) = \widehat{\chi}_{(0)}^{\mathrm{Sp}(1)}(\mathbf{m}_{\mathrm{Sp}(1)}, q) \xi_{(0)}^{3, G_2}(\mathbf{m}_{G_2}, v, q) \\
+ \widehat{\chi}_{(1)}^{\mathrm{Sp}(1)}(\mathbf{m}_{\mathrm{Sp}(1)}, q) \xi_{(1)}^{3, G_2}(\mathbf{m}_{G_2}, v, q).
\end{aligned}
\tag{6.25}
$$

We find that the $\mathbf{m}_{G_2}$ dependent pieces of the elliptic genus admit the following expansions:

$$
\begin{aligned}
\xi_{(0)}^{3, G_2}(\mathbf{m}_{G_2}, v, q) = \sum_{n=0}^{\infty} \widehat{\chi}_{(0n)}^{G_2}(\mathbf{m}_{G_2}, q) \left[\frac{v^{-2(n+1)} q^{-(n+1)^2}}{\eta(q)} - \frac{v^{2n} q^{-n^2}}{\eta(q)} \right] \\
+ \sum_{n=0}^{\infty} \widehat{\chi}_{(1n)}^{G_2}(\mathbf{m}_{G_2}, q) \left[\frac{v^{-2n} q^{-n^2}}{\eta(q)} - \frac{v^{2(n+2)} q^{-(n+2)^2}}{\eta(q)} \right]
\end{aligned}
\tag{6.26}
$$

and

$$
\begin{aligned}
\xi_{(1)}^{3, G_2}(\mathbf{m}_{G_2}, v, q) = \sum_{n=0}^{\infty} \widehat{\chi}_{(0n)}^{G_2}(\mathbf{m}_{G_2}, q) \left[\frac{v^{-1+2(n+2)} q^{-\frac{(-1+2(n+2))^2}{4}}}{\eta(q)} - \frac{v^{-1-2(n-1)} q^{-\frac{(-1-2(n-1))^2}{4}}}{\eta(q)} \right] \\
+ \sum_{n=0}^{\infty} \widehat{\chi}_{(1n)}^{G_2}(\mathbf{m}_{G_2}, q) \left[\frac{v^{-1+2(n+1)} q^{-\frac{(-1+2(n+1))^2}{4}}}{\eta(q)} - \frac{v^{-1-2(n+1)} q^{-\frac{(-1-2(n+1))^2}{4}}}{\eta(q)} \right],
\end{aligned}
\tag{6.27}
$$

which we have verifed to hold to $\mathcal{O}(q^{13})$ and $\mathcal{O}(v^{22})$. The level -3 characters that appear in these expressions, which we have computed by making use of Kazhdan-Lusztig polynomials, are reported in table 19.

We interpret these results, as well as the ones obtained in section 5.3 for $G = \mathrm{Sp}(1)$ and $\mathrm{Sp}(2)$, as an indication that the 6d gauge symmetry, at least for the theories with $n \neq h_G^\vee$, is realized in the chiral algebra of the $\mathbf{h}_n^G$ CFTs as an affine Kac-Moody algebra.[27,28] A similar phenomenon has recently been observed in a related context by Gukov and Dedushenko [204], who argued that the chiral algebra of the $\mathcal{N} = (0, 2)$ gauge theory with $\mathrm{SU}(2)$ gauge group and four chiral multiplets is the $\mathrm{SO}(8)$ affine Kac-Moody algebra at level -2; note that the $\mathbf{h}_4^{\mathrm{SO}(8)}$ CFT admits a similar UV realization as a $(0, 4)$ gauge theory with $\mathrm{SU}(2)$ gauge group and four hypermultiplets. Whether an extension of the techniques of [204] can be used to obtain the chiral algebra of the $\mathbf{h}_4^{\mathrm{SO}(8)}$ CFT from first principles is currently under investigation [195].

The $\mathrm{SU}(2)_v$ symmetry should also in principle be realized in terms of an affine Kac-Moody algebra; however, the generators of the reduced moduli space of one G instanton $\widetilde{\mathcal{M}}_{G,1}$ are simultaneously charged under G and $\mathrm{SU}(2)_v$, and the $\mathrm{SU}(2)_v$ affine symmetry is broken by the expansion (6.18) which makes the affine G symmetry manifest. In the expansion one nevertheless still clearly recognizes the characters of the Verma modules of the $\mathrm{U}(1)_{n-h_G^\vee}^v$ affine Kac-Moody algebra associated to the Cartan subgroup of $\mathrm{SU}(2)_R$. Moreover, the value

$$
R^2 = h_G^\vee - n
\tag{6.28}
$$

---

[27] We thank J. de Boer and M. Dedushenko for discussions on this point.

[28] In section 9, on the other hand, will see that the gauge symmetry for the theories $\mathbf{h}_2^{\mathrm{SU}(2)}$ and $\mathbf{h}_3^{\mathrm{SU}(3)}$, for which $n = h_G^\vee$, appears to be realized differently.

is precisely the coefficient multiplying the ϵ_+^2 term of the anomaly polynomial of the string worldsheet theory upon substituting $c_2(R) \to -\epsilon_+^2$ and $c_2(I) \to -\epsilon_+^2$, see the discussion in section 8 below.

Unlike the $\mathbf{h}_4^{\mathrm{SO}(8)}$ CFT, many of the $\mathbf{h}_n^G$ theories do not have a known UV $(0,4)$ gauge theory realization; nevertheless, our results suggest that the chiral algebras of these theories all have a similar structure which is strongly constrained by the global symmetries of the CFTs. A more detailed study of the chiral algebras of these theories is also currently under investigation and will be addressed in [195]. Finally, we remark that a different realization of chiral algebras of $\mathcal{N} = (0,4)$ CFTs along the lines of [199] has been proposed in [205], which may also shed light on the chiral algebras of this class of CFTs.

6.4 NS-R elliptic genus, spectral flow, and low energy spectrum

In section 5.4 we found of a spectral flow which relates the states that contribute to the NS-R and R-R elliptic genera for the $\mathbf{h}_1^{\mathrm{Sp}(N)}$ theories. We conjecture that the same transformation

$$v \to q^{1/2}/v \tag{6.29}$$

leads to a spectral flow for arbitrary $\mathbf{h}_n^G$ theories, and as a consequence, up to a possible overall sign,

$$\mathcal{E}_n^G(\mathbf{m}_G, \mathbf{m}_F, v, q) = q^{\frac{k_{\epsilon_+}}{4}} v^{-k_{\epsilon_+}} \mathbb{E}_n^G(\mathbf{m}_G, \mathbf{m}_F, q^{1/2}/v, q), \tag{6.30}$$

where $k_{\epsilon_+} = n - h_G^\vee$.

Of course, this spectral flow symmetry cannot be a property of the full spectrum of the $\mathbf{h}_n^G$ CFT. In particular, the $\mathrm{SU}(2)_v$ global symmetry which is used to perform the spectral flow acts nontrivially on the bosonic superfields in the NLSM and therefore is not a chiral symmetry. Nevertheless, in light of the discussion of the previous section we are led naturally to propose that the spectral flow symmetry is a symmetry of the *chiral algebra* of the $\mathbf{h}_n^G$ CFTs, which in particular explains the relation between elliptic genera with NS-R and R-R boundary conditions on the left-moving degrees of freedom.

We are able to verify the symmetry under spectral flow explicitly whenever a UV Lagrangian description for $\mathbf{h}_n^G$ is known. Rather than providing an extensive list of checks, for brevity we content ourselves here with demonstrating relation (6.30) for the infinite family of theories $\mathbf{h}_4^{\mathrm{SO}(8+2N)}$. The R-R elliptic genus is given by [70]:

$$\mathbb{E}_4^{\mathrm{SO}(8+2N)}(\mathbf{m}_{\mathrm{SO}(8+2N)}, \mathbf{m}_{\mathrm{Sp}(2N)}, v, q) = \int \frac{dz}{2\pi i z} \frac{\theta_1(v^2, q)}{\eta(q)} \tag{6.31}$$

$$\times \prod_{s=\pm 1} \left[\theta_1(z^{2\,s}, q) \theta_1(v^2 z^{2\,s}, q) \frac{\eta(q)^7 \prod_{j=1}^N \theta_1(z^s m_{\mathrm{Sp}(N)}^j, q) \theta_1(z^s \tilde{m}_{\mathrm{Sp}(N)}^j, q)}{\prod_{j=1}^{4+N} \theta_1(v z^s m_{\mathrm{SO}(8+4N)}^j, q) \theta_1(v z^s / m_{\mathrm{SO}(8+4N)}^j), q)} \right],$$

where the integral is along a suitable contour determined by the Jeffrey-Kirwan prescription [206], and $m_{\mathrm{Sp}(N)}^j$ and $\tilde{m}_{\mathrm{Sp}(N)}^j$ are exponentiated chemical potentials for the regular maximal $\mathrm{Sp}(N) \times \mathrm{Sp}(N)$ subalgebra of the $\mathrm{Sp}(2N)$ flavor symmetry. The shift $v \to q^{1/2}/v$

only affects the terms

$$\prod_{s=\pm 1} \frac{\prod_{j=1}^{N} \theta_1(z^s m_{\mathrm{Sp}(N)}^j, q)\theta_1(z^s \tilde{m}_{\mathrm{Sp}(N)}^j, q)}{\prod_{j=1}^{4+N} \theta_1(vz^s m_{\mathrm{SO}(8+4N)}^j, q)\theta_1(vz^s/m_{\mathrm{SO}(8+4N)}^j), q)} \tag{6.32}$$

in the integrand, since under this shift the remainder of the integrand only transforms up to an overall phase. Furthermore, it is possible to absorb the shift by redefining $z \to q^{1/2}z$. Then, the denominator is also only transformed up to a phase, while the numerator transforms to

$$\prod_{s=\pm 1} \prod_{j=1}^{N} \theta_3(z^s m_{\mathrm{Sp}(N)}^j, q)\theta_3(z^s \tilde{m}_{\mathrm{Sp}(N)}^j, q) \tag{6.33}$$

which corresponds to imposing NS boundary conditions on the chiral fermions appearing in (0,4) the Fermi multiplets, or in other words to computing the NS-R elliptic genus.

As a further check of the spectral flow symmetry, the strings of the minimal SCFTs with $n > 3$ and no matter content, whose spectrum does not include chiral fermions, should have an elliptic genus invariant under the transformation (6.29). Indeed, invariance under this transformation was already observed in [83].

The same considerations that were made in section 5.4 for the $\mathbf{h}_1^{\mathrm{Sp}(N)}$ case on the low energy spectrum in the NS-R sector also apply in general: imposing NS boundary conditions on the chiral fermions implies that the set of vacua of the theory are in one-to-one correspondence with the unique fermionic vacuum, tensored with gauge neutral combinations of the zero-modes of the bosonic fields of the SCFT, modulo relations; these operators are counted by the Hilbert series of the reduced moduli space of one G-instanton, $\widetilde{\mathcal{M}}_{G,1}$, leading to the prediction

$$\mathcal{E}_n^G(\mathbf{m}_G, \mathbf{m}_F, v, q)\Big|_{q^{-\frac{c_L}{24}}} = v^{h_G^\vee - 1} \sum_{k=0}^{\infty} v^{2k} \chi_{k\cdot\theta_G}^G(\mathbf{m}_G), \tag{6.34}$$

which indeed always turns out to be case in examples whose elliptic genus is known. Here, θ_G indicates the highest weight of the adjoint representation of G; the overall factor of $v^{h_G^\vee - 1}$ multiplying the Hilbert series will be discussed in section 7.

At the next excited level $L_0 = 1/2$, if a UV Lagrangian description is available one can again in analogy with the $\mathbf{h}_1^{\mathrm{Sp}(N)}$ theories construct the states with $\mathrm{U}(1)_v$ charge

$$1 + (h_G^\vee - 1) \tag{6.35}$$

by taking the 2d gauge group-invariant combinations of zero modes of bosonic fields and $L_0 = 1/2$ modes of chiral fermionic fields; such fields transform in the representations of $G \times F$ of the 6d matter hypermultiplets (see the last column in tables 20 and 21). If the 6d matter transforms in a direct sum of representations

$$\bigoplus_{i=1}^{r} (R_i^G, R_i^F), \tag{6.36}$$

we observe that the $L_0 = 1/2$ sector in the NS-R elliptic genus is captured by a straightforward generalization of equation (5.48):

$$\mathcal{E}_n^G(\mathbf{m}_G, \mathbf{m}_F, v, q)\Big|_{q^{-\frac{c_L}{24}+\frac{1}{2}}} = -v^{h_G^\vee - 1} \sum_{k=0}^{\infty} v^{2k+1} \sum_{i=1}^{r} \chi_{\lambda_i^G + k \cdot \theta_G}^G(\mathbf{m}_G) \chi_{\lambda_i^F}^F(\mathbf{m}_F), \qquad (6.37)$$

which we have verified hold for a large number of theories $\mathbf{h}_n^G$.[29] Here, λ_i^G and λ_i^F denote respectively the highest weights of the irreducible representation R_i^G.

We find that the central charge of c_L for the left moving degrees of freedom of the $\mathbf{h}_n^G$ CFT can be expressed as follows in terms of the central charges of the F, G, and $\mathrm{U}(1)_v$ current algebras:

$$c_L = c_F + c_G + c_v - 24 \frac{(h_G^\vee - 1)^2}{4(n - h_G^\vee)}. \qquad (6.38)$$

which we have verified to hold for all the theories whose elliptic genus is known. The last term on the right hand side of equation (6.38) has the following explanation: the NS-R ground state does not belong to the vacuum Verma module of the $\mathrm{U}(1)_v$ WZW model, but rather to the one with

$$h_{\ell,n}^v = \frac{(h_G^\vee - 1)^2}{4(n - h_G^\vee)}. \qquad (6.39)$$

The values of ℓ and m that correspond to this module are $\ell = 2n - h_G^\vee - 1$ and $m = -1$ for all theories $\mathbf{h}_n^G$, with the exceptions of $\mathbf{h}_3^{G_2}$ for which $\ell = -1$ and $m = -2$ and of $\mathbf{h}_4^{\mathrm{SO}(8)}$ for which $\ell = -3$ and $m = -2$. For precisely these two theories the generic assignment $\ell = 2n - h_G^\vee - 1$ is outside of the bound

$$2(n - h_G^\vee) + 1 \le \ell \le 0, \qquad (6.40)$$

and therefore one must shift $\ell \to \ell - 2(n - h_G^\vee)$ (and $m \to m - 1$) in order to bring ℓ within the appropriate range in equation (6.18).

Equation (6.38) also provides an independent way to compute the central charge c_F given G and n, which in section 6.5 we will use as a consistency check that the flavor symmetry F has been correctly identified.

The currents of the flavor symmetry of the chiral algebra are chiral operators of conformal weight 1, which we expect to see in the elliptic genera of the $\mathbf{h}_n^G$ CFTs; indeed, expanding the NS-R elliptic genera and extracting the states that contribute at energy $(1, 0)$ above the NS-R vacuum,[30] we find that

$$\mathcal{E}_n^G(\mathbf{m}_G, \mathbf{m}_F, v, q)\Big|_{q^{-\frac{c_L}{24}+1} v^{h_G^\vee - 1}} = \chi_{\theta_F}^F(\mathbf{m}_F) + \chi_{\theta_G}^G(\mathbf{m}_G) + 1; \qquad (6.41)$$

---

[29]To be precise, in the majority of cases we have performed checks after setting $\mathbf{m}_G = \mathbf{m}_F = \mathbf{1}$ for semplicity, and for a smaller number of cases we have also performed checks for arbitrary $\mathbf{m}_G$ and $\mathbf{m}_F$.

[30]In the NS-R sector we still we have an infinitely-degenerate set of vacua. The vacua contribute to the elliptic genus as the Hilbert series of one $\mathrm{Sp}(N)$ instanton, $q^{-\frac{c_L}{24}} v^{h_G^\vee - 1} \sum_{k=0}^{\infty} v^{2k} \chi_{k \cdot \theta_G}^G(\mathbf{m}_G)$. Here we focus on the ground state which is in the trivial representation of G.

the last term is the contribution from the current of the $\mathrm{U}(1)_v \subset \mathrm{SU}(2)_R$ algebra. We find that this expression holds for almost all $\mathbf{h}_n^G$ theories; notably, however, the $\mathbf{h}_1^{\mathrm{Sp}(N)}$ theories with $N \geq 2$ are an exception: for them turns out that

$$\chi_{\theta_{\mathrm{Sp}(N)}}^{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}) \to \chi_{\theta_{\mathrm{Sp}(N)}}^{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}) + \chi_{(010\ldots0)}^{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}), \tag{6.42}$$

in equation (6.41). The second term on the right hand side is the character of the antisymmetric, $2N^2 - N - 1$ dimensional representation of $\mathrm{Sp}(N)$; notice that the two representations combine into the adjoint representation of $\mathrm{SU}(2N)$, of which $\mathrm{Sp}(N)$ is a maximal special subalgebra. We will provide more details about this special case in section 7.2.3.

The 6d SCFT $\mathbf{2}_{\mathrm{SU}(2)}$ enjoys a flavor symmetry $F = \mathrm{SO}(8)$ on the tensor branch, but at the superconformal fixed point this symmetry is reduced to a $\mathrm{SO}(7)$ subgroup [20]. One can verify (as we do in section 9.2) that indeed for the $\mathbf{h}_2^{\mathrm{SU}(2)}$ theory equation (6.41) holds with $F = \mathrm{SO}(7)$ rather than $\mathrm{SO}(8)$. This also suggests that the BPS string elliptic genus conveys nontrivial information about the superconformal fixed point.[31]

6.5 Current algebra realization of F (II)

In this section we provide more detailed information about the flavor symmetry F of the $\mathbf{h}_n^G$ CFTs and discuss the subtleties that arise for some of these theories. The matter content of rank 1 6d (1,0) SCFTs on the tensor branch is given in table 6; the flavor symmetry of the SCFTs that is suggested by the matter content is also provided there, and is in general a product of simple and abelian factors

$$F = \prod_i F_i, \tag{6.43}$$

to which we associate a current algebra of total central charge

$$c_F = \sum_{i\,|\,F_i \text{ simple}} \frac{\dim(F_i) k_i}{h_{F_i}^\vee + k_i} + \sum_{i\,|\,F_i \text{ abelian}} 1. \tag{6.44}$$

From equation (6.38) we also obtain an independent prediction for the central charge of the F sector:

$$c_F = c_L + \frac{n \dim(G)}{h_G^\vee - n} - 1 + 6 \frac{(h_G^\vee - 1)^2}{n - h_G^\vee}, \tag{6.45}$$

which provides a consistency check on the flavor symmetry of the SCFT. The two expressions match for most theories, but we encounter a few outliers which we discuss at the end of this section. We collect the information about the flavor symmetry current algebras of the $\mathbf{h}_n^G$ theories in tables 20 and 21. It is interesting to remark that our computations confirm the flavor symmetries one would compute from field theory for these models. In particular, since the levels correspond to intersection numbers, whenever the former are different from 1 we predict that the corresponding noncompact flavor divisor intersects the gauge theory divisor in the F-theory base in a non-transverse fashion, which has to be contrasted with the assumptions in [157] where transverse intersections were assumed.[32]

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[31]Nevertheless, we still find that the elliptic genus is written more naturally in terms of level 1 SO(8)

n	G	F	c_F	(R^G, R^F)
12	E_8	—	0	—
8	E_7	—	0	—
7	E_7	—	0	$(\mathbf{56}, \mathbf{1})$
6	E_6	—	0	—
6	E_7	$\mathrm{SO}(2)_{12} = \mathrm{U}(1)_{12}$	1	$(\mathbf{56}, \mathbf{2})$
5	F_4	—	0	—
5	E_6	$\mathrm{SU}(1)_6 \times \mathrm{U}(1)_6 = \mathrm{U}(1)_6$	1	$(\mathbf{27}, \overline{\mathbf{1}}_{-1}) \oplus \mathrm{c.c.}$
5	E_7	$\mathrm{SO}(3)_{12}$	$\frac{36}{13}$	$(\mathbf{56}, \mathbf{3})$
4	$\mathrm{SO}(N),\, N \geq 8$	$\mathrm{Sp}(N-8)_1$	$\frac{(N-8)(2N-15)}{N-6}$	$(\mathbf{N}, \mathbf{2(N-8)})$
4	F_4	$\mathrm{Sp}(1)_3$	$\frac{9}{5}$	$(\mathbf{26}, \mathbf{2})$
4	E_6	$\mathrm{SU}(2)_6 \times \mathrm{U}(1)_{12}$	$\frac{13}{4}$	$(\mathbf{27}, \overline{\mathbf{2}}_{-1}) \oplus \mathrm{c.c.}$
4	E_7	$\mathrm{SO}(4)_{12}$	$\frac{36}{7}$	$(\mathbf{56}, \mathbf{4})$
3	$\mathrm{SU}(3)$	—	0	—
3	$\mathrm{SO}(7)$	$- \times \mathrm{Sp}(2)_1$	$\frac{5}{2}$	$(\mathbf{8}, \mathbf{4})$
3	$\mathrm{SO}(8)$	$\mathrm{Sp}(1)_1^a \times \mathrm{Sp}(1)_1^b \times \mathrm{Sp}(1)_1^c$	3	$(\mathbf{8^v}, \mathbf{2}^a) \oplus (\mathbf{8^s}, \mathbf{2}^b) \oplus (\mathbf{8^c}, \mathbf{2}^c)$
3	$\mathrm{SO}(9)$	$\mathrm{Sp}(2)_1^a \times \mathrm{Sp}(1)_2^b$	4	$(\mathbf{9}, \mathbf{4}^a) \oplus (\mathbf{16}, \mathbf{2}^b)$
3	$\mathrm{SO}(10)$	$\mathrm{Sp}(3)_1^a \times (\mathrm{SU}(1)_4 \times \mathrm{U}(1)_4)^b$	$\frac{26}{5}$	$(\mathbf{10}, \mathbf{6}^a) \oplus [(\mathbf{16^s}, \mathbf{1}_1^b) \oplus \mathrm{c.c.}]$
3	$\mathrm{SO}(11)$	$\mathrm{Sp}(4)_1^a \times \mathrm{Ising}^b$	$\frac{13}{2}$	$(\mathbf{11}, \mathbf{8}^a) \oplus (\mathbf{32}, \mathbf{1}_{\mathbf{s}}^b)$
3	$\mathrm{SO}(12)$	$\mathrm{Sp}(5)_1$	$\frac{55}{7}$	$(\mathbf{12}, \mathbf{10}) \oplus (\mathbf{32^s}, \mathbf{1})$
3	G_2	$\mathrm{Sp}(1)_1$	1	$(\mathbf{7}, \mathbf{2})$
3	F_4	$\mathrm{Sp}(2)_3$	5	$(\mathbf{26}, \mathbf{4})$
3	E_6	$\mathrm{SU}(3)_6 \times \mathrm{U}(1)_{18}$	$\frac{19}{3}$	$(\mathbf{27}, \overline{\mathbf{3}}_{-1}) \oplus \mathrm{c.c.}$
3	E_7	$\mathrm{SO}(5)_{12}$	8	$(\mathbf{56}, \mathbf{5})$

Table 20. Flavor symmetry F of the theories $\mathbf{h}_n^G$ for $n \geq 3$. For each theory, we denote the simple non-abelian factors in the current algebra by F_k, where k is the level. The abelian factors are denoted by $\mathrm{U}(1)_{R^2}$, where R is the radius of the compact boson that realizes the current algebra. When F includes a product of several non-abelian factors, we use superscripts $a, b, \dots$ to distinguish more easily between them and their representations. The last column displays the representation (R^G, R^F) of $G \times F$ in which the matter of the parent 6d SCFT transforms. If F includes a $\mathrm{U}(1)$ factor, we indicate the charge with respect to it by a subscript. For the Ising model, $\mathbf{1_s}$ indicates that the Virasoro primary with $h = \frac{1}{16}$ (whose lowest order component is just a single state).

We can perform a simple consistency check on the central charges listed in tables 20 and 21: since the current algebras are realized in terms of the chiral fermions of the NLSM, their central charges cannot be greater than the contribution to the left-moving central

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characters, with a certain specialization of fugacities — see equations (9.4)–(9.7).

[32]The transverse intersection requirement is a necessary condition for the corresponding flavor symmetry to be gauged, but not all flavor symmetries need to give rise to consistent models upon gauging hence there is no contradiction in letting the corresponding noncompact flavor curves to intersect non-transversally.

n	G	F	c_F	(R^G, R^F)
2	SU(2)	$SO(8)_1 \to SO(7)_1 \times \text{Ising}$	4	$(\mathbf{2}, \mathbf{8^v}) \to (\mathbf{2}, \mathbf{8^s} \times \mathbf{1_s})$
2	$SU(N), N > 2$	$SU(2N)_1$	2N-1	$(\mathbf{N}, \overline{\mathbf{2N}}) \oplus \text{c.c.}$
2	SO(7)	$Sp(1)_1^a \times Sp(4)_1^b$	7	$(\mathbf{7}, \mathbf{2}^a) \oplus (\mathbf{8}, \mathbf{8}^b)$
2	SO(8)	$Sp(2)_1^a \times Sp(2)_1^b \times Sp(2)_1^c$	$\frac{15}{2}$	$(\mathbf{8^v}, \mathbf{4}^a) \oplus (\mathbf{8^s}, \mathbf{4}^b) \oplus (\mathbf{8^c}, \mathbf{4}^c)$
2	SO(9)	$Sp(3)_1^a \times Sp(2)_2^b$	$\frac{41}{5}$	$(\mathbf{9}, \mathbf{6}^a) \oplus (\mathbf{16}, \mathbf{4}^b)$
2	SO(10)	$Sp(4)_1^a \times (SU(2)_4 \times U(1)_8)^b$	9	$(\mathbf{10}, \mathbf{8}^a) \oplus [(\mathbf{16^s}, \mathbf{2}_1^b) \oplus \text{c.c.}]$
2	SO(11)	$Sp(5)_1^a \times ???^b$	$\frac{69}{7}$	$(\mathbf{11}, \mathbf{10}^a) \oplus (\mathbf{32}, \mathbf{2}^b)$
2	$SO(12)_a$	$Sp(6)_1^a \times SO(2)_8$	$\frac{43}{4}$	$(\mathbf{12}, \mathbf{12}^a) \oplus (\mathbf{32^s}, \mathbf{2}^b)$
2	$SO(12)_b$	$Sp(6)_1^a \times \text{Ising}^b \times \text{Ising}^c$	$\frac{43}{4}$	$(\mathbf{12}, \mathbf{12}^a) \oplus (\mathbf{32^s}, \mathbf{1_s}^b) \oplus (\mathbf{32^c}, \mathbf{1_s}^c)$
2	SO(13)	$Sp(7)_1$	$\frac{35}{3}$	$(\mathbf{13}, \mathbf{14}) \oplus (\mathbf{64}, \mathbf{1})$
2	G_2	$Sp(4)_1$	6	$(\mathbf{7}, \mathbf{8})$
2	F_4	$Sp(3)_3$	9	$(\mathbf{26}, \mathbf{6})$
2	E_6	$SU(4)_6 \times U(1)_{24}$	10	$(\mathbf{27}, \overline{\mathbf{4}}_{-1}) \oplus \text{c.c.}$
2	E_7	$SO(6)_{12}$	$\frac{45}{4}$	$(\mathbf{56}, \mathbf{6})$
1	$Sp(N)$	$SO(4N+16)_1$	2N+8	$(\mathbf{2N}, \mathbf{4N+16})$
1	SU(3)	$SU(12)_1$	11	$(\mathbf{3}, \overline{\mathbf{12}}) \oplus \text{c.c.}$
1	SU(4)	$SU(12)_1^a \times SU(2)_1^b$	12	$[(\mathbf{4}, \overline{\mathbf{12}}^a) \oplus \text{c.c.}] \oplus (\mathbf{6}, \mathbf{2}^b)$
1	$SU(N), N \geq 4$	$SU(N+8)_1 \times U(1)_{2N(N-1)(N+8)}$	$N+8$	$[(\mathbf{N}, \overline{\mathbf{N+8}}_{-N+4}) \oplus (\mathbf{\Lambda^2}, \mathbf{1}_{-N-8})] \oplus \text{c.c.}$
1	$SU(6)_*$	$SU(15)_1 \times SO(1)_6 = SU(15)_1$	14	$[(\mathbf{6}, \overline{\mathbf{15}}) \oplus \text{c.c.}] \oplus (\mathbf{20}, \mathbf{1})$
1	SO(7)	$Sp(2)_1^a \times Sp(6)_1^b$	$\frac{49}{4}$	$(\mathbf{7}, \mathbf{4}^a) \oplus (\mathbf{8}, \mathbf{12}^b)$
1	SO(8)	$Sp(3)_1^a \times Sp(3)_1^b \times Sp(3)_1^c$	$\frac{63}{5}$	$(\mathbf{8^v}, \mathbf{6}^a) \oplus (\mathbf{8^s}, \mathbf{6}^b) \oplus (\mathbf{8^c}, \mathbf{6}^c)$
1	SO(9)	$Sp(4)_1^a \times Sp(3)_2^b$	13	$(\mathbf{9}, \mathbf{8}^a) \oplus (\mathbf{16}, \mathbf{6}^b)$
1	SO(10)	$Sp(5)_1^a \times (SU(3)_4 \times U(1)_{12})^b$	$\frac{94}{7}$	$(\mathbf{10}, \mathbf{10}^a) \oplus [(\mathbf{16^s}, \mathbf{3}_1^b) \oplus \text{c.c.}]$
1	SO(11)	$Sp(6)_1^a \times ???^b$	$\frac{111}{8}$	$(\mathbf{11}, \mathbf{12}^a) \oplus (\mathbf{32}, \mathbf{3}^b)$
1	$SO(12)_a$	$Sp(7)_1^a \times SO(3)_8^b$	$\frac{43}{3}$	$(\mathbf{12}, \mathbf{14}^a) \oplus (\mathbf{32^s}, \mathbf{3}^b)$
1	$SO(12)_b$	$Sp(7)_1^a \times ???^b \times ???^c$	$\frac{43}{3}$	$(\mathbf{12}, \mathbf{14}^a) \oplus (\mathbf{32^s}, \mathbf{2}^b) \oplus (\mathbf{32^c}, \mathbf{1}^c)$
1	G_2	$Sp(7)_1$	$\frac{35}{3}$	$(\mathbf{7}, \mathbf{14})$
1	F_4	$Sp(4)_3$	$\frac{27}{2}$	$(\mathbf{26}, \mathbf{8})$
1	E_6	$SU(5)_6 \times U(1)_{30}$	$\frac{155}{11}$	$(\mathbf{27}, \overline{\mathbf{5}}_{-1}) \oplus \text{c.c.}$
1	E_7	$SO(7)_{12}$	$\frac{252}{17}$	$(\mathbf{56}, \mathbf{7})$

Table 21. Flavor symmetry F of the theories $\mathbf{h}_n^G$ for $n = 1$ and 2. For the $\mathbf{h}_1^{SU(N)}$ theories, $\mathbf{\Lambda^2}$ symbolizes the $\frac{N(N-1)}{2}$-dimensional antisymmetric representation of $SU(N)$. We indicate by ??? the cases where we do not have a good understanding of the worldsheet realization of the flavor symmetry.

charge from the chiral fermions, which according to table 16 is given by

$$c_L^f = 2h_G^\vee + 6(2-n). \tag{6.46}$$

Indeed, for all $\mathbf{h}_n^G$ CFTs listed in tables 20 and 21 we find:

$$c_F \le c_L^f, \tag{6.47}$$

the bound being saturated only for the $\mathbf{h}_1^{\mathrm{Sp}(N)}$ theories (for which we have indeed shown in section 5 that the chiral fermions provide a free field realization of the $\mathrm{SO}(16+4N)$ WZW model), for the $\mathbf{h}_2^{\mathrm{SU}(2)}$ theory, and trivially for the theories $\mathbf{h}_3^{\mathrm{SU}(3)}$, $\mathbf{h}_4^{\mathrm{SO}(8)}$, $\mathbf{h}_5^{F_4}$, $\mathbf{h}_6^{E_6}$, $\mathbf{h}_8^{E_7}$, and $\mathbf{h}_{12}^{E_8}$ corresponding to 6d SCFTs without matter. Moreover, one can check by looking at tables 20 and 21 that the central charge c_F decreases monotonically as one moves along the edges of the Higgsing trees of section 2.4.

A further consistency check is obtained by requiring that the general form of the elliptic genus in terms of affine characters (equations (6.17) and (6.18)) is consistent with the correct representations appearing at $L_0 = 1/2$ in the NS-R elliptic genus, as in equation (6.37). This is in particular helpful for checking abelian factors $\mathrm{U}(1)_{R^2}$ of the flavor symmetry, for which $c_F \equiv 1$ but the spectrum of primaries depends on the value of R.

In the remainder of this section we comment on subtleties that arise for certain $\mathbf{h}_n^G$ theories. A first subtlety concerns the way $\mathrm{U}(N)$ flavor symmetry factors are realized in the CFT. For instance, the $\mathbf{n}_{E_6}$ SCFTs possess a $\mathrm{U}(6-n)$ flavor symmetry which rotates the $6-n$ hypermultiplets in the $\mathbf{26}$ of E_6. The flavor symmetry is realized on the string worldsheet in terms of a

$$\mathrm{SU}(6-n)_6 \times \mathrm{U}(1)_{6(6-n)} \tag{6.48}$$

current algebra; interestingly, the levels of the two factors are inconsistent with the ones of the $\mathrm{U}(6-n)_k$ current algebra. Indeed, the latter is given by a $\mathbb{Z}_{6-n}$ quotient of the

$$\mathrm{SU}(6-n)_k \times \mathrm{U}(1)_{(6-n)(k+6-n)} \tag{6.49}$$

current algebra, which is not compatible with (6.48); moreover, the $\mathrm{U}(N)_k$ WZW model is only well defined for odd k [207, 208], whereas for the $\mathbf{h}_n^{E_6}$ CFTs we find an even level. We encounter a similar situation for the $\mathrm{U}(4-n)$ flavor symmetry that rotates the spinorial matter in the $\mathbf{n}_{\mathrm{SO}(10)}$ SCFTs. At the level of current algebra, we find that this is captured on the string by a

$$\mathrm{SU}(4-n)_4 \times \mathrm{U}(1)_{4(4-n)} \tag{6.50}$$

current algebra, and not by a $\mathrm{U}(4-n)_k$ current algebra.

Next, we discuss some peculiarities we encounter for $\mathbf{h}_n^G$ theories whose flavor symmetry includes orthogonal groups of small rank:

- For the theory $\mathbf{h}_6^{E_7}$, we find $F = \mathrm{SO}(2)_{12}$, which is equivalent to the $\mathrm{U}(1)$ WZW model at radius $R = \sqrt{12}$ (see appendix E).

- For certain theories the flavor symmetry F includes a factor of 'SO(1)', which may or may not correspond to a trivial WZW model according to the following reasoning: recall that the central charge for a generic $SO(K)_k$ WZW model is given by:

$$c_{SO(K)_k} = \frac{K(K-1)k}{2(K-2+k)}.$$ (6.51)

Setting k to be any level except for 1, and analytically continuing the rank to $K = 1$ one finds that

$$\lim_{K \to 1} c_{SO(K)_k} = 0, \qquad \text{for } k \neq 1,$$ (6.52)

so that the CFT for $k \neq 1$ is trivial. On the other hand, setting $k = 1$ first and taking the limit gives

$$\lim_{K \to 1} c_{SO(K)_1} = \frac{1}{2}.$$ (6.53)

This is of course consistent with the well-known fact that the expressions for the characters of the $SO(K)$ WZW model at level 1 reduce to the characters of the Ising model when $K = 1$, which is simply the theory of one free chiral fermion [209].

By computing the central charge (6.45), we find that the SO(1) factor in F contributes as the trivial $c = 0$ theory for the following theories: $\mathbf{h}_1^{SU(6)_*}$, $\mathbf{h}_2^{SO(13)}$, $\mathbf{h}_3^{SO(12)}$, and $\mathbf{h}_7^{E_7}$. On the other hand, for $\mathbf{h}_3^{SO(11)}$ we find that the SO(1) factor contributes $c = 1/2$ to the central charge and is therefore captured by the Ising model. By the methods of section 8 we are able to check that the elliptic genera of these theories are consistent with this identification of the SO(1) factor.

- For the three theories $\mathbf{h}_3^{SO(12)}$, $\mathbf{h}_2^{SO(12)}$, and $\mathbf{h}_1^{SO(12)}$, there is a priori an ambiguity in the chirality of the SO(12) spinors that contribute to the matter of the 6d SCFT, which may in principle involve n_s hypermultiplets the spinor representation $\mathbf{32^s}$ of SO(12) and n_c hypermultiplets in the conjugate spinor representation $\mathbf{32^c}$. Exchanging n_s and n_c leads to the same elliptic genus up to a redefinition of the $\mathbf{m}_G$ fugacities, since the two choices are related by an outer automorphism of SO(12).

For $n = 3$, up to this redefinition there is a unique theory with $n_s = 1$ and $n_c = 0$; the flavor symmetry rotating the spinors is SO(1), which is realized as the Ising model as discussed above.

For $n = 2$, there are potentially two distinct theories. The first, $\mathbf{h}_2^{SO(12)a}$, has $n_s = 2$ and $n_c = 0$, and that the flavor symmetry rotating the spinors is expected to be $SO(2)_8 = U(1)_8$; the second, $\mathbf{h}_2^{SO(12)b}$, has $n_s = 1$ and $n_c = 1$, and it is expected that the flavor symmetry rotating the spinors is $SO(1) \times SO(1)$. From (6.45), one expects the central charge of this component of the flavor symmetry to be $c = 1$, which is the correct value for $SO(2)_8$ in the former case, and is consistent with SO(1) being realized as the Ising model in the latter. We will look at these theories in more detail in section 8.3 and will find that both choices lead to seemingly consistent elliptic genera.

Finally, for $n = 1$ there are also two possible choices: a first choice, $\mathbf{h}_1^{SO(12)a}$ with $n_s = 3$ and $n_c = 0$ which contributes a $SO(3)_8 = SU(2)_{16}$ factor to the flavor symmetry, and a second choice, $\mathbf{h}_1^{SO(12)b}$ with $n_s = 2$ and $n_c = 1$, with expected flavor

symmetry $\mathrm{SO}(2)_8 \times \mathrm{SO}(1)$. On the other hand, in the second case $\mathrm{SO}(2)_8 \times \mathrm{SO}(1)$ would have central charge 1 or 3/2 depending on whether the $\mathrm{SO}(1)$ factor is realized as the trivial theory or as the Ising model. Neither choice is consistent with the prediction from equation (6.45) that the central charge be 8/3, and there does not seem to be an obvious way to recover the correct central charge, even by allowing for enhancements of the flavor symmetry (other than allowing for the possibility that $\mathrm{SO}(2)_8 \times \mathrm{SO}(1)$ enhances to $\mathrm{SO}(3)_8$, so that the elliptic genus reduces to the $\mathbf{h}_1^{\mathrm{SO}(12)_a}$ theory). Therefore we are not able to fit the $\mathbf{h}_1^{\mathrm{SO}(12)_b}$ theory within our framework, possibly indicating an inconsistency in the $\mathbf{h}_1^{\mathrm{SO}(12)_b}$ theory.

- For the theories $\mathbf{h}_2^{\mathrm{SO}(11)}$ and $\mathbf{h}_1^{\mathrm{SO}(11)}$, equation (6.45) predicts the central charge of the flavor symmetry chiral algebra, c_F, to be respectively $\frac{69}{7}$ and $\frac{111}{8}$. In particular, the factor of the flavor symmetry that rotates the spinorial matter is expected to have central charge respectively 2 and $\frac{33}{8}$. This is inconsistent with the naive expectation that the flavor symmetry associated to the spinorial matter be respectively $\mathrm{SO}(2)_8$ and $\mathrm{SO}(3)_8$, for which respectively $c = 1$ and $c = 8/3$. There also does not appear to be any obvious alternative choice of flavor symmetry that correctly captures these central charges, and therefore somewhat unexpectedly we have not been able to fit these theories within our framework.

Finally, recall from section 2.1 that the $\mathcal{O}(-n) \to \mathbb{P}^1$ models with $n = 9$, 10, and 11 have $G = E_8$ gauge algebra coupled respectively to 3, 2, and 1 additional E-string subsectors. Amusingly, one may ask whether these models also admit a description in terms of some effective theory with $n_T = 1$ tensor multiplet and an ordinary BPS string sector; however, we immediately run into a problem when calculating c_F for the string of this putative effective theory: from equation (6.45) we find $c_F = -1$ for $n = 9$ and 11, and to $c_F = -\frac{13}{10}$ for $n = 10$; suggesting that such an effective description is inconsistent.

7 Five-dimensional limit

In this section we discuss a relation between the lowest energy states in the elliptic genus and the one-instanton part of the 5d Nekrasov partition function, which arises by taking the zero-size limit of the 6d circle in the $(S^1 \times S^1 \times \mathbb{R}^4)_{\epsilon_1, \epsilon_2}$ partition function

$$Z_{T^2 \times \mathbb{R}^4}^{6d}(\varphi, \mathbf{m}_G, \mathbf{m}_F, x, v, q) = Z_{\mathrm{pert}}(\mathbf{m}_G, \mathbf{m}_F, x, v, q) \times \sum_{k \geq 0} e^{-k\,\Phi} \, \mathbb{E}_{(k)}(\mathbf{m}_G, \mathbf{m}_F, x, v, q).$$
(7.1)

where, as remarked in section 4,

$$\Phi = \varphi + \text{ linear combinations of } \tau, \mathbf{m}_G, \text{ and } \mathbf{m}_F.$$
(7.2)

As a first step, it is useful to recall the compactification of the 6d theory on a circle of finite size, which gives rise to a five-dimensional theory at finite gauge coupling. Namely, place the 6d SCFT with simple gauge group G and matter content in the representation $\mathcal{R}$ of G on the tensor branch on the 6d Omega background; this can alternatively be interpreted

as a 5d theory with gauge group $U(1) \times G$ on the 5d Omega background $(S^1 \times \mathbb{R}^4)_{\epsilon_1, \epsilon_2}$, where the $U(1)$ gauge field $A_{U(1)}$ originates from the reduction of the two-form field B in the tensor multiplet. By a well known argument that can be found in [210] the coupling $\frac{g_{U(1)}^2}{4\pi}$ of this $U(1)$ gauge field is identified with the radius of the 6d circle, R_{6d}. The KK momentum along the 6d circle is identified with the instanton charge with respect to the gauge $U(1)$; the $U(1)_C$ isometry of the 6d circle becomes a global symmetry in 5d, to which a background $U(1)$ gauge field C is associated. Notice that the latter couples to the gauge $U(1)$ field strength via an interaction $\int C \wedge F_{U(1)} \wedge F_{U(1)}$. Hence the KK charge, which is the conserved charge associated to the $U(1)_C$ isometry in 6d, is also identified with the $U(1)$ gauge field instanton charge. Finally, the 6d gauge multiplet for G reduces trivially to a 5d gauge multiplet. We denote the real adjoint-valued scalar in the 5d gauge multiplet by a_G. The 5d gauge coupling is given by

$$\frac{8\pi^2}{g_{G,5d}^2} = R_{6d}\frac{8\pi^2}{g_{G,6d}^2} = R_{6d}\,\phi_{6d} = \sqrt{R_{6d}}\phi_{5d}, \qquad (7.3)$$

where ϕ_{6d} is the vev of the scalar in the 6d tensor multiplet, which reduces in 5d to the scalar partner ϕ_{5d} of the $U(1)$ gauge field. In equation (7.1) the combination $\varphi = \phi_{6d} + i\int_{T^2} B$ appears.

The complex modulus of the torus gets identified with

$$\tau = \frac{4\pi i}{g_{U(1)}^2}R_{5d} + \int_{S_{5d}^1} C, \qquad (7.4)$$

where R_{5d} is the radius of the circle in the 5d Omega background.

The 5d $U(1)$ vector multiplet scalar, ϕ_{5d}, combines with the $A_{U(1)}$ holonomy on the 5d circle into a complex scalar:

$$\varphi_{5d} = \phi_{5d} + 2\pi i \int_{S_{5d}^1} A_{U(1)}. \qquad (7.5)$$

Notice that the imaginary component can also be viewed as a flux $\int_{T^2} B$ for the 6d tensor multiplet two-form field, and appears here due to the 6d $\int B \wedge \mathrm{Tr}\, F_G \wedge F_G$ interaction term. Finally, the fugacities $\mathbf{m}_G, \mathbf{m}_F$ are interpreted respectively as Coulomb branch parameters for G and as hypermultiplet masses, complexified respectively by Wilson lines for the G gauge potential and of the F background potential.

In the limit of zero size of the 6d circle, $\tau \to i\infty$ and the $U(1)$ gauge field decouples. The $U(1)$ symmetry then gets identified with the 5d topological $U(1)$ global symmetry associated to the G instantons. In this limit we project the spectrum of the 6d theory to the subsector with lowest $U(1)_C$ charge. At a naive level, one expects the 5d limit of the 6d theory to be a 5d theory with gauge group G and matter content in the same representation $\mathcal{R}$ as in 6d, plus the decoupled $U(1)$ vector multiplet. However, we will see that the 5d limit in fact depends on whether $n \geq 3$, $n = 2$, or $n = 1$. We discuss these three cases in turn, after providing a rationale for these different limits from the perspective of geometric engineering.

7.1 Remarks on 5d theories and M-theory geometry

The reduction of six-dimensional $\mathcal{N} = (1,0)$ SCFTs on S^1 discussed above is essentially correct, provided the self-intersection of the curve in the F-theory base is sufficiently high. At the level of Nekrasov partition functions, we find the following trichotomy:

$$(\mathbf{n}_G^{\mathcal{R}})_{6d} \quad \longrightarrow \quad \begin{cases} (G, \mathcal{R})_{5d} & n \geq 3 \\ (G, \mathcal{R})_{5d} \oplus \mathfrak{R}_{2,G} & n = 2 \\ \text{Free hyper} & n = 1 \end{cases} \tag{7.6}$$

where the notation is as follows: the six-dimensional theory is denoted by $\mathbf{n}_G^{\mathcal{R}}$ where $\mathcal{R}$ is a representation of G corresponding to the matter content from table 6, the five-dimensional theory $(G, \mathcal{R})_{5d}$ is a 5d gauge theory with gauge group G and matter in the $\mathcal{R}$ representation, and the five-dimensional theory $\mathfrak{R}_{2,G}$ is a residual decoupled system that arises only when $n = 2$.[33] For $n = 1$ the lowest energy degrees of freedom consist simply of a free hypermultiplet, and any degrees of freedom charged under G only appear at higher levels in the KK tower and therefore decouple in the $R_{6d} \to 0$ limit.

This has the following geometrical interpretation. The reduction from 6d to 5d is realized within geometric engineering by F-theory/M-theory duality. F-theory on $S_{6d}^1 \times \widehat{X}$, where $\widehat{X}$ is the elliptic Calabi-Yau associated to the tensor branch of the SCFT, is equivalent to M-theory on $\widehat{X}$. In the M-theory picture, M2 branes wrapped on holomorphic two-cycles in $\widehat{X}$ give rise to BPS particles in 5d. The radius of S_{6d}^1 in the F-theory background is inversely proportional to the volume of the elliptic fiber of $\widehat{X}$ on the M-theory side, so one reaches a genuine 5d theory when the volume of the elliptic fiber is sent to infinite size. For rank-one models with a nontrivial gauge symmetry the elliptic fiber along the compact curve in the base $\widehat{B}$ has a nontrivial Kodaira type. The relevant geometry is shown schematically in figure 16 — see e.g. [211]. Within M-theory, the geometry involves a collection of ruled surfaces S_i, $i = 0, \ldots, r$. This gives rise to a fibration which is given by a collection of -2 curves that combine into an elliptic fiber of Kodaira type, drawn in blue in the figure. We label this collection of curves $\widehat{G}_r$, as it can be viewed as an affinization of the Dynkin graph of the corresponding gauge group G. The combination of 2-cycles associated to the imaginary root δ of $\widehat{G}_r$ gives rise in the M-theory geometric engineering to the extra U(1) gauge symmetry with gauge field $A_{\mathrm{U}(1)}$ which becomes global in the limit $R_{6d} \to 0$. As discussed above, this is the global U(1) corresponding to the current

$$j_{\mathrm{top}} = *_{5d} \, \mathrm{tr} \, F_G \wedge F_G \tag{7.7}$$

whose charge is the G instanton number. The surfaces $S_0, \ldots, S_r$ are all compact as long as the radius of the 6d F-theory circle S_{6d}^1 is finite. There are several inequivalent ways to send the volume of the elliptic fiber E of $\widehat{X}$ to infinity, which are related by flop transitions in the

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[33]The $n = 2$ case overlaps with the class of models studied in [20], as these theories are all Higgsable to 6d A_1 $(2,0)$ theories. It would be interesting to probe the conjecture of [20] which predicts that $\mathfrak{R}_{2,G}$ should consist of an SU(2) SYM sector. This is out of the scope of the present work, but can be addressed with our techinques and we leave it for the future.
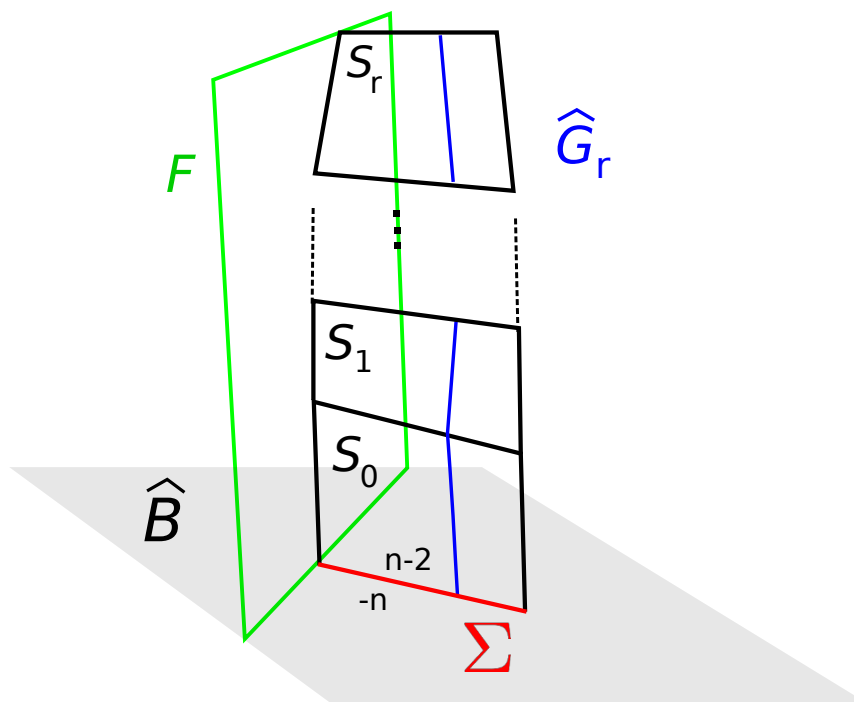
Figure 16. Schematic description of the generic geometry for rank-one models.

extended Kähler cone of $\widehat{X}$, see e.g. [41]. The limit we discuss in this section involves taking the volume τ of the elliptic fiber to infinity, while keeping the parameters $\mathbf{m}_G$ associated to the 2-cycles corresponding to the real roots of the Dynkin diagram finite. This is achieved by sending to infinite volume the -2 curve that provides the ruling of the surface S_0 while keeping the volumes of the analogous curves for the $S_1, \ldots, S_r$ surfaces finite.

Let us also comment on how to read off the $U(1)_C$ charge from geometry. It is clear that it is proportional to the class of the elliptic fiber E. Notice that $E = F + \cdots$ and therefore we can use F to read off the $U(1)_C$ charge of a wrapped M2 brane.

The intersection of the base $\widehat{B}$ with S_0 is precisely the rational curve Σ, which controls the tensor branch vev in 6d and the 6d gauge coupling (in red in figure 16). Within the base $\widehat{B}$, this curve has self intersection

$$\Sigma \cdot_{\widehat{B}} \Sigma = -n \,. \tag{7.8}$$

Locally, the CY condition amounts to the requirement that the normal bundle to Σ has the form

$$\mathcal{O}(-n) \oplus \mathcal{O}(n-2) \to \mathbb{P}^1 \qquad \Sigma \simeq \mathbb{P}^1 \,, \tag{7.9}$$

so that within S_0 the curve Σ has self-intersection

$$\Sigma \cdot_{S_0} \Sigma = n - 2 \,. \tag{7.10}$$

Moreover, S_0 viewed as a ruled surface has a generic fiber $F \simeq \mathbb{P}^1$ which is also a rational curve with

$$F \cdot_{S_0} F = 0 \tag{7.11}$$

with normal bundle

$$\mathcal{O}(0) \oplus \mathcal{O}(-2) \to \mathbb{P}^1 \, ; \tag{7.12}$$

finally, we have the following intersection between the two curves

$$F \cdot_{S_0} \Sigma = 1. \tag{7.13}$$

The curve F is the -2 curve of the Kodaira fiber of $\widehat{X}$ that corresponds to the affine node, so, based on the previous discussion, the 5d limit is achieved by sending the volume of the curve F of the S_0 divisor to infinity.

In the case $n > 2$, the curve Σ has a positive self-intersection in S_0, and therefore it is not contractible. As the volume of F, and hence of S_0, is scaled to infinity the volume of Σ necessarily also scales to infinite size. Moreover, any state with nonzero $\mathrm{U}(1)_C$ charge arises from M2 branes wrapped on the elliptic fiber E, and by sending $\mathrm{vol}(F) \to \infty$ acquires an infinite mass. In this way, we are left with the collection of contractible surfaces $S_1, \ldots, S_r$ that correspond to the 5d gauge theory with gauge group G and matter in a representation $\mathcal{R}$. This is precisely the naive behavior we have argued for above. An additional subtlety arises in the case $n = 2$: because in this case $\Sigma \cdot_{S_0} \Sigma = 0$, the curve Σ itself provides a second ruling of the surface S_0, which means that the curve can be moved within S_0. Therefore there is a moduli space of solutions involving M2 branes wrapping Σ, that are neutral with respect to $\mathrm{U}(1)_C$ and, as we will see in section 7.2.2, lead to extra degrees of freedom in the 5d Nekrasov partition function.

The behavior is markedly different in the case

$$\Sigma \cdot_{\widehat{B}} \Sigma = -1. \tag{7.14}$$

By the same argument as above, in this case Σ has negative self-intersection within S_0 and therefore is not constrained to scale up when S_0 does. For definiteness, consider the case in which S_0 is a Hirzebruch surface of type $\mathbb{F}_1$. This is not always the case, as in the presence of matter one typically encounters more general surfaces, e.g. blow-ups of Hirzebruch surfaces, but an analogous argument carries over in that case. The divisors S_0 and S_1 intersect along the curve $\Sigma' = \Sigma + F$ which has self-intersection $+1$. Therefore, the curve's volume scales to infinity along with F, and any M2 brane that is wrapped on a two-cycle which has a component along this curve becomes infinitely heavy in the limit $\mathrm{vol}(F) \to \infty$; this in particular means that any degree of freedom corresponding to the gauge group and matter in 6d which also carries BPS string charge becomes infinitely massive (this does not include the W-bosons of G which arise from M2 branes wrapped on the curves in $S_1, \ldots, S_r$ associated to the finite nodes of $\widehat{G}_r$, which however contribute to Z_{pert} in equation (7.1)). An analogous way to rephrase this is that this class of BPS M2 branes carry positive $\mathrm{U}(1)_C$ charge since they wrap on curves with a component in $\Sigma' = \Sigma + F$, and therefore differently from the case $n \neq 1$ they do not contribute to lowest

energy in the KK tower of states. On the other hand, the local geometry of Σ is the total space of the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ which gives rise to the 5d theory of one free hypermultiplet. This analysis is also compatible with the results of [212] that enlarges the list of consistent models found in [211]. Indeed, for the gauge groups and matter content that arise for 6d $n = 1$ SCFTs, a putative 5d gauge theory obtained by naive dimensional reduction would possess too much matter to be consistent; this class of models would possess a negative definite 5d metric everywhere along the Coulomb branch to start with, and therefore cannot have a good UV definition in 5d.

7.2 5d Nekrasov partition functions

7.2.1 Circle reduction of 6d models with $n \geq 3$

In 5d limit $q \to 0$ the coupling of the U(1) gauge field is turned off, and the 6d Nekrasov partition function reduces to the partition function of $\mathcal{N} = 1$ G Yang-Mills theory in the 5d Omega background, with matter content determined by KK reduction of the 6d (1,0) multiplets. The g_G-independent term Z_{pert} in equation (7.1) captures the perturbative piece of this partition function (as well as additional decoupled degrees of freedom), while the sum over elliptic genera captures the instanton terms:

$$\sum_{k \geq 0} e^{-k\,\Phi}\, \mathbb{E}_{(k)}(\mathbf{m}_G, \mathbf{m}_F, x, v, q) \xrightarrow{q \to 0} \sum_{k \geq 0} Q_{5d}^k\, Z_k(\mathbf{m}_G, \mathbf{m}_F, x, v). \tag{7.15}$$

The precise identification of the 5d limit involves some subtleties to which we now turn.

In the zero-size limit of the 6d circle, the NLSM for k strings is expected to reduce to an ADHM-like quantum mechanics on the moduli space of k instantons in the presence of matter. The states of the string that contribute to the quantum mechanics are the ground states in the Ramond-Ramond sector of the BPS string CFT, which as discussed in section 6.1, have

$$H_L = -k\frac{n-2}{2} \tag{7.16}$$

(where now, unlike in (6.11), we include the center of mass hypermultiplet). In other words, we expect that

$$\mathbb{E}_{(k)}(\mathbf{m}_G, \mathbf{m}_F, x, v, q)\bigg|_{q^{-k\frac{n-2}{2}}} = Z_k(\mathbf{m}_G, \mathbf{m}_F, x, v), \tag{7.17}$$

where $Z_k(\mathbf{m}_G, \mathbf{m}_F, x, v)$ is proportional to the 5d, or K-theoretic, Nekrasov partition function [136], up to an overall factor of v which is theory dependent. For instance, for the 6d theories with no matter content $\mathbf{h}_3^{SU(3)}$, $\mathbf{h}_4^{SO(8)}$, $\mathbf{h}_5^{F_4}$, $\mathbf{h}_6^{E_6}$, $\mathbf{h}_8^{E_7}$, and $\mathbf{h}_{12}^{E_8}$, one finds that [83]

$$Z_k(\mathbf{m}_G, \mathbf{m}_F, x, v) = v^{k\,h_G^\vee} Z_{k-inst}(\mathbf{m}_G, \mathbf{m}_F, x, v) = v^{k\,h_G^\vee} \mathcal{H}_{\mathcal{M}_{G,k}}(\mathbf{m}_G, \mathbf{m}_F, x, v), \tag{7.18}$$

where the k-instanton piece of the Nekrasov partition function for 5d $\mathcal{N} = 1$ G SYM, Z_{k-inst}, is the same as the Hilbert series $\mathcal{H}_{\mathcal{M}_{G,k}}$ of the moduli space of k G-instantons [213–215]. The overall factor of v is natural from various points of view. First of all, it follows from the fact that

$$\mathbb{E}_{(k)}(\mathbf{m}_G, \mathbf{m}_F, x, v^{-1}, q) = \mathbb{E}_{(k)}(\mathbf{m}_G, \mathbf{m}_F, x, v, q) \tag{7.19}$$

(up to a possible overall minus sign), which is a manifestation of the $\text{Weyl}[\text{SU}(2)_R] = \mathbb{Z}_2$ symmetry of the elliptic genus and is a basic property of Jacobi forms (see appendix B). Moreover, in [216], it was observed for the 5d $\mathcal{N} = 1$ super-Yang-Mills theory with $G = \text{SU}(N)$ that it is natural to include this factor (with $h^\vee_{\text{SU}(N)} = N$) in the definition of the k-instanton piece of the 5d Nekrasov partition function; there, it has the interpretation of the contribution from half of the canonical bundle of $\mathcal{M}_{\text{SU}(N),k}$ to the k-instanton piece of the partition function, which arises naturally if one defines this as the index of the Dirac operator on $\mathcal{M}_{\text{SU}(N),k}$ rather than the index of the Dolbeault operator. See also [217] for related comments on the appearance of a v-dependent prefactor.

The identification (7.17) also implies the relation

$$e^{-\Phi} = q^{\frac{n-2}{2}} Q_{5d}, \tag{7.20}$$

which was observed in the context of minimal SCFTs in [70], see also [85]. The 5d limit is taken by keeping Q_{5d} fixed while sending $q \to 0$.

7.2.2 Circle reduction of 6d models with $n = 2$

We next turn to the theories with $n = 2$. As anticipated in section 7.1, we expect extra degrees of freedom to arise from M2 branes wrapping a curve in the homology class of Σ in S_0: a representative corresponding to a generic nonzero section clearly has vanishing interesection number with all compact divisors in the Calabi-Yau, and M2 branes wrapped on these holomorphic curves lead to BPS states which are neutral with respect to G. These BPS states give extra contributions to the topological string partition function, which appear alongside the Nekrasov partition function for G with matter content determined from KK reduction of the 6d hypermultiplets. For $G = \text{SU}(N), N_f = 2N$ this effect has been explained in [140], where it was argued that the topological string partition function factorizes as the $\text{SU}(N), N_f = 2N$ Nekrasov partition function times a decoupled extra factor:

$$Z_{\text{top}} = Z_{\text{extra}} \times Z_{\text{SU}(N), N_f=2N}. \tag{7.21}$$

Therefore, we expect the following slight modification of equation (7.15) to hold:

$$\sum_{k \geq 0} e^{-k\Phi} \mathbb{E}_{(k)}(\mathbf{m}_G, \mathbf{m}_F, x, v, q) \xrightarrow{q \to 0} \sum_{k \geq 0} Q_{5d}^k Z'_{k-inst}(\mathbf{m}_G, \mathbf{m}_F, x, v), \tag{7.22}$$

where $Q_{5d} = e^{-\Phi}$ in accordance with equation (7.20), and $Z'_{k-inst}(\mathbf{m}_F, \mathbf{m}_G, x, v)$ differs from the naive k-instanton piece of the Nekrasov partition function due to the presence of extra, gauge neutral states.

For instance, for the 6d SCFT $\mathbf{2}_{\text{SU}(2)}$ the sum over elliptic genera reduces to the instanton piece of the 5d Nekrasov partition function of $G = \text{SU}(2)$ SYM with $N_f = 4$ flavors, times an extra factor of[34]

$$\prod_{j,k \geq 1} (1 - Q_{5d}\, e^{2\pi i\, (j-1)\epsilon_1} e^{2\pi i\, k\, \epsilon_2})^{-1}(1 - Q_{5d}\, e^{2\pi i\, j\, \epsilon_1} e^{2\pi i\, (k-1)\epsilon_2})^{-1}, \tag{7.23}$$

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[34]This factor differs slightly from the one given in equation (4.69) of [140]. Namely, the 5d partition function has flavor symmetry SO(8), and therefore depends on an additional fugacity in comparison with the 6d SCFT. Setting this extra fugacity to zero gives (7.23).

where the two infinite products are associated with the two $\mathcal{O}(0) \to \mathbb{P}^1$ line bundles which one obtains after sending the volume of the adjoint node in the Kodaira fiber to infinity. At the level of the one-instanton partition function, one has:

$$Z'_{1-\text{inst}}(\mathbf{m}_G, \mathbf{m}_F, x, v) = Z_{1-\text{inst}}(\mathbf{m}_G, \mathbf{m}_F, x, v) + (v^{-1} + v)\frac{v}{(1 - v\,x)(1 - v/x)}, \quad (7.24)$$

where

$$Z_{1-\text{inst}}(\mathbf{m}_G, \mathbf{m}_F, x, v) = v\frac{-(1 + \chi^F_{(100)})\left(\sum_{n=0}^{\infty}\chi^G_{(2n)}v^{2n+1}\right) + \chi^F_{(001)}\left(\sum_{n=0}^{\infty}\chi^G_{(2n+1)}v^{2n+2}\right)}{(1 - v\,x)(1 - v/x)}$$

$$(7.25)$$

is the 5d Nekrasov partition function, expressed here as an infinite sum in terms of the characters of the 6d flavor and gauge symmetries, $F = \text{SO}(7)$ and $G = \text{SU}(2)$.

7.2.3 Circle reduction of 6d models with $n = 1$

Finally, let us turn to the case $n = 1$; we have seen in section 7.1 that in the limit $q \to 0$ the local geometry reduces simply to the conifold geometry $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ over the base curve, and all exceptional divisors in the elliptic fiber decouple. In other words, the geometry suggests that the strings of the $n = 1$ SCFTs in the 5d limit should simply contribute as a free hypermultiplet to the 5d theory. In the following we discuss how this phenomenon is realized from the perspective of the strings' CFTs. It is convenient to begin by discussing the more familiar case of the E-string, and then generalize to arbitrary $\mathbf{h}_1^G$ theories.

The 6d E-string SCFT is realized in M-theory as the theory of one M5 brane probing an M9 plane, where the M9 brane provides the E_8 flavor symmetry of the E-string SCFT.[35] It famously also admits a realization as the UV fixed point of the 5d Sp(1) gauge theory with 8 hypermultiplets. As explained in [137], in order to arrive at this realization one must turn on a Wilson line along the compactification circle, which breaks E_8 to SO(16). In the presence of the Wilson line, the 5d U(1) vector multiplet arising from compactification of the tensor multiplet enhances at the origin of the Coulomb branch to Sp(1) and the W-bosons arise from wrapped strings [137, 218] (see also [219]). The Sp(1) gauge theory with $N_f \leq 7$ hypermultiplets has flavor symmetry SO($2N_f$) on the Coulomb branch, which enhances to the exceptional group E_{N_f+1} at the fixed point; for $N_f = 8$, the enhancement is to the affine E_8 algebra. The Sp(1) gauge theory on the Coulomb branch is the worldvolume theory supported of the D4 brane in the Type I' configuration of table 22, where the D8 branes all sit on top of the $O8^-$ plane.

However, in the limit we are considering in the present work the Wilson line is not turned on. In the Type I' framework, varying the value of the Wilson line back to 0 corresponds to moving one of the D8 branes away from the O8 plane to a critical point; as a consequence, the profile of the dilaton field varies in the interval between the displaced D8 brane and the remaining configuration of O8+D8 branes. The enhancement to the exceptional algebra occurs when the position of the displaced D8 brane is tuned so that

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[35]To be more precise, the M5-M9 brane system also includes a decoupled theory of one free 6d hypermultiplet that reflects the freedom to move the M5 brane along the M9 plane.

	0	1	2	3	4	5	6	7	8	9
D4	X	X	X	X	X	–	–	–	–	–
$O8^-$	X	X	X	X	X	X	X	X	X	–
8 D8	X	X	X	X	X	X	X	X	X	–

Table 22. Type I' brane configuration for the 5d $Sp(1)$ gauge theory with 8 flavors.

the dilaton diverges at the orientifold plane [193]; from the point of view of the D4 brane worldvolume theory this leads to the enhancement of the flavor symmetry group [218]. In this strongly coupled background the description of the D4 brane worldvolume theory as a $Sp(1)$ gauge theory breaks down. It turns out that in order to obtain a finite partition function the sensible limit to take is

$$Z_\varphi(\phi, \mathbf{m}_{SO(16)}, x, v, q) \rightarrow Z_\varphi^{(0)}(\varphi, \mathbf{m}_{SO(16)}, x, v) = \sum_{k=0}^{\infty} \mathcal{Q}_{5d}^k \lim_{q \to 0} q^{k/2} \mathbb{E}_{(k)}(\mathbf{m}_{SO(16)}, x, v, q), \tag{7.26}$$

where

$$\mathcal{Q}_{5d} = q^{-1/2} e^{-\Phi}. \tag{7.27}$$

As expected from the geometry, this limit turns out to be very simple: $Z_\varphi^{(0)}(\varphi, \mathbf{m}_{SO(16)}, x, v)$ is just the 5d Nekrasov partition function for a free hypermultiplet of mass $m = -\log(\mathcal{Q}_{5d})$:

$$Z_\varphi^{(0)}(\varphi, \mathbf{m}_{SO(16)}, x, v) = PE\left[\frac{v\,\mathcal{Q}_{5d}}{(1 - v\,x)(1 - v/x)}\right], \tag{7.28}$$

and any $\mathbf{m}_{SO(16)}$-dependent states that couple to the E_8 degrees of freedom only appear at higher order in the q expansion.

It is instructive to look at this behavior from the perspective of the BPS string CFT. We adopt the description of the CFT of one E-string (without center of mass piece) as eight chiral fermions with Ramond boundary conditions, coupled to a $O(1)$ gauge field. From this perspective, the elliptic genus receives contributions from the states with $\overline{L}_0 = 0$; the level matching condition for a string wrapped once around the 6d circle is simply

$$L_0 - \overline{L}_0 = n, \tag{7.29}$$

where $n \in \mathbb{Z}$ reflects the possibility of giving strings momentum around the 6d circle. The $L_0 = 0$ states in the Ramond sector include the degenerate Ramond vacuum, which provides a set of states in the $\mathbf{128^s}$ spinor representation of SO(16). However, due to the coupling to $O(1)$ which leads to summing over different periodicities for the fermions, one is also lead to include states built out of the NS vacuum, which from this perspective has $L_0 = -1$ and we therefore denote as $|-1\rangle$. The appearance of this state is allowed due to the level matching condition (7.29); if 6d circle were decompactified, on the other hand, this state would be interpreted as a tachyon but would be projected out by the more stringent level-matching condition $L_0 - \overline{L}_0 = 0$. The five-dimensional limit 7.26 singles out precisely the $L_0 = -1$ and decouples all states with $L_0 \geq 0$.

Besides the Ramond ground states, then, at $L_0 = 0$ one also finds the **120** vector representation of SO(16) which arises from acting on the NS vacuum with a pair of fermionic operators:

$$|\mathbf{120}\rangle = \left\{ \psi_{1/2}^i \psi_{1/2}^j | -1 \rangle \right\}_{j \neq i}^{i,j=1,\dots,16}. \tag{7.30}$$

In other words, the at $L_0 = 0$ one finds the adjoint representation $\mathbf{248} = \mathbf{128^s} + \mathbf{120}$ of E_8, but the lowest energy state at $L_0 = -1$ is just a singlet, which gives rise to the $\mathcal{Q}_{5d}^1$ term in the series expansion of equation (7.28).

Similar considerations can be applied to the $\mathbf{h}_1^{\mathrm{Sp}(N)}$ theories. On the one hand, upon turning on a suitable Wilson line it is expected that the 6d SCFT with $n = 1$ and $G = \mathrm{Sp}(N)$ can be realized in 5d as the $\mathrm{Sp}(N+1)$ theory with $8 + 2N$ flavors [220]. This equivalence has been analyzed in detail for the case $G = \mathrm{Sp}(1)$ in [80], where it was shown that the $T^2 \times \mathbb{R}^4$ partition function of the 6d SCFT agrees with the partition function of the 5d $\mathcal{N} = 1$ $\mathrm{Sp}(2)$ theory with 10 flavors. On the other hand, if one does not turn on any Wilson lines one finds again a single free hypermultiplet in the spectrum of the theory in the zero size limit of the 6d circle; this can again be studied from the perspective of a single string by exploiting the free field realization of the $\mathbf{h}_1^{\mathrm{Sp}(N)}$ CFTs detailed in section 5. Recall that the elliptic genus (5.1) is given by the following sum over $O(1)$ holonomies:

$$\mathbb{E}_1^{\mathrm{Sp}(N)}(\mathbf{m}_{\mathrm{Sp}(N)}, \mathbf{m}_{\mathrm{SO}(16+4N)}, v, q) = \frac{1}{2} \sum_{a \in \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}} \left(\prod_{i=1}^{8+2N} \frac{\theta_1(e^{2\pi i a} m_{\mathrm{SO}(16+4N)}^i, q)}{\eta(q)} \right) \tag{7.31}$$
$$\times \left(\prod_{i=1}^{N} \frac{\eta(q)^2}{\theta_1(e^{2\pi i a} v\, m_{\mathrm{Sp}(N)}^i, q) \theta_1(e^{2\pi i a} v / m_{\mathrm{Sp}(N)}^i, q)} \right).$$

For the $a = 0$ and $a = 1/2$ terms the $O(1)$ holonomy does not affect the boundary conditions of the periodic fermions. Their contribution to the $L_0 = 0$ sector in the elliptic genus is

$$Z_R^{\mathrm{Sp}(N)} = \frac{1}{2} \tag{7.32}$$
$$\times \sum_{s = \pm 1} \frac{\prod_{i=1}^{8+2N}((m_{\mathrm{SO}(16+4N)}^i)^{-1/2} + s(m_{\mathrm{SO}(16+4N)}^i)^{1/2})}{\prod_{i=1}^{N}((v\, m_{\mathrm{Sp}(N)}^i)^{-1/2} + s(v\, m_{\mathrm{Sp}(N)}^i)^{1/2})((v^{-1} m_{\mathrm{Sp}(N)}^i)^{-1/2} + s(v^{-1} m_{\mathrm{Sp}(N)}^i)^{1/2}))}.$$

This coincides with what would be the one-instanton partition function of the 5d $\mathrm{Sp}(N)$ theory with $8 + 2N$ flavors, which can easily be obtained from [221]:

$$Z_R^{\mathrm{Sp}(N)} = \left[\frac{v}{(1 - v\, x)(1 - v/x)} \right]^{-1} Z_{1-\mathrm{inst}}. \tag{7.33}$$

Of course, genuine 5d theories with such a high number of multiplets are inconsistent, as the metric on the 5d Coulomb phase would become negative definite. It is reassuring that the corresponding one-instanton partition function appears here as a component of the 6d Nekrasov partition function and cannot be isolated by taking the 5d limit as is the case for the 6d SCFTs considered in section 7.2.1.

The $a = \frac{\tau}{2}$ and $a = \frac{1+\tau}{2}$ terms in equation (7.31) can be thought of contributions from $16 + 4N$ antiperiodic fermions ψ^i in the vector representation of $F = \mathrm{SO}(16 + 4N)$ and $2N$

antiperiodic bosons $\phi^{\pm,i}$ transforming in the fundamental representations of $G = \mathrm{Sp}(N)$ and $\mathrm{SU}(2)_R$. Their lowest-energy contributions to the elliptic genus consist of a singlet state $|-1\rangle$ at $L_0 = -1$, which is the only surviving state in the limit 7.26 and can be viewed as being a component of the Nekrasov partition function of a free hypermultiplet. At $L_0 = 0$, one finds a number of states obtained by acting on the singlet state with the $L_0 = 1/2$ components $\phi^{\pm,i}_{1/2}, \psi^i_{1/2}$ of the free fermionic and bosonic operators. Specifically, one finds: a set of states in the adjoint representation of F coming from states of the form

$$\psi^{[i}_{1/2}\psi^{j]}_{1/2}|-1\rangle; \tag{7.34}$$

a set of states in the bifundamental representation of $G \times F$ and in the $j = 1/2$ representation of $\mathrm{SU}(2)_R$, of the form

$$\psi^i_{1/2}\phi^{\pm,j}_{1/2}|-1\rangle; \tag{7.35}$$

a set of states in the adjoint of G and in the $j = 1$ representation of $\mathrm{SU}(2)_R$ coming from symmetric combinations

$$\phi^{+,(i}_{1/2}\phi^{+,j)}_{1/2}|-1\rangle, \qquad \phi^{-,(i}_{1/2}\phi^{-,j)}_{1/2}|-1\rangle, \text{and} \qquad \phi^{+,(i}_{1/2}\phi^{-,j)}_{1/2}|-1\rangle, \tag{7.36}$$

and finally a set of states in the trivial representation of $\mathrm{SU}(2)_R$, in the anti-symmetric representation of G plus a singlet, coming from the following states:

$$\phi^{+,[i}_{1/2}\phi^{-,j]}_{1/2}|-1\rangle. \tag{7.37}$$

All in all, these states contribute the following terms to the $L_0 = 0$ sector of the elliptic genus:

$$Z_{\mathrm{extra}} = \chi^{\mathrm{SU}(2)_R}_{(2)}(v)\chi^{\mathrm{Sp}(N)}_{(20\ldots0)}(\mathbf{m}_{\mathrm{Sp}(N)}) - \chi^{\mathrm{SU}(2)_R}_{(1)}(v)\chi^{\mathrm{Sp}(N)}_{(10\ldots0)}(\mathbf{m}_{\mathrm{Sp}(N)})\chi^{\mathrm{SO}(16+4N)}_{(10\ldots0)}(\mathbf{m}_{\mathrm{SO}(16+4N)})$$
$$+ (\chi^{\mathrm{Sp}(N)}_{(01\ldots0)}(\mathbf{m}_{\mathrm{Sp}(N)}) + 1). \tag{7.38}$$

If on the other hand we expand the $\mathrm{SU}(2)_R$ characters in powers of v, we find that the v^0 coefficient is given by

$$\chi^{\mathrm{Sp}(N)}_{(20\ldots0)}(\mathbf{m}_{\mathrm{Sp}(N)}) + \chi^{\mathrm{Sp}(N)}_{(01\ldots0)}(\mathbf{m}_{\mathrm{Sp}(N)}) + 1 \tag{7.39}$$

as already remarked in section 6.3.

For other $\mathbf{h}^G_1$ CFTs we do not have a weakly coupled description, but we still expect the elliptic genus to take the same form:

$$\mathbb{E}^G_1(\mathbf{m}_G, \mathbf{m}_F, v, q) = q^{\frac{1}{2}-\frac{1}{6}}\left(q^{-1} + q^0\left(Z_{\mathrm{extra}}(\mathbf{m}_G, \mathbf{m}_F, v)\right.\right. \tag{7.40}$$
$$\left.\left.+ \left[\frac{v}{(1 - v\,x)(1 - v/x)}\right]^{-1} Z_{1-\mathrm{inst}}(\mathbf{m}_G, \mathbf{m}_F, v)\right) + \mathcal{O}(q)\right),$$

where $Z_{1-\mathrm{inst}}(\mathbf{m}_G, \mathbf{m}_F, v)$ is the Nekrasov partition function of the would-be 5d $\mathcal{N} = 1$ theory with gauge group G and matter content obtained by compactification from the 6d

hypermultiplets, whereas $Z_{\text{extra}}(\mathbf{m}_G, \mathbf{m}_F, v)$ comes from acting on the $L_0 = -1$ singlet with various combinations of operators as in the $G = \text{Sp}(N)$ case.

The appearance of a singlet at $L_0 = -1$ follows also from the spectral flow symmetry (6.30) that maps its contribution to the R-R elliptic genus to contribution of the vacuum in the NS-R elliptic genus (6.34) that is neutral under G. Summing up the contributions from arbitrary numbers of strings, one again expects the instanton piece of the 5d Nekrasov partition function to coincide with that of a free hypermultiplet of mass $m = -\log(\mathcal{Q}_{5d})$, in accord with the expectations from geometric engineering.

8 Modular bootstrap of the elliptic genera

In this section we exploit modularity of the BPS strings' elliptic genera to formulate a general Ansatz in terms of Jacobi forms that captures the elliptic genus for all $\mathbf{h}_n^G$ theories. In section 8.1 we formulate the Ansatz, which depends only on n and G and fixes the elliptic genus up to a finite number of undetermined coefficients. In section 8.2 we state a number of constraints on the elliptic genus that arise from the universal properties of section 6. In section 8.3 we then use these constraints to fix the undetermined coefficients in the elliptic genera for the theories with $\text{rank}(G) \leq 7$; this restriction on the rank covers all exceptional theories whose elliptic genus is not yet known. In most cases, we are able to completely fix the elliptic genus, while in a few cases we are left with a small number of undetermined coefficients. In section 8.4 we use these results to compute the one-instanton piece of the 5d partition functions that arise from circle compactification of the 6d theory as described in section 7.

8.1 The Ansatz

In this section we aim to express the elliptic genus of an arbitrary $\mathbf{h}_n^G$ theory as an element of a finite-dimensional vector space of meromorphic Jacobi of the correct weight and index. The Ansatz we obtain is a straightforward generalization of the one presented in [83] for BPS strings of 6d SCFTs without matter.

In order to make the computations tractable and to keep the discussion succint, we make some simplifying choices. Namely, we only focus on the case of a single BPS string, and we choose to turn off the chemical potentials for G and F, so that the elliptic genus of the $\mathbf{h}_n^G$ theory only depends on the parameters v and q. The dependence of the elliptic genus on these fugacities is captured in terms of Weyl-invariant Jacobi forms, which have been employed recently in [91] to study the elliptic genera of the strings of the 6d SCFT without matter, and in [93] for the case at hand of theories with matter.[36] Turning off these fugacities has the benefit of greatly reducing the vector space of Jacobi forms one needs to consider.

The elliptic genus is a torus partition function, which by its behavior under large diffeomorphisms is expected to transform as a Jacobi form of modular weight zero, that is,

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[36]The authors of [93] were in particular recently able to compute the elliptic genera of the $\mathbf{h}_3^{G_2}$ and $\mathbf{h}_3^{\text{SO}(7)}$ theories by these techniques, which also fall into the discussion of this section.

as a function of the variables[37] $(\tau, \epsilon_+) \in \mathbb{H} \times \mathbb{C}$ such that (up to possibly an overall sign)

$$\mathbb{E}_n^G(\epsilon_+/\tau, -1/\tau) = e^{2\pi i \, \epsilon_+^2 \, \rho/\tau} \, \mathbb{E}_n^G(\epsilon_+, \tau), \tag{8.1}$$

where the index ρ is the coefficient of ϵ_+^2 in the modular anomaly polynomial. The latter is obtained from the anomaly polynomial of the 2d $(0,4)$ theory in Equation (3.29) by an equivariant integral which amounts to performing the substitution

$$c_2(L) \mapsto -\varepsilon_-^2, \quad c_2(R) \mapsto -\varepsilon_+^2, \quad c_2(I) \mapsto -\varepsilon_+^2, \quad p_1(M_2) \mapsto 0. \tag{8.2}$$

The resulting modular anomaly (for a generic bound state $\mathbf{S} = \sum Q^I \Sigma_I$ of BPS strings) is

$$\begin{aligned}
f &= \frac{1}{2} A_{IJ} Q^I Q^J \left(-\varepsilon_-^2 + \varepsilon_+^2 \right) \\
&\quad + Q^I \left(\frac{1}{4} A_{Ia} \mathrm{Tr}(F^{(a)})^2 - \frac{1}{4}(2 - A_{II})(2\varepsilon_-^2 + 2\varepsilon_+^2) - h_{G_I}^\vee \varepsilon_+^2 \right) \\
&= -\frac{\varepsilon_-^2}{2} \left((2 - A_{II}) Q^I + A_{IJ} Q^I Q^J \right) \\
&\quad - \frac{\varepsilon_+^2}{2} \left(\left((2 - A_{II}) + 2h_{G_I}^\vee \right) Q^I - A_{IJ} Q^I Q^J \right) \\
&\quad + Q^I A_{Ia} \frac{1}{4} \mathrm{Tr}(F^{(a)})^2
\end{aligned} \tag{8.3}$$

Specializing the above to the case of a rank-one theory with $A_{11} = n$, we obtain

$$\begin{aligned}
f &= -\frac{\varepsilon_-^2}{2} \left((2-n)Q + nQ^2 \right) \\
&\quad - \frac{\varepsilon_+^2}{2} \left(\left((2-n) + 2h_{G_i}^\vee \right) Q^i - nQ^2 \right) \\
&\quad + Q^i A_{ia} \frac{1}{4} \mathrm{Tr}(F^{(a)})^2 \, .
\end{aligned} \tag{8.4}$$

In particular, for $Q = 1$, we have

$$f = -\varepsilon_-^2 + \varepsilon_+^2 (n - h_G^\vee - 1) - \frac{1}{4} A_{1a} \mathrm{Tr}(F^{(a)})^2 \, . \tag{8.5}$$

Setting to zero the flavor fugacities and subtracting the contribution of the center of mass hypermultiplet one is left with

$$f = \varepsilon_+^2 (n - h_G^\vee), \tag{8.6}$$

so that for the $\mathbf{h}_n^G$ theories ρ is given by

$$\rho = n - h_G^\vee. \tag{8.7}$$

In order to motivate our Ansatz, let us start by briefly recalling the case of the BPS strings for 6d SCFTs without matter that was considered in [83], that is: $\mathbf{h}_3^{\mathrm{SU}(3)}$, $\mathbf{h}_4^{\mathrm{SO}(8)}$,

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[37]In this section, unlike in the rest of the paper, we find it more convenient to write all expressions in terms non-exponentiated variables.

$\mathbf{h}_5^{F_4}$, $\mathbf{h}_6^{E_6}$, $\mathbf{h}_8^{E_7}$, and $\mathbf{h}_{12}^{E_8}$. For these theories $h_G^\vee = 3(n-2)$, so that the index is given by $6 - 2n$. Moreover the elliptic genus takes the following form:

$$\frac{\mathcal{N}(2\,\epsilon_+, \tau)}{\eta(\tau)^{4(h_G^\vee - 1)} \varphi_{-2,1}(2\,\epsilon_+, \tau)^{\frac{\dim(G) - \mathrm{rank}(G)}{2}}}, \tag{8.8}$$

where $\eta(\tau)$ is the weight-1/2 Dedekind eta function and $\varphi_{-2,1}(2\epsilon_+, \tau)$ is a holomorphic Jacobi form of weight -2 and index 1 with respect to $2\,\epsilon_+$ (i.e. of index 4 with respect to ϵ_+), whose explicit formula is given by:

$$\varphi_{-2,1}(2\epsilon_+, \tau) = -\frac{\theta_1(2\epsilon_+, \tau)^2}{\eta^6}. \tag{8.9}$$

The factors of $\eta(\tau)$ are responsible for the leading order singularity in the q-expansion of the elliptic genus, which is $q^{-\frac{h_G^\vee - 1}{6}}$. Furthermore, the factors of $\varphi_{-2,1}(2\epsilon_+, \tau)$ in the denominator can be interpreted as contributions to the elliptic genus from the bosonic generators of the moduli space of one G-instanton. The denominator terms are responsible for the meromorphic behavior of the elliptic genus, which diverges when $2\,\epsilon_+$ takes values at the lattice points

$$p_1 \cdot 1/2 + p_2 \cdot \tau/2, \ (p_1, p_2) \in \mathbb{Z}^2. \tag{8.10}$$

The numerator is a holomorphic Jacobi form of even weight

$$m = 2(h_G^\vee - 1) - (\dim(G) - \mathrm{rank}(G)) \quad \in \quad 2\mathbb{Z} \tag{8.11}$$

and index

$$k = \frac{1 - h_G^\vee/3 + \dim(G) - \mathrm{rank}(G)}{2} \quad \in \quad \mathbb{Z}_+/2 \tag{8.12}$$

with respect to $2\,\epsilon_+$. In [83] it was observed that only even powers of $v = e^{2\pi i \epsilon_+}$ appear in the elliptic genera. Therefore, the numerator can also be captured by a holomorphic Jacobi function of integer or half-integer index with respect to $2\,\epsilon_+$, as follows from a lemma of Gritsenko [222]. The numerator is then an element of the finite-dimensional vector space of holomorphic Jacobi forms $J_{m,k}^{1/2}(2\,\epsilon_+, \tau)$ of weight m and index k, which is in turn a component of the bi-graded ring

$$J_{*,*}^{1/2}(2\,\epsilon_+, \tau) = \bigoplus_{M \in \mathbb{Z}} \bigoplus_{K \in \mathbb{Z}_+/2} J_{M,K}^{1/2}(2\,\epsilon_+, \tau). \tag{8.13}$$

of Jacobi forms of integer weight and integer or half-integer index. As we review in section B, $J_{*,*}^{1/2}(2\,\epsilon_+, \tau)$ is a polynomial ring over the ring of $\mathrm{SL}(2,\mathbb{Z})$ modular forms (generated by the weight 4 and 6 Eisenstein series $E_4(\tau), E_6(\tau)$), which itself is freely generated by the three Jacobi forms

$$\varphi_{M,K}(2\,\epsilon_+, \tau) \in J_{M,K}^{1/2}(2\,\epsilon_+, \tau), \qquad (M, K) = (0,1), (-2,1), \text{ or } (0,3/2). \tag{8.14}$$

As a matter of fact, when gauge fugacities $\mathbf{m}_G$ are switched off it follows from the structure of the ring of Jacobi forms that cancellations must occur between the numerator and denominator, and the elliptic genus takes the form

$$\frac{\mathcal{N}'(2\,\epsilon_+, \tau)}{\eta(\tau)^{4(h_G^\vee - 1)} \varphi_{-2,1}(2\,\epsilon_+, \tau)^{h_G^\vee - 1}}, \tag{8.15}$$

where $\mathcal{N}'(2\,\epsilon_+, \tau)$ is now a holomorphic Jacobi form of weight 0 and index

$$k' = \frac{5}{6}h_G^\vee - \frac{1}{2} \tag{8.16}$$

with respect to $2\,\epsilon_+$. As is the case for the Hilbert series of the moduli space of one instanton [139], the presence of a factor of $\varphi_{-2,1}(2\,\epsilon_+, \tau)^{h_G^\vee-1}$ in the denominator is consistent with the fact that the quaternionic dimension of $\widetilde{\mathcal{M}}_{G,1}$ is $h_G^\vee - 1$.

The generalization to arbitrary $\mathbf{h}_n^G$ theories is simple. The $\eta(\tau)$ factor is fixed by the leading order singularity of the elliptic genus, equation (6.10), to be:

$$\eta(\tau)^{-12(n-2)+4+24\,\delta_{n,1}}. \tag{8.17}$$

The remaining terms in the denominator of the elliptic genus again capture the bosonic degrees of freedom of the CFT, which from the discussion in section 6 is a sigma model with target space the moduli space of one G-instanton. Therefore, in analogy with the theories without matter, one can write the following Ansatz:

$$\mathbb{E}_n^G(\epsilon_+, \tau) = \frac{\mathcal{N}_n^G(\epsilon_+, \tau)}{\eta(\tau)^{12(n-2)-4+24\,\delta_{n,1}}\varphi_{-2,1}(2\,\epsilon_+, \tau)^{h_G^\vee-1}}. \tag{8.18}$$

The numerator factor $\mathcal{N}_n^G(\epsilon_+, \tau)$ is now a holomorphic Jacobi form of even weight

$$m_n^G = 6(n-2) - 2 + 12\,\delta_{n,1} - 2(h_G^\vee - 1) \quad \in \quad 2\mathbb{Z} \tag{8.19}$$

and index

$$k_n^G = n + 3\,h_G^\vee - 4 \quad \in \quad \mathbb{Z}_+ \tag{8.20}$$

with respect to ϵ_+. The presence of matter implies that it is no longer possible to express the numerator in terms of even powers of v only, as can be seen already in the case of the M-string, for which

$$\mathbb{E}_1^{\mathrm{SU}(2)}(\epsilon_+, \tau) = \eta(\tau)^4 \varphi_{-2,1}(\epsilon_+, \tau). \tag{8.21}$$

Therefore the numerator must be expressed in terms of Jacobi forms with elliptic parameter ϵ_+ and no longer $2\,\epsilon_+$.

The bi-graded ring of such Jacobi forms of arbitrary integral weight and positive integral index,

$$J_{*,*}(\epsilon_+, \tau) = \bigoplus_{m\in\mathbb{Z}} \bigoplus_{k\in\mathbb{Z}_+} J_{m,k}(\epsilon_+, \tau). \tag{8.22}$$

is the subring of the polynomial ring $J_{*,*}^{1/2}(\epsilon_+, \tau)$ over modular forms of $\mathrm{SL}(2,\mathbb{Z})$ which is freely generated by

$$\varphi_{-2,0}(\epsilon_+, \tau) \qquad \text{and} \qquad \varphi_{0,0}(\epsilon_+, \tau). \tag{8.23}$$

For fixed weight and index the vector space of Jacobi forms is finite-dimensional. It follows from the structure of the ring (8.22) of Jacobi forms that the numerator can always be written in the form

$$\mathcal{N}_n^G(\epsilon_+, \tau) = \varphi_{-2,1}(\epsilon_+, \tau)^{h_G^\vee - 3\,n + 6(1-\delta_{n,1})} p_n^G(E_4(\tau), E_6(\tau), \varphi_{0,1}(\tau), \varphi_{-2,1}(\tau)), \tag{8.24}$$

where

$$p_n^G(E_4(\tau), E_6(\tau), \varphi_{0,1}(\tau), \varphi_{-2,1}(\tau)) \in J_{0,4n+2h_G^\vee-10+6\,\delta_{n,1}}(\epsilon_+, \tau) \qquad (8.25)$$

is a Jacobi form of integer weight and integer index with respect to ϵ_+, which can be expressed as the following finite sum:

$$\begin{aligned}
p_n^G(E_4(\tau), E_6(\tau), \varphi_{0,1}(\tau), \varphi_{-2,1}(\tau)) = \\
= \sum_{\{a,b,c,d\}\in\mathcal{I}_n^G} r_{a,b,c,d}\, E_4(\tau)^a E_6(\tau)^b \varphi_{-2,1}(\epsilon_+, \tau)^c \varphi_{-2,1}(\epsilon_+, \tau)^d,
\end{aligned} \qquad (8.26)$$

where $r_{a,b,c,d}$ are a set of undetermined numerical coefficients, and $\mathcal{I}_n^G$ is the set of tuples of non-negative integers a, b, c, d such that

$$4\,a + 6\,b - 2\,c = 0, \qquad (8.27)$$
$$c + d = 4n + 2h_G^\vee - 10 + 6\,\delta_{n,1}, \qquad (8.28)$$

so that the Jacobi form possesses the correct weight and index.

In section 8.2 we will make use of the universal features of the $\mathbf{h}_n^G$ CFTs discussed in section 6 to impose constraints on the numerical coefficients $r_{a,b,c,d}$. For the 6d SCFTs with no matter, $\mathcal{N}_n^G(\epsilon_+, \tau)$ can also be obtained easily from the results presented in appendix A.1 of [83]. There, the numerator factor was determined as a Jacobi form with elliptic parameter $2\,\epsilon_+$; one can express such a Jacobi form in terms of Jacobi forms with elliptic parameter ϵ_+ by making use of identities (B.21)–(B.23).

We end this section with a brief remark on the flavor symmetry. The F dependence, which is suppressed in equation (8.18), is realized in terms of chiral fermions and therefore can only enter in the numerator of the elliptic genus. From our discussion in section 6.2, it follows that the numerator should be given by a linear combination of Kac-Moody characters of F, at the level given in tables 20 and 21, multiplied by the components of a vector-valued Jacobi form of suitable weight and index that captures the dependence of the numerator on ϵ_+ (provided that we do not turn on the chemical potentials $\mathbf{m}_G$). The components of the vector-valued Jacobi must transform under a modular transformation in such a way that the combination with the Kac-Moody characters for F is as an ordinary (scalar) Jacobi form. It would be interesting to see whether this additional structure on the numerator can be used to further simplify the computation of elliptic genera.

8.2 Constraints on the Ansatz

The general features of the $\mathbf{h}_n^G$ theories that were discussed in section 6 can be used to impose constraints on the coefficients of the elliptic genus and to fix the undetermined coefficients $r_{a,b,c,d}$ in the Ansatz (8.18). In particular, one can easily formulate the following constraints:

- **Constraint C1:** in section 6.4 we found that the lowest energy and next-to-lowest energy states in the elliptic genus with NS boundary conditions take a universal form, given respectively in equations (6.34) and (6.37), which is completely determined in terms of the gauge symmetry, flavor symmetry, and matter content of the theory.

Thanks to the spectral flow relation (6.30), this implies that an infinite number of coefficients in the Ramond elliptic genus are also known. In order to illustrate this it is convenient to express the elliptic genus as the following sum:

$$\mathbb{E}_n^G(v,q) = q^{\frac{1}{6} - \frac{n-2}{2} - \delta_{1,n}} v^{1-n} \sum_{j,k \geq 0} b_{jk}(q/v^2)^j v^k. \tag{8.29}$$

The condition $j \geq 0$ arises from the fact that the energy of the states in the Ramond elliptic genus is bounded from below as in equation (6.10) and is automatically satisfied by the Ansatz. On the other hand, the condition $k \geq 0$ is a consequence spectral flow and of the fact that the ground state energy in the NS sector is $-\frac{c_L}{24}$. This condition is not satisfied automatically for a generic choice of coefficients in the Ansatz, and therefore one can obtain a first set of constraints by imposing

$$b_{jk} = 0 \qquad \text{for } k < 0. \tag{8.30}$$

The coefficients b_{jk} are bilinear in the dimensions of representations of F and G. In particular, from equation (6.34) it follows that

$$b_{n-2+\delta_{1,n}+\ell,0} = \dim(R_{\ell \cdot \theta_G}^G) \qquad \text{for } \ell \geq 0, \tag{8.31}$$

where $\lambda_{\ell \cdot \theta_G}^G$ is the irreducible representation of G whose highest weight is ℓ times the highest weight of the adjoint representation, θ_G. Additionally, from (6.37) it follows that

$$b_{n-1+\delta_{1,n}+\ell,1} = -\sum_{i=1}^{r} \dim(R_{\lambda_i^G + \ell \cdot \theta}^G) \dim(R_i^F) \qquad \text{for } \ell \geq 0, \tag{8.32}$$

where we denote by λ_i^G the highest weights of the representation R_i^G of G. Equations (8.30)–(8.32) lead to an infinite number of constraints, a finite number of which are independent from each other.

- **Constraint C2:** in sections 6.2 and 6.3 we have seen that the elliptic genus organizes itself in terms of characters of Kac-Moody algebras, according to equations (6.17) and (6.18). Turning off the chemical potentials $\mathbf{m}_F$ and $\mathbf{m}_G$, this implies that the elliptic genus takes the form

$$\mathbb{E}_n^G(v,q) = \sum_{\lambda} \sum_{R_\nu \in \text{Rep}(G)} \sum_{\ell = 2(n-h_G^\vee)+1}^{0} \sum_{m \in \mathbb{Z}} n_{\nu,\ell,m}^\lambda \, \widehat{\chi}_\lambda^F(\mathbf{1},q) \, \frac{q^{-\frac{c_G}{24} + h_\nu^G} \dim(R_\nu)}{\prod_{j=1}^{\infty}(1-q^j)^{\dim(G)}}$$
$$\times \frac{q^{-\frac{c_v}{24} + h_{\ell,m}^v} v^{\ell + 2(n-h_G^\vee)m}}{\prod_{j=1}^{\infty}(1-q^j)}. \tag{8.33}$$

Multiplying the elliptic genus by a factor of $\prod_{j=1}^{\infty}(1-q^j)^{\dim(G)+\dim(F)+1}$ removes all denominators in the right hand side that arise from bosonic oscillators (including the ones that appear in the characters of the Kac-Moody algebra for F). After multiplying by this factor and expanding equation (8.33) in the same fashion as in equation (8.29), it turns out that for most $\mathbf{h}_n^G$ elliptic genera several of the b_{jk}

vanish, since none of the terms appearing in the sums on the right hand side of equation (8.33) can contribute to them. We use the vanishing of these coefficients to impose an additional set of constraints on the elliptic genera; for an explicit example of this procedure, see appendix F.

Incidentally, if one pursues this approach for the elliptic genera of the 6d SCFTs without matter, one finds that necessarily

$$b_{jk} = 0 \qquad \text{if} \quad j < h_G^\vee/3 \quad \text{or} \quad k < h_G^\vee/3, \tag{8.34}$$

a fact which was observed in [83] and employed there to compute the elliptic genera for these theories.

- **Constraint C3:** for $n = 1$, we can impose additional constraints on all $\mathbf{h}_1^G$ theories by requiring that

$$b_{00} = 1 \qquad \text{and} \qquad b_{0j} = 0 \quad \text{for} \quad j > 0, \tag{8.35}$$

which follows from the results of section 7.2.3.

8.3 Computational results

One can employ the constraints of the previous section to reduce the number of unfixed coefficients of the Ansatz (8.18) for any $\mathbf{h}_n^G$ theory. As there are infinite numbers of $\mathbf{h}_n^G$ theories, we choose to restrict our analysis to a suitable subset of theories. From the review of section C, the elliptic genera of almost all of the $\mathbf{h}_n^G$ theories for which $\mathrm{rank}(G) \geq 8$ have been computed by localization. The only exceptions are the theories $\mathbf{h}_4^{\mathrm{SO}(9+2N)}$ whose elliptic genus we compute by localization in section D, and the theory $\mathbf{h}_{12}^{E_8}$ which, although it does not have a known Lagrangian realization, has also been computed in [83]. We therefore restrict our attention to the theories with $\mathrm{rank}(G) \leq 7$. We also choose not to compute the elliptic genus for the $\mathbf{h}_8^{E_7}$ theory by our techniques, since it has already been computed in [83] and the Ansatz involves a very large number of unfixed coefficients. Finally, we omit the M-string and E-string theories which have trivial gauge group and are already well studied.

This leaves an additional 72 $\mathbf{h}_n^G$ theories, which we analyze in this section. For 42 of them, to the best of our knowledge the elliptic genus has not yet appeared in the literature. For all theories with $n \geq 2$, we successively employ constraints **C1** and **C2** to fix the Ansatz, while for the theories with $n = 1$ we successively employ constraints **C1**, **C2**, and **C3**. We present the results of our computation in tables 23 and 24. The constraints we identified are sufficient to completely fix the elliptic genus for 59 out of the 72 theories, including 31 theories whose elliptic genus was not previously known.

We have performed several consistency checks on the coefficients of the elliptic genera thus determined. First, in case the elliptic genus has already been calculated (see appendix C), we can directly check whether our answer is consistent with the results available in the literature. Second, we can verify whether the condition

$$b_{n-1+\delta_{1,n},2} = \dim(G) + \dim(F) + 1, \tag{8.36}$$

n	G	Number of unfixed coefficients		
		Ansatz	**C1**	**C1+C2**
*7	E_7	271	48	–
6	E_6	140	21	–
*6	E_7	234	40	–
5	F_4	80	10	–
*5	E_6	140	16	–
*5	E_7	200	33	–
4	SO(8)	37	3	–
*4	SO(9)	44	4	–
4	SO(10)	52	5	–
*4	SO(11)	81	7	–
4	SO(12)	70	8	–
*4	SO(13)	80	10	–
4	SO(14)	91	12	–
*4	SO(15)	102	14	–
*4	F_4	61	7	–
*4	E_6	91	12	–
*4	E_7	169	27	9
3	SU(3)	10	–	–
3	SO(7)	19	1	–
*3	SO(8)	24	1	–
*3	SO(9)	30	2	–
*3	SO(10)	37	3	–
*3	SO(11)	44	4	–
*3	SO(12)	52	5	–

n	G	Number of unfixed coefficients		
		Ansatz	**C1**	**C1+C2**
3	G_2	14	–	–
*3	F_4	44	4	–
*3	E_6	70	8	–
*3	E_7	140	21	11
2	SU(2)	2	–	–
2	SU(3)	4	–	–
2	SU(4)	7	–	–
2	SU(5)	10	–	–
2	SU(6)	14	–	–
2	SU(7)	19	1	–
2	SU(8)	24	1	–
*2	SO(7)	10	–	–
*2	SO(8)	14	–	–
*2	SO(9)	19	1	–
*2	SO(10)	24	1	–
*2	SO(11)	30	2	2
*2	SO(12)$_a$	37	3	2
*2	SO(12)$_b$	37	3	1
*2	SO(13)	44	4	–
*2	G_2	7	–	–
*2	F_4	30	2	–
*2	E_6	52	5	3
*2	E_7	114	16	9

Table 23. Fixing the Ansatz for $n \geq 2$. For each theory, we first list the total number of unfixed coefficients in the Ansatz, and in the remaining columns the number of coefficients left unfixed after imposing successively the constraints **C1** and **C2**. For the theory $\mathbf{h}_2^{\mathrm{SO}(11)}$ we cannot impose constraint **C2** since we do not understand the flavor symmetry F. We highlight in red the affected table entry. We prefix by an asterisk the theories for which our results are novel.

which follows from equation (6.41), is satisfied. We have already remarked that this condition does not hold for the theories $\mathbf{h}_1^{\mathrm{Sp}(N)}$ with $N > 2$, for which $\dim(\mathrm{Sp}(N))$ is replaced by $\dim(\mathrm{SU}(2N))$ in equation (8.36); for all other elliptic genera we compute we find that this relation holds. Finally, the coefficients b_{jk} of the elliptic genus must be linear combinations with integer coefficients of specific products of dimensions of representations of G and F that can be read off from the expansion (8.33). The coefficients count sets of states transforming in given representations of $G \times F \times \mathrm{U}(1)_v$, with a sign; from inspection on

n	G	Number of unfixed coefficients			
		Ansatz	**C1**	**C1+C2**	**C1+C2+C3**
1	SU(2)	2	–	–	–
1	SU(3)	6	–	–	–
1	SU(4)	9	–	–	–
1	SU(5)	13	1	–	–
1	SU(6)	18	2	–	–
1	SU(7)	23	3	–	–
1	SU(8)	29	4	1	–
*1	SU(6)$_*$	18	2	–	–
1	Sp(2)	6	–	–	–
1	Sp(3)	9	–	–	–
1	Sp(4)	13	1	–	–
1	Sp(5)	18	2	–	–
1	Sp(6)	23	3	–	–

n	G	Number of unfixed coefficients			
		Ansatz	**C1**	**C1+C2**	**C1+C2+C3**
1	Sp(7)	29	4	–	–
1	G_2	9	–	–	–
*1	F_4	36	6	4	1
*1	E_6	60	11	6	3
*1	E_7	126	26	19	15
*1	SO(7)	13	1	–	–
*1	SO(8)	18	2	–	–
*1	SO(9)	23	3	2	–
*1	SO(10)	29	4	–	–
*1	SO(11)	36	6	6	1
*1	SO(12)$_a$	43	7	3	1
*1	SO(12)$_b$	43	7	7	1

Table 24. Fixing the Ansatz for $n = 1$. For each theory, we list the total number of unfixed coefficients in the Ansatz, and in the remaining columns the number of coefficients left unfixed after imposing successively the constraints **C1**,**C2**, and **C3**. For the $\mathbf{h}_1^{\mathrm{SO}(11)}$ and $\mathbf{h}_2^{\mathrm{SO}(12)_b}$ theories we cannot impose condition **C2** since we do not have a good enough understanding of the flavor symmetry F. We highlight in red the affected table entries. We prefix by an asterisk the theories for which our results are novel.

the known elliptic genera, moreover, these coefficients always turn out to be small integers: we have not encountered any example where any such coefficient is greater than 3. For instance, for $\mathbf{h}_2^{F_4}$ the first few b_{jk} we find are given by:

$k\backslash j$	0	1	2	3	4	5
0	1	0	0	0	0	-14
1	52	-156	74	56	90	126

whereas (8.36) leads to the prediction that these b_{jk} coefficients are given by:

$k\backslash j$	0	1	2	3	4	5
0	$n_{0,0}^{(1)}$	0	$n_{0,2}^{(1)}$	0	0	$14\,n_{0,5}^{(1)}$
1	$52\,n_{1,0}^{(1)}+364\,n_{1,0}^{(2)}+84\,n_{1,0}^{(3)}$	$156\,n_{1,1}^{(1)}+126\,n_{1,1}^{(2)}$	$21\,n_{1,2}^{(1)}+90\,n_{1,2}^{(2)}+52\,n_{1,2}^{(3)}+n_{1,2}^{(4)}$	$56\,n_{1,3}^{(1)}$	$21\,n_{1,4}^{(1)}+90\,n_{1,4}^{(2)}$	$156\,n_{1,5}^{(1)}+126\,n_{1,5}^{(2)}$

There is essentially a unique solution for which all undetermined coefficients $n_{jk}^{(m)}$ are small integers:

$$n_{0,0}^{(1)} = -n_{0,0}^{(1)} = n_{1,0}^{(1)} = -n_{1,1}^{(1)} = n_{1,2}^{(1)} = n_{1,2}^{(3)} = n_{1,2}^{(4)} = n_{1,3}^{(1)} = n_{1,4}^{(1)} = n_{1,5}^{(2)} = 1;$$
$$n_{0,2}^{(1)} = n_{1,0}^{(2)} = n_{1,0}^{(3)} = n_{1,1}^{(2)} = n_{1,2}^{(2)} = n_{1,4}^{(1)} = n_{1,5}^{(1)} = 0. \tag{8.37}$$

We have performed similar checks for the entire set of 59 theories; for many theories it turns out that several b_{jk} only involve linear combinations of small numbers of terms,

which makes this check easy to perform. On the other hand, for theories where F includes abelian factors (which as Kac-Moody algebras have large numbers of integrable highest weight representations) it turns out that the b_{jk} tend to involve large numbers of terms, and it is more challenging to perform this check except for a small set of b_{jk} coefficients.

Of the set of 72 theories we considered, 13 remain whose elliptic genus cannot be fixed by applying the constraints **C1**, **C2**, and **C3** alone. One might still wonder if it is possible to fix the remaining undetermined coefficients in the Ansatz by applying additional constraints, such as:

(a) Imposing the condition

$$b_{n-1+\delta_{1,n},2} = \dim(G) + \dim(F) + 1, \tag{8.38}$$

which follows from equation (6.41);

(b) Looking for relations between coefficients b_{jk} that follow from the fact that the F dependence is captured by affine Kac-Moody algebra characters as in equation (8.33);

(c) Requiring that the coefficients b_{jk} can be written in terms of small linear combinations of $n_{j,k}^{(m)}$ coefficients as in the $\mathbf{h}_2^{F_4}$ example given above.

On the other hand, we do not obtain additional constraints from assuming that the dependence of the elliptic genus on the 6d gauge symmetry is captured in terms of the irreducible characters of the G affine Kac-Moody algebra, as opposed to just the Verma modules.

We now discuss the various remaining theories in turn:

- For $\mathbf{h}_1^{\mathrm{SO}(12)_a}$, $\mathbf{h}_2^{\mathrm{SO}(12)_a}$, and $\mathbf{h}_1^{F_4}$, only one coefficient in the Ansatz was left undetermined after applying constraints **C1**, **C2**, and **C3**. By applying a combination of constraints **(a)**, **(b)**, and **(c)**, we are able to fix the remaining coefficient with a good degree of confidence; namely, after imposing constraint **(a)**, we can verify that conditions **(b)** and **(c)** are also satisfied by the elliptic genera of these theories.

- For $\mathbf{h}_2^{\mathrm{SO}(12)_b}$, imposing condition **(a)** to fix the single undetermined coefficient leads to non-integer values of certain b_{jk} coefficients. Notably, however, we find that if we slightly modify condition **(a)** and take the $b_{1,2}$ coefficient in the expansion of the elliptic genus to be equal to $\dim(G) + \dim(F) = 144$ instead of 145 the elliptic genus we find is completely consistent with conditions **(b)** and **(c)**. Moreover, the elliptic genus we find differs from the one of the $\mathbf{h}_2^{\mathrm{SO}(12)_a}$ theory, supporting the possibility that the parent 6d SCFT $\mathbf{2}_{\mathrm{SO}(12)_b}$ might be a consistent theory distinct from the $\mathbf{2}_{\mathrm{SO}(12)_a}$ SCFT.

- For $\mathbf{h}_1^{\mathrm{SO}(11)}$ and $\mathbf{h}_1^{\mathrm{SO}(12)_b}$, by setting $b_{n-1+\delta_{1,n},2} = \dim(G) + \dim(F) + 1$ we are able to completely fix the elliptic genus, but since we do not fully understand the flavor symmetry we are not able to check the validity of our result to a satisfying degree of confidence.

- For $\mathbf{h}_2^{\mathrm{SO}(11)}$ the number of undetermined coefficients is two, and we do not understand the flavor symmetry; we could not find a way to fix the elliptic genus to a sufficient level of confidence.

- For the theories $\mathbf{h}_1^{E_7}$, $\mathbf{h}_2^{E_7}$, $\mathbf{h}_3^{E_7}$, and $\mathbf{h}_4^{E_7}$ the number of unfixed coefficients, respectively 15, 9, 11, and 9, is so high that we have not been able to find a criterion to fix them.

- Finally, for the theories $\mathbf{h}_2^{E_6}$ and $\mathbf{h}_1^{E_6}$ only 3 coefficients are unfixed, and condition **(a)** can be used to fix one. Unfortunately, however, the number of distinct affine characters of the flavor symmetry F that are allowed to appear in the elliptic genus is extremely high (respectively 1092 and 3360), and so conditions **(b)** and **(c)** are not very predictive. As a consequence, we are unable to fix the E_6 elliptic genera for these theories with a good degree of confidence.

It is of course possible that one may be able to identify additional constraints on the elliptic genus that may suffice to completely fix the yet undetermined elliptic genera.

In appendix G we display our results for the elliptic genera of the 59 theories which we fixed with conditions **C1**, **C2**, **C3**, as well as the ones for the $\mathbf{h}_1^{\mathrm{SO}(12)a}$, $\mathbf{h}_1^{F_4}$, $\mathbf{h}_2^{\mathrm{SO}(12)a}$, and $\mathbf{h}_2^{\mathrm{SO}(12)b}$ theories. In order to present our computational results in a concise and useful way, in appendix G we have chosen to tabulate several of the b_{jk} coefficients in their series expansion (8.29). The tables of coefficients make it easy to compare our results with computations performed by other techniques, such as localization; furthermore, it is easy to verify directly from the tables that various of the universal properties of BPS string CFTs discussed in section 6 are satisfied for these theories.

On the other hand, the expressions we obtain for the elliptic genera as meromorphic Jacobi forms are quite unwieldy; rather than including them in the text, we have chosen to attach a Mathematica file to this paper as supplementary material, where we include these expressions for all $59 + 4$ theories whose elliptic genus is completely fixed, as well as for the remaining 9 theories for which some of the coefficients are still undetermined.

8.4 5d Nekrasov partition functions

In section 7 we argued that the elliptic genus for the $\mathbf{h}_n^G$ theories for $n \geq 2$ reduces in the $q \to 0$ limit to

$$\lim_{q \to 0} q^{-1/6 + \frac{n-2}{2}} \mathbb{E}_n^G(\mathbf{m}_G, \mathbf{m}_F, v, q) = \left[\frac{v}{(1 - v\,x)(1 - v/x)} \right]^{-1} Z_{1-\mathrm{inst}}(\mathbf{m}_G, \mathbf{m}_F, x, v), \quad (8.39)$$

where we have removed the center of mass contribution on the right hand side. For $n \geq 3$, the term appearing on the right hand side is the one-instanton piece of the Nekrasov partition function for the $\mathcal{N} = 1$ theory with gauge group G and matter hypermultiplets in the same representation as the 6d matter. For instance, the $\mathbf{h}_7^{E_7}$ theory gives in the $q \to 0$ limit the one-instanton piece of the Nekrasov partition function of the 5d $G = E_7$ theory with one half-hypermultiplet in the **56** representation of E_7. For $n = 2$, as discussed

in section 7.2.2, the right hand side contains extra states in comparison to the Nekrasov partition function coming from a sector decoupled from the 5d gauge theory.

Our results for the 6d elliptic genus include many cases that reduce in 5d to theories whose 1-instanton partition function is not yet known. By taking the limit (8.39) we can straightforwardly obtain the 5d one-instanton partition functions from the elliptic genera, in the limit where $\mathbf{m}_G \to \mathbf{1}$ and $\mathbf{m}_F \to \mathbf{1}$. For instance, for $G = F_4$ with two hypermultiplets in the **26** representation, which descends from the 6d SCFT with $n = 3$ and $G = F_4$, we obtain:

$$Z_{1-\text{inst}}(\mathbf{1},\mathbf{1},x,v) = \frac{v^7\left(5 + 80v + 268v^2 - 1232v^3 + 2142v^4 - 1232v^5 + 268v^6 + 80v^7 + 5v^8\right)}{(1 - v\,x)(1 - v/x)(-1 + v)^4(1 + v)^{16}},$$
(8.40)

where we only expect our answer to differ from the equivariant calculation of the index by an overall factor of v, as remarked in section 7. We have collected our results for the 5d one-instanton partition functions, with $\mathbf{m}_G$ and $\mathbf{m}_F$ fugacities switched off, in the tables of appendix H.2.

It is also of course desirable to recover the full dependence of the 5d partition function on all chemical potentials. For several theories, we are able to achieve this by comparing the expression for $Z_{1-\text{inst}}$ with F and G chemical potentials switched off to the leading order terms in the q-expansion of equation (6.17). We explain this approach with a detailed example in section H.1. In many instances have been able to infer all-order expansions in v for the elliptic genus, which capture the full dependence on the $\mathbf{m}_F$ and $\mathbf{m}_G$ fugacities. These results are presented in appendix H.2.

9 The exceptional cases $\mathbf{h}_3^{\mathbf{SU(3)}}$ and $\mathbf{h}_2^{\mathbf{SU(2)}}$

In section 6.3 we found evidence that the G dependence of the elliptic genera of the $\mathbf{h}_n^G$ theories is captured in terms of a Kac-Moody algebra at level $-n$, and for $n \neq h_G^\vee$ we found that the elliptic genus admits an expansion given by equation (6.18). In this section we discuss the two nontrivial cases $\mathbf{h}_3^{\mathrm{SU(3)}}$ and $\mathbf{h}_2^{\mathrm{SU(2)}}$ for which $n = h_G^\vee$ (the remaining, trivial case being the E-string for which $G = \mathrm{Sp}(0)$). We find an expansion of the elliptic genus of these theories similar to equation (6.18), and comment on the lack of an obvious relation to the irreducible characters of the affine Kac-Moody algebra for G at critical level.

9.1 The $\mathbf{h}_3^{\mathbf{SU(3)}}$ case

In the case of the $\mathbf{h}_3^{\mathrm{SU(3)}}$ CFT the flavor symmetry F is trivial, and the elliptic genus is given by a single function

$$\mathbb{E}_3^{\mathrm{SU(3)}}(\mathbf{m}_{\mathrm{SU(3)}}, v, q) = \xi_0^{3,\mathrm{SU(3)}}(\mathbf{m}_{\mathrm{SU(3)}}, v, q).$$
(9.1)

Starting from the explicit expression for the elliptic genus obtained by localization in [82], we find the following expansion:

$$\xi_0^{3,\mathrm{SU(3)}}(\mathbf{m}_{\mathrm{SU(3)}}, v, q) = \left[q^{8/24}\widetilde{\Delta}_{\mathrm{SU(3)}}(\mathbf{m}_{\mathrm{SU(3)}}, q)\right]^{-1}\sum_{n=1}^{\infty}\chi_{(n-1,n-1)}^{\mathrm{SU(3)}}(\mathbf{m}_{\mathrm{SU(3)}})\frac{v^{2n} + q^n v^{-2n}}{1 - q^n},$$
(9.2)

(a similar expression with fugacities $\mathbf{m}_{\mathrm{SU}(3)} \to \mathbf{1}$ was found in [83]). Note that this expression is explicitly invariant under $v \to q^{1/2}/v$.

Unlike in the noncritical cases, the SU(3) dependence does not appear to be captured in terms of characters of the SU(3) Kac-Moody algebra at the critical level $k = -3$ in a straightforward way. For an affine Lie algebra at critical level, a formula for the irreducible characters is available [223]. In particular, the irreducible character corresponding to the SU(3) representation with Dynkin label $(n_1 n_2)$ is given (up to an overall power of q) by:

$$\widehat{\chi}^{\mathrm{SU}(3)}_{(n_1 n_2)}(\mathbf{m}_{\mathrm{SU}(3)}, q) = \frac{\prod_{j=1}^{\infty}(1 - q^j)^2}{\widetilde{\Delta}_{\mathrm{SU}(3)}(\mathbf{m}_{\mathrm{SU}(3)}, q)} \cdot \frac{\chi^{\mathrm{SU}(3)}_{(n_1 n_2)}(\mathbf{m}_{\mathrm{SU}(3)}, q)}{(1 - q^{n_1+1})(1 - q^{n_2+1})(1 - q^{n_1+n_2+2})}, \quad (9.3)$$

which bears no clear relation to the SU(3) dependence of the elliptic genus (9.2).

9.2 The $\mathbf{h}_2^{\mathrm{SU}(2)}$ case

The elliptic genus of the $\mathbf{h}_2^{\mathrm{SU}(2)}$ theory is known from [70]. Although the flavor symmetry of the 6d SCFT is $F = \mathrm{SO}(7)$ instead of the naive SO(8) [20], it turns out that it is still more natural to write the elliptic genus in terms of level 1 SO(8) characters. The embedding of SO(7) into SO(8) is such that the SO(7) spinor representation is identified with the SO(8) vector representation. At the level of fugacities, embedding SO(7) into SO(8) implies the following relations:

$$m^1_{\mathrm{SO}(8)} = \frac{m^1_{\mathrm{SO}(7)} + m^2_{\mathrm{SO}(7)} + m^3_{\mathrm{SO}(7)}}{2}; \quad (9.4)$$

$$m^2_{\mathrm{SO}(8)} = \frac{-m^1_{\mathrm{SO}(7)} + m^2_{\mathrm{SO}(7)} + m^3_{\mathrm{SO}(7)}}{2}; \quad (9.5)$$

$$m^3_{\mathrm{SO}(8)} = \frac{m^1_{\mathrm{SO}(7)} - m^2_{\mathrm{SO}(7)} + m^3_{\mathrm{SO}(7)}}{2}; \quad (9.6)$$

$$m^4_{\mathrm{SO}(8)} = \frac{m^1_{\mathrm{SO}(7)} + m^2_{\mathrm{SO}(7)} - m^3_{\mathrm{SO}(7)}}{2}. \quad (9.7)$$

With this assignment of SO(8) fugacities, the affine level 1 SO(8) characters decompose as follows:

$$\widehat{\chi}^{\mathrm{SO}(8)}_{\mathbf{0}}(\mathbf{m}_{\mathrm{SO}(8)}, q) = \widehat{\chi}^{\mathrm{SO}(7)}_{\mathbf{0}}(\mathbf{m}_{\mathrm{SO}(7)}, q)\widehat{\chi}^{\mathrm{SO}(1)}_{\mathbf{0}}(q) + \widehat{\chi}^{\mathrm{SO}(7)}_{\mathbf{v}}(\mathbf{m}_{\mathrm{SO}(7)}, q)\widehat{\chi}^{\mathrm{SO}(1)}_{\mathbf{v}}(q), \quad (9.8)$$

$$\widehat{\chi}^{\mathrm{SO}(8)}_{\mathbf{c}}(\mathbf{m}_{\mathrm{SO}(8)}, q) = \widehat{\chi}^{\mathrm{SO}(7)}_{\mathbf{0}}(\mathbf{m}_{\mathrm{SO}(7)}, q)\widehat{\chi}^{\mathrm{SO}(1)}_{\mathbf{v}}(q) + \widehat{\chi}^{\mathrm{SO}(7)}_{\mathbf{v}}(\mathbf{m}_{\mathrm{SO}(7)}, q)\widehat{\chi}^{\mathrm{SO}(1)}_{\mathbf{0}}(q), \quad (9.9)$$

$$\widehat{\chi}^{\mathrm{SO}(8)}_{\mathbf{v}}(\mathbf{m}_{\mathrm{SO}(8)}, q) = \widehat{\chi}^{\mathrm{SO}(8)}_{\mathbf{s}}(\mathbf{m}_{\mathrm{SO}(7)}, q) = \widehat{\chi}^{\mathrm{SO}(7)}_{\mathbf{s}}(\mathbf{m}_{\mathrm{SO}(7)}, q)\widehat{\chi}^{\mathrm{SO}(1)}_{\mathbf{s}}(q), \quad (9.10)$$

where $\widehat{\chi}^{\mathrm{SO}(1)}_{\mathbf{0}}(q)$, $\widehat{\chi}^{\mathrm{SO}(1)}_{\mathbf{v}}(q)$, and $\widehat{\chi}^{\mathrm{SO}(1)}_{\mathbf{s}}(q)$ are the three Ising model characters given in equations (E.12).

We find that the $\mathbf{h}_2^{\mathrm{SU}(2)}$ elliptic genus can be written as follows:

$$\begin{aligned}
\mathbb{E}^{\mathrm{SU}(2)}_2(\mathbf{m}_{\mathrm{SU}(2)}, v, q) &= \widehat{\chi}^{\mathrm{SO}(8)}_{\mathbf{0}}(\mathbf{m}_{\mathrm{SO}(8)}, q)\xi^{2,\mathrm{SU}(2)}_{\mathbf{0}}(\mathbf{m}_{\mathrm{SU}(2)}, v, q) \\
&\quad + \widehat{\chi}^{\mathrm{SO}(8)}_{\mathbf{c}}(\mathbf{m}_{\mathrm{SO}(8)}, q)\xi^{2,\mathrm{SU}(2)}_{\mathbf{c}}(\mathbf{m}_{\mathrm{SU}(2)}, v, q) \\
&\quad + \widehat{\chi}^{\mathrm{SO}(8)}_{\mathbf{v}}(\mathbf{m}_{\mathrm{SO}(8)}, q)\xi^{2,\mathrm{SU}(2)}_{\mathbf{v}}(\mathbf{m}_{\mathrm{SU}(2)}, v, q),
\end{aligned} \quad (9.11)$$

where

$$\xi_{\mathbf{0}}^{2,\mathrm{SU}(2)}(\mathbf{m}_{\mathrm{SU}(2)}, v, q) = \tag{9.12}$$

$$= \frac{1}{q^{4/24} \prod_{j=0}^{\infty}(1-q^j)\widetilde{\Delta}_{\mathrm{SU}(2)}(\mathbf{m}_{\mathrm{SU}(2)}, q)} \sum_{k \geq 0} \frac{q^{k+1/2}(v^{2k+1}+v^{-2k-1})}{1-q^{2k+1}} \chi_{(2k)}^{\mathrm{SU}(2)}(\mathbf{m}_{\mathrm{SU}(2)}),$$

$$\xi_{\mathbf{c}}^{2,\mathrm{SU}(2)}(\mathbf{m}_{\mathrm{SU}(2)}, v, q) = \tag{9.13}$$

$$= -\frac{1}{q^{4/24} \prod_{j=0}^{\infty}(1-q^j)\widetilde{\Delta}_{\mathrm{SU}(2)}(\mathbf{m}_{\mathrm{SU}(2)}, q)} \sum_{k \geq 0} \frac{v^{2k+1}+q^{2k+1}v^{-2k-1}}{1-q^{2k+1}} \chi_{(2k)}^{\mathrm{SU}(2)}(\mathbf{m}_{\mathrm{SU}(2)}),$$

$$\xi_{\mathbf{v}}^{2,\mathrm{SU}(2)}(\mathbf{m}_{\mathrm{SU}(2)}, v, q) = \tag{9.14}$$

$$= \frac{1}{q^{4/24} \prod_{j=0}^{\infty}(1-q^j)\widetilde{\Delta}_{\mathrm{SU}(2)}(\mathbf{m}_{\mathrm{SU}(2)}, q)} \sum_{k \geq 0} \frac{v^{2k+2}-q^{k+1}v^{-2k-2}}{1+q^{k+1}} \chi_{(2k+1)}^{\mathrm{SU}(2)}(\mathbf{m}_{\mathrm{SU}(2)}).$$

As a side remark, note that under the spectral flow transformation $v \to q^{1/2}/v$

$$\xi_{\mathbf{0}}^{2,\mathrm{SU}(2)}(\mathbf{m}_{\mathrm{SU}(3)}, v, q) \leftrightarrow -\xi_{\mathbf{c}}^{2,\mathrm{SU}(2)}(\mathbf{m}_{\mathrm{SU}(2)}, v, q) \tag{9.15}$$

while

$$\xi_{\mathbf{v}}^{2,\mathrm{SU}(2)}(\mathbf{m}_{\mathrm{SU}(3)}, v, q) \to -\xi_{\mathbf{v}}^{2,\mathrm{SU}(2)}(\mathbf{m}_{\mathrm{SU}(3)}, v, q). \tag{9.16}$$

Let us now look more in detail at how the flavor symmetry is realized at the level of the elliptic genus. Recall that the $b_{1,2}$ coefficient in the series expansion (8.29) of the elliptic genus is given (with a few exceptions noted in section 8.3) by

$$\chi_{\theta_G}^G(\mathbf{m}_G) + \chi_{\theta_F}^F(\mathbf{m}_F) + 1. \tag{9.17}$$

Computing this coefficient for the $\mathbf{h}_2^{\mathrm{SU}(2)}$ theory, we find

$$b_{1,2} = \chi_{\theta_{\mathrm{SO}(8)}}^{\mathrm{SO}(8)}(\mathbf{m}_{\mathrm{SO}(8)}) + \chi_{\theta_{\mathrm{SU}(2)}}^{\mathrm{SU}(2)}(\mathbf{m}_{\mathrm{SU}(8)}) + 2 - \chi_{\mathbf{c}}^{\mathrm{SO}(8)}(\mathbf{m}_{\mathrm{SO}(8)}). \tag{9.18}$$

Using

$$\chi_{\theta_{\mathrm{SO}(8)}}^{\mathrm{SO}(8)}(\mathbf{m}_{\mathrm{SO}(8)}) = \chi_{\theta_{\mathrm{SO}(7)}}^{\mathrm{SO}(7)}(\mathbf{m}_{\mathrm{SO}(7)}) + \chi_{\mathbf{v}}^{\mathrm{SO}(7)}(\mathbf{m}_{\mathrm{SO}(7)}) \tag{9.19}$$

and

$$\chi_{\mathbf{c}}^{\mathrm{SO}(8)}(\mathbf{m}_{\mathrm{SO}(8)}) = \chi_{\mathbf{v}}^{\mathrm{SO}(7)}(\mathbf{m}_{\mathrm{SO}(7)}) + 1, \tag{9.20}$$

we obtain

$$b_{1,2} = \chi_{\theta_{\mathrm{SO}(7)}}^{\mathrm{SO}(7)}(\mathbf{m}_{\mathrm{SO}(7)}) + \chi_{\theta_{\mathrm{SU}(2)}}^{\mathrm{SU}(2)}(\mathbf{m}_{\mathrm{SU}(2)}) + 1, \tag{9.21}$$

indicating that the BPS string CFT indeed knows about the reduction of flavor symmetry from SO(8) to SO(7) at the 6d superconformal fixed point!

Finally, we remark that the irreducible characters of the affine SU(2) Kac-Moody algebra at critical level $k = -2$ are given by:

$$\widehat{\chi}_{(n)}^{\mathrm{SU}(2)}(\mathbf{m}_{\mathrm{SU}(2)}, q) = \frac{\prod_{j=1}^{\infty}(1-q^j)}{\widetilde{\Delta}_{\mathrm{SU}(2)}(\mathbf{m}_{\mathrm{SU}(2)}, q)} \cdot \frac{1}{1-q^{n+1}}. \tag{9.22}$$

One may ask whether the elliptic genus of the $\mathbf{h}_2^{\mathrm{SU}(2)}$ CFT is captured in terms of these characters, as was the case for the theories with $n \neq h_G^\vee$ studied in section 6. On the one hand, the SU(2) dependence in equations (9.12) and (9.13) bears some similarity to equation (9.22) (in fact, they just differ by a factor of $\prod_{j=1}^{\infty}(1-q^j)^2$); on the other hand, the SU(2) dependence in equation (9.14) cannot easily be captured in terms of the irreducible characters (9.22). Also, expressing the dependence of the elliptic genus in terms of SO(7) characters rather than SO(8) characters does not resolve this mismatch. It would be interesting to understand in more detail the chiral algebra of this CFT as well as the $\mathbf{h}_3^{\mathrm{SU}(3)}$ CFT and the way the flavor symmetries are realized in it [195].

10 Conclusions and future directions

In this paper we have studied the properties of a BPS string of an arbitrary six-dimensional SCFT on the tensor branch from the point of view of the worldsheet $\mathcal{N}=(0,4)$ CFT that lives on it. The upshot is that a uniform description of these CFTs emerges, various features of which are fixed by a minimal amount of data: the Dirac pairing of the string $-n$, the 6d gauge symmetry G, and the 6d flavor symmetry F, which with a few exceptions is uniquely determined once n and G are specified.

In the first half of the paper we found a uniform realization of the string worldsheet CFTs in the context of F-theory. We argued that the strings' CFTs can be obtained by compactifying the worldvolume theory of a D3 brane in the presence of defects preserving (0,4) supersymmetry in 2 dimensions; the compactification involves a topological twist which is a hybrid of the β-twist [166] and the topological duality twist [135]. Two kinds of defects appear: generalized chiral defects and folding defects, which are related respectively to the matter and gauge content of the 6d SCFT; from the details of the compactification we are able to infer certain properties of these defects, including their contribution to an effective shift of the R-charge in the curved rigid supersymmetric backgrounds. A precise study of such curved supersymmetric backgrounds requires a more detailed analysis which we leave to future work [163]. The compactified theory flows in the IR to the CFT on the string, whose anomalies and central charges we compute by relating them via 2d $\mathcal{N}=(0,4)$ Ward identities to the coefficients of the 2d anomaly polynomial determined by anomaly inflow from 6d after [31, 82].

In [224] a deep interconnection was found between the chiral algebras of 4d $\mathcal{N}=2$ SCFTs [199, 225, 226] and the geometry of their Higgs branches, which provides a physical realization of Arakawa's results associating a vertex operator algebra to any hyperkähler variety $\mathcal{M}$, see [227] for a review. In particular, the Schur index of the 4d SCFT [228] is identified with the vacuum character of the chiral algebra. However, not all instances studied by Arakawa appear to be realized directly in the context of $\mathcal{N}=2$ SCFT; in particular the cases where $\mathcal{M}$ is the moduli space of one F_4 or one G_2 instanton appear to be excluded [229–232]. For the class of $(0,4)$ theories studied in our previous work [83], which arise from compactification of 4d $\mathcal{N}=2$ theories without the insertion of generalized chiral defects, we were able to recover the Schur index of the 4d $\mathcal{N}=2$ SCFT out of a particular function, $L_G(v,q)$, which arose in the computation of the elliptic genus. The

Schur index is obtained by specializing this function to the value $L_G(q^{1/4}, q)$; for the cases corresponding to simply-laced gauge group, the Schur index in turn also corresponds to the vacuum character of the chiral algebra of the 4d $\mathcal{N} = 2$ SCFT [199]. As we have discussed here, one can realize the corresponding BPS string starting from the 4d $\mathcal{N} = 2$ theory $H_{E_6}^{Q=1}$ (i.e., the E_6 Minahan-Nemeschansky theory) by inserting folding defects. This construction does give rise to a 2d $(0, 4)$ theory whose infrared limit is a NLSM with target the moduli space of one F_4 instanton. From the elliptic genus of the 2d theory, one can determine the $L_{F_4}(v, q)$ function, which was done in [83]. When specialized to $L_{F_4}(q^{1/4}, q)$, the function coincides with $\widehat{\chi}_0^{(F_4)-5/2}$, the vacuum character of the chiral algebra of F_4 type that in the construction of Arakawa is associated to the hyperkähler manifold $\mathcal{M}_{F_4,1}$. This suggests a potential generalization of the results of [224] to more general classes of chiral algebras by considering β-twisted compactifications on $T^2 \times S^2$ backgrounds of 4d $\mathcal{N} = 2$ theories with or without the insertions of 2d $(0, 4)$ surface defects; note in particular that the twisted theory flows to a NLSM with target space the Higgs branch of the corresponding 4d $\mathcal{N} = 2$ theory [165, 166] for models that admit a β-twist.

In the second half of the paper we studied the realization of the BPS string CFTs as $\mathcal{N} = (0, 4)$ nonlinear sigma models on the moduli space of one instanton of the 6d gauge group. This description correctly accounts for the central charges of the CFT and implies that the 6d flavor symmetry group is realized on the string as a chiral current algebra, whose level we were able to infer from the anomaly polynomial. We also found a simple, uniform realization for the G dependence of the elliptic genus. Namely, we found that (at least for theories for which $n \neq h_G^\vee$) this symmetry is realized as an affine Kac-Moody algebra at level $-n$, and the elliptic genus is expressed in terms of the irreducible characters of this algebra; the theories $\mathbf{h}_2^{\mathrm{SU}(2)}$ and $\mathbf{h}_3^{\mathrm{SU}(3)}$, for which the level of the Kac-Moody algebra is the critical one, $k = -h_G^\vee$, appear to be an exception to this statement and are worthy of further study. We also found that at the level of chiral algebra the $\mathrm{SU}(2)_v$ global symmetry can be exploited to perform a spectral flow that interpolates between different boundary conditions for the chiral fermions. This implies a relation between the elliptic genera computed with R-R and NS-R boundary conditions. From simple considerations about the structure of the NLSM we were able to determine the spectrum of low energy operators contributing to the NS-R elliptic genus, which in turn provides information about the R-R elliptic genus.

Of course, ultimately the motivation for studying the string CFTs is that they provide an original angle to probe the 6d SCFT to which they belong. In this paper we have encountered several instances which demonstrate the usefulness of this approach: first of all, imposing $c_L \geq 0$, which is required by the unitarity of the string CFT, implies that the gauge algebra of the 6d SCFT must be nontrivial for $n \geq 3$, consistent with the finding in F-theory that tensor multiplets with $n \geq 3$ are always paired to a nontrivial gauge group [150]. We have also found that the string CFT is a useful tool for understanding the flavor symmetry of the 6d SCFT. For example, the flavor symmetry of the 6d $\mathbf{2}_{\mathrm{SU}(2)}$ SCFT is SO(8) on the tensor branch but is reduced to SO(7) at the superconformal fixed point [20]; in section 9 we found that this is reflected explicitly in the elliptic genus of the $\mathbf{h}_2^{\mathrm{SU}(2)}$ CFT: the spectrum organizes itself naturally in terms of SO(8) characters, but

among the operators of conformal dimensions $(h_L, h_R) = (1, 0)$ above the NS-R vacuum one finds a set of operators transforming in the adjoint representation of SO(7) rather than SO(8). Moreover, in section 6.5 we have encountered some difficulties in realizing the flavor symmetry of the $\mathbf{h}_1^{\mathrm{SO}(11)}$, $\mathbf{h}_2^{\mathrm{SO}(11)}$, $\mathbf{h}_1^{\mathrm{SO}(12)_b}$, and $\mathbf{h}_2^{\mathrm{SO}(12)_b}$ CFTs in terms of WZW models, which might point to potential subtleties with the parent 6d SCFTs.

The most direct connection to 6d physics however comes from the fact that the R-R elliptic genus of the $\mathbf{h}_n^G$ string captures the one-instanton piece of the Nekrasov partition function of the 6d $\mathbf{n}_G$ SCFT. By exploiting modularity and various constraints that arise from the universal features of the string CFTs we were able to determine the elliptic genera of a number of string CFTs of 6d SCFTs with exceptional gauge symmetry or matter content, leading to a wealth of novel results for the associated 6d Nekrasov partition functions (albeit with gauge symmetries fugacities switched off). Moreover, by considering the 5d limit of 6d SCFTs with $n \geq 2$ we also obtained new results for the one-instanton pieces of the Nekrasov partition functions of the resulting 5d $\mathcal{N} = 1$ theories, and using the geometric engineering of the 6d SCFTs in F-theory, we were also able to clarify their behavior upon compactification to 5d and explain the different limits obtained for theories with $n = 1$, $n = 2$, and $n > 2$.

The work presented in this paper lends itself naturally to a number of extensions. At the computational level, we found that we could determine the elliptic genera of several 2d CFTs based on simple considerations on their low energy spectrum in the NS-R sector. Understanding the spectrum of low energy operators in more detail may lead to additional constraints and help determine the elliptic genera of the remaining CFTs. It would also be desirable to obtain expressions for the elliptic genera with the fugacities for the 6d gauge symmetry G reinstated, extending the results of [91, 94]. Moreover, while in this paper we have largely focused on the CFTs describing a single BPS string, several of our results can naturally be extended to study bound states of strings associated to the same gauge factor or to different gauge factors; the CFT of such bound states will be described by nonlinear sigma models on the moduli spaces of higher numbers of instantons. The calculation of central charges and anomaly polynomials easily generalizes to bound states of strings, and likewise one still expects the 6d flavor symmetry to be captured by current algebras at higher level fixed by the anomaly polynomials we computed in this paper. It would be interesting to understand whether other universal features we found for single-string CFTs carry over to bound states. For instance, it would be desirable to know whether the spectral flow symmetry observed in the single string CFT carries over to bound states of strings and leads to relations between R-R and NS-R elliptic genera. Understanding the large N behavior of bound states of strings might also be worthwile. It would also be interesting to obtain a clearer picture of the mechanism by which the strings bind to each other from the point of view of their CFTs, and study whether the classification of 6d SCFTs obtained in [2, 5, 10] from geometric considerations also follows from e.g. unitarity constraints on the CFTs describing bound states of strings.

One of the salient features of the backgrounds corresponding to frozen singularities in F-theory is that the one-to-one correspondence between simple factors of the gauge group and the tensor multiplets is lost. Therefore the BPS strings corresponding to these theories

have features that are slightly different from the ones that we have considered in this paper, in particular the target space of the corrersponding NLSM is no longer the one-instanton moduli space for a simple gauge group. Generalizing our results to this class of geometries is an obvious future direction.

Lastly, it should be possible to interpret various properties of the BPS strings, including the appearance of a G Kac-Moody algebra at negative level and the existence of a spectral flow symmetry, most naturally as features of their chiral algebra. Some of the string CFTs are essentially free theories or admit UV $N = (0, 4)$ supersymmetric Lagrangian realizations, and it may therefore be possible to study them by an extension of the techniques of [204] or [205]. While most theories are not known to admit such simple realizations, the results of this paper suggest that their chiral algebras bear many similarities to the chiral algebras of the ones that do. One may therefore hope for a uniform description of the chiral algebras of all 6d string CFTs. Among these, it would be important to understand better the exceptional cases for which $n = h_G^\vee$, which as discussed in section 9 appear to have different properties from the rest. These matters are currently under investigation [195].

Acknowledgments

A Lie algebras and representations

A.1 Simple Lie algebras

In this paper we use the standard labeling for the nodes of the Dynkin diagrams associated to simple Lie algebras indicated in figure 17. There, we also indicate the comark $a_i^\vee$ associated to each node. The dual Coxeter number of a simple Lie algebra $\mathfrak{g}$ is defined as the sum of all the comarks:

$$h_\mathfrak{g}^\vee = \sum_{i=1}^{\mathrm{rank}(\mathfrak{g})} a_i^\vee. \tag{A.1}$$

To the i-th node of the Dynkin diagram of any Lie algebra $\mathfrak{g}$ one can associate a simple root α_i; the total number of nodes equals the rank of the algebra, $\mathrm{rank}(\mathfrak{g})$. We denote the set of positive roots by $\Delta_+^\mathfrak{g}$. The Killing form induces a scalar product $\langle \cdot, \cdot \rangle$ on the complexification of the dual Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{h}_\mathbb{C}^* \simeq \mathbb{C}^{\mathrm{rank}(\mathfrak{g})}$. By making use of
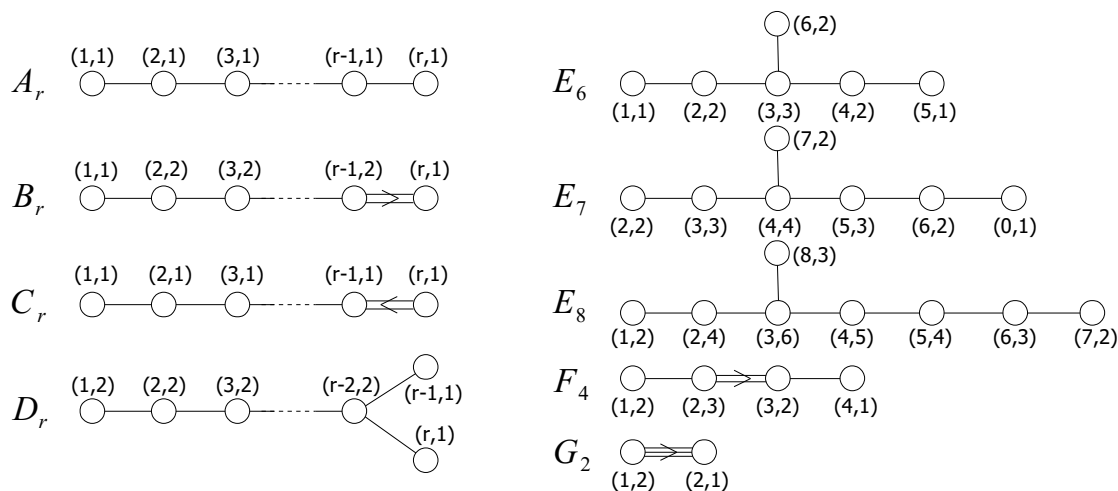
Figure 17. Dynkin diagrams associated to simple Lie algebras. We label the i-th node by $(i, a_i^\vee)$, where $a_i^\vee$ are the *comarks*.

this scalar product one can define the coroots $\alpha_i^\vee$:

$$\alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}, \tag{A.2}$$

where the scalar product is normalized so that it equals 2 for long roots. One furthermore defines the fundamental weights via the relation

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}; \tag{A.3}$$

the fundamental weights form a basis of the weight space of the Lie algebra.

We denote the set of irreducible representations of $\mathfrak{g}$ as $\text{Rep}(\mathfrak{g})$. An irreducible representation R_λ of a Lie algebra can be labeled in terms of its highest weight λ, which we expand in terms of the fundamental weights:

$$\lambda = \sum_{i=1}^{\text{rank}(\mathfrak{g})} a_i\, \omega_i. \tag{A.4}$$

In this paper we denote such a representation in terms of its Dynkin labels a_i:

$$\lambda \simeq (a_1 a_2 \ldots a_{\text{rank}(\mathfrak{g})}). \tag{A.5}$$

We separate the entries by commas in cases where ambiguity might arise from this notation.

In some parts of the paper we also find it convenient to label representations in terms of their dimension. Thus for example the adjoint representation of $\mathfrak{e}_8$ is also denoted by **248**. For $\mathfrak{g} = \mathfrak{so}_{2N}$ use a superscript **s**, **c** to distinguish between spinor and conjugate spinor representations, and for $\mathfrak{g} = \mathfrak{so}_8$ we denote by $\mathbf{8^v}$, $\mathbf{8^s}$, and $\mathbf{8^c}$ respectively the vector, spinor and conjugate spinor representations. Alternatively, we sometimes for convenience use the notation $\mathbf{v}, \mathbf{s}, \mathbf{c}$ to denote the vector, spinor and conjugate spinor representations of $\mathfrak{so}_{2N}$.

Also for $\mathfrak{g} = \mathfrak{su}_N$ we denote respectively by $\mathbf{N}$ and $\overline{\mathbf{N}}$ the $(10\ldots0)$ representation and its conjugate representation $(0\ldots01)$, and by $\mathbf{\Lambda^2}$ and $\overline{\mathbf{\Lambda^2}}$ respectively the representations $(010\ldots0)$ and $(0\ldots010)$.

To any irreducible representation one can associate its quadratic Casimir invariant

$$C_2(R_\lambda) = \langle \lambda, \lambda + 2\rho_\mathfrak{g} \rangle, \tag{A.6}$$

where the Weyl vector $\rho_\mathfrak{g}$ is defined as one half of the sum of positive roots, and has Dynkin labels $(11\ldots1)$. Another relevant quantity associated to a representation is its character, which we always evaluate at a specific point $\mathbf{m}_\mathfrak{g} = e^{2\pi i \mu_\mathfrak{g}}$, where $\mu_\mathfrak{g} \in \mathfrak{h}^*_\mathbb{C}$:

$$\chi^\mathfrak{g}_\lambda(\mathbf{m}_\mathfrak{g}) = \sum_{\nu \in R_\lambda} \mathrm{mult}(\nu)\, e^{2\pi i (\mu_\mathfrak{g}, \nu)}, \tag{A.7}$$

where the sum runs over the weights ν in the representation R_λ, which we view here as points in $\mathfrak{h}^*_\mathbb{C}$, and $\mathrm{mult}(\nu)$ denotes the multiplicity with which ν appears in the representation. In the limit $\mu_\mathfrak{g} \to 0$ (which we sometimes denote alternatively by $\mathbf{m}_\mathfrak{g} \to \mathbf{1}$), the character reduces to the dimension of the representation:

$$\chi^\mathfrak{g}_\lambda(\mathbf{1}) = \dim(R_\lambda). \tag{A.8}$$

At various points in the paper we specialize the parameter $\mathbf{m}_\mathfrak{g}$ to specific values. For a root α of $\mathfrak{g}$, we find it convenient to define

$$m_\alpha = e^{2\pi i \langle \mu_\mathfrak{g}, \alpha \rangle}. \tag{A.9}$$

Furthermore, in computations we find it convenient to expand $\mathbf{m}_\mathfrak{g}$ in the following way:

- For $\mathfrak{g} = \mathfrak{su}_N$, define

$$e_1 = \omega_1, \quad e_2 = \omega_2 - \omega_1, \quad e_3 = \omega_3 - \omega_2, \quad \ldots, \quad e_{N-1} = \omega_{N-1} - \omega_{N-2}, \quad e_N = -\omega_{N-1}, \tag{A.10}$$

and define

$$m^i_{\mathfrak{su}_N} = e^{2\pi i \langle \mu_{\mathfrak{su}_N}, e_i \rangle}, \tag{A.11}$$

so that the $\mathbf{N}$ representation has character

$$\chi^{\mathfrak{su}_N}_{(10\ldots0)}(\mathbf{m}_{\mathfrak{su}_N}) = \sum_{i=1}^{N} m^i_{\mathfrak{su}_N}, \tag{A.12}$$

where

$$m^N_{\mathfrak{su}_N} = \frac{1}{\prod_{i=1}^{N-1} m^i_{\mathfrak{su}_N}}. \tag{A.13}$$

- For $\mathfrak{g} = \mathfrak{so}_{2N}$, define

$$e_1 = \omega_1,\ e_2 = \omega_2 - \omega_1,\ e_3 = \omega_3 - \omega_2,\ \ldots,\ e_{N-1} = \omega_{N-1} + \omega_N - \omega_{N-2},\ e_N = \omega_N - \omega_{N-1}, \tag{A.14}$$

and define

$$m^i_{\mathfrak{so}_{2N}} = e^{2\pi i\langle\mu_{\mathfrak{so}_{2N}},e_i\rangle}, \tag{A.15}$$

so that the **2 N** representation has character

$$\chi^{\mathfrak{so}_{2N}}_{(10\ldots0)}(\mathbf{m}_{\mathfrak{so}_{2N}}) = \sum_{i=1}^{N}\left[m^i_{\mathfrak{so}_{2N}} + \frac{1}{m^i_{\mathfrak{so}_{2N}}}\right]. \tag{A.16}$$

- For $\mathfrak{g} = \mathfrak{so}_{2N+1}$, define

$$e_1 = \omega_1,\ e_2 = \omega_2 - \omega_1,\ e_3 = \omega_3 - \omega_2,\ \ldots,\ e_{N-1} = \omega_{N-1} - \omega_{N-2},\ e_N = 2\omega_N - \omega_{N-1}, \tag{A.17}$$

and define

$$m^i_{\mathfrak{so}_{2N+1}} = e^{2\pi i\langle\mu_{\mathfrak{so}_{2N+1}},e_i\rangle}, \tag{A.18}$$

so that the **2 N + 1** representation has character

$$\chi^{\mathfrak{so}_{2N+1}}_{(10\ldots0)}(\mathbf{m}_{\mathfrak{so}_{2N+1}}) = 1 + \sum_{i=1}^{N}\left[m^i_{\mathfrak{so}_{2N+1}} + \frac{1}{m^i_{\mathfrak{so}_{2N+1}}}\right]. \tag{A.19}$$

- For $\mathfrak{g} = \mathfrak{sp}_N$, define

$$e_1 = \omega_1,\quad e_2 = \omega_2 - \omega_1,\quad e_3 = \omega_3 - \omega_2,\quad \ldots,\quad e_N = \omega_N - \omega_{N-1}, \tag{A.20}$$

and define

$$m^i_{\mathfrak{sp}_N} = e^{2\pi i\langle\mu_{\mathfrak{sp}_N},e_i\rangle}, \tag{A.21}$$

so that the 2**N** representation has character

$$\chi^{\mathfrak{sp}_N}_{(10\ldots0)}(\mathbf{m}_{\mathfrak{sp}_N}) = \sum_{i=1}^{N}\left[m^i_{\mathfrak{sp}_N} + \frac{1}{m^i_{\mathfrak{sp}_N}}\right]. \tag{A.22}$$

In this paper representations of the abelian algebra $\mathfrak{u}(1)$ also appear. An irreducible representation of U(1) is determined by its charge q, and we choose a normalization such that the vector representation of $\mathfrak{so}(2)$ is equivalent to the sum of two $\mathfrak{u}(1)$ representations with charge respectively $+1$ and -1.

Finally, we remark that in several parts of the text we find it convenient to phrase our discussion in terms of Lie groups G rather than their Lie algebras $\mathfrak{g}$, and we adapt the notation there such that for example by Δ_+^G we intend the set of positive roots of the Lie algebra, $\Delta_+^{\mathfrak{g}}$.

A.2 Affine Lie algebras

In the case of affine Lie algebras, we label the additional simple root by α_0, corresponding to the 0 node in figure 18. Recall that for any simple lie algebra one can construct an affine Lie algebra $\widehat{\mathfrak{g}}$ given by a $\mathfrak{u}(1) \otimes \mathfrak{u}(1)$ extension of the loop algebra $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$:

$$\widehat{\mathfrak{g}} = L(\mathfrak{g}) \oplus \mathbb{C}\widehat{k} \oplus \mathbb{C}L_0. \tag{A.23}$$
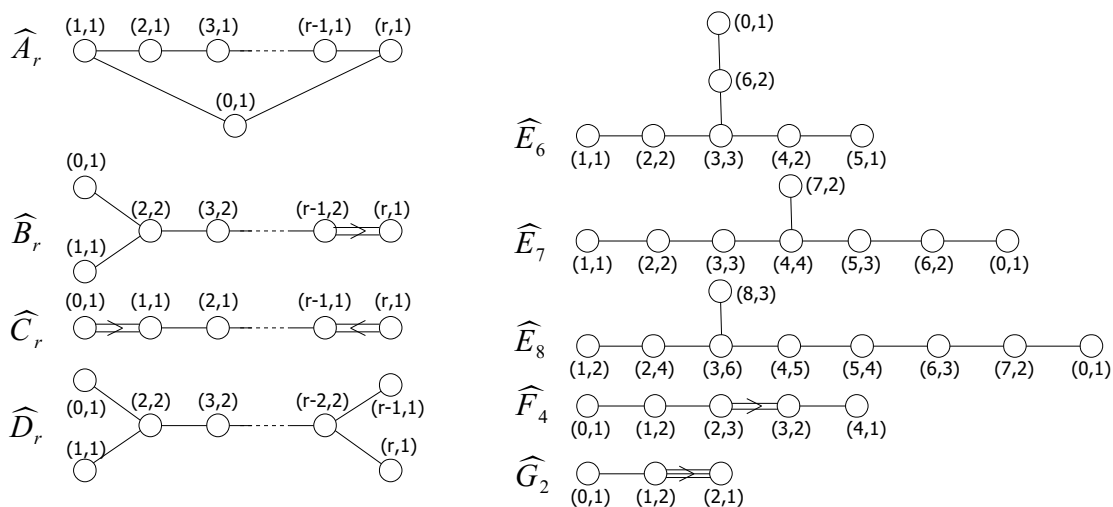
Figure 18. Dynkin labels associated to affine Lie algebras. We label the i-th node by $(i, a_i^\vee)$, where $a_i^\vee$ are the comarks.

Affine weights are labeled by a weight of $\mathfrak{g}$ as well as its eigenvalues under operators $\widehat{k}$ and $-L_0$:

$$\widehat{\lambda} = (\lambda; k; n). \tag{A.24}$$

In this notation, the simple roots of the $\mathfrak{g}$ subalgebra are identified with $(\alpha; 0; 0)$. Furthermore, defining

$$\delta = (0; 0; 1) \tag{A.25}$$

to be the imaginary root, one has

$$\alpha_0 = (-\theta_\mathfrak{g}; 0; 0) + \delta, \tag{A.26}$$

where $\theta_\mathfrak{g}$ is the highest root of $\mathfrak{g}$. The corresponding affine coroots are given by:

$$\alpha_0^\vee = \alpha_0, \qquad \alpha_i^\vee = (\alpha_i^\vee; 0; 0). \tag{A.27}$$

The scalar product between affine weights is defined as:

$$\langle (\lambda; k_\lambda; n_\lambda), (\mu; k_\mu; n_\mu) \rangle = \langle \lambda, \mu \rangle + k_\lambda n_\mu + k_\mu n_\lambda. \tag{A.28}$$

Fundamental weights are chosen to have zero L_0 eigenvalue and are again fixed by the requirement

$$\langle \alpha_i, \widehat{\omega}_j \rangle = \delta_{ij}. \tag{A.29}$$

Explicitly, one has:

$$\widehat{\omega}_0 = (0; 1; 0), \qquad \widehat{\omega}_i = (\omega_i; a_i^\vee; 0), \qquad i = 1, \ldots, r. \tag{A.30}$$

The positive integers $a_i^\vee$ are the comarks associated to the nodes of the Dynkin diagrams, which are given in figure 18.

An affine representation $R_{\widehat{\lambda}}$ is characterized in terms by its highest weight $\widehat{\lambda} = (\lambda; k; n)$, which has an expansion

$$\widehat{\lambda} = \sum_{i=0}^{r} \lambda_i \widehat{\omega}_i + n\, \delta. \tag{A.31}$$

Its level k is given by

$$k = \sum_{i=0}^{r} a_i^{\vee} \lambda_i. \tag{A.32}$$

Another common notation (which does not keep track of the value of n) is

$$\widehat{\lambda} = (\lambda_0 \lambda_1 \dots \lambda_r). \tag{A.33}$$

The level k integrable highest-weight representations are those representations for which the Dynkin labels are all non-negative integers subject to the condition (A.32). Moreover, as the value of n is inessential we specialize $n = 0$.

We define the normalized character of $R_{\widehat{\lambda}}$ (evaluated at a point $(\mu_{\mathfrak{g}}; \tau; 0)$, where $\mu_{\mathfrak{g}} \in \mathfrak{h}_{\mathbb{C}}^*$ and $\tau \in \mathbb{H}$) as

$$\widehat{\chi}_{\lambda}^{\mathfrak{g}}(\mathbf{m}_{\mathfrak{g}}, q) = q^{-\frac{1}{24}\frac{\dim(\mathfrak{g})k}{h_{\mathfrak{g}}^{\vee}+k} + \frac{C_2(R_{\lambda})}{2(h_{\mathfrak{g}}^{\vee}+k)}} \sum_{\widehat{\nu}} \operatorname{mult}(\widehat{\nu}) e^{-2\pi i \tau \langle \widehat{\nu}, \widehat{\omega}_0 \rangle} e^{2\pi i \langle \nu, m \rangle}, \tag{A.34}$$

where $q = e^{2\pi i \tau}$ and the sum runs over the weights $\widehat{\nu}$ in the representation, whose finite part we denote by ν. We choose to label the character in terms of the finite component λ of the highest weight, keeping in mind that for a given level the remaining component λ_0 is determined from equation (A.32).

For an integrable highest-weight representation, the character is given by the Weyl-Kac character formula:

$$\widehat{\chi}_{\lambda}^{\mathfrak{g}}(\mathbf{m}_{\mathfrak{g}}, q) = q^{-\frac{1}{24}\frac{\dim(\mathfrak{g})k}{h_{\mathfrak{g}}^{\vee}+k} + \frac{C_2(R_{\lambda})}{2(h_{\mathfrak{g}}^{\vee}+k)}} \frac{\sum_{w \in W_{\widehat{\mathfrak{g}}}} \epsilon(w) e^{2\pi i \langle w(\widehat{\rho}+\widehat{\lambda}), (\mu_{\mathfrak{g}};\tau;0)\rangle}}{\sum_{w \in W_{\widehat{\mathfrak{g}}}} \epsilon(w) e^{2\pi i \langle w(\widehat{\rho}), (\mu_{\mathfrak{g}};\tau;0)\rangle}}, \tag{A.35}$$

where $W_{\widehat{\mathfrak{g}}}$ is the affine Weyl group of $\widehat{\mathfrak{g}}$, $\epsilon(w) = (-1)^{\ell(w)}$ is the length of the Weyl group element w, and $\widehat{\rho}$ is the affine Weyl vector, defined as the element of the affine weight lattice with Dynkin labels $(111\dots 1)$. Thanks to the Macdonald-Weyl identity, the character can be rewritten in the form

$$\widehat{\chi}_{\lambda}^{\mathfrak{g}}(\mathbf{m}_{\mathfrak{g}}, q) = q^{-\frac{1}{24}\frac{\dim(\mathfrak{g})k}{h_{\mathfrak{g}}^{\vee}+k} + \frac{C_2(R_{\lambda})}{2(h_{\mathfrak{g}}^{\vee}+k)}} \frac{\sum_{w \in W_{\widehat{\mathfrak{g}}}} \epsilon(w) e^{2\pi i \langle w(\widehat{\rho}+\widehat{\lambda})-\widehat{\rho}, (\mu_{\mathfrak{g}};\tau;0)\rangle}}{\Delta_{\mathfrak{g}}(\mathbf{m}_{\mathfrak{g}}, q)}, \tag{A.36}$$

where

$$\Delta_{\mathfrak{g}}(\mathbf{m}_{\mathfrak{g}}, q) = \prod_{j=1}^{\infty} (1-q^j)^{\operatorname{rank}(\mathfrak{g})} \prod_{\alpha \in \Delta_{+}^{\mathfrak{g}}} (1-q^{j-1}m_{\alpha})(1-q^j m_{\alpha}^{-1}). \tag{A.37}$$

Moreover, the sum in the numerator admits an expansion in positive powers of q, with coefficients given in terms of linear combinations of characters of the finite part $\mathfrak{g}$ of the affine Lie algebra. In particular, to leading order in the q expansion,

$$\widehat{\chi}_{\lambda}^{\mathfrak{g}}(\mathbf{m}_{\mathfrak{g}}, q) = q^{-\frac{1}{24}\frac{\dim(\mathfrak{g})k}{h_{\mathfrak{g}}^{\vee}+k} + \frac{C_2(R_{\lambda})}{2(h_{\mathfrak{g}}^{\vee}+k)}} (\chi_{\lambda}^{\mathfrak{g}}(\mathbf{m}_{\mathfrak{g}}) + \mathcal{O}(q)). \tag{A.38}$$

In the limit $\mathbf{m_g} \to \mathbf{1}$, the characters associated to the integrable representations at a given level become elements of a vector-valued, weight zero modular form [233].

B Modular and Jacobi forms

In this appendix we collect some useful information about various modular and Jacobi forms we make use of in the text. First of all, the Dedekind eta function is defined as

$$\eta(\tau) = q^{1/24} \prod_{j=1}^{\infty} (1 - q^j), \tag{B.1}$$

where $\tau \in \mathbb{H}$ and $q = e^{2\pi i \tau}$, is a weight-1/2 modular form with nontrivial multiplier system, which under an S transformation behaves as follows:

$$\eta(-1/\tau) = \sqrt{-i\tau}\, \eta(\tau). \tag{B.2}$$

Another important class of functions are the Jacobi theta functions:

$$\theta_1(\zeta, \tau) = iq^{1/8} z^{-1/2}(1 - z) \prod_{j=1}^{\infty}(1 - q^j)(1 - q^j z)(1 - q^j z^{-1}), \tag{B.3}$$

$$\theta_2(\zeta, \tau) = q^{1/8} z^{1/2}(1 + z) \prod_{j=1}^{\infty}(1 - q^j)(1 + q^j z)(1 + q^j z^{-1}), \tag{B.4}$$

$$\theta_3(\zeta, \tau) = \prod_{j=1}^{\infty}(1 - q^j)(1 + q^{j-1/2} z)(1 + q^{j-1/2} z^{-1}), \tag{B.5}$$

$$\theta_4(\zeta, \tau) = \prod_{j=1}^{\infty}(1 - q^j)(1 - q^{j-1/2} z)(1 - q^{j-1/2} z^{-1}), \tag{B.6}$$

where $z = e^{2\pi i \zeta}$.[38] The function $\theta_1(\zeta, \tau)$ is a Jacobi form of weight $1/2$ and index $1/2$, also with a nontrivial multiplier system:

$$\theta_1(\zeta/\tau, -1/\tau) = -i\sqrt{-i\tau} e^{2\pi i \frac{1}{2} \frac{\zeta^2}{\tau}} \theta_1(\zeta, \tau); \tag{B.7}$$

on the other hand, the functions $\theta_2(\zeta, \tau)$, $\theta_3(\zeta, \tau)$, and $\theta_4(\zeta, \tau)$ transform as components of a vector-valued Jacobi form, also of weight $1/2$ and index $1/2$:

$$\begin{pmatrix} \theta_2(\zeta/\tau, -1/\tau) \\ \theta_3(\zeta/\tau, -1/\tau) \\ \theta_4(\zeta/\tau, -1/\tau) \end{pmatrix} = \sqrt{-i\tau} e^{2\pi i \frac{1}{2} \frac{\zeta^2}{\tau}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \theta_2(\zeta, \tau) \\ \theta_3(\zeta, \tau) \\ \theta_4(\zeta, \tau) \end{pmatrix}. \tag{B.8}$$

Under a shift $\zeta \to \zeta + \tau$ of the elliptic parameter, the functions $\theta_3(\zeta, \tau)$ and $\theta_4(\zeta, \tau)$ are invariant, while $\theta_1(\zeta, \tau)$ and $\theta_2(\zeta, \tau)$ transform back to themselves up to a minus sign. Also,

---

[38]In most of the text, for notational ease we use the exponentiated variables z, q as the arguments of modular and Jacobi forms, rather than ζ and τ.

$\theta_1(-\zeta, \tau) = -\theta_1(\zeta, \tau)$, while for $\ell = 2$, 3, and 4 one has $\theta_\ell(-\zeta, \tau) = \theta_\ell(\zeta, \tau)$. Moreover, under a shift $\zeta \to \frac{\tau}{2} - \zeta$, one has:

$$
\begin{pmatrix} \theta_1(\tau/2 - \zeta, \tau) \\ \theta_2(\tau/2 - \zeta, \tau) \\ \theta_3(\tau/2 - \zeta, \tau) \\ \theta_4(\tau/2 - \zeta, \tau) \end{pmatrix} = e^{-\pi i \tau/4} e^{\pi i z} \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \theta_1(\zeta, \tau) \\ \theta_2(\zeta, \tau) \\ \theta_3(\zeta, \tau) \\ \theta_4(\zeta, \tau) \end{pmatrix}.
\tag{B.9}
$$

We also resort to structure theorems on the spaces of holomorphic modular forms. First of all, a holomorphic $\mathrm{SL}(2, \mathbb{Z})$ modular form $f(\tau)$ of weight m is a function from $\mathbb{H}$ to $\mathbb{C}$ that satisfies

$$
f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^m f(\tau) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})
\tag{B.10}
$$

and admits a Fourier expansion

$$
f(\tau) = \sum_{n \geq 0} a_n q^n.
\tag{B.11}
$$

The ring of holomorphic modular forms $M_* = \bigoplus_{m \geq 0} M_m$, which admits an obvious grading in terms of the weight m, is generated freely by the two Eisenstein series

$$
E_4(\tau) = 1 + 240 \sum_{k=1}^{\infty} \frac{n^3 q^k}{1 - q^k},
\tag{B.12}
$$

$$
E_6(\tau) = 1 - 504 \sum_{k=1}^{\infty} \frac{n^5 q^k}{1 - q^k},
\tag{B.13}
$$

which have respectively weight 4 and 6.

On the other hand, a holomorphic Jacobi form of index $k \geq 0$ and weight $m \in 2\mathbb{Z}$ is defined as a holomorphic function $\varphi(\zeta, \tau)$ from $\mathbb{C} \times \mathbb{H}$ to $\mathbb{C}$ such that

$$
\varphi\left(\frac{\zeta}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^m e^{2\pi i k \frac{c\zeta^2}{c\tau + d}} \varphi(\zeta, \tau) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})
\tag{B.14}
$$

and

$$
\varphi(\zeta + a\tau + b, \tau) = e^{-2\pi i k(a^2 \tau + 2az)} \varphi(\zeta, \tau) \qquad \forall a, b \in \mathbb{Z},
\tag{B.15}
$$

which admits a Fourier expansion

$$
\varphi(\zeta, \tau) = \sum_{n \geq 0} \sum_{r \in \mathbb{Z}} c_{n,r} q^n z^r,
\tag{B.16}
$$

such that

$$
c_{n,r} = 0 \qquad \text{if} \qquad r^2 \geq 4 \, k \, n.
\tag{B.17}
$$

The ring of all such functions, $J_{*,*}^{\text{even}} = \bigoplus_{m,k} J_{m,k}^{\text{even}}$, is a polynomial ring over the ring of holomorphic modular forms which is freely generated by the two functions $\varphi_{-2,1}(\zeta, \tau)$, and

$\varphi_{0,1}(\zeta, \tau)$. These functions are given in terms of the Dedekind eta function and Jacobi theta functions defined above as:

$$\varphi_{-2,1}(\zeta, \tau) = -\frac{\theta_1(\zeta, \tau)^2}{\eta(\tau)^6}, \tag{B.18}$$

$$\varphi_{0,1}(\zeta, \tau) = 4 \sum_{j=2}^{4} \frac{\theta_j(\zeta, \tau)^2}{\theta_j(0, \tau)^2}. \tag{B.19}$$

In the paper we also make use of the larger ring $J_{*,*}^{\text{even},1/2} = \bigoplus_{m,k} J_{m,k}^{\text{even},1/2}$, of weak Jacobi forms of even weight and half/integer index. It has been shown in [222] that this ring is again a polynomial ring over the ring of holomorphic modular forms, which is freely generated by the functions

$$\varphi_{-2,1}(\zeta, \tau), \qquad \varphi_{0,1}(\zeta, \tau), \quad \text{and} \qquad \varphi_{0,3/2}(\zeta, \tau) = \frac{\theta_1(2\zeta, \tau)}{\theta_1(\zeta, \tau)}. \tag{B.20}$$

Note also that the functions $\varphi_{-2,1}(\zeta, \tau)$ and $\varphi_{0,1}(\zeta, \tau)$ are even functions of ζ, while $\varphi_{0,3/2}(\zeta, \tau)$ is odd.

Finally, one can show that the following relations hold between the generators of weak Jacobi forms with elliptic variable 2ζ and those with elliptic variable ζ:

$$\varphi_{-2,1}(2\zeta, \tau) = \frac{1}{216} \varphi_{-2,1}(\zeta, \tau)^4 E_6(\tau) - \frac{1}{144} \varphi_{-2,1}(\zeta, \tau)^3 \varphi_{0,1}(\zeta, \tau) E_4(\tau)$$
$$+ \frac{1}{432} \varphi_{-2,1}(\zeta, \tau) \varphi_{0,1}(\zeta, \tau)^3; \tag{B.21}$$

$$\varphi_{0,1}(2\zeta, \tau) = \frac{1}{192} \varphi_{-2,1}(\zeta, \tau)^4 E_4(\tau)^2 - \frac{1}{108} \varphi_{-2,1}(\zeta, \tau)^3 \varphi_{0,1}(\zeta, \tau) E_6(\tau)$$
$$+ \frac{1}{288} \varphi_{-2,1}(\zeta, \tau)^2 \varphi_{0,1}(\zeta, \tau)^2 E_4(\tau) + \frac{1}{1728} \varphi_{0,1}(\zeta, \tau)^4; \tag{B.22}$$

$$\varphi_{0,\frac{3}{2}}(2\zeta, \tau) = \varphi_{-2,1}(\zeta, \tau)^6 \left(\frac{1}{55296} E_4(\tau)^3 - \frac{1}{46656} E_6(\tau)^2 \right)$$
$$+ \frac{1}{62208} \varphi_{-2,1}(\zeta, \tau)^5 \varphi_{0,1}(\zeta, \tau) E_4(\tau) E_6(\tau) - \frac{5}{165888} \varphi_{-2,1}(\zeta, \tau)^4 \varphi_{0,1}(\zeta, \tau)^2 E_4(\tau)^2$$
$$+ \frac{5}{186624} \varphi_{-2,1}(\zeta, \tau)^3 \varphi_{0,1}(\zeta, \tau)^3 - \frac{5}{497664} \varphi_{-2,1}(\zeta, \tau)^2 \varphi_{0,1}(\zeta, \tau)^4$$
$$+ \frac{1}{1492992} \varphi_{0,1}(\zeta, \tau)^6. \tag{B.23}$$

C A catalogue of BPS string elliptic genera

In this appendix we collect a list of references to other works where the elliptic genera of various BPS string CFTs are presented in a form which may be readily compared to our results. We warn the reader that in what follows we make no attempt to provide a comprehensive review of the literature.

- The CFT for one E-string has been understood since the 1990s to consist of the E_8 level 1 current algebra and a decoupled center of mass hypermultiplet [61, 137, 234].
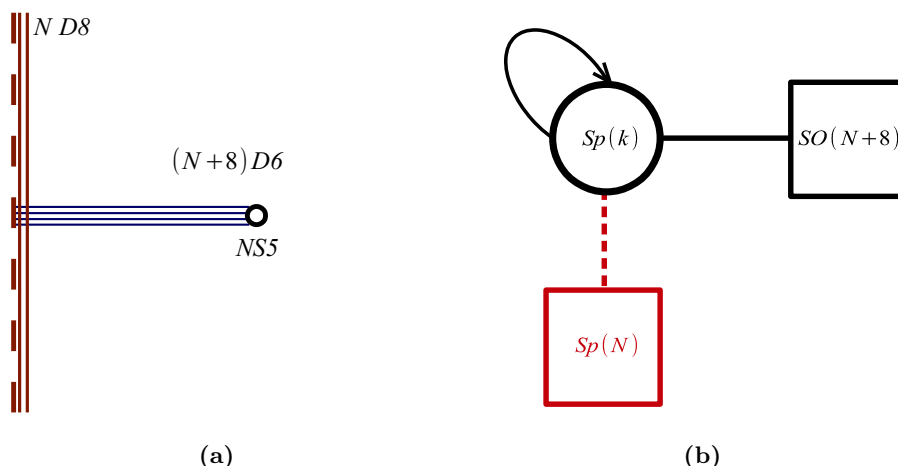
Figure 19. (a): brane construction and (b): 2d $(0,4)$ quiver for k BPS strings of the 6d $(1,0)$ SCFT with $(n, G) = (4, \mathrm{SO}(8 + N))$. The dashed and continuous edge symbolizes respectively bifundamental Fermi and hypermultiplets, while the gauge node supports an adjoint vector multiplet and a hypermultiplet in the antisymmetric representation of $\mathrm{Sp}(k)$, symbolized by the arrow.

In [185] it was argued that the CFT for k strings included a subsector consisting of a E_8 current algebra at level k. A UV gauge theory realization of the E-string CFT for arbitrary numbers of strings, with full dependence on the Omega background parameters, was ultimately given in [68] and was employed by the authors to compute the elliptic genera of the CFTs using localization. A UV quiver gauge theory description for the infinite sequence of theories $\mathbf{h}_1^{\mathrm{Sp}(N)}$ which Higgs to the E-string was given implicitly in [73]; the elliptic genus (for one and two strings) was computed in [79]. On the other hand, the UV gauge theories for the $\mathbf{h}_1^{\mathrm{SU}(N)}$ theories and their multiple-string generalizations, as well as their elliptic genera, were obtained in [76].

- The CFT of M-strings was first studied in [63], where the elliptic genus was computed using the refined topological vertex [235] as well as by an index computation. The CFTs of the infinite sequence of $\mathbf{h}_2^{\mathrm{SU}(N)}$ theories (the A_n orbifolds of the M-string) were studied in [64] and [65] and their elliptic genera computed via the topological vertex [64, 65], by supersymmetric localization in [64] and with an index computation in [65].

- The elliptic genera for the $\mathbf{h}_3^{\mathrm{SU}(3)}$ CFT and its multiple-string generalization have been computed via supersymmetric localization in [82].

- The elliptic genera for the $\mathbf{h}_4^{\mathrm{SO}(8)}$ CFT and its multiple-string generalization have been computed via supersymmetric localization in [70], and so have the elliptic genera for the $\mathbf{h}_4^{\mathrm{SO}(8+2N)}$ CFTs. The ones for the $\mathbf{h}_4^{\mathrm{SO}(9+2N)}$ CFTs have not appeared explicitly in the literature but may be obtained for instance from the brane configuration in figure 19(a), which leads to the 2d $(0,4)$ quiver gauge theory of figure 19(b), from

which one may compute the elliptic genera via localization;[39] this computation is performed in appendix D.

- The elliptic genera for the $\mathbf{h}_5^{F_4}$, $\mathbf{h}_6^{E_6}$, $\mathbf{h}_8^{E_7}$, and $\mathbf{h}_{12}^{E_8}$ CFTs (for one string) have been computed in [83] exploiting modularity of the elliptic genus, in the limit where the fugacities $\mathbf{m}_F, \mathbf{m}_G$ are switched off and for a single string. The elliptic genera for $\mathbf{h}_6^{E_6}$ and $\mathbf{h}_8^{E_7}$ have also been computed by localization, with all fugacities turned on, respectively in [165] and [236].

- UV quiver gauge theories and elliptic genera for the $\mathbf{h}_3^{G_2}$ and $\mathbf{h}_3^{\mathrm{SO}(7)}$ CFTs and their multiple-string generalizations have recently been computed by supersymmetric localization [94]; the elliptic genus for the $\mathbf{h}_3^{G_2}$ string has also been computed in [93] by exploiting modularity.

- For the remaining theories ($G = \mathrm{SU}(6)_*$ for $n = 1$, $G = \mathrm{SO}(8),\ldots,\mathrm{SO}(12)$ for $n = 1, 2, 3$, $G = \mathrm{SO}(13)$ for $n = 2$, $G = F_4$ for $n = 1,\ldots,4$, $G = E_6$ for $n = 1,\ldots,5$, and $G = E_7$ for $n = 1,\ldots,7$) no approach to computing the elliptic genera at all orders in q is has been found at the time of writing of this paper.

D   Elliptic genera of the $n = 4$, $G = \mathrm{SO}(2M + 1)$ BPS strings

In this appendix we compute the elliptic genera of the strings of the $\mathbf{4}_{\mathrm{SO(2M+1)}}$ SCFTs (with $M \geq 4$). The brane configuration corresponding to this class of 6d SCFTs is given in figure 19(a) (with $N = 2M - 7$), while the 2d (0,4) quiver of their strings is shown in figure 19(b). The computation of the elliptic genus is a slight modification of the one for $G = \mathrm{SO}(2M)$ in [70], which we can easily adapt to the present case.

In [70] the elliptic genus for $G = \mathrm{SO}(2M)$ was constructed starting from a brane configuration in which only a $\mathrm{Sp}(M - 4) \times \mathrm{Sp}(M - 4)$ is visible, but the quiver theory displays the full $F = \mathrm{Sp}(2M - 8)$ flavor symmetry. For the case $G = \mathrm{SO}(2M + 1)$ under consideration here, the field content is essentially the same as for $G = \mathrm{SO}(2M)$, the only difference between the two cases being that for k strings the hypermultiplets that couple to G transform in the bifundamental of $\mathrm{Sp}(k) \times \mathrm{SO}(2M + 1)$ rather than $\mathrm{Sp}(k) \times \mathrm{SO}(2M)$, and the Fermi multiplets transform in the bifundamental of $\mathrm{Sp}(k) \times \mathrm{Sp}(2M - 7)$ rather than $\mathrm{Sp}(k) \times \mathrm{Sp}(2M - 8)$. This allows us to straightforwardly write down the elliptic genus

---

[39]The possibility of constructing a Lagrangian model for the $n = 4$, $G = \mathrm{SO}(2N + 9)$ strings by a simple modification of the $G = \mathrm{SO}(2N + 8)$ model was already remarked to us by C. Vafa in 2014.

for k strings of the $G = \mathrm{SO}(2M+1)$ SCFT:

$$\mathbb{E}_{(k)}(\mathbf{m}_{\mathrm{SO}(2M+1)}, \mathbf{m}_{\mathrm{Sp}(2M-7)}, x, v, q) = \tag{D.1}$$

$$= \frac{1}{2^n\, n!} \left[\frac{\eta(q)^2}{\theta_1(vx, q)\theta_1(v/x, q)} \right]^k \int \prod_{i=1}^{k} \left[d\zeta_i\, \eta(q)^2 \frac{\theta_1(v^2, q)}{\eta(q)} \prod_{s=\pm} \frac{\theta_1(z_i^{2s}, q)\theta_1(v^2 z_i^{2s}, q)}{\eta(q)^2} \right]$$

$$\times \prod_{i<j}^{k} \prod_{s_1=\pm,\, s_2=\pm} \left[\frac{\theta_1(z_i^{s_1} z_j^{s_2}, q)\theta_1(v^2 z_i^{s_1} z_j^{s_2}, q)}{\theta_1(v\, x\, z_i^{s_1} z_j^{s_2}, q)\theta_1(v\, x^{-1} z_i^{s_1} z_j^{s_2}, q)} \right]$$

$$\times \prod_{i=1}^{k} \frac{\prod_{j=1}^{2M-7} \prod_{s=\pm} \theta_1(z_i^s\, m_{\mathrm{Sp}(2M-7)}^j, q)/\eta(q)}{\theta_1(v\, z_i, q)\theta_1(v\, z_i^{-1}, q) \prod_{j=1}^{M} \prod_{s_1=\pm} \prod_{s_2=\pm} \theta_1(v\, z_i^{s_1}\, (m_{\mathrm{SO}(2M+1)}^j)^{s_2}, q)/\eta(q)},$$

where $\zeta_i = \frac{\log(z_i)}{2\pi i}$ are holonomies of the 2d $\mathrm{Sp}(k)$ gauge field, the fugacities $m_{\mathrm{Sp}(2M-7)}^i$ and $m_{\mathrm{SO}(2M+1)}^i$ are defined as in appendix A.1, and the integral is computed by the Jeffrey-Kirwan prescription [206, 237]. The only difference from the $G = \mathrm{SO}(2M)$ case, given in equations (3.13)-(3.19) of [70], is in the last line of equation (D.1).

For the case of $k = 1$ string, the integration reduces to summing over residues at

$$z_1 = v\, m_{\mathrm{SO}(2M+1)}^i \qquad \text{and} \qquad z_1 = v/m_{\mathrm{SO}(2M+1)}^i, \tag{D.2}$$

while the residue at $z_1 = v$ vanishes. This leads to the following result:

$$\mathbb{E}_4^{\mathrm{SO}(2M+1)}(\mathbf{m}_{\mathrm{SO}(2M+1)}, \mathbf{m}_{\mathrm{Sp}(2M-7)}, v, q) = \left[\sum_{i=1}^{M} \frac{\theta_1(v^2 (m_{\mathrm{SO}(2M+1)}^i)^2, q)\theta_1(v^4 (m_{\mathrm{SO}(2M+1)}^i)^2, q)}{\theta_1(m_{\mathrm{SO}(2M+1)}^i, q)\theta_1(v^2 m_{\mathrm{SO}(2M+1)}^i, q)} \right.$$

$$\times \frac{\eta(q)^{10} \prod_{j=1}^{2M-7} \prod_{s=\pm} \theta_1(v\, m_{\mathrm{SO}(2M+1)}^i (m_{\mathrm{Sp}(2M-7)}^j)^s, q)}{\prod_{\substack{j=1 \\ j \neq i}}^{M} \prod_{s_1=\pm} \prod_{s_2=\pm} \theta_1(v (v\, m_{\mathrm{SO}(2M+1)}^i)^{s_1} (m_{\mathrm{SO}(2M+1)}^j)^{s_2}, q)}$$

$$\left. + \left(m_{\mathrm{SO}(2M+1)}^i \to 1/m_{\mathrm{SO}(2M+1)}^i \right) \right], \tag{D.3}$$

which indeed agrees with our results obtained exploiting modularity in section 8 for $G = \mathrm{SO}(9), \ldots, \mathrm{SO}(15)$.

E WZW models

In this section we collect a few basic properties of the WZW models, mostly following the discussion in [209]. The spectrum of the (chiral half of the) unitary WZW CFT associated to a Lie group F (which we assume to be simple) includes a set of chiral currents $j_F^a(z)$ of conformal dimension 1, with $a = 1, \ldots, \dim(F)$, which satisfy the OPE

$$j_F^a(z) j_F^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \sum_c i f_{abc} \frac{j_F^c(w)}{z-w}. \tag{E.1}$$

The modes that appear in the Laurent expansion of the currents,

$$j_F^a(z) \sum_{n \in \mathbb{Z}} z^{-n-1} j_F^a[n], \tag{E.2}$$

satisfy the commutation relations of the affine Lie algebra $\widehat{\mathfrak{f}}$ of F at level k:

$$[j_F^a[n], j_F^b[m]] = \sum_c i f_{abc} J_F^c[n+m] + k\, n\, \delta_{ab}\delta_{n+m,0}, \tag{E.3}$$

where f_{abc} are the structure constants of $\mathfrak{f}$.

The energy-momentum tensor of the theory takes the Sugawara form

$$T(z) = \frac{1}{2(h_F^\vee + k)} \sum_a\ :\! j^a j^a\!:\! (z), \tag{E.4}$$

and its OPE with itself is given by:

$$T(z)T(w) \sim \frac{c_F/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{E.5}$$

where the central charge is given by

$$c_F = \frac{\dim(F)k}{h_F^\vee + k}, \tag{E.6}$$

where $\mathfrak{f}$ is the Lie algebra of F.

The states of the CFT can be organized in terms of the extended chiral algebra $\widehat{\mathfrak{f}}$. of the theory. In particular, one can define a set of WZW primary fields $|\lambda\rangle$ such that

$$\begin{aligned} j_F^a[0]\,|\lambda\rangle &= -t_\lambda^a\,|\lambda\rangle, \\ j_F^a[n]\,|\lambda\rangle &= 0 \text{ for } n > 0, \end{aligned} \tag{E.7}$$

where t_λ^a are the matrices in the representation R_λ of $\widehat{\mathfrak{f}}$ corresponding to the roots of the Lie algebra $\mathfrak{f}$, so that WZW primaries organize themselves in terms of representations R_λ. It can furthermore be shown that for a given k only the choices of λ which correspond to integrable highest-weight representations of $\widehat{\mathfrak{f}}$ give rise to states in the CFT, in the sense that any other choice of λ leads to states whose correlators with other states in the CFT all vanish. The conformal dimension of the primary field $|\lambda\rangle$ is

$$h_{\widehat{\lambda}} = \frac{C_2(R_\lambda)}{2(h_F^\vee + k)}, \tag{E.8}$$

and the extended chiral algebra descendants of the WZW primary are states of the form

$$j_F^a[-n_1] j_F^b[-n_2]\ldots|\lambda\rangle, \qquad n_1, n_2, \cdots \geq 0. \tag{E.9}$$

The chiral component of the Hilbert space of the WZW model organizes itself into a number of sectors $H_\lambda^{WZW_F}$, corresponding to the families of states associated to each WZW primary. For each sector one can define a flavored character which counts the states in that sector:

$$\widehat{\chi}_\lambda^F(\mathbf{m}_F, q) = \mathrm{Tr}_{H_\lambda^{WZW_F}} q^{L_0 - \frac{c_F}{24}} \prod_{j=1}^{\mathrm{rank}(F)} e^{2\pi i \langle \mu_F, K_F^j \rangle}, \tag{E.10}$$

where K_F^j are generators of the Cartan subalgebra of $\mathfrak{f}$. These characters coincide with the characters (A.35) of the corresponding integrable highest-weight representations R_λ of $\widehat{\mathfrak{f}}$.

For certain WZW models it is possible to give closed formulas for the characters in terms of Jacobi theta functions. For instance, the $SO(2N)$ WZW model at level 1 has four conformal primaries: the trivial, vector, spinor and conjugate spinor representations, whose highest weights have Dynkin labels $(10\ldots0)$, $(01\ldots0)$, $(0\ldots10)$, and $(0\ldots01)$ respectively; we denote these representations concisely as $\mathbf{1}$, $\mathbf{v}$, $\mathbf{s}$, and $\mathbf{c}$. The corresponding characters are given by:

$$\widehat{\chi}_{\mathbf{1}}^{SO(2N)}(\mathbf{m}_{SO(2N)}, q) = \frac{1}{2}\left(\prod_{i=1}^N \frac{\theta_3(m^i_{SO(2N)}, q)}{\eta(q)} + \prod_{i=1}^N \frac{\theta_4(m^i_{SO(2N)}, q)}{\eta(q)}\right),$$

$$\widehat{\chi}_{\mathbf{v}}^{SO(2N)}(\mathbf{m}_{SO(2N)}, q) = \frac{1}{2}\left(\prod_{i=1}^N \frac{\theta_3(m^i_{SO(2N)}, q)}{\eta(q)} - \prod_{i=1}^N \frac{\theta_4(m^i_{SO(2N)}, q)}{\eta(q)}\right),$$

$$\widehat{\chi}_{\mathbf{s}}^{SO(2N)}(\mathbf{m}_{SO(2N)}, q) = \frac{1}{2}\left(\prod_{i=1}^N \frac{\theta_2(m^i_{SO(2N)}, q)}{\eta(q)} + e^{\pi i n/2}\prod_{i=1}^N \frac{\theta_1(m^i_{SO(2N)}, q)}{\eta(q)}\right),$$

$$\widehat{\chi}_{\mathbf{c}}^{SO(2N)}(\mathbf{m}_{SO(2N)}, q) = \frac{1}{2}\left(\prod_{i=1}^N \frac{\theta_2(m^i_{SO(2N)}, q)}{\eta(q)} - e^{-\pi i n/2}\prod_{i=1}^N \frac{\theta_1(m^i_{SO(2N)}, q)}{\eta(q)}\right), \quad \text{(E.11)}$$

where the fugacities m^i_{SO2N} are given in equation (A.15).

Analogous formulas can also be obtained for $F = SO(2N+1)$. Here we just remark that the characters of the 'SO(1)' WZW model at level 1, which has $c = 1/2$, are given by:

$$\widehat{\chi}_{\mathbf{1}}^{SO(1)}(\mathbf{m}_{SO(1)}, q) = \frac{1}{2}\left(\sqrt{\frac{\theta_3(1, q)}{\eta(q)}} + \sqrt{\frac{\theta_4(1, q)}{\eta(q)}}\right),$$

$$\widehat{\chi}_{\mathbf{v}}^{SO(1)}(\mathbf{m}_{SO(1)}, q) = \frac{1}{2}\left(\sqrt{\frac{\theta_3(1, q)}{\eta(q)}} - \sqrt{\frac{\theta_4(1, q)}{\eta(q)}}\right),$$

$$\widehat{\chi}_{\mathbf{s}}^{SO(1)}(\mathbf{m}_{SO(1)}, q) = \sqrt{\frac{\theta_2(1, q)}{2\eta(q)}}. \quad \text{(E.12)}$$

The unique level 1 character for $F = E_8$ also has a simple closed form expression: it is given by the following sum of $SO(16)$ level 1 characters:

$$\widehat{\chi}_{\mathbf{1}}^{E_8}(\mathbf{m}_{E_8}, q) = \widehat{\chi}_{\mathbf{1}}^{SO(16)}(\mathbf{m}_{SO(16)}, q) + \widehat{\chi}_{\mathbf{s}}^{SO(16)}(\mathbf{m}_{SO(16)}, q) = \frac{\frac{1}{2}\sum_{\ell=1}^4 \prod_{i=1}^8 \theta_\ell(m^i_{SO(16)}, q)}{\eta(q)^8},$$
$$\text{(E.13)}$$

where the regular embedding of $SO(16)$ into E_8 fixes the precise relation between $\mathbf{m}_{E_8}$ and $\mathbf{m}_{SO(16)}$ fugacities.

The two level 1 characters of $SU(2)$ are given in terms of the following theta series:

$$\widehat{\chi}_{\mathbf{1}}^{SU(2)}(\mathbf{m}_{SU(2)}, q) = \frac{\sum_{l\in\mathbb{Z}} q^{l^2}(m^1_{SU(2)})^{2l}}{\eta(q)}; \quad \text{(E.14)}$$

$$\widehat{\chi}_{\mathbf{2}}^{SU(2)}(\mathbf{m}_{SU(2)}, q) = \frac{\sum_{l+1/2\in\mathbb{Z}} q^{l^2}(m^1_{SU(2)})^{2l}}{\eta(q)}. \quad \text{(E.15)}$$

The abelian case $F = \mathrm{U}(1)$, which coincides with the $c = 1$ theory of a boson compactified on a circle or radius R, falls slightly outside of the discussion above. We denote the CFT at radius R as $\mathrm{U}(1)_{R^2}$. When the square of the radius of the circle is a rational number

$$R = \sqrt{\frac{2p'}{p}}, \tag{E.16}$$

the CFT is rational and the spectrum can be organized into a finite number of characters of an extended algebra. The $2\,p\,p' + 1$ primaries under the extended algebra can be labeled by an integer ℓ in the range $-pp' + 1 \leq \ell \leq pp'$, and have conformal dimension

$$h_\ell = \frac{\ell^2}{4pp'} \tag{E.17}$$

and character

$$\widehat{\chi}_\ell^{\mathrm{U}(1)}(m_{\mathrm{U}(1)}, q) = \frac{1}{\eta(\tau)} \sum_{k \in \mathbb{Z}} q^{\frac{(2\,p\,p'k+\ell)^2}{4\,p\,p'}} m_{\mathrm{U}(1)}^{2\,p\,p'k+\ell}. \tag{E.18}$$

At $R^2 = 2$, the $\mathrm{U}(1)_{R^2}$ CFT coincides with the $\mathrm{SU}(2)$ WZW model at level 1.

For the several cases for which no simple closed formula for the characters is available, we have computed the characters to sufficiently high powers in q using the Sage platform.

We conclude this appendix by remarking that for low rank exceptional isomorphisms of Lie algebras lead to equivalences between WZW models:

$$\begin{aligned}
\mathrm{Sp}(1)_k &\simeq \mathrm{SU}(2)_k, \\
\mathrm{SO}(3)_k &\simeq \mathrm{SU}(2)_{2k}, \\
\mathrm{SO}(4)_k &\simeq \mathrm{SU}(2)_k \times \mathrm{SU}(2)_k, \\
\mathrm{SO}(5)_k &\simeq \mathrm{Sp}(2)_k, \\
\mathrm{SO}(6)_k &\simeq \mathrm{SU}(4)_k.
\end{aligned} \tag{E.19}$$

Moreover, the $\mathrm{SO}(2)_k$ WZW model coincides with the $\mathrm{U}(1)_k$ model.

F An explicit example of $\xi_\lambda^{n,G}$ functions

In this appendix we provide some details about the functions $\xi_\lambda^{n,G}(\mathbf{m}_G, v, q)$ that we compute for a nontrivial example: the $\mathbf{h}_3^{F_4}$ CFT. We also show how the form of these functions leads to constraints on the elliptic genus.

The elliptic genus is expected, according to equation (8.29), to take the form

$$\mathbb{E}_3^{F_4}(v, q) = q^{-\frac{1}{3}} v^{-2} \sum_{j,k \geq 0} b_{jk}(q/v^2)^j v^k = \sum_\lambda \widehat{\chi}_\lambda^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)}, q) \xi_\lambda^{3, F_4}(\mathbf{m}_{F_4}, v, q), \tag{F.1}$$

where $\widehat{\chi}_\lambda^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)}, q)$ are the characters corresponding to the ten integrable representations of $F = \mathrm{Sp}(2)$ at level 3:

$$R_{(00)}^{\mathrm{Sp}(2)}, \quad R_{(01)}^{\mathrm{Sp}(2)}, \quad R_{(10)}^{\mathrm{Sp}(2)}, \quad R_{(03)}^{\mathrm{Sp}(2)}, \quad R_{(30)}^{\mathrm{Sp}(2)}, \quad R_{(12)}^{\mathrm{Sp}(2)}, \quad R_{(02)}^{\mathrm{Sp}(2)}, \quad R_{(20)}^{\mathrm{Sp}(2)}, \quad R_{(21)}^{\mathrm{Sp}(2)}, \quad R_{(11)}^{\mathrm{Sp}(2)}. \tag{F.2}$$

We keep the terms on the right hand side of equation (F.1) which may contribute to the b_{jk} coefficients for a suitable range of j and k, which we choose to be $0 \leq j \leq 3$ and $0 \leq k \leq 6$.

First, we compute the $\mathrm{Sp}(2) = \mathrm{SO}(5)$ characters to the desired power in the q expansion. We find:

$$\widehat{\chi}_\lambda^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)}, q) = \frac{1}{\widetilde{\Delta}_{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)}, q)} \widetilde{\chi}_\lambda^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)}, q),\tag{F.3}$$

where

$$\frac{1}{\widetilde{\Delta}_{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)}, q)} = 1 + q\chi_{(20)}^{\mathrm{Sp}(2)} + q^2(\chi_{(00)}^{\mathrm{Sp}(2)} + \chi_{(01)}^{\mathrm{Sp}(2)} + \chi_{(20)}^{\mathrm{Sp}(2)} + \chi_{(02)}^{\mathrm{Sp}(2)} + \chi_{(40)}^{\mathrm{Sp}(2)})\tag{F.4}$$
$$+ q^3(\chi_{(00)}^{\mathrm{Sp}(2)} + \chi_{(01)}^{\mathrm{Sp}(2)} + 4\chi_{(20)}^{\mathrm{Sp}(2)} + \chi_{(02)}^{\mathrm{Sp}(2)} + 2\chi_{(21)}^{\mathrm{Sp}(2)} + \chi_{(40)}^{\mathrm{Sp}(2)} + \chi_{(22)}^{\mathrm{Sp}(2)} + \chi_{(60)}^{\mathrm{Sp}(2)})$$

plus higher order terms, and the rescaled characters $\widetilde{\chi}_\lambda^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)}, q)$ are given to the desired order in the q expansion by:

$$\widetilde{\chi}_{(00)}^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)}, q) = q^{-5/24};\tag{F.5}$$

$$\widetilde{\chi}_{(01)}^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)}, q) = q^{1/8}\chi_{(01)}^{\mathrm{Sp}(2)};\tag{F.6}$$

$$\widetilde{\chi}_{(10)}^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)}, q) = \chi_{(10)}^{\mathrm{Sp}(2)};\tag{F.7}$$

$$\widetilde{\chi}_{(03)}^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)}, q) = q^{31/24}(\chi_{(03)}^{\mathrm{Sp}(2)} - q\,\chi_{(23)}^{\mathrm{Sp}(2)});\tag{F.8}$$

$$\widetilde{\chi}_{(30)}^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)}, q) = q^{2/3}(\chi_{(30)}^{\mathrm{Sp}(2)} - q\,\chi_{(50)}^{\mathrm{Sp}(2)});\tag{F.9}$$

$$\widetilde{\chi}_{(12)}^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)}, q) = q^1(\chi_{(12)}^{\mathrm{Sp}(2)} - q\,\chi_{(32)}^{\mathrm{Sp}(2)});\tag{F.10}$$

$$\widetilde{\chi}_{(02)}^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)}, q) = q^{5/8}\chi_{(02)}^{\mathrm{Sp}(2)};\tag{F.11}$$

$$\widetilde{\chi}_{(20)}^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)}, q) = q^{7/24}(\chi_{(20)}^{\mathrm{Sp}(2)} - q^2\,\chi_{(60)}^{\mathrm{Sp}(2)});\tag{F.12}$$

$$\widetilde{\chi}_{(21)}^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)}, q) = q^{19/24}(\chi_{(21)}^{\mathrm{Sp}(2)} - q\,\chi_{(41)}^{\mathrm{Sp}(2)});\tag{F.13}$$

$$\widetilde{\chi}_{(11)}^{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)}, q) = 0.\tag{F.14}$$

We also need to expand the $\xi_\lambda^{3,F_4}(\mathbf{m}_{F_4}, v, q)$ functions according to (6.18) and keep all terms that may contribute to equation (F.1) for $0 \leq j \leq 3$ and $0 \leq k \leq 6$.

The range of ℓ in the sum of equation (6.18) is $-11 \leq \ell \leq 0$. We define the rescaled functions

$$\widetilde{\xi}_\lambda^{3,F_4}(\mathbf{m}_{F_4}, v, q) = \xi_\lambda^{3,F_4}(\mathbf{m}_{F_4}, v, q) \cdot \widetilde{\Delta}_{F_4}(\mathbf{m}_{F_4}, q) \prod_{j=1}^\infty (1 - q^j)\tag{F.15}$$

in order to remove the terms in the denominator of equation (6.18). Explicitly, we find:

$$\widetilde{\xi}^{3,F_4}_{(00)}(\mathbf{m}_{F_4},v,q) = q^{7/8}\bigg((v^{-2}n^{(00)}_{(0000),-2,0} + v^2 n^{(00)}_{(0000),-10,-1}\chi^{F_4}_{(0000)} + \dots) \tag{F.16}$$

$$+ q(v^{-2}n^{(00)}_{(1000),-4,0}\chi^{F_4}_{(1000)} + n^{(00)}_{(0001),-2,0}\chi^{F_4}_{(0001)} + v^4 n^{(00)}_{(0001),-10,-1}\chi^{F_4}_{(0001)} + \dots)$$

$$+ q^2(v^{-2}n^{(00)}_{(2000),-6,0}\chi^{F_4}_{(2000)} + n^{(00)}_{(0010),-2,0}\chi^{F_4}_{(0010)} + \dots) + \dots \bigg);$$

$$\widetilde{\xi}^{3,F_4}_{(01)}(\mathbf{m}_{F_4},v,q) = q^{-11/24}\bigg(v^4 n^{(01)}_{(0000),-6,-1}\chi^{F_4}_{(0000)} \tag{F.17}$$

$$+ q(v^{-2}n^{(01)}_{(0100),-6,0}\chi^{F_4}_{(0100)} + n^{(00)}_{(0002),-4,0}\chi^{F_4}_{(0002)} + v^4 n^{(01)}_{(1000),0,0}\chi^{F_4}_{(0001)} + \dots) + \dots \bigg);$$

$$\widetilde{\xi}^{3,F_4}_{(10)}(\mathbf{m}_{F_4},v,q) = q^{2/3}\bigg((v^3 n^{(10)}_{(0000),-9,-1} + \dots) + q(v^{-1}n^{(10)}_{(0001),-3,0}\chi^{F_4}_{(0001)} + \dots) \tag{F.18}$$

$$+ q^2(v^{-1}n^{(10)}_{(1001),-5,0}\chi^{F_4}_{(1001)} + v n^{(10)}_{(0010),-3,0}\chi^{F_4}_{(0010)} + \dots) + \dots \bigg);$$

$$\widetilde{\xi}^{3,F_4}_{(03)}(\mathbf{m}_{F_4},v,q) = q^{3/8}\bigg((v^{-2}n^{(03)}_{(0000),-4,0} + \dots) + q(n^{(03)}_{(0001),-4,0}\chi^{F_4}_{(0001)} + \dots) + \dots \bigg); \tag{F.19}$$

$$\widetilde{\xi}^{3,F_4}_{(30)}(\mathbf{m}_{F_4},v,q) = q^1\bigg((v^1 n^{(30)}_{(0000),-1,0} + v^3 n^{(30)}_{(0000),-11,-1} + \dots) \tag{F.20}$$

$$+ q(v^{-1}n^{(30)}_{(0010),-5,0}\chi^{F_4}_{(0010)} + v^3 n^{(30)}_{(0001),-1,0}\chi^{F_4}_{(0001)} + \dots) + \dots \bigg);$$

$$\widetilde{\xi}^{3,F_4}_{(12)}(\mathbf{m}_{F_4},v,q) = q^{2/3}\bigg((v^{-1}n^{(12)}_{(0000),-3,0} + \dots) + q(v n^{(12)}_{(0001),-3,0}\chi^{F_4}_{(0001)} + \dots) + \dots \bigg); \tag{F.21}$$

$$\widetilde{\xi}^{3,F_4}_{(02)}(\mathbf{m}_{F_4},v,q) = q^{25/24}\bigg((v^2 n^{(02)}_{(0000),0,0} + \dots) + q(v^4 n^{(02)}_{(0001),0,0}\chi^{F_4}_{(0001)} + \dots) + \dots \bigg); \tag{F.22}$$

$$\widetilde{\xi}^{3,F_4}_{(20)}(\mathbf{m}_{F_4},v,q) = q^{3/8}\bigg((v^4 n^{(02)}_{(0000),-8,-1} + \dots) + q(v^{-2}n^{(20)}_{(0001),-4,0}\chi^{F_4}_{(0001)} + \dots) \tag{F.23}$$

$$+ q^2(n^{(20)}_{(0010),-4,0}\chi^{F_4}_{(0010)} + v^2 n^{(20)}_{(1000),-2,0}\chi^{F_4}_{(1000)}) + \dots \bigg);$$

$$\widetilde{\xi}^{3,F_4}_{(21)}(\mathbf{m}_{F_4},v,q) = q^{7/8}\bigg((n^{(21)}_{(0000),-2,0} + v^4 n^{(21)}_{(0000),-10,-1} + \dots) \tag{F.24}$$

$$+ q(n^{(21)}_{(1000),-4,0}\chi^{F_4}_{(1000)} + v^2 n^{(21)}_{(0001),-4,0}\chi^{F_4}_{(0001)} + \dots) + \dots \bigg);$$

$$\widetilde{\xi}^{3,F_4}_{(11)}(\mathbf{m}_{F_4},v,q) = q^{13/4}\bigg(\dots \bigg), \tag{F.25}$$

where the ... indicate higher order terms which do not contribute to the elliptic genus for the specified range of j, k. There are overall 33 undetermined integral coefficients $n^\lambda_{\nu,\ell,m}$.

As reported in table 23, after imposing constraint **C1** the modular Ansatz still has 4 unfixed coefficients which we would like to fix by exploiting equation (6.18). In order to impose constraint **C2**, we first rescale the elliptic genus as:

$$\widetilde{\mathbb{E}}^{F_4}_3(v,q) = \widetilde{\Delta}_{\mathrm{Sp}(2)}(\mathbf{m}_{\mathrm{Sp}(2)},q)\widetilde{\Delta}_{F_4}(\mathbf{m}_{F_4},q)\prod_{j=1}^{\infty}(1-q^j) \times \mathbb{E}^{F_4}_3(v,q) \tag{F.26}$$

$k \backslash j$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	*
1	*	0	0	0	*	*	*
2	*	*	*	*	*	*	*
3	*	*	*	*	*	*	*

Table 25. Coefficients $\widetilde{b}_{jk}$ in equation (F.1). The coefficients which are identically zero are indicated by a '0', while the rest are indicated by an asterisk.

and expand it as

$$\widetilde{\mathbb{E}}_3^{F_4}(v,q) = q^{-\frac{1}{3}}v^{-2} \sum_{j,k \geq 0} \widetilde{b}_{jk}(q/v^2)^j v^k. \tag{F.27}$$

Equation (6.18) then implies that

$$\sum_{j,k \geq 0} \widetilde{b}_{jk}(q/v^2)^j v^k = \sum_\lambda \widetilde{\chi}_\lambda^{F_4}(\mathbf{m}_{\mathrm{Sp}(2)}, q)\widetilde{\xi}_\lambda^{3,F_4}(\mathbf{m}_{F_4}, v, q). \tag{F.28}$$

Expanding the right hand side in terms of the functions (F.16)–(F.25), we find that certain coefficients $\widetilde{b}_{jk}$ are necessarily zero; we indicate these in the table 25 by a 0, while we indicate by an asterisk the ones which are not necessarily vanishing.

In the modular Ansatz, the coefficients b_{00}, b_{01}, and b_{10} are already guaranteed to vanish as a consequence of imposing constraint **C1**. On the other hand, the six coefficients $b_{02}, \ldots, b_{05}$, b_{12}, and b_{13} do not automatically vanish. This gives six constraints on the four undetermined coefficients of the Ansatz, which is sufficient to uniquely fix them. After fixing these undetermined coefficients, one finds the following values of $\widetilde{b}_{jk}$ coefficients in the expansion (F.27) of the elliptic genus:

$k \backslash j$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	5
1	1	0	0	0	0	-4	-10
2	52	-104	0	20	14	0	-9
3	1053	-4212	4350	0	-910	-56	364

Having determined the elliptic genus with fugacities turned off, we can compare the numerical values for $\widetilde{b}_{jk}$ with equation (F.28) to determine the values of the 33 $n_{\nu,\ell,m}^\lambda$ coefficients in equations (F.16)–(F.25).

For instance, for $j=3, k=6$ one finds:

$$364 = 260n_{(1000),0,0}^{(01)} + 364n_{(0001),0,0}^{(02)}, \tag{F.29}$$

from which the only solution for which the $n_{\nu,\ell,m}^\lambda$ take reasonably small values is:

$$n_{(1000),0,0}^{(01)} = 0 \text{ and } n_{(0001),0,0}^{(02)} = 1. \tag{F.30}$$

Proceeding similarly for the remaining coefficients, we are easily able to fix all remaining coefficients in equations (F.16)–(F.25). This allows us to fix the functions $\xi_\lambda^{3,F_4}(\mathbf{m}_{F_4}, v, q)$ completely:

$$\xi_{(00)}^{3,F_4}(\mathbf{m}_{F_4}, v, q) = q^{7/8}\left(v^{-2} + q(v^{-2}\chi_{(1000)}^{F_4} + v^4\chi_{(0001)}^{F_4}) + q^2v^{-2}\chi_{(2000)}^{F_4}\right); \tag{F.31}$$

$$\xi_{(01)}^{3,F_4}(\mathbf{m}_{F_4}, v, q) = q^{-11/24}\left(v^4\chi_{(0000)}^{F_4} + q(v^{-2}\chi_{(0100)}^{F_4} + \chi_{(0002)}^{F_4})\right); \tag{F.32}$$

$$\xi_{(10)}^{3,F_4}(\mathbf{m}_{F_4}, v, q) = -q^{2/3}\left(v^3 + q\,v^{-1}\chi_{(0001)}^{F_4} + q^2v^{-1}\chi_{(1001)}^{F_4}\right); \tag{F.33}$$

$$\xi_{(30)}^{3,F_4}(\mathbf{m}_{F_4}, v, q) = q\,v; \tag{F.34}$$

$$\xi_{(02)}^{3,F_4}(\mathbf{m}_{F_4}, v, q) = q^{49/24}v^4\chi_{(0001)}^{F_4} \tag{F.35}$$

$$\xi_{(20)}^{3,F_4}(\mathbf{m}_{F_4}, v, q) = q^{3/8}\left(-v^4 + q^2\chi_{(0010)}^{F_4}\right); \tag{F.36}$$

$$\xi_{(21)}^{3,F_4}(\mathbf{m}_{F_4}, v, q) = q^{7/8}\left(-v^4 - q\,v^2\chi_{(0001)}^{F_4}\right); \tag{F.37}$$

$$\xi_{(03)}^{3,F_4}(\mathbf{m}_{F_4}, v, q) = \xi_{(12)}^{3,F_4}(\mathbf{m}_{F_4}, v, q) = \xi_{(11)}^{3,F_4}(\mathbf{m}_{F_4}, v, q) = 0. \tag{F.38}$$

Note that in this example all coefficients turn out to be either 0, 1, or -1.

Via equation (F.1), this calculation allows us to determine the $\mathbf{m}_{F_4}$ and $\mathbf{m}_{Sp(2)}$ dependence of the elliptic genus up to a reasonable order in its series expansion.

G Tables of elliptic genus coefficients

By exploiting modularity, in section 8 we have obtained explicit expressions for the one-string elliptic genera of 63 out of the 72 known six-dimensional SCFTs with one tensor multiplet and rank(G) up to rank 7, with fugacities $\mathbf{m}_F, \mathbf{m}_G$ turned off. Except for the cases of smallest rank, these expressions tend to be quite lengthy. Rather than including them here, we provide them in the supplementary material. In this appendix, on the other hand, we provide tables of coefficients of the elliptic genera, which display several of the universal features that are outlined in section 6 and can be easily employed to compare our results with other calculations of the elliptic genera.

The tables are to be read as follows: for any theory $\mathbf{h}_n^G$ we expand the elliptic genus as a power series:

$$\mathbb{E}_n^G(v, q) = q^{\frac{1}{6} - \frac{n-2}{2} - \delta_{1,n}}v^{1-n}\sum_{j,k\geq 0} b_{jk}(q/v^2)^j v^k. \tag{G.1}$$

The tables list the coefficients b_{jk} for a suitable range of j and k.

Of particular relevance are the following sequences of coefficients:

- The $j = 0$ row gives the v-expansion of the one-instanton piece of the 5d Nekrasov partition function for the theory obtained from the 6d SCFT in the zero radius limit of the 6d circle (see section 7); note that for certain theories the $j = 0$ coefficients are

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	3	-40	194	472	-1021	1536	-2048	2560	-3072
2	5	-80	582	-2480	4998	9808	-23994	36784	-49147
3	7	-120	970	-4960	18057	-46952	61836	108696	-293239
4	9	-160	1358	-7440	30095	-96976	255442	-515968	532246
5	11	-200	1746	-9920	42133	-145464	429150	-1102944	2427308
6	13	-240	2134	-12400	54171	-193952	600810	-1658256	4124353
7	15	-280	2522	-14880	66209	-242440	772470	-2211008	5778395
8	17	-320	2910	-17360	78247	-290928	944130	-2763760	7429365

Table 26. Table of b_{jk} coefficients for $\mathbf{h}_1^{\mathrm{SU}(2)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	8	-72	152	440	520	-4608	13312	-28160	50688
2	27	-360	2008	-5400	2870	17128	6104	-129384	401435
3	64	-1008	7272	-31400	86464	-134568	13424	330328	-20032
4	125	-2160	17648	-91000	331506	-890200	1716080	-1956032	-224024
5	216	-3960	34840	-197640	818720	-2644712	6863760	-14236480	22201384
6	343	-6552	60552	-364760	1628830	-5797656	17159008	-43080128	91730079
7	512	-10080	96488	-605800	2842560	-10748584	34317680	-95072832	231514304
8	729	-14688	144352	-934200	4540634	-17897048	60055632	-176828608	464904985

Table 27. Table of b_{jk} coefficients for $\mathbf{h}_1^{\mathrm{SU}(3)}$.

only nonzero for large values of k, so all the coefficients displayed in the tables below vanish. Nevertheless, in appendix H we provide expressions for the $j = 0$ coefficients which are valid to all orders in v.

- The $k = 0$ column gives the one-instanton piece of the Nekrasov partition function of the 5d pure G $\mathcal{N} = 1$ SYM theory, as in equation (8.31), while the $k = 1$ column captures information about the bifundamental matter of the 6d SCFT as in equation (8.32).

- For the majority of theories, the $j = n - 1 + \delta_{1,n}$, $k = 2$ coefficient is equal to $\dim(G) + \dim(F) + 1$ as in equation (8.38). The exceptions are the theories $\mathbf{h}_1^{\mathrm{Sp}(N)}$ for $N \geq 2$ and $\mathbf{h}_1^{\mathrm{SO}(12)_b}$ (see the discussion in section 8.3).

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	15	-108	162	276	783	128	-16896	78720	-235264
2	84	-992	4710	-10080	3880	11424	29286	-2016	-638892
3	300	-4440	29568	-114500	270420	-340548	48128	246204	565332
4	825	-13920	110160	-542016	1838865	-4440512	7371600	-7028640	668338
5	1911	-35140	309210	-1740060	7041927	-21715464	52206882	-96895180	130916670
6	3920	-76608	724542	-4447264	20024268	-70656000	202708158	-480625344	939588888
7	7344	-150192	1496880	-9766260	47260200	-181837516	580179072	-1573241940	3667470216
8	12825	-271680	2817648	-19236480	98071385	-400941408	1374258288	-4061540576	10524196914

Table 28. Table of b_{jk} coefficients for $\mathbf{h}_1^{\mathrm{SU}(4)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	24	-150	194	202	408	1824	-3328	-55840	394368
2	200	-2170	9296	-17670	7308	12538	-1968	117126	-120632
3	1000	-14130	88900	-320572	693592	-798690	158856	599838	-1024680
4	3675	-60830	469520	-2232060	7219992	-16360848	25145904	-22328790	2401696
5	10976	-202020	1774150	-9903410	39366600	-117671424	270049808	-471489438	594085944
6	28224	-560700	5385184	-33433218	151129508	-529617726	1489166800	-3410975826	6353004528
7	64800	-1363740	13990536	-93720760	462937608	-1801792122	5749054872	-15391101594	34958597160
8	136125	-2997060	32344800	-229438392	1209644592	-5074442320	17667293616	-52433918862	134819632032

Table 29. Table of b_{jk} coefficients for $\mathbf{h}_1^{\mathrm{SU}(5)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	35	-198	232	214	-21	1876	1680	-14196	-183464
2	405	-4128	16464	-29024	11706	23104	-36848	86720	260373
3	2695	-36750	221724	-760940	1557101	-1701634	329808	1476798	-2348563
4	12740	-207648	1571724	-7283392	22792635	-49588416	72864988	-62207872	6191064
5	47628	-876708	7680764	-42580510	167016850	-488713026	1088867280	-1833431804	2223625278
6	149940	-3015936	29288196	-183247456	830327589	-2895984224	8037320060	-18018269536	32605072278
7	413820	-8906436	93392628	-637920024	3198608868	-12557442856	40106780532	-106586786534	238358008812
8	1029105	-23364000	260031948	-1898849760	10266844476	-43920568384	154842653924	-461713031136	1183204470747

Table 30. Table of b_{jk} coefficients for $\mathbf{h}_1^{\mathrm{SU}(6)}$.

$k \backslash j$	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	48	-252	274	252	-228	952	4536	-5400
2	735	-7140	27012	-45108	16998	39708	-61212	29708
3	6272	-83160	485898	-1608180	3164712	-3327324	554116	3147540
4	37044	-596232	4443488	-20194020	61696002	-130515228	186131376	-154246344
5	169344	-3116400	27239436	-150165288	583139508	-1680840084	3671120772	-6038635992
6	640332	-13006224	127370376	-801499944	3638271504	-12649513968	34805722028	-76951883316
7	2090880	-45821160	488830188	-3389732136	17195257296	-67982653816	217486465392	-575678320620
8	6073353	-141381240	1612755936	-12049911720	66458923968	-288778622328	1028844925920	-3082991552840

Table 31. Table of b_{jk} coefficients for $\mathbf{h}_1^{\mathrm{SU}(7)}$.

$k \backslash j$	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	63	-312	320	296	-317	312	3384	7128
2	1232	-11520	41844	-67072	23760	60160	-78332	-13312
3	13104	-169848	967648	-3115184	5955360	-6081608	894144	5790064
4	94500	-1505280	11078496	-49586432	148778157	-308381440	430533600	-349049088
5	518364	-9535680	83176620	-456460824	1759089020	-5014874080	10800110640	-17482263048
6	2317392	-47443968	467719164	-2956466688	13440933264	-46636945152	127582632164	-279460228864
7	8833968	-196499160	2125800864	-14921076984	76406098800	-303879888192	974147978784	-2573467183328
8	29630601	-704301312	8198155680	-62406567168	349797709668	-1539653616384	5535251144160	-16668756397056

Table 32. Table of b_{jk} coefficients for $\mathbf{h}_1^{\mathrm{SU}(8)}$.

$k \backslash j$	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	35	-200	260	20	875	-1260	10640	-36148
2	405	-4140	16660	-30580	19826	-8660	63700	-184140
3	2695	-36792	222480	-767560	1595125	-1864780	894008	-176780
4	12740	-207760	1573880	-7303652	22917795	-50166500	75016480	-68986440
5	47628	-876960	7685860	-42630980	167346410	-490325716	1095236580	-1854745160
6	149940	-3016440	29298780	-183356600	831071605	-2899794380	8053101980	-18073750672
7	413820	-8907360	93412620	-638132880	3200110228	-12565415380	40141093720	-106712365860
8	1029105	-23365584	260067060	-1899233400	10269626220	-43935780840	154910202524	-461968529260

Table 33. Table of b_{jk} coefficients for $\mathbf{h}_1^{\mathrm{SU}(6)_*}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	10	-96	292	-96	2058	-8192	20480	-40960	71680
2	35	-480	2856	-9248	15750	-17440	84776	-287200	688163
3	84	-1344	9964	-45280	139132	-294112	420936	-580832	1810300
4	165	-2880	23888	-126112	477778	-1378080	3088720	-5331200	7208280
5	286	-5280	46900	-269664	1139320	-3793888	10352600	-23514880	44314066
6	455	-8736	81272	-493856	2231390	-8074272	24479904	-63786240	144368959
7	680	-13440	129276	-816608	3861620	-14751968	47758440	-134923008	338101988
8	969	-19584	193184	-1255840	6137642	-24359712	82476016	-245743872	656643625

Table 34. Table of b_{jk} coefficients for $\mathbf{h}_1^{\mathrm{Sp}(2)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	21	-168	414	-168	21	8192	-49152	172032	-458752
2	126	-1568	8310	-23520	35380	-23520	-40842	522720	-2457474
3	462	-7056	49476	-209720	590850	-1132152	1433624	-960120	-1866750
4	1287	-22176	180900	-931392	3393705	-9244928	19233468	-30427680	35281510
5	3003	-56056	503250	-2910600	12242019	-39948720	104819022	-224759080	394299510
6	6188	-122304	1173354	-7335328	33832986	-123332160	370343706	-938417088	2030226408
7	11628	-239904	2416680	-15975960	78591240	-308819560	1013000076	-2851819800	7005006096
8	20349	-434112	4539816	-31301760	161497175	-670004832	2339316720	-7080221792	18930661794

Table 35. Table of b_{jk} coefficients for $\mathbf{h}_1^{\mathrm{Sp}(3)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	36	-256	560	-256	36	0	32768	-262144	1179648
2	330	-3840	18880	-49408	70152	-49408	18880	-265984	3146058
3	1716	-25344	170976	-692736	1854572	-3379968	4158912	-3379968	3034220
4	6435	-109824	884928	-4477440	15915024	-41951232	83982912	-128196864	147998752
5	19448	-366080	3310736	-19219200	80612532	-260032512	667972992	-1390912768	2361125988
6	50388	-1018368	9992320	-63734528	298296856	-1094429952	3275080128	-8186561280	17317499520
7	116280	-2480640	25867200	-176743424	894572796	-3590016768	11914381632	-33590839040	81815642556
8	245157	-5457408	59659392	-429646848	2306412576	-9889224704	35372882496	-108612850944	291735314784

Table 36. Table of b_{jk} coefficients for $\mathbf{h}_1^{\mathrm{Sp}(4)}$.

$k\backslash j$	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	55	-360	730	-360	55	0	0	131072
2	715	-7920	36950	-91920	126426	-91920	36950	-7920
3	5005	-72072	472670	-1855920	4807235	-8507400	10318308	-8507400
4	24310	-411840	3287570	-16425552	57414335	-148243920	290209230	-434652960
5	92378	-1750320	15915900	-92646840	388001900	-1243035816	3154283110	-6461977200
6	293930	-6046560	60363940	-390904800	1850251975	-6828272880	20425933880	-50725384512
7	817190	-17907120	191829700	-1344351840	6954608778	-28379585520	95172862070	-269335998360
8	2042975	-47070144	532878940	-3969489120	21972846970	-96681303840	352734033924	-1097369959920

Table 37. Table of b_{jk} coefficients for $\mathbf{h}_1^{\mathrm{Sp}(5)}$.

$k\backslash j$	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	78	-480	924	-480	78	0	0	0
2	1365	-14560	65352	-156960	211682	-156960	65352	-14560
3	12376	-174720	1121484	-4303520	10894872	-18917856	22752072	-18917856
4	75582	-1272960	10087584	-49926240	172447626	-439115872	847529088	-1255045824
5	352716	-6718400	61351368	-357913920	1497819674	-4778545824	12035999976	-24419800768
6	1352078	-28217280	285599184	-1871746240	8940106812	-33168338112	99330679448	-245966456736
7	4457400	-99845760	1093039752	-7816186560	41146639248	-170223959360	576190947696	-1638255055008
8	13037895	-309046400	3599439648	-27550297920	156304791772	-702349866816	2605397698176	-8201957116288

Table 38. Table of b_{jk} coefficients for $\mathbf{h}_1^{\mathrm{Sp}(6)}$.

$k\backslash j$	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	105	-616	1142	-616	105	0	0	0
2	2380	-24640	107366	-251328	334520	-251328	107366	-24640
3	27132	-376992	2379272	-8972040	22337812	-38280088	45787152	-38280088
4	203490	-3410880	26872104	-132030976	452040449	-1139797120	2178876280	-3202899392
5	1144066	-21886480	200550700	-1172148912	4904012610	-15604903776	39120480630	-78892132088
6	5200300	-109830336	1124196332	-7439811456	35803341980	-133468775616	400437497964	-990651116224
7	20058300	-457626400	5100038216	-37076890928	198003321740	-828627305968	2828123668328	-8080816527696
8	67863915	-1647455040	19644950440	-153762470400	890244245282	-4070880504768	15316806055432	-48733092166528

Table 39. Table of b_{jk} coefficients for $\mathbf{h}_1^{\mathrm{Sp}(7)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	14	-98	120	350	910	-1216	-11520	63168	-198016
2	77	-896	4110	-7840	-1370	17248	40974	-89472	-335795
3	273	-4004	26250	-98630	217015	-208614	-182788	447258	970679
4	748	-12544	98370	-477120	1579725	-3639776	5394622	-3128800	-5100524
5	1729	-31654	276870	-1544970	6175245	-18676588	43438192	-75584530	87019530
6	3542	-68992	649740	-3965920	17722815	-61876640	174768900	-404326720	757815968
7	6630	-135240	1343580	-8731310	42038360	-160673842	508042238	-1360145630	3111214236
8	11571	-244608	2530620	-17225600	87500420	-356097504	1213415216	-3558427040	9123119434

Table 40. Table of b_{jk} coefficients for $\mathbf{h}_1^{G_2}$.

$k\backslash j$	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	52	-208	89	120	308	792	3072	1704
2	1053	-8424	23204	-18512	-11350	-10704	6820	54936
3	12376	-143208	689364	-1685272	1866152	-39608	-765216	-1659000
4	100776	-1478048	9711260	-36937160	86395300	-116741552	59692724	41427000
5	627912	-10883808	87746010	-432815080	1437418628	-3300129352	5065163264	-4423934120
6	3187041	-62581896	584202348	-3435586440	14178101676	-43106676088	97983239544	-163533566872
7	13748020	-297457160	3102350628	-20743270824	99578268180	-363361928088	1037554927552	-2340760733472
8	51949755	-1214825040	13822964948	-101985799864	547895844084	-2278100342760	7591910100408	-20670151476576

Table 41. Table of b_{jk} coefficients for $\mathbf{h}_1^{F_4}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	21	-124	110	260	789	1152	-8704	-14464	223488
2	168	-1764	7024	-10588	-4116	10916	50032	46364	-503640
3	825	-11412	69276	-233688	434927	-283380	-389832	226572	1676015
4	3003	-48972	369840	-1700184	5202264	-10629024	12838416	-3438780	-12574496
5	8918	-162344	1404746	-7682004	29644377	-84687744	180329328	-273616588	246274245
6	22848	-450072	4276384	-26181428	116169412	-396796620	1075184112	-2327385492	3947607216
7	52326	-1093848	11130168	-73807664	359942835	-1377924132	4300502952	-11166392996	24266871171
8	109725	-2402664	25763232	-181348656	947203248	-3927734240	13475440272	-39235439724	98330686560

Table 42. Table of b_{jk} coefficients for $\mathbf{h}_1^{\mathrm{SO}(7)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	28	-144	92	216	588	1848	-2464	-29176	56976
2	300	-2880	10094	-11904	-8104	4416	41454	144384	-260564
3	1925	-25200	142134	-431280	670691	-218856	-657052	-299112	1736931
4	8918	-141120	1023484	-4444992	12511959	-22394112	20688612	3535296	-21767636
5	32928	-592704	5039944	-26818680	99212344	-265762776	512485572	-658521264	396776184
6	102816	-2032128	19302276	-117375552	512196412	-1696845696	4376235864	-8782492416	13252883352
7	282150	-5987520	61719588	-412790112	2015818938	-7649920824	23361506350	-58392693816	119647476102
8	698775	-15681600	172164168	-1236951360	6558107850	-27385792896	93662349144	-268572416448	653514465071

Table 43. Table of b_{jk} coefficients for $\mathbf{h}_1^{\mathrm{SO}(8)}$.

$k\backslash j$	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	36	-168	94	168	468	1456	2016	-16560
2	495	-4440	14244	-14712	-9066	-1848	28452	134504
3	4004	-50160	267438	-752280	1044204	-232104	-770540	-778344
4	22932	-353808	2482496	-10310520	27272082	-44523144	34688784	11092752
5	102816	-1831200	15328908	-79686672	284855316	-726010296	1301494452	-1488757776
6	383724	-7593696	71975304	-434396496	1866929904	-6028515168	14949532076	-28319747544
7	1241460	-26634960	276957516	-1861197264	9079256232	-34150691824	102358483944	-248103044904
8	3581721	-81921840	915368256	-6673637520	35737932576	-149790689136	510200930592	-1443615699152

Table 44. Table of b_{jk} coefficients for $\mathbf{h}_1^{\mathrm{SO}(9)}$.

k\j	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	45	-196	110	120	391	1152	2124	-792
2	770	-6560	19842	-19680	-7110	-7872	17290	115648
3	7644	-92484	472780	-1264044	1660818	-441172	-694716	-1197432
4	52920	-799680	5465616	-21949344	55603779	-85982720	63374700	13067584
5	282744	-4989600	41220300	-210290508	732226820	-1800471036	3076437870	-3303090628
6	1241460	-24577344	232368444	-1392936192	5911557018	-18710825952	45065511608	-81985256736
7	4671810	-101080980	1057754016	-7129660956	34722608970	-129616856208	382781764860	-906421301260
8	15520791	-360456096	4083433068	-30103638048	162385463460	-682190154720	2315054813436	-6481392386112

Table 45. Table of b_{jk} coefficients for $\mathbf{h}_1^{\mathrm{SO}(10)}$.

k\j	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	66	-264	175	0	459	160	3030	1704
2	1638	-13024	37125	-39480	10059	-35688	49689	-37040
3	23100	-266112	1290081	-3286304	4346958	-2380224	1097942	-3954696
4	222156	-3256704	21474365	-82748424	200841537	-303068832	251722053	-82582152
5	1613898	-28087488	227685777	-1132610688	3818763138	-9040729320	14897234001	-15981701160
6	9447438	-187098912	1762140237	-10462560680	43669190334	-134850391440	314361285071	-550479506568
7	46562373	-1020323304	10776662897	-72955273248	354577272357	-1310825156872	3800990387739	-8759070969360
8	199377750	-4740241792	54822956331	-410877059736	2240474903238	-9448968696288	31934354072709	-88265081159848

Table 46. Table of b_{jk} coefficients for $\mathbf{h}_1^{\mathrm{SO}(12)_a}$.

k\j	0	1	2	3	4	5	6	7	8
0	1	0	-7	16	-24	32	-40	48	-56
1	3	-16	25	0	-71	176	-285	384	-480
2	5	-32	96	-176	189	0	-467	1136	-1872
3	7	-48	160	-384	756	-1136	1069	0	-2323
4	9	-64	224	-576	1300	-2592	4323	-5648	4859
5	11	-80	288	-768	1820	-3936	7640	-13184	19674
6	13	-96	352	-960	2340	-5248	10752	-20304	35430
7	15	-112	416	-1152	2860	-6560	13824	-27136	50218
8	17	-128	480	-1344	3380	-7872	16896	-33920	64638

Table 47. Table of b_{jk} coefficients for $\mathbf{h}_2^{\mathrm{SU}(2)}$.

k\j	0	1	2	3	4	5	6	7	8
0	1	0	-1	-20	90	-232	470	-828	1330
1	8	-36	44	20	-44	-340	1664	-4448	9160
2	27	-180	532	-864	629	376	-719	-3400	17416
3	64	-504	1908	-4720	8344	-10024	5732	4264	-7404
4	125	-1080	4616	-13496	30681	-56120	80498	-82280	39853
5	216	-1980	9100	-29160	74816	-162148	299772	-466628	587968
6	343	-3276	15804	-53680	148015	-351756	735192	-1357888	2204377
7	512	-5040	25172	-89024	257544	-648824	1458180	-2955744	5429088
8	729	-7344	37648	-137160	410669	-1077232	2540628	-5461016	10790865

Table 48. Table of b_{jk} coefficients for $\mathbf{h}_2^{\mathrm{SU}(3)}$.

$k \backslash j$	0	1	2	3	4	5	6	7	8
0	1	0	-1	0	-70	448	-1610	4352	-9870
1	15	-64	79	0	-9	64	-2297	13312	-46520
2	84	-576	1725	-2816	2343	-448	-61	2816	-42223
3	300	-2560	10381	-26624	47165	-57216	42406	-13312	-298
4	825	-8000	38091	-120320	283087	-517504	733480	-773888	551586
5	1911	-20160	106139	-378880	1038969	-2316544	4288284	-6584320	8226740
6	3920	-43904	247689	-958720	2896091	-7268480	15604884	-28983296	46563419
7	7344	-86016	510425	-2093056	6760273	-18359936	43330056	-90327040	167511165
8	12825	-155520	959191	-4107264	13933995	-40043136	100932416	-227385344	462669790

Table 49. Table of b_{jk} coefficients for $\mathbf{h}_2^{\mathrm{SU}(4)}$.

$k \backslash j$	0	1	2	3	4	5	6	7	8
0	1	0	-1	0	0	-252	2100	-9648	32550
1	24	-100	124	0	-124	352	-24	-12648	91200
2	200	-1400	4251	-7000	5849	0	-7949	19648	-4251
3	1000	-9000	38176	-100800	180876	-220000	156700	9648	-247900
4	3675	-38500	196625	-655200	1592224	-2951200	4191075	-4382800	2833250
5	10976	-127400	733524	-2818200	8140100	-18716904	35134624	-54080096	67220100
6	28224	-352800	2209949	-9354800	30258051	-79590896	175861625	-331388496	534403300
7	64800	-856800	5714176	-25956000	90975724	-262542800	646529076	-1385477104	2608959500
8	136125	-1881000	13168251	-63117600	234934576	-725674500	1931317325	-4526691104	9463184100

Table 50. Table of b_{jk} coefficients for $\mathbf{h}_2^{\mathrm{SU}(5)}$.

$k \backslash j$	0	1	2	3	4	5	6	7	8
0	1	0	-1	0	0	0	-924	9504	-53130
1	35	-144	179	0	-179	144	889	0	-66198
2	405	-2880	8848	-14688	12329	0	-12329	5184	57350
3	2695	-25200	110400	-298240	542332	-663824	473955	0	-420825
4	12740	-141120	757625	-2626400	6558060	-12349568	17672137	-18531808	12124497
5	47628	-592704	3639181	-14757120	44403755	-104978608	200384283	-311062016	387847460
6	149940	-2032128	13734112	-62102208	211781479	-579406720	1315368329	-2521066912	4103016876
7	413820	-5987520	43500960	-213190656	795881912	-2413262544	6166189029	-13563009536	25980535867
8	1029105	-15681600	120559275	-628689600	2513967480	-8230614528	22928202815	-55635665568	119281776315

Table 51. Table of b_{jk} coefficients for $\mathbf{h}_2^{\mathrm{SU}(6)}$.

$k \backslash j$	0	1	2	3	4	5	6	7
0	1	0	-1	0	0	0	0	-3432
1	48	-196	244	0	-244	196	-48	3432
2	735	-5292	16416	-27440	23127	0	-23127	27440
3	6272	-60368	271020	-745584	1371952	-1690696	1211084	0
4	37044	-428064	2386349	-8529136	21779225	-41624520	60092816	-63314272
5	169344	-2222640	14329168	-60571056	188335812	-456080240	884841460	-1387541232
6	640332	-9236304	66157056	-314791680	1119685084	-3165617364	7362858187	-14351595104
7	2090880	-32443488	251776944	-1308916224	5137991040	-16228172800	42806145984	-96426865392
8	6073353	-99891792	825805035	-4599601776	19474888644	-66881396736	193668842592	-484485941600

Table 52. Table of b_{jk} coefficients for $\mathbf{h}_2^{\mathrm{SU}(7)}$.

$k\backslash j$	0	1	2	3	4	5	6	7
0	1	0	-1	0	0	0	0	0
1	63	-256	319	0	-319	256	-63	0
2	1232	-8960	28017	-47104	39839	0	-39839	47104
3	13104	-129024	590513	-1648640	3064689	-3800576	2731504	0
4	94500	-1128960	6482476	-23740416	61757311	-119587840	174133728	-184414208
5	518364	-7096320	47542572	-207728640	663462436	-1639975680	3229234547	-5115265024
6	2317392	-35126784	263461716	-1305722880	4806514188	-13969951744	33191990195	-65722116096
7	8833968	-144967680	1185759828	-6463328256	26431333236	-86379245568	234193247568	-538964647936
8	29630601	-518259456	4541329287	-26674053120	118344386556	-422961896448	1266010130448	-3253134753792

Table 53. Table of b_{jk} coefficients for $\mathbf{h}_2^{\mathrm{SU}(8)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	1	0	0	-8	-35	336	-1317	3688	-8519
1	14	-56	51	48	7	-344	-630	8512	-35036
2	77	-512	1443	-2016	753	1552	63	-7024	-5171
3	273	-2288	9030	-22128	35799	-34272	5900	26176	420
4	748	-7168	33606	-103872	236250	-406976	511339	-393624	17393
5	1729	-18088	94281	-332304	895446	-1945024	3452071	-4921600	5296513
6	3542	-39424	220857	-847776	2533905	-6267968	13179345	-23719936	36229900
7	6630	-77280	456204	-1859760	5964231	-16050592	37418381	-76689152	138787518
8	11571	-139776	858636	-3660672	12356022	-35286720	88234055	-196702208	394629950

Table 54. Table of b_{jk} coefficients for $\mathbf{h}_2^{G_2}$.

$k\backslash j$	0	1	2	3	4	5	6	7
0	1	0	0	0	0	-14	-70	-216
1	52	-156	74	56	90	126	448	-1348
2	1053	-6318	14117	-11544	-2065	616	6660	12996
3	12376	-107406	406432	-838148	908054	-278910	-260850	-162008
4	100776	-1108536	5658107	-17462484	35136029	-45104518	30486813	513540
5	627912	-8162856	50812932	-200047368	550927748	-1099670598	1573235888	-1487730508
6	3187041	-46936422	337074079	-1568026824	5274548964	-13507533022	26869094301	-41293163340
7	13748020	-223092870	1785727008	-9392648100	36394268948	-110078146326	267922966188	-531517036124
8	51949755	-911118780	7943335361	-45932548116	197938648545	-675704802882	1892326957877	-4433982679452

Table 55. Table of b_{jk} coefficients for $\mathbf{h}_2^{F_4}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	1	0	0	-2	-27	-20	1027	-5958	22057
1	21	-78	61	54	49	-118	-1263	368	34148
2	168	-1106	2994	-3770	849	2376	1799	-3518	-29724
3	825	-7146	28646	-69034	104604	-85566	-451	54720	33333
4	3003	-30646	151258	-480798	1088896	-1798160	2061990	-1304906	-193235
5	8918	-101556	571431	-2132702	5925554	-12891132	22247190	-29845102	28859157
6	22848	-281484	1734236	-7198126	22716939	-57851398	122249718	-216042714	316189540
7	52326	-684012	4505004	-20174310	69522117	-196427868	470616039	-972117806	1740019295
8	109725	-1502292	10414332	-49382314	181545392	-552475098	1443393448	-3304130206	6697776060

Table 56. Table of b_{jk} coefficients for $\mathbf{h}_2^{\mathrm{SO}(7)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	-15	-64	287	1728	-19215
1	28	-96	59	64	75	96	-1220	-2816	16768
2	300	-1920	4865	-5184	-196	2816	5190	3456	-45241
3	1925	-16800	66525	-152960	207284	-123744	-55740	58496	173940
4	8918	-94080	473935	-1505344	3304770	-5059840	4944037	-1892160	-1639248
5	32928	-395136	2321410	-8904000	24887830	-53072672	86903151	-105405184	83317010
6	102816	-1354752	8863302	-38555776	125315359	-322157568	672406592	-1144825280	1563120876
7	282150	-3991680	28285110	-134717184	486175437	-1413958112	3425954142	-7027986048	12249028858
8	698775	-10454400	78795090	-401969280	1567387743	-4980406272	13375882455	-31002012608	62669671935

Table 57. Table of b_{jk} coefficients for $\mathbf{h}_2^{\mathrm{SO}(8)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	-5	-44	-95	1456	429
1	36	-118	68	60	84	134	-340	-3608	-4280
2	495	-3114	7606	-7520	-508	2680	6096	8776	-16074
3	4004	-35156	138200	-308904	393184	-199732	-114164	56560	223684
4	22932	-247884	1267072	-4024108	8660300	-12647208	11215914	-2977932	-4261158
5	102816	-1282680	7774740	-30412512	85418344	-179833016	284507040	-323208660	223496044
6	383724	-5318280	36369804	-163762680	544151025	-1410138574	2921674928	-4853150808	6325588310
7	1241460	-18651996	139612836	-696342168	2602763528	-7744363160	18952660776	-38752829440	66373180536
8	3581721	-57364164	460669440	-2484244224	10136320143	-33326857120	91557798580	-214613330920	433765167251

Table 58. Table of b_{jk} coefficients for $\mathbf{h}_2^{\mathrm{SO}(9)}$.

$k\backslash j$	0	1	2	3	4	5	6	7
0	1	0	0	0	-3	-16	-103	-48
1	45	-144	86	48	88	160	-243	-736
2	770	-4800	11538	-11312	351	1968	6427	12304
3	7644	-67536	264946	-584752	727803	-369776	-146259	39264
4	52920	-583296	3018522	-9610752	20491020	-29251696	24994161	-6373856
5	282744	-3636864	22590900	-89797440	253668779	-530960352	824437119	-904981216
6	1241460	-17905536	126761316	-586487136	1984425114	-5184363184	10713955577	-17548934352
7	4671810	-73616400	575268738	-2975442624	11439020010	-34693369952	85735276641	-175328820640
8	15520791	-262451904	2216098170	-12488323584	52844569869	-178688324304	500546068987	-1186103984576

Table 59. Table of b_{jk} coefficients for $\mathbf{h}_2^{\mathrm{SO}(10)}$.

$k\backslash j$	0	1	2	3	4	5	6	7
0	1	0	0	0	-2	0	-65	144
1	66	-208	146	0	132	64	71	0
2	1638	-10176	24541	-26000	8744	-4992	13695	4832
3	23100	-206976	818012	-1814784	2323365	-1486288	262759	-371200
4	222156	-2525952	13369697	-43113840	92344902	-132562176	118728676	-49678624
5	1613898	-21745152	140310401	-573281280	1645528555	-3461643088	5361177380	-5889533952
6	9447438	-144665664	1078909832	-5205925296	18159108739	-48318072064	100506127656	-163975921808
7	46562373	-788186256	6569279717	-35898720000	144212523466	-451656029968	1138839165448	-2348714494976
8	199377750	-3659225856	33314037614	-200620394832	897738323185	-3173937344512	9188877126670	-22247675867120

Table 60. Table of b_{jk} coefficients for $\mathbf{h}_2^{\mathrm{SO}(12)_a}$.

$k\backslash j$	0	1	2	3	4	5	6	7
0	1	0	0	0	0	-24	89	-560
1	66	-208	144	24	0	480	-519	-1632
2	1638	-10176	24519	-25712	6892	2800	-10023	57408
3	23100	-206976	817882	-1812960	2310583	-1426224	49569	233744
4	222156	-2525952	13369147	-43105680	92284010	-132254664	117539906	-45924864
5	1613898	-21745152	140308531	-573252240	1645300811	-3460428560	5356188890	-5872654176
6	9447438	-144665664	1078904398	-5205837648	18158392835	-48314083616	100488952236	-163914717136
7	46562373	-788186256	6569265703	-35898486624	144210551342	-451644637904	1138788176396	-2348525112416
8	199377750	-3659225856	33314004724	-200619831984	897733427591	3173908187384	9188742332120	-22247157728176

Table 61. Table of b_{jk} coefficients for $\mathbf{h}_2^{\mathrm{SO}(12)_b}$.

$k\backslash j$	0	1	2	3	4	5	6	7
0	1	0	0	0	0	-14	13	0
1	78	-246	184	0	0	414	-78	-2016
2	2275	-14170	34491	-37984	14824	0	-12545	55428
3	37400	-337650	1343860	-3011060	3949818	-2740344	625970	147420
4	415701	-4789070	25623440	-83363256	180151060	-262494856	244659096	-119438520
5	3461458	-47509462	311410944	-1288483040	3734402980	-7918507302	12377186244	-13852158216
6	23046023	-361266906	2750438847	-13502044556	47742957054	-128332589450	268924830830	-441474787956
7	128271000	-2232770540	19083881388	-106586370072	435987698822	-1385009164420	3529169247240	-7331469476188
8	616258500	-11677308500	109474843920	-676675858040	3096495676326	-11151643938096	32758441385150	-80174635432992

Table 62. Table of b_{jk} coefficients for $\mathbf{h}_2^{\mathrm{SO}(13)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	1	0	8	0	27	0	64
1	1	0	0	0	9	0	64	0	216
2	8	0	9	0	0	0	53	0	360
3	27	0	64	0	53	0	0	0	245
4	64	0	216	0	360	0	245	0	0
5	125	0	512	0	1188	0	1600	0	971
6	216	0	1000	0	2816	0	5211	0	6168
7	343	0	1728	0	5500	0	12288	0	19818
8	512	0	2744	0	9504	0	24000	0	46528

Table 63. Table of b_{jk} coefficients for $\mathbf{h}_3^{\mathrm{SU}(3)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	5	-32	140	-448	1218
1	1	0	0	0	-1	0	132	-864	3610
2	21	-32	32	0	-5	0	-32	0	1985
3	168	-448	820	-864	560	0	-132	64	-560
4	825	-2880	6936	-11424	14536	-12800	6900	0	-2006
5	3003	-12320	34884	-71712	120340	-161056	173040	-136800	66927
6	8918	-40768	128576	-303072	597548	-986016	1391040	-1644160	1595550
7	22848	-112896	384664	-995904	2185760	-4110656	6791808	-9824256	12442264
8	52326	-274176	990192	-2745792	6507886	-13398112	24596472	-40391648	59697986

Table 64. Table of b_{jk} coefficients for $\mathbf{h}_3^{\mathrm{SO}(7)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	1	8	-96	560	-2270
1	1	0	0	0	-2	0	38	256	-3369
2	28	-48	38	8	0	-16	-76	48	708
3	300	-960	1736	-1728	764	256	96	-512	-1500
4	1925	-8400	21480	-36688	44240	-34104	11464	3720	3369
5	8918	-47040	147070	-325760	554910	-730688	724192	-482880	145530
6	32928	-197568	705796	-1834728	3795484	-6422728	8965812	-10156784	8939168
7	102816	-677376	2662464	-7727552	18169840	-35873856	60568904	-87710080	108129264
8	282150	-1995840	8430912	-26536608	68393217	-150204712	287427296	-483941880	718911028

Table 65. Table of b_{jk} coefficients for $\mathbf{h}_3^{\mathrm{SO}(8)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	4	6	-268	2116
1	1	0	0	0	-1	-4	0	212	170
2	36	-68	50	12	0	-12	-56	-144	-36
3	495	-1788	3296	-3220	1155	680	0	-412	-1325
4	4004	-20152	54730	-96324	114160	-81396	17166	19384	-2116
5	22932	-141960	483808	-1133924	1977545	-2594628	2464000	-1453892	190621
6	102816	-734160	2912364	-8157240	17667168	-30634032	42847730	-47462412	39305640
7	383724	-3042864	13468416	-42692664	106715805	-219628604	379624320	-553494568	674483511
8	1241460	-10669032	51315660	-178281720	494121760	-1145255280	2275563774	-3922608352	5884380680

Table 66. Table of b_{jk} coefficients for $\mathbf{h}_3^{\mathrm{SO}(9)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	14	-32	-674
1	1	0	0	0	-1	0	-14	0	1000
2	45	-92	68	12	0	-12	-68	124	-1045
3	770	-3040	5784	-5728	2010	1152	0	-1152	-1336
4	7644	-42588	122056	-221480	263676	-183280	30196	50668	0
5	52920	-366912	1342068	-3298080	5902644	-7798880	7284468	-4017248	116592
6	282744	-2284128	9854040	-29328852	66178359	-117613440	165998096	-182425360	146015652
7	1241460	-11233728	54650376	-186062592	490131700	-1046847936	1852644768	-2731167520	3321256884
8	4671810	-46151820	246102516	-926394084	2730649548	-6631573704	13633247000	-24046382136	36526686612

Table 67. Table of b_{jk} coefficients for $\mathbf{h}_3^{\mathrm{SO}(10)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	48	-255
1	1	0	0	0	-1	0	0	-48	0
2	55	-120	92	8	0	-8	-92	120	200
3	1144	-4840	9540	-9752	3744	1488	0	-1488	-3744
4	13650	-82160	246848	-462088	560130	-394984	71728	100144	0
5	112200	-847440	3286136	-8415368	15467387	-20735176	19424424	-10568584	56403
6	703494	-6238320	28806624	-90213120	211137472	-384543328	550176564	-606292456	479544108
7	3586440	-35864400	188273448	-680093568	1874297920	-4139175384	7496610080	-11204712064	13686616296
8	15520791	-170450280	987592320	-3972386088	12339695160	-31218639928	66197493824	-119379939624	183922406640

Table 68. Table of b_{jk} coefficients for $\mathbf{h}_3^{\mathrm{SO}(11)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	165
1	1	0	0	0	-1	0	0	0	-165
2	66	-152	122	0	0	0	-122	152	-66
3	1638	-7328	14960	-15904	6910	1408	0	-1408	-6910
4	23100	-147840	462956	-893664	1111286	-814040	191700	158136	0
5	222156	-1795200	7311216	-19418784	36649600	-50094752	47714700	-26720160	1272584
6	1613898	-15402816	75236832	-246081792	595164394	-1110311664	1615203928	-1799004312	1431701106
7	9447438	-102232416	571319892	-2169449568	6218563110	-14154709888	26217103880	-39798057376	49064889198
8	46562373	-556049208	3448909464	-14667256032	47666004672	-125023810032	272698290096	-502378917392	785763764522

Table 69. Table of b_{jk} coefficients for $\mathbf{h}_3^{\mathrm{SO}(12)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	2	-7	28	-64	154	-286
1	1	0	0	0	0	32	-126	448	-1075
2	14	-14	18	0	-1	-2	0	306	-1274
3	77	-128	252	-224	184	0	-18	-32	0
4	273	-572	1386	-1894	2576	-2114	1392	0	-177
5	748	-1792	4914	-8256	13986	-17120	19424	-15008	8638
6	1729	-4522	13464	-25578	49308	-73180	103908	-117596	119780
7	3542	-9856	31122	-64160	134706	-224672	363384	-494400	631352
8	6630	-19320	63756	-139342	310843	-560794	988548	-1504986	2188872

Table 70. Table of b_{jk} coefficients for $\mathbf{h}_3^{G_2}$.

$k \backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	5	20	-282
1	1	0	0	0	0	-4	-10	0	445
2	52	-104	63	20	14	0	-9	-292	-630
3	1053	-4212	7626	-6552	1169	1204	1246	520	-735
4	12376	-71604	204724	-356372	382158	-205296	-12984	38052	47996
5	100776	-739024	2771158	-6798428	11772934	-14419412	11556342	-4309188	-1268849
6	627912	-5441904	24520358	-74594556	168305172	-291899816	389737341	-386624816	254747324
7	3187041	-31290948	161203486	-570136684	1532228038	-3279299588	5702866566	-8070216388	9126753214
8	13748020	-148728580	848959748	-3360036076	10246505556	-25358941140	52337158336	-91211114924	134417094168

Table 71. Table of b_{jk} coefficients for $\mathbf{h}_3^{F_4}$.

$k \backslash j$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	-2
1	1	0	0	0	0	0	-8	-16
2	78	-162	88	20	27	20	2	108
3	2430	-10368	18382	-14256	1898	464	2430	3184
4	43758	-277992	818262	-1383868	1336426	-559926	-55982	-57612
5	537966	-4447872	17830722	-44652160	75077496	-83838128	55183076	-11828160
6	4969107	-49691070	246665562	-794475864	1824658484	-3087757604	3820192146	-3254775622
7	36685506	-424030464	2466530352	-9503406640	26897792168	-58618543536	100133981910	-133439345056
8	225961450	-2928460392	19265962144	-85010356288	280290468042	-728384278466	1533138581700	-2642650203582

Table 72. Table of b_{jk} coefficients for $\mathbf{h}_3^{E_6}$.

$k \backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	-1	0	-28	0	-300
1	0	0	0	0	0	0	-29	0	-707
2	1	0	0	0	0	0	-2	0	-464
3	28	0	29	0	2	0	0	0	-58
4	300	0	707	0	464	0	58	0	0
5	1925	0	6999	0	9947	0	5365	0	928
6	8918	0	42889	0	92391	0	101850	0	49775
7	32928	0	193102	0	544786	0	894198	0	843165
8	102816	0	699762	0	2392663	0	5096487	0	7032993

Table 73. Table of b_{jk} coefficients for $\mathbf{h}_4^{\mathrm{SO}(8)}$.

$k \backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	-2	9	-72	231
1	0	0	0	0	0	0	0	-76	360
2	1	0	0	0	0	0	-1	0	9
3	36	-18	40	0	0	2	0	0	-40
4	495	-462	1314	-684	855	-18	-9	76	-9
5	4004	-5148	17028	-15396	25740	-13788	12880	-612	-324
6	22932	-36036	132704	-159192	316053	-278280	357390	-195624	152777
7	102816	-185640	741780	-1064232	2378100	-2707254	4173660	-3593760	3929652
8	383724	-767448	3271140	-5319840	12980880	-17413812	30382730	-33166980	43785090

Table 74. Table of b_{jk} coefficients for $\mathbf{h}_4^{\mathrm{SO}(9)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	-5	40	-279
1	0	0	0	0	0	0	1	0	-245
2	1	0	0	0	0	0	-2	0	56
3	45	-40	56	0	-5	0	10	0	-112
4	770	-1280	2580	-2040	1617	0	-245	-40	490
5	7644	-17640	44110	-58368	74235	-53872	32253	0	-5877
6	52920	-150528	436254	-752640	1222155	-1403648	1443189	-982352	499765
7	282744	-931392	3009720	-6156288	11723738	-17082864	22655787	-23612416	21452859
8	1241460	-4561920	16035624	-37022832	79084500	-134489856	209693627	-272934384	320594252

Table 75. Table of b_{jk} coefficients for $\mathbf{h}_4^{\mathrm{SO}(10)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	-14	154
1	0	0	0	0	0	0	0	6	-11
2	1	0	0	0	0	0	-1	-6	0
3	55	-66	77	0	-14	0	14	14	-66
4	1144	-2574	4815	-4466	2954	154	-889	0	735
5	13650	-42900	104855	-160006	194845	-153076	75845	9482	-29491
6	112200	-437580	1279047	-2533102	4109862	-5095904	5078920	-3556784	1470238
7	703494	-3197700	10645448	-24989690	48638496	-76538462	101986291	-110913242	98067660
8	3586440	-18290844	67269411	-178497858	395474962	-727883200	1160357220	-1583817950	1869442995

Table 76. Table of b_{jk} coefficients for $\mathbf{h}_4^{\mathrm{SO}(11)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	-42
1	0	0	0	0	0	0	0	0	27
2	1	0	0	0	0	0	-1	0	-27
3	66	-96	103	0	-27	0	27	0	-61
4	1638	-4480	8382	-8448	5129	576	-2187	0	2187
5	23100	-88704	224913	-371712	455928	-369408	164314	44544	-93897
6	222156	-1064448	3298922	-7096320	11893791	-15372032	15378627	-10761216	3823551
7	1613898	-9060480	32465070	-83128320	169434325	-279711744	381972903	-424986624	374859639
8	9447438	-59799168	239370417	-695844864	1629505845	-3163425408	5222290225	-7355904000	8835469032

Table 77. Table of b_{jk} coefficients for $\mathbf{h}_4^{\mathrm{SO}(12)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	-1	0	0
3	78	-130	134	0	-44	0	44	0	-134
4	2275	-7150	13697	-14560	8464	1430	-4466	0	4466
5	37400	-165750	441602	-772200	966322	-797160	330054	136864	-242946
6	415701	-2309450	7638640	-17484220	30403198	-40556386	41062240	-28619032	8937641
7	3461458	-22632610	87499854	-239745220	511877560	-878703202	1232023366	-1396956444	1236834410
8	23046023	-170600430	742996397	-2323025120	5738161356	-11650207270	19899034288	-28804097088	35248051027

Table 78. Table of b_{jk} coefficients for $\mathbf{h}_4^{\mathrm{SO}(13)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	-1	0	0
3	91	-168	170	0	-65	0	65	0	-170
4	3080	-10752	21229	-23464	13340	2912	-8125	0	8125
5	58344	-288288	806857	-1475488	1888770	-1580096	621416	339808	-548665
6	741741	-4612608	16213223	-39055744	70329257	-96394144	98968389	-68803056	19148888
7	7014007	-51531480	213488275	-619114080	1379233375	-2449921344	3522152982	-4063960800	3619939648
8	52676624	-439735296	2065623560	-6869599880	17806412780	-37609387296	66284300602	-98357989120	122549026660

Table 79. Table of b_{jk} coefficients for $\mathbf{h}_4^{\mathrm{SO}(14)}$.

$k\backslash j$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	-1	0
3	105	-210	211	0	-90	0	90	0
4	4080	-15470	31500	-35910	20197	5250	-13635	0
5	88179	-474810	1390500	-2639882	3455235	-2926728	1107683	736470
6	1270815	-8641542	32068764	-80692710	150002760	-210610862	219308082	-152371128
7	13537524	-109120648	480435165	-1463068362	3385778433	-6199988298	9122011644	-10698480114
8	113859200	-1046081400	5249688444	-18417344428	49833498435	-109031193264	197720128902	-300138572958

Table 80. Table of b_{jk} coefficients for $\mathbf{h}_4^{\mathrm{SO}(15)}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	-1	-4	78
1	0	0	0	0	0	0	0	0	-56
2	1	0	0	0	0	0	0	0	0
3	52	-52	56	4	0	0	0	0	0
4	1053	-2106	3731	-2912	1652	218	-78	0	0
5	12376	-35802	84240	-117728	131768	-87200	35020	6920	-4692
6	100776	-369512	1050413	-1969448	3034239	-3459820	3121573	-1846224	599891
7	627912	-2720952	8868288	-20048236	37681436	-56343664	70792176	-70785988	56282564
8	3187041	-15645474	56590569	-146074812	315602586	-561134470	860525796	-1115562760	1236729117

Table 81. Table of b_{jk} coefficients for $\mathbf{h}_4^{F_4}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	-3
1	0	0	0	0	0	0	0	0	3
2	1	0	0	0	0	0	-1	0	0
3	78	-108	83	8	5	0	-5	-8	-83
4	2430	-6912	11232	-8964	3137	664	475	0	-475
5	43758	-185328	449847	-679360	676806	-380160	65948	31360	24607
6	537966	-2965248	9350289	-19752000	30317218	-33588632	25630241	-10991832	472219
7	4969107	-33127380	125981856	-331568640	660173886	-1020970848	1231123076	-1122978824	715378105
8	36685506	-282686976	1239006483	-3822563316	9111294504	-17436087864	27282831582	-34914675072	35993355287

Table 82. Table of b_{jk} coefficients for $\mathbf{h}_4^{E_6}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	1	0	52
1	0	0	0	0	0	0	0	0	53
2	0	0	0	0	0	0	0	0	0
3	1	0	0	0	0	0	0	0	-1
4	52	0	53	0	0	0	-1	0	0
5	1053	0	2432	0	1484	0	0	0	-53
6	12376	0	44980	0	59996	0	28824	0	0
7	100776	0	495872	0	1023464	0	1034880	0	434010
8	627912	0	3856722	0	10670660	0	16410602	0	13979280

Table 83. Table of b_{jk} coefficients for $\mathbf{h}_5^{F_4}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0
3	1	0	0	0	0	0	0	0	-2
4	78	-54	80	2	0	0	0	0	0
5	2430	-3456	6942	-4320	3320	160	-54	0	0
6	43758	-92664	229500	-266886	315822	-180634	95744	7342	-5024
7	537966	-1482624	4288284	-6905472	10600767	-10709856	9800076	-5240480	2163562
8	4969107	-16563690	54041130	-107374410	197183493	-266577744	325272780	-296589906	233061387

Table 84. Table of b_{jk} coefficients for $\mathbf{h}_5^{E_6}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0
3	1	0	0	0	0	0	0	0	-1
4	133	-168	137	0	9	0	0	0	-9
5	7371	-19440	30305	-23016	9590	-504	1233	0	0
6	238602	-960336	2223053	-3136608	2930978	-1601088	455166	-69048	96568
7	5248750	-28217280	86065839	-172425912	247054003	-251275536	176091905	-74826408	16416162
8	85709988	-571399920	2142790550	-5459137152	10330301397	-14884282272	16397030227	-13416536880	7652991703

Table 85. Table of b_{jk} coefficients for $\mathbf{h}_5^{E_7}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	-1	0	-78
1	0	0	0	0	0	0	0	0	0	0	-79
2	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	1
4	1	0	0	0	0	0	0	0	0	0	-2
5	78	0	79	0	0	0	-1	0	2	0	0
6	2430	0	5512	0	3239	0	0	0	-79	0	158
7	43758	0	157221	0	201292	0	90990	0	0	0	-3083
8	537966	0	2644707	0	5283549	0	5048576	0	1959755	0	0
9	4969107	0	30843384	0	83526287	0	122611239	0	97616584	0	34418248
10	36685506	0	273370383	0	928768412	0	1830734165	0	2205133146	0	1549515786

Table 86. Table of b_{jk} coefficients for $\mathbf{h}_6^{E_7}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0
4	1	0	0	0	0	0	0	0	0
5	133	-112	135	0	2	0	0	0	0
6	7371	-12960	22496	-15120	9315	0	270	-112	-2
7	238602	-640224	1450211	-1804928	1851930	-1041456	434236	0	21558
8	5248750	-18811520	52210138	-90994800	128684164	-128191680	100480265	-48760896	15278490
9	85709988	-380933280	1241505600	-2711923200	4800711340	-6529223840	7273882755	-6188140800	4078768380

Table 87. Table of b_{jk} coefficients for $\mathbf{h}_6^{E_7}$.

$k\backslash j$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0
5	1	0	0	0	0	0	0	0	0
6	133	-56	134	0	0	0	0	0	0
7	7371	-6480	17822	-7504	9179	0	0	0	0
8	238602	-320112	987714	-817248	1220807	-514024	428130	0	-1539
9	5248750	-9405760	31972668	-38373008	67041156	-52636272	56941290	-23950960	15283035
10	85709988	-190466640	703332500	-1081109120	2141810853	-2350286320	3073034328	-2305492512	2031727950
11	1101296924	-2895092928	11485138392	-21144148480	46448671100	-63533253056	96029039582	-97944233008	106985485698

Table 88. Table of b_{jk} coefficients for $\mathbf{h}_7^{E_7}$.

H One-instanton component of $Z_{S^1\times\mathbb{R}^4}^{5d}$

In this section we describe the computation of one-instanton component of 5d Nekrasov partition functions starting from the elliptic genera of the $\mathbf{h}_n^G$ theories. In section H.1 we discuss our approach, and in section H.2 we provide the results of our computations.

G	Matter	
E_7	$m \cdot \mathbf{56}$	$m = 1^*, 2^*, 3^*$
E_6	$m \cdot (\mathbf{27} + \overline{\mathbf{27}})$	$m = 1^*, 2, 3$
F_4	$m \cdot \mathbf{26}$	$m = 1^*, 2^*, \underline{3}^*$
G_2	$m \cdot \mathbf{7}$	$m = \underline{4}^*$
SO(7)	$(3 - m) \cdot \mathbf{7} \oplus 2(4 - m) \cdot \mathbf{8}$	$m = \underline{2}^*$
SO(8)	$(4 - m) \cdot \mathbf{8^v} \oplus (4 - m) \cdot \mathbf{8^s} \oplus (4 - m) \cdot \mathbf{8^c}$	$m = 2, 3$
SO(9)	$(5 - m) \cdot \mathbf{9} \oplus (4 - m) \cdot \mathbf{16}$	$m = \underline{2}, 3^*$
SO(10)	$(6 - m) \cdot \mathbf{10} \oplus (4 - m) \cdot \mathbf{16^s} \oplus (4 - m) \cdot \mathbf{16^c}$	$m = \underline{2}, 3^*$
SO(11)	$(7 - m) \cdot \mathbf{11} \oplus (4 - m) \cdot \mathbf{32}$	$m = 3^*$
SO(12)$_a$	$(8 - m) \cdot \mathbf{12} \oplus (4 - m) \cdot \mathbf{32}^s$	$m = \underline{2}, 3$
SO(12)$_b$	$(8 - m) \cdot \mathbf{12} \oplus \left(\frac{4-m}{2}\right) \cdot \mathbf{32}^s \oplus \left(\frac{4-m}{2}\right) \cdot \mathbf{32}^c$	$m = \underline{2}$
SO(13)	$(9 - m) \cdot \mathbf{13} \oplus \left(\frac{4-m}{2}\right) \cdot \mathbf{64}$	$m = \underline{2}$

Table 89. Choices of gauge and matter content of 5d theories for which we obtain novel results for the one-instanton partition function. For the values of m which are underlined, the one-instanton partition function contains extra gauge-neutral states in comparison to the Nekrasov partition function (as discussed in section 7). We denote by an asterisk the theories for which we are able to determine the one-instanton partition function to all orders in v with chemical potentials $\mathbf{m}_F$ and $\mathbf{m}_G$ turned on; for the remaining theories we are only able to obtain partial information about the dependence on $\mathbf{m}_F$ and $\mathbf{m}_G$.

H.1 Computing $Z_{1-\text{inst}}$

From the discussion of section 7 it follows that for $n \geq 3$ the leading order terms in the q-expansion of the 6d $T^2 \times \mathbb{R}^4$ partition function gives the Nekrasov partition function for the 5d $\mathcal{N} = 1$ theory with gauge group G and matter hypermultiplets in the same representations as the 6d matter, up to an overall power of v. For $n = 2$, the leading order terms give the Nekrasov partition function with the addition of some decoupled states which are neutral under the gauge group.

By exploiting modularity as in section 8, we have computed the one-string elliptic genus for 42 theories with $n \geq 2$; from these computations we can readily obtain the one-instanton piece of the corresponding 5d partition function, $Z_{1-\text{inst}}$, with fugacities $\mathbf{m}_G$ and $\mathbf{m}_F$ switched off. To the best of our knowledge, this leads to new results for the 5d theories in table 89.[40]

By using equations (6.17) and (6.18) as in appendix F, one can in fact reconstruct the dependence of the one-instanton functions $Z_{1-\text{inst}}(\mathbf{m}_G, \mathbf{m}_F, v, q)$, written as a power series in v. For several theories, we are able to perform such a computation to very high orders in v, and this allows us to conjecture an exact expression for the one-instanton piece of the

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[40]While this work was under completion, the papers [93, 94, 238] appeared with similar results for the theories with $G = \text{SO}(7)$ and G_2, with respectively two hypermultiplets in the spinor representation and one hypermultiplet in the $\mathbf{7}$ representation.

Nekrasov partition function of the corresponding 5d theory, with all fugacities turned on. This also leads to several novel results for the theories listed in table 89.

We illustrate this idea by means of a specific example: the theory with $G = G_2$ and 4 hypermultiplets in the $\mathbf{7}$ representation, which transform under $F = \mathrm{Sp}(4)$. Expanding our result for $Z_{1-\mathrm{inst}}$ to a suitable power in v, we find

$$\left[\frac{v}{(1-vx)(1-v/x)}\right]^{-1} Z_{1-\mathrm{inst}}(\mathbf{1},\mathbf{1},x,v) = \mathbb{E}_2^{G_2}(\mathbf{1},\mathbf{1},v,q)\Big|_{q^{1/6}} \tag{H.1}$$

$$= \frac{(-1+v)^2\left(1+8v+30v^2+64v^3+30v^4+8v^5+v^6\right)}{v(1+v)^6} = v^{-1}-8v^2-35v^3+336v^4-1317v^5+\dots$$

On the other hand, by expanding equation (6.17) we obtain:

$$\mathbb{E}_2^{G_2}(\mathbf{m}_{G_2},\mathbf{m}_{\mathrm{Sp}(4)},v,q)\Big|_{q^{1/6}} = n^{(00)}_{(0000),-1,0}v^{-1} + v\, n^{(00)}_{(0000),-3,-1} + v^2\, n^{(00)}_{(1000),-2,-1}\chi_{\mathbf{8}}^F$$

$$+ v^3(n^{(10)}_{(0000),-1,-1}\chi_{\mathbf{7}}^G + n^{(00)}_{(0001),-1,-1}\chi_{\mathbf{42}}^F) + n^{(10)}_{(0010),0,-1}v^3\chi_{\mathbf{48}}^F\chi_{\mathbf{7}}^G$$

$$+ v^4(n^{(20)}_{(0100),-3,-2}\chi_{\mathbf{27}}^F\chi_{\mathbf{27}}^G + n^{(01)}_{(0001),-3,-2}\chi_{\mathbf{42}}^F\chi_{\mathbf{14}}^G) + \dots, \tag{H.2}$$

and in order to restore the $\mathbf{m}_{G_2}$, $\mathbf{m}_{\mathrm{Sp}(4)}$ dependence we need to find small integers $n^{\lambda}_{\nu,\ell,m}$ for which equations (H.1) and (H.2) agree. Several of these coefficients are uniquely determined:

$$n^{(00)}_{(0000),-1,0} = 1; \qquad n^{(00)}_{(0000),-3,-1} = 0; \qquad n^{(00)}_{(1000),-2,-1} = -1; \qquad n^{(10)}_{(0010),0,-1} = 1. \tag{H.3}$$

We can also easily determine the remaining coefficients with high confidence. For example, for the v^4 term we need to impose

$$729\, n^{(20)}_{(0100),-3,-2} + 588\, n^{(01)}_{(0001),-3,-2} = -1317. \tag{H.4}$$

The generic solution for integer coefficients is:

$$n^{(20)}_{(0100),-3,-2} = 195 - 196c; \qquad n^{(01)}_{(0001),-3,-2} = -244 + 243c; c \in \mathbb{Z}. \tag{H.5}$$

The only choice that leads to small coefficients is $c = 1$, which gives

$$n^{(20)}_{(0100),-3,-2} = n^{(01)}_{(0001),-3,-2} = -1. \tag{H.6}$$

Likewise, for the v^3 coefficient the only reasonable choice is

$$n^{(10)}_{(0000),-1,-1} = 1 \quad \text{and} \quad n^{(00)}_{(0001),-1,-1} = -1. \tag{H.7}$$

One can easily automate such computations, and pushing the calculation to higher order one begins to recognize a pattern in the representations that appear in the one-instanton partition function:

$$\left[\frac{v}{(1-vx)(1-v/x)}\right]^{-1} Z_{1-\mathrm{inst}}(\mathbf{1},\mathbf{1},x,v) = \tag{H.8}$$

$$= v^{-1} - v^2\chi^F_{(1000)} + v^3(\chi^G_{(10)} - \chi^F_{(0001)}) + v^4\chi^G_{(10)}\chi^F_{(0010)} - v^5(\chi^G_{(20)}\chi^F_{(0100)} + \chi^G_{(01)}\chi^F_{(0001)})$$

$$+ v^6(\chi^G_{(30)}\chi^F_{(1000)} + \chi^G_{(11)}\chi^F_{(0010)}) - v^7(\chi^G_{(40)} + \chi^G_{(21)}\chi^F_{(0100)} + \chi^G_{(02)}\chi^F_{(0001)})$$

$$+ v^8(\chi^G_{(31)}\chi^F_{(1000)} + \chi^G_{(12)}\chi^F_{(0010)}) - v^9(\chi^G_{(41)} + \chi^G_{(22)}\chi^F_{(0100)} + \chi^G_{(03)}\chi^F_{(0001)}) + \mathcal{O}(v^{10})$$

(in fact one can easily push the computation to higher orders with little effort). This leads us to conjecture that

$$\left[\frac{v}{(1-v\,x)(1-v/x)}\right]^{-1} Z_{1-\text{inst}}(\mathbf{1},\mathbf{1},x,v) = \tag{H.9}$$

$$= v^{-1} - v^2 \chi^F_{(1000)} + v^3 \chi^G_{(10)} - \sum_{n=0}^{\infty} v^{3+2n} \chi^G_{0n} \chi^F_{(0001)} + \sum_{n=0}^{\infty} v^{4+2n} \chi^G_{(1n)} \chi^F_{(0010)}$$

$$- \sum_{n=0}^{\infty} v^{5+2n} \chi^G_{2n} \chi^F_{(0100)} + \sum_{n=0}^{\infty} v^{6+2n} \chi^G_{(3n)} \chi^F_{(1000)} - \sum_{n=0}^{\infty} v^{7+2n} \chi^G_{(4n)}.$$

As a consistency check, setting the fugacities $\mathbf{m}_{G_2}$, and $\mathbf{m}_{\text{Sp}(N)}$ to zero and plugging in the formulas for the dimensions of the representations of G_2 and $\text{Sp}(4)$ that appear in equation (H.9), that is:

$$\dim(R^{G_2}_{(n_1,n_2)}) = \frac{(n_1+1)(n_2+1)(n_1+n_2+2)(n_1+2n_2+3)(n_1+3n_2+4)(2n_1+3n_2+5)}{120}, \tag{H.10}$$

and

$$\dim(R^{\text{Sp}(4)}_{(1000)}) = 8; \quad \dim(R^{\text{Sp}(4)}_{(0100)}) = 27; \quad \dim(R^{\text{Sp}(4)}_{(0010)}) = 48; \quad \dim(R^{\text{Sp}(4)}_{(0001)}) = 42, \tag{H.11}$$

one finds from equation (H.9) that

$$Z_{1-\text{inst}}(\mathbf{1},\mathbf{1},v) = v^{-1} + \sum_{n=0}^{\infty} \frac{1}{15}(-1)^n n(76 - 70n^2 + 9n^4)v^n, \tag{H.12}$$

which indeed is the series expansion of

$$\frac{(-1+v)^2 \left(1 + 8v + 30v^2 + 64v^3 + 30v^4 + 8v^5 + v^6\right)}{v(1+v)^6}. \tag{H.13}$$

H.2 One-instanton partition functions

In tables 90–92 we provide the one-instanton partition functions of the 5d $\mathcal{N} = 1$ theories we obtain from compactification of the 6d SCFTs, with fugacities $\mathbf{m}_F$ and $\mathbf{m}_G$ turned off. In the text that follows we present our results with chemical potentials restored. We omit to discuss the cases without matter and the two families of theories with $n = 2$, $G = \text{SU}(N)$ and $n = 4$, $G = \text{SO}(8 + N)$ since their 5d partition function is already well studied. For the remaining theories, the matter content of the 5d theory can be read off from tables 20 and 21. In this entire section we have factored out the center of mass term $\frac{v}{(1-v\,x)(1-v/x)}$ from $Z_{1-\text{inst}}$.

In the paper we have not kept track of the overall sign of the elliptic genus and this also affects the overall sign of the one-instanton partition function we compute. Here we make a choice of sign so that the leading order coefficient in the v-expansion of $Z_{1-\text{inst}}$ has positive sign when the fugacities $\mathbf{m}_F$ and $\mathbf{m}_G$ are set to $\mathbf{1}$.

n	G	$Z_{1-\text{inst}}$
2	SU(2)	$v^{-1}(1+v)^{-2}(1-v)^2[1+4v+v^2]$
2	SU(3)	$v^{-1}(1+v)^{-4}(1-v)^2[1+6v+16v^2+6v^3+v^4]$
2	SU(4)	$v^{-1}(1+v)^{-6}(1-v)^2[1+8v+29v^2+64v^3+29v^4+8v^5+v^6]$
2	SU(5)	$v^{-1}(1+v)^{-8}(1-v)^2[1+10v+46v^2+130v^3+256v^4+130v^5+46v^6+10v^7+v^8]$
2	SU(6)	$v^{-1}(1+v)^{-10}(1-v)^2[1+12v+67v^2+232v^3+562v^4+1024v^5+562v^6+232v^7$ $+67v^8+12v^9+v^{10}]$
2	SU(7)	$v^{-1}(1+v)^{-12}(1-v)^2[1+14v+92v^2+378v^3+1093v^4+2380v^5+4096v^6+2380v^7$ $+1093v^8+378v^9+92v^{10}+14v^{11}+v^{12}]$
2	SU(8)	$v^{-1}(1+v)^{-14}(1-v)^2[1+16v+121v^2+576v^3+1941v^4+4944v^5+9949v^6$ $+16384v^7+9949v^8+4944v^9+1941v^{10}+576v^{11}+121v^{12}+16v^{13}+v^{14}]$
2	SO(7)	$v^{-1}(1+v)^{-8}(1-v)^2[1+10v+47v^2+138v^3+256v^4+138v^5+47v^6+10v^7+v^8]$
2	SO(8)	$v^{-1}(1+v)^{-10}(1-v)^2[1+12v+68v^2+244v^3+615v^4+1024v^5+615v^6+244v^7$ $+68v^8+12v^9+v^{10}]$
2	SO(9)	$v^{-1}(1+v)^{-12}(1-v)^2[1+14v+93v^2+392v^3+1181v^4+2658v^5+4106v^6+2658v^7$ $+1181v^8+392v^9+93v^{10}+14v^{11}+v^{12}]$
2	SO(10)	$v^{-1}(1+v)^{-14}(1-v)^2[1+16v+122v^2+592v^3+2060v^4+5472v^5+11287v^6+16496v^7$ $+11287v^8+5472v^9+2060v^{10}+592v^{11}+122v^{12}+16v^{13}+v^{14}]$
2	SO(12)$_a$	$v^{-1}(1+v)^{-18}(1-v)^2[1+20v+192v^2+1180v^3+5226v^4+17804v^5+48575v^6$ $+108512v^7+197370v^8+267144v^9+197370v^{10}+108512v^{11}+48575v^{12}$ $+17804v^{13}+5226v^{14}+1180v^{15}+192v^{16}+20v^{17}+v^{18}]$
2	SO(12)$_b$	$v^{-1}(1+v)^{-18}(1-v)^2[1+20v+192v^2+1180v^3+5228v^4+17820v^5+48633v^6$ $+108640v^7+197566v^8+267368v^9+197566v^{10}+108640v^{11}+48633v^{12}$ $+17820v^{13}+5228v^{14}+1180v^{15}+192v^{16}+20v^{17}+v^{18}]$
2	SO(13)	$v^{-1}(1+v)^{-20}(1-v)^2[1+22v+233v^2+1584v^3+7780v^4+29466v^5+89645v^6$ $+224944v^7+471813v^8+818552v^9+1077376v^{10}+818552v^{11}+471813v^{12}$ $+224944v^{13}+89645v^{14}+29466v^{15}+7780v^{16}+1584v^{17}+233v^{18}+22v^{19}+v^{20}]$
2	G_2	$v^{-1}(1+v)^{-6}(1-v)^2[1+8v+30v^2+64v^3+30v^4+8v^5+v^6]$
2	F_4	$v^{-1}(1+v)^{-16}(1-v)^2[1+18v+155v^2+852v^3+3369v^4+10240v^5+24825v^6+47834v^7$ $+66180v^8+47834v^9+24825v^{10}+10240v^{11}+3369v^{12}+852v^{13}+155v^{14}+18v^{15}+v^{16}]$

Table 90. 5d limit of the one-string elliptic genus for the theories $\mathbf{h}_n^G$ with $n=2$, with chemical potentials $\mathbf{m}_F$, $\mathbf{m}_G$ turned off. The partition function is displayed with the center of mass factor $\frac{v}{(1-v\,x)(1-v/x)}$ removed.

n	G	$Z_{1-\text{inst}}$
3	SU(3)	$(1-v)^{-4}(1+v)^{-4}[v^2+4v^4+v^6]$
3	SO(7)	$(1-v)^{-4}(1+v)^{-8}v^4[5-12v+22v^2-12v^3+5v^4]$
3	SO(8)	$(1-v)^{-4}(1+v)^{-10}v^4[1+14v-37v^2+68v^3-37v^4+14v^5+v^6]$
3	SO(9)	$(1-v)^{-4}(1+v)^{-12}2v^5[2+19v-62v^2+106v^3-62v^4+19v^5+2v^6]$
3	SO(10)	$(1-v)^{-4}(1+v)^{-14}2v^6[7+54v-210v^2+344v^3-210v^4+54v^5+7v^6]$
3	SO(11)	$(1-v)^{-4}(1+v)^{-16}v^7[48+321v-1436v^2+2302v^3-1436v^4+321v^5+48v^6]$
3	SO(12)	$(1-v)^{-4}(1+v)^{-18}11v^8[15+90v-451v^2+716v^3-451v^4+90v^5+15v^6]$
3	G_2	$(1-v)^{-4}(1+v)^{-6}v^3[2-3v+8v^2-3v^3+2v^4]$
3	F_4	$(1-v)^{-4}(1+v)^{-16}v^6[5+80v+268v^2-1232v^3+2142v^4-1232v^5+268v^6+80v^7+5v^8]$
3	E_6	$(1-v)^{-4}(1+v)^{-22}2v^7[1+28v+356v^2+2045v^3+1583v^4-19638v^5+36572v^6$ $-19638v^7+1583v^8+2045v^9+356v^{10}+28v^{11}+v^{12}]$
4	SO(8)	$(1-v)^{-10}(1+v)^{-10}v^5[1+18v^2+65v^4+65v^6+18v^8+v^{10}]$
4	SO(9)	$(1-v)^{-10}(1+v)^{-12}v^6[2-5v+36v^2-46v^3+130v^4-90v^5+130v^6-46v^7$ $+36v^8-5v^9+2v^{10}]$
4	SO(10)	$(1-v)^{-10}(1+v)^{-14}v^7[5-20v+99v^2-184v^3+370v^4-360v^5+370v^6$ $-184v^7+99v^8-20v^9$ $+5v^{10}]$
4	SO(11)	$(1-v)^{-10}(1+v)^{-16}v^8[14-70v+306v^2-651v^3+1180v^4-1278v^5+1180v^6-651v^7$ $+306v^8-70v^9+14v^{10}]$
4	SO(12)	$(1-v)^{-10}(1+v)^{-18}v^9[42-240v+999v^2-2264v^3+3947v^4-4464v^5+3947v^6-2264v^7$ $+999v^8-240v^9+42v^{10}]$
4	SO(13)	$(1-v)^{-10}(1+v)^{-20}v^{10}[132-825v+3366v^2-7898v^3+13550v^4-15642v^5+13550v^6$ $-7898v^7+3366v^8-825v^9+132v^{10}]$
4	SO(14)	$(1-v)^{-10}(1+v)^{-22}13v^{11}[33-220v+891v^2-2136v^3+3640v^4-4248v^5+3640v^6$ $-2136v^7+891v^8-220v^9+33v^{10}]$
4	SO(15)	$(1-v)^{-10}(1+v)^{-24}2v^{12}[715-5005v+20241v^2-49231v^3+83680v^4-98280v^5$ $+83680v^6-49231v^7+20241v^8-5005v^9+715v^{10}]$
4	F_4	$(1-v)^{-10}(1+v)^{-16}v^7[1+10v-49v^2+266v^3-549v^4+1068v^5-1110v^6+1068v^7$ $-549v^8+266v^9-49v^{10}+10v^{11}+v^{12}]$
4	E_6	$(1-v)^{-10}(1+v)^{-22}v^9[3+44v+33v^2-1052v^3+6513v^4-17404v^5+31905v^6-37432v^7$ $+31905v^8-17404v^9+6513v^{10}-1052v^{11}+33v^{12}+44v^{13}+3v^{14}]$

Table 91. 5d limit of the one-string elliptic genus for the theories $\mathbf{h}_n^G$ with $n=3$ and 4, with chemical potentials $\mathbf{m}_F$, $\mathbf{m}_G$ turned off.

n	G	$Z_{1-\text{inst}}$
5	F_4	$(1-v)^{-16}(1+v)^{-16}v^8[1+36v^2+341v^4+1208v^6+1820v^8+1208v^{10}+341v^{12}+36v^{14}+v^{16}]$
5	E_6	$(1-v)^{-16}(1+v)^{-22}v^{10}[1+8v-43v^2+456v^3-1436v^4+5116v^5-9848v^6+19504v^7-24164v^8$ $+30016v^9-24164v^{10}+19504v^{11}-9848v^{12}+5116v^{13}-1436v^{14}+456v^{15}-43v^{16}+8v^{17}+v^{18}]$
5	E_7	$(1-v)^{-16}(1+v)^{-34}v^{14}[7+126v+307v^2-4096v^3+8014v^4+121428v^5-898477v^6+3512818v^7$ $-9100043v^8+17425312v^9-25317468v^{10}+28704184v^{11}-25317468v^{12}+17425312v^{13}$ $-9100043v^{14}+3512818v^{15}-898477v^{16}+121428v^{17}+8014v^{18}-4096v^{19}+307v^{20}+126v^{21}+7v^{22}]$
6	E_6	$(1-v)^{-22}(1+v)^{-22}v^{11}[1+56v^2+945v^4+6776v^6+23815v^8+43989v^{10}+43989v^{12}$ $+23815v^{14}+6776v^{16}+945v^{18}+56v^{20}+v^{22}]$
6	E_7	$(1-v)^{-22}(1+v)^{-34}v^{15}[2+24v-43v^2+52v^3+8027v^4-53360v^5+279039v^6-950972v^7$ $+2698740v^8-5898532v^9+10988680v^{10}-16600348v^{11}+21616127v^{12}-23243264v^{13}$ $+21616127v^{14}-16600348v^{15}+10988680v^{16}-5898532v^{17}+2698740v^{18}-950972v^{19}$ $+279039v^{20}-53360v^{21}+8027v^{22}+52v^{23}-43v^{24}+24v^{25}+2v^{26}]$
7	E_7	$(1-v)^{-28}(1+v)^{-34}v^{16}[1+6v-13v^2+764v^3-3200v^4+24172v^5-76952v^6+317380v^7$ $-758576v^8+2097116v^9-3826888v^{10}+7681284v^{11}-10844990v^{12}+16441672v^{13}-18077202v^{14}$ $+21130252v^{15}-18077202v^{16}+16441672v^{17}-10844990v^{12}+7681284v^{11}-3826888v^{10}$ $+2097116v^9-758576v^8+317380v^7-76952v^6+24172v^5-3200v^4+764v^3-13v^2+6v+1]$

Table 92. 5d limit of the one-string elliptic genus for the theories $\mathbf{h}_n^G$ with $n=5,6,7$, with chemical potentials $\mathbf{m}_F$, $\mathbf{m}_G$ turned off.

- **n = 2, G = SO(7), F = Sp(1)$_\mathbf{a}$ × Sp(4)$_\mathbf{b}$.** We find that the first 11 terms in the v expansion in $Z_{1-\text{inst}}$ are given by:

$$
\begin{aligned}
Z_{1-\text{inst}} &= v^{-1} - v^2\chi^F_{(1)_a} - v^3\chi^F_{(0100)_b} + v^4(\chi^G_{(001)}\chi^F_{(1000)_b} - \chi^F_{(1)_a\otimes(0001)_b}) \\
&+ v^5(\chi^G_{(100)}\chi^F_{(0001)_b} - \chi^G_{(002)} + \chi^G_{(001)}\chi^F_{(1)_a\otimes(0010)_b}) \\
&- v^6(\chi^G_{(101)}\chi^F_{(0010)_b} + \chi^G_{(010)}\chi^F_{(1)_a\otimes(0001)_b} + \chi^G_{(002)}\chi^F_{(1)_a\otimes(0100)_b}) \\
&+ v^7(\chi^G_{(110)}\chi^F_{(0001)_b} + \chi^G_{(102)}\chi^F_{(0100)_b} + \chi^G_{(011)}\chi^F_{(1)_a\otimes(0010)_b} + \chi^G_{(003)}\chi^F_{(1)_a\otimes(1000)_b}) \\
&- v^8(\chi^G_{(111)}\chi^F_{(0010)_b} + \chi^G_{(103)}\chi^F_{(1000)_b} + \chi^G_{(020)}\chi^F_{(1)_a\otimes(0001)_b} + \chi^G_{(012)}\chi^F_{(1)_a\otimes(0100)_b} + \chi^G_{(004)}\chi^F_{(1)_a}) \\
&+ v^9(\chi^G_{(120)}\chi^F_{(0001)_b} + \chi^G_{(112)}\chi^F_{(0100)_b} + \chi^G_{(104)} + \chi^G_{(021)}\chi^F_{(1)_a\otimes(0010)_b} + \chi^G_{(013)}\chi^F_{(1)_a\otimes(1000)_b}) \\
&+ \mathcal{O}(v^{10}).
\end{aligned}
\tag{H.14}
$$

We have pushed this computation to $\mathcal{O}(v^{16})$ and conjecture that the Nekrasov partition function to all orders in v is given by:

$$
\begin{aligned}
Z_1 &= v^{-1} - v^2\chi^F_{(1)_a} - v^3\chi^F_{(0100)_b} + v^4\chi^G_{(001)}\chi^F_{(1000)_b} - v^5\chi^G_{(002)} - \sum_{n=0}^{\infty}v^{4+2n}\chi^G_{(0n0)}\chi^F_{(1)_a\otimes(0001)_b} \\
&+ \sum_{n=0}^{\infty}v^{5+2n}(\chi^G_{(1n0)}\chi^F_{(0001)_b} + \chi^G_{(0n1)}\chi^F_{(1)_a\otimes(0010)_b}) \\
&- \sum_{n=0}^{\infty}v^{6+2n}(\chi^G_{(1n1)}\chi^F_{(0010)_b} + \chi^G_{(0n2)}\chi^F_{(1)_a\otimes(0100)_b})
\end{aligned}
$$

$$+ \sum_{n=0}^{\infty} v^{7+2n}(\chi_{(1n2)}^{G}\chi_{(0100)_b}^{F} + \chi_{(0n3)}^{G}\chi_{(1)_a\otimes(1000)_b}^{F})$$

$$- \sum_{n=0}^{\infty} v^{8+2n}(\chi_{(1n3)}^{G}\chi_{(1000)_b}^{F} + \chi_{(0n4)}^{G}\chi_{(1)_a}^{F}) + \sum_{n=0}^{\infty} v^{9+2n}\chi_{(1n4)}^{G}. \tag{H.15}$$

As a further check, upon specializing $\mathbf{m}_G, \mathbf{m}_F \to 1$ this expression matches with the result in table 90.

- **n = 2, G = SO(8), F = Sp(2)$_\mathbf{a}$ × Sp(2)$_\mathbf{b}$ × Sp(2)$_\mathbf{c}$.** In this case we are not able to completely fix the Nekrasov partition function due to the appearance of many representations of $F = \mathrm{Sp}(2)^3$ of the same dimension. We make the assumption that the combinations of representations that appear as coefficients in the Nekrasov partition function must be triality invariant. Still, it is sometimes possible to find more than one triality-invariant combination of representations. For instance, the combinations of representations

$$\chi_{(1000)}^{G}\chi_{(10)_a\otimes(10)_a}^{F} + \chi_{(0010)}^{G}\chi_{(10)_b\otimes(10)_b}^{F} + \chi_{(0001)}^{G}\chi_{(10)_c\otimes(10)_c}^{F} \tag{H.16}$$

and

$$\chi_{(1000)}^{G}\chi_{(10)_b\otimes(10)_c}^{F} + \chi_{(0010)}^{G}\chi_{(10)_a\otimes(10)_c}^{F} + \chi_{(0001)}^{G}\chi_{(10)_a\otimes(10)_b}^{F} \tag{H.17}$$

are both triality invariant and have the same dimension. For cases such as these where we are not able to determine the precise representations appearing we use the following notation:

$$(\chi_{(1000)}^{G} + \chi_{(0010)}^{G} + \chi_{(0001)}^{G})\chi_{(10)_{(abc)}\otimes(10)_{(abc)}}^{F}. \tag{H.18}$$

With this proviso, by studying the partition function up to $\mathcal{O}(v^{15})$ we have found the following expression, which we conjecture to hold at all orders in v:

$$Z_{1-\mathrm{inst}} = v^{-1} - v^3(\chi_{(01)_a}^{F} + \chi_{(01)_b}^{F} + \chi_{(01)_c}^{F}) - v^5((\chi_{(1000)}^{G} + \chi_{(0010)}^{G} + \chi_{(0001)}^{G})\chi_{(10)_{(abc)}\otimes(10)_{(abc)}}^{F}$$

$$+ \chi_{(0100)}^{G}) - v^6(\chi_{(1010)}^{G}\chi_{(10)_c}^{F} + \chi_{(1001)}^{G}\chi_{(10)_b}^{F} + \chi_{(0011)}^{G}\chi_{(10)_a}^{F}) + v^7\chi_{(1011)}^{G}$$

$$- \sum_{n=0}^{\infty} v^{5+2n}\chi_{(0n00)}^{G}\chi_{(01)_a\otimes(01)_b\otimes(01)_c}^{F}$$

$$+ \sum_{n=0}^{\infty} v^{6+2n}(\chi_{(1n00)}^{G} + \chi_{(0n10)}^{G} + \chi_{(0n01)}^{G})\chi_{(01)_{(abc)}\otimes(01)_{(abc)}\otimes(10)_{(abc)}}^{F}$$

$$- \sum_{n=0}^{\infty} v^{7+2n}\bigg((\chi_{(1n10)}^{G} + \chi_{(1n01)}^{G} + \chi_{(0n11)}^{G})\chi_{(01)_{(abc)}\otimes(10)_{(abc)}\otimes(10)_{(abc)}}^{F}$$

$$+ (\chi_{(2n00)}^{G} + \chi_{(0n20)}^{G} + \chi_{(0n02)}^{G})\chi_{(01)_{(abc)}\otimes(01)_{(abc)}}^{F}\bigg)$$

$$+ \sum_{n=0}^{\infty} v^{8+2n}\bigg(\chi_{(1n11)}^{G}\chi_{(10)_a\otimes(10)_b\otimes(10)_c}^{F}$$

$$+ (\chi_{(2n10)}^{G} + \chi_{(2n01)}^{G} + \chi_{(1n20)}^{G} + \chi_{(1n02)}^{G} + \chi_{(0n21)}^{G} + \chi_{(0n12)}^{G})\chi_{(10)_{(abc)}\otimes(01)_{(abc)}}^{F}$$

$$-\sum_{n=0}^{\infty} v^{9+2n}\Big(\big(\chi^G_{(2n20)}\chi^F_{(01)_c} + \chi^G_{(2n02)}\chi^F_{(01)_b} + \chi^G_{(0n22)}\chi^F_{(01)_a}\big)$$
$$+ \big(\chi^G_{(2n11)} + \chi^G_{(1n21)} + \chi^G_{(1n12)}\big)\chi^F_{(10)_{(abc)}\otimes(10)_{(abc)}}\Big) \tag{H.19}$$
$$+\sum_{n=0}^{\infty} v^{10+2n}\big(\chi^G_{(2n21)}\chi^F_{(10)_c} + \chi^G_{(2n12)}\chi^F_{(10)_b} + \chi^G_{(1n22)}\chi^F_{(10)_a}\big) - \sum_{n=0}^{\infty} v^{11+2n}\chi^G_{(2n22)}.$$

As a further check, upon specializing $\mathbf{m}_G$, $\mathbf{m}_F \to 1$ this expression matches with the result in table 92.

- **n = 2, G = SO(9), F = Sp(3)$_\mathbf{a}$ × Sp(2)$_\mathbf{b}$.** We are able to determine the first 8 coefficients of $Z_{1-\mathrm{inst}}$ unequivocally:

$$Z_{1-\mathrm{inst}} = v^{-1} - v^3\chi^F_{(01)_b} - v^4\big(\chi^F_{(100)_a\otimes(01)_b} + \chi^F_{(001)_a}\big) + v^5\big(\chi^G_{(1000)}\chi^F_{(01)_b} - \chi^F_{(010)_a\otimes(20)_b}\big)$$
$$+ v^6\big(\chi^G_{(0100)}\chi^F_{(100)_a\otimes(20)_b} + \chi^G_{(0100)}\chi^F_{(100)_a} + \chi^G_{(0001)}\chi^F_{(010)_a\otimes(10)_b} - \chi^F_{(001)_a\otimes(02)_b}\big)$$
$$+ \mathcal{O}(v^7). \tag{H.20}$$

At higher powers of v, large numbers of representations are allowed to contribute, and it becomes difficult to identify the correct representations.

- **n = 2, G = SO(10), F = Sp(4)$_\mathbf{a}$ × SU(2)$_\mathbf{b}$ × U(1)$_\mathbf{c}$.** In this case we are only able to unambiguously identify the first few coefficients of the Nekrasov partition function:

$$Z_{1-\mathrm{inst}} = v^{-1} - v^3\chi^F_{(2)_b} - v^4\chi^F_{(1000)_a\otimes((2)_c\oplus(-2)_c)} + \mathcal{O}(v^5). \tag{H.21}$$

At higher powers of v, large numbers of representations are allowed to contribute, and it becomes difficult to identify the correct representations.

- **n = 2, G = SO(12)$_\mathbf{a}$, F = Sp(6)$_\mathbf{a}$ × U(1)$_\mathbf{b}$.** We have determined the Nekrasov partition function up to $\mathcal{O}(v^9)$. We find:

$$Z_{1-\mathrm{inst}} = v^{-1} - v^3\chi^F_{(2)_b\oplus(-2)_b} - v^5\chi^F_{(010000)_a} + v^6\chi^G_{(100000)}\chi^F_{(100000)_a}$$
$$- v^7\big(\chi^G_{(200000)}\chi^F_{(01)_b} + \chi^F_{(000100)_a\otimes((2)_b\oplus(-2)_b)} + \chi^F_{(000001)_a}\big)$$
$$+ v^8\big(\chi^G_{(100000)}\chi^F_{(001000)_a\otimes((2)_b\oplus(-2)_b)}\big) + \mathcal{O}(v^9). \tag{H.22}$$

- **n = 2, G = SO(12)$_\mathbf{b}$, F = Sp(6).** We have determined the Nekrasov partition function up to $\mathcal{O}(v^{10})$. We find:

$$Z_{1-\mathrm{inst}} = v^{-1} - 2v^5\chi^F_{(100000)} + v^6\big(2\chi^G_{(100000)} + \chi^F_{(010000)}\big) - v^7\big(\chi^G_{(100000)}\chi^F_{(100000)} + 2\chi^F_{(001000)}\big)$$
$$+ v^8\big(\chi^G_{(200000)} + 2\chi^G_{(200000)}\chi^F_{(010000)} - \chi^F_{(000001)}\big) - v^9\big(2\chi^G_{(200000)}\chi^F_{(100000)} + 2\chi^F_{(000010)}\big)$$
$$+ \mathcal{O}(v^{10}). \tag{H.23}$$

- **n = 2, G = SO(13), F = Sp(7).** We have determined the first several coefficients of the Nekrasov partition function:

$$
\begin{aligned}
Z_{1-\text{inst}} &= v^{-1} - v^4 \chi^F_{(1000000)} + v^5 \chi^G_{(10000)_a} - v^7 \chi^G_{(0001000)} + v^8(\chi^G_{(10000)} \chi^F_{(0010000)} - \chi^F_{(0000001)}) \\
&- v^9 \chi^G_{(200000)} \chi^F_{(0100000)} + v^{10}(\chi^G_{(300000)} \chi^F_{(1000000)} + \chi^G_{(000001)} \chi^F_{(0000010)} + \chi^G_{(010000)} \chi^F_{(0000100)})) \\
&+ v^{11}(\chi^G_{(400000)} + \chi^G_{(100001)} \chi^F_{(0000100)} + \chi^G_{(001000)} \chi^F_{(0000010)} + \chi^G_{(000100)} \chi^F_{(0001000)}) \\
&+ v^{12}(\chi^G_{(101000)} \chi^F_{(0000100)} + \chi^G_{(200001)} \chi^F_{(0001000)} + \chi^G_{(210000)} \chi^F_{(0010000)} - \chi^G_{(000010)} \chi^F_{(0000001)}) \\
&+ \mathcal{O}(v^{13}). \tag{H.24}
\end{aligned}
$$

- **n = 2, G = G$_2$, F = Sp(4).** We find the following all-order expression for $Z_{1-\text{inst}}$:

$$
\begin{aligned}
Z_{1-\text{inst}} &= v^{-1} - v^2 \chi^F_{(1000)} + v^3 \chi^G_{(10)} - \sum_{n=0}^{\infty} v^{3+2n} \chi^G_{0n} \chi^F_{(0001)} + \sum_{n=0}^{\infty} v^{4+2n} \chi^G_{(1n)} \chi^F_{(0010)} \\
&- \sum_{n=0}^{\infty} v^{5+2n} \chi^G_{2n} \chi^F_{(0100)} + \sum_{n=0}^{\infty} v^{6+2n} \chi^G_{(3n)} \chi^F_{(1000)} - \sum_{n=0}^{\infty} v^{7+2n} \chi^G_{(4n)}. \tag{H.25}
\end{aligned}
$$

Upon specializing $\mathbf{m}_G, \mathbf{m}_F \to 1$ this expression matches with the result in table 90.

- **n = 2, G = F$_4$, F = Sp(3).** We have determined the coefficients of $Z_{1-\text{inst}}$ up to $\mathcal{O}(v^{20})$ and conjecture that the Nekrasov partition function to all orders in v is given by:

$$
\begin{aligned}
Z_{1-\text{inst}} &= v^{-1} - v^4 \chi^F_{(001)} - v^5 \chi^F_{(101)} - v^6 \chi^F_{(201)} + v^7(\chi^G_{(1000)} \chi^F_{(010)} + \chi^G_{(0001)} \chi^F_{(101)} - \chi^F_{(030)}) \\
&+ v^8(\chi^G_{(1000)} \chi^F_{(300)} - \chi^G_{(0010)} \chi^F_{(001)} + \chi^G_{(0001)} \chi^F_{(120)}) \\
&- v^9(\chi^G_{(2000)} + \chi^G_{(1001)} \chi^F_{(200)} + \chi^G_{(0002)} \chi^F_{(020)} + \chi^G_{(0010)} \chi^F_{(210)}) \\
&+ v^{10}(\chi^G_{(1010)} \chi^F_{(100)} + \chi^G_{(0100)} \chi^F_{(300)} + \chi^G_{(0011)} \chi^F_{(110)}) \\
&- v^{11}(\chi^G_{(1100)} + \chi^G_{(0101)} \chi^F_{(200)} + \chi^G_{(0020)} \chi^F_{(010)}) + v^{12} \chi^G_{(0110)} \chi^F_{(100)} - v^{13} \chi^G_{(0200)} \\
&- \sum_{n=0}^{\infty} v^{8+2n} \chi^G_{(n,0,0,0)} \chi^F_{(0,0,3)} + \sum_{n=0}^{\infty} v^{9+2n} \chi^G_{(n,0,0,1)} \chi^F_{(0,1,2)} \\
&- \sum_{n=0}^{\infty} v^{10+2n} \left(\chi^G_{(n,0,1,0)} \chi^F_{(0,2,1)} + \chi^G_{(n,0,0,2)} \chi^F_{(1,0,2)} \right) \\
&+ \sum_{n=0}^{\infty} v^{11+2n} \left(\chi^G_{(n,1,0,0)} \chi^F_{(0,3,0)} + \chi^G_{(n,0,0,3)} \chi^F_{(0,0,2)} + \chi^G_{(n,0,1,1)} \chi^F_{(1,1,1)} \right) \\
&- \sum_{n=0}^{\infty} v^{12+2n} \left(\chi^G_{(n,0,1,2)} \chi^F_{(0,1,1)} + \chi^G_{(n,0,2,0)} \chi^F_{(2,0,1)} + \chi^G_{(n,1,0,1)} \chi^F_{(1,2,0)} \right) \\
&+ \sum_{n=0}^{\infty} v^{13+2n} \left(\chi^G_{(n,1,0,2)} \chi^F_{(0,2,0)} + \chi^G_{(n,0,2,1)} \chi^F_{(1,0,1)} + \chi^G_{(n,1,1,0)} \chi^F_{(2,1,0)} \right) \\
&- \sum_{n=0}^{\infty} v^{14+2n} \left(\chi^G_{(n,2,0,0)} \chi^F_{(3,0,0)} + \chi^G_{(n,0,3,0)} \chi^F_{(0,0,1)} + \chi^G_{(n,1,1,1)} \chi^F_{(1,1,0)} \right)
\end{aligned}
$$

$$+ \sum_{n=0}^{\infty} v^{15+2n} \left(\chi^G_{(n,1,2,0)} \chi^F_{(0,1,0)} + \chi^G_{(n,2,0,1)} \chi^F_{(2,0,0)} \right)$$

$$- \sum_{n=0}^{\infty} v^{16+2n} \chi^G_{(n,2,1,0)} \chi^F_{(1,0,0)} + \sum_{n=0}^{\infty} v^{17+2n} \chi^G_{(n,3,0,0)}. \tag{H.26}$$

As a check, upon specializing $\mathbf{m}_G, \mathbf{m}_F \to 1$ this expression matches with the result in table 90.

- **n = 3, G = SO(7), F = Sp(1)$_\mathbf{a}$ × Sp(4)$_\mathbf{b}$.** We have determined the coefficients in $Z_{1-\text{inst}}$ up to $\mathcal{O}(v^{20})$ and conjecture the following all-order expression:

$$Z_{1-\text{inst}} = \sum_{n=0}^{\infty} v^{4+2n} \chi^G_{(0n0)} \chi^F_{(01)} - \sum_{n=0}^{\infty} v^{5+2n} \chi^G_{(0n1)} \chi^F_{(10)} + \sum_{n=0}^{\infty} v^{6+2n} \chi^G_{(0n2)}. \tag{H.27}$$

As a check, upon specializing $\mathbf{m}_G, \mathbf{m}_F \to 1$ this expression matches with the result in table 91.

- **n = 3, G = SO(8), F = Sp(1)$_\mathbf{a}$ × Sp(1)$_\mathbf{b}$ × Sp(1)$_\mathbf{c}$.** As in the $n = 2$, $G = $ SO(8) case, we assume invariance of the elliptic genus spectrum under triality to help us determine the representations that appear in the Nekrasov partition function. However, we are not able to completely fix the answer. We find:

$$Z_{1-\text{inst}} = v^4 + \sum_{n=0}^{\infty} v^{5+2n} \chi^G_{(0n00)} \chi^F_{(1)_a \otimes (1)_b \otimes (1)_c}$$

$$- \sum_{n=0}^{\infty} v^{6+2n} (\chi^G_{(1n00)} + \chi^G_{(0n10)} + \chi^G_{(0n01)}) \chi^F_{(1)_{abc} \otimes (1)_{abc}} \tag{H.28}$$

$$+ \sum_{n=0}^{\infty} v^{7+2n} (\chi^G_{(1n10)} \chi^F_{(1)_c} + \chi^G_{(1n01)} \chi^F_{(1)_b} + \chi^G_{(0n11)} \chi^F_{(1)_c}) - \sum_{n=0}^{\infty} v^{8+2n} \chi^G_{(1n11)}.$$

- **n = 3, G = SO(9), F = Sp(2)$_\mathbf{a}$ × Sp(1)$_\mathbf{b}$.** We have determined the coefficients of $Z_{1-\text{inst}}$ up to $\mathcal{O}(v^{14})$ and conjecture the following all-order expression:

$$Z_{1-\text{inst}} = v^5 \chi^F_{(10)_a} - v^6 \chi^G_{(1000)}$$

$$+ \sum_{n=0}^{\infty} v^{6+2n} \chi^G_{(0n00)} \chi^F_{(01)_a \otimes (2)_b} - \sum_{n=0}^{\infty} v^{7+2n} (\chi^G_{(1n00)} \chi^F_{(10)_a \otimes (2)_b} + \chi^G_{(0n01)} \chi^F_{(01)_a \otimes (1)_b})$$

$$+ \sum_{n=0}^{\infty} v^{8+2n} (\chi^G_{(0n10)} \chi^F_{(01)_a} + \chi^G_{(2n00)} \chi^F_{(2)_b} + \chi^G_{(1n01)} \chi^F_{(10)_a \otimes (1)_b})$$

$$- \sum_{n=0}^{\infty} -v^{9+2n} (\chi^G_{(2n01)} \chi^F_{(1)_b} + \chi^G_{(1n10)} \chi^F_{(10)_a}) + \sum_{n=0}^{\infty} v^{10+2n} \chi^G_{(2n10)}. \tag{H.29}$$

As a check, upon specializing $\mathbf{m}_G, \mathbf{m}_F \to 1$ this expression matches with the result in table 91.

• **n = 3, G = SO(10), F = Sp(3)$_a$ × U(1)$_b$.** We have determined the coefficients of $Z_{1-\text{inst}}$ up to $\mathcal{O}(v^{14})$ and conjecture the following all-order expression:

$$
\begin{aligned}
Z_{1-\text{inst}} = {}& v^6 \chi^F_{(010)_a} - v^7 \chi^G_{(100000)} \chi^F_{(100)_a} + v^8 \chi^G_{(20000)} + \sum_{n=0}^{\infty} v^{7+2n} \chi^G_{(0n000)} \chi^F_{(001)_a \otimes ((2)_b \oplus (-2)_b)} \\
& - \sum_{n=0}^{\infty} v^{8+2n} (\chi^G_{(1n000)} \chi^F_{(010)_a \otimes ((2)_b \oplus -2)_b)} + \chi^G_{(0n001)} \chi^F_{(001)_a \otimes ((1)_b \oplus (-1)_b)}) \\
& + \sum_{n=0}^{\infty} v^{9+2n} (\chi^G_{(2n000)} \chi^F_{(100)_a \otimes ((2)_b \oplus (-2)_b)} + \chi^G_{(1n001)} \chi^F_{(010)_a \otimes ((1)_b \oplus (-1)_b)} + \chi^G_{(0n100)} \chi^F_{(001)_a}) \\
& - \sum_{n=0}^{\infty} v^{10+2n} (\chi^G_{(3n000)} \chi^F_{(2)_b \oplus (-2)_b} + \chi^G_{(2n001)} \chi^F_{(100)_a \otimes ((1)_b \oplus (-1)_b)} + \chi^G_{(1n100)} \chi^F_{(010)_a}) \\
& + \sum_{n=0}^{\infty} v^{11+2n} (\chi^G_{(3n001)} \chi^F_{\otimes ((1)_b \oplus (-1)_b)} + \chi^G_{(2n100)} \chi^F_{(100)_a}) - \sum_{n=0}^{\infty} v^{12+2n} \chi^G_{(3n100)}. \quad \text{(H.30)}
\end{aligned}
$$

As a check, upon specializing $\mathbf{m}_G, \mathbf{m}_F \to 1$ this expression matches with the result in table 91.

• **n = 3, G = SO(11), F = Sp(4).** We have determined the coefficients of $Z_{1-\text{inst}}$ up to $\mathcal{O}(v^{16})$ and conjecture the following all-order expression:

$$
\begin{aligned}
Z_{1-\text{inst}} = {}& v^7 \chi^F_{(0010)} - v^8 \chi^G_{(10000)} \chi^F_{(0100)} + v^9 \chi^G_{(20000)} \chi^F_{(1000)} - v^{10} \chi^G_{(30000)} \\
& + \sum_{n=0}^{\infty} v^{8+2n} \chi^G_{(0n000)} \chi^F_{(0001)} - \sum_{n=0}^{\infty} v^{9+2n} (\chi^G_{(1n000)} \chi^F_{(0010)} + \chi^G_{(0n001)} \chi^F_{(0001)}) \\
& + \sum_{n=0}^{\infty} v^{10+2n} (\chi^G_{(2n000)} \chi^F_{(0100)} + \chi^G_{(1n001)} \chi^F_{(0010)} + \chi^G_{(0n100)} \chi^F_{(0001)}) \\
& - \sum_{n=0}^{\infty} v^{11+2n} (\chi^G_{(3n000)} \chi^F_{(1000)} + \chi^G_{(2n001)} \chi^F_{(0100)} + \chi^G_{(1n100)} \chi^F_{(0010)}) \\
& + \sum_{n=0}^{\infty} v^{12+2n} (\chi^G_{(4n000)} + \chi^G_{(3n001)} \chi^F_{(1000)} + \chi^G_{(2n100)} \chi^F_{(0100)}) \\
& - \sum_{n=0}^{\infty} v^{13+2n} (\chi^G_{(4n001)} + \chi^G_{(3n100)} \chi^F_{(1000)}) + \sum_{n=0}^{\infty} v^{14+2n} \chi^G_{(4n100)}. \quad \text{(H.31)}
\end{aligned}
$$

As a check, upon specializing $\mathbf{m}_G, \mathbf{m}_F \to 1$ this expression matches with the result in table 91.

• **n = 3, G = SO(12), F = Sp(5).** The matter content consists of one spinor half-hypermultiplet of definite chirality. We have determined the coefficients of $Z_{1-\text{inst}}$ up to $\mathcal{O}(v^{18})$, except for the fact that we are not able to determine the spinor labels appearing

in the various coefficients. We find the following all-order expression:

$$
\begin{aligned}
Z_{1-\text{inst}} = {}& v^8 \chi^F_{(00010)} - v^9 \chi^G_{(100000)}\chi^F_{(00100)} + v^{10}\chi^G_{(200000)}\chi^F_{(01000)} \\
& - v^{11}\chi^G_{(300000)}\chi^F_{(10000)} + v^{12}\chi^G_{(400000)} \\
& - \sum_{n=0}^{\infty} v^{10+2n}\chi^G_{(0n00,a,1-a)}\chi^F_{(00001)} + \sum_{n=0}^{\infty} v^{11+2n}(\chi^G_{(1n00,a,1-a)}\chi^F_{(00010)} \\
& + \chi^G_{(0n1000)}\chi^F_{(00001)}) \\
& - \sum_{n=0}^{\infty} v^{12+2n}(\chi^G_{(2n00,a,1-a)}\chi^F_{(00100)} + \chi^G_{(1n1000)}\chi^F_{(00010)}) \\
& + \sum_{n=0}^{\infty} v^{13+2n}(\chi^G_{(3n00,a,1-a)}\chi^F_{(01000)} + \chi^G_{(2n1000)}\chi^F_{(00100)}) \\
& - \sum_{n=0}^{\infty} v^{14+2n}(\chi^G_{(4n00,a,1-a)}\chi^F_{(10000)} + \chi^G_{(3n1000)}\chi^F_{(01000)}) \\
& + \sum_{n=0}^{\infty} v^{15+2n}(\chi^G_{(5n00,a,1-a)} + \chi^G_{(4n1000)}\chi^F_{(10000)}) - \sum_{n=0}^{\infty} v^{16+2n}\chi^G_{(5n1000)}, \quad \text{(H.32)}
\end{aligned}
$$

where $a = 0$ or 1 for each term.

As a check, upon specializing $\mathbf{m}_G, \mathbf{m}_F \to 1$ this expression matches with the result in table 91.

• **n = 3, G = G$_2$, F = Sp(1).** We have determined the coefficients of $Z_{1-\text{inst}}$ up to $\mathcal{O}(v^{20})$ and conjecture the following all-order expression:

$$
Z_{1-\text{inst}} = \sum_{n=0}^{\infty} v^{3+2n}\chi^G_{(0n)}\chi^F_{(1)} - \sum_{n=0}^{\infty} v^{4+2n}\chi^G_{(1n)}. \qquad \text{(H.33)}
$$

As a check, upon specializing $\mathbf{m}_G, \mathbf{m}_F \to 1$ this expression matches with the result in table 91.

• **n = 3, G = F$_4$, F = Sp(2).** We have determined the coefficients of $Z_{1-\text{inst}}$ up to $\mathcal{O}(v^{16})$ and conjecture the following all-order expression:

$$
\begin{aligned}
Z_{1-\text{inst}} = {}& v^6 \chi^F_{(01)} + v^7 \chi^F_{(30)} - v^8(\chi^G_{(1000)} + \chi^G_{(0001)}\chi^F_{(20)}) + v^9 \chi^G_{(0010)}\chi^F_{(10)} - v^{10}\chi^G_{(0100)} \\
& + \sum_{n=0}^{\infty} v^{8+2n}\chi^G_{(n000)}\chi^F_{(03)} - \sum_{n=0}^{\infty} v^{9+2n}\chi^G_{(n001)}\chi^F_{(12)} \\
& + \sum_{n=0}^{\infty} v^{10+2n}(\chi^G_{(n010)}\chi^F_{(21)} + \chi^G_{(n002)}\chi^F_{(02)}) \\
& - \sum_{n=0}^{\infty} v^{11+2n}(\chi^G_{(n100)}\chi^F_{(30)} + \chi^G_{(n011)}\chi^F_{(11)})
\end{aligned}
$$

$$+ \sum_{n=0}^{\infty} v^{12+2n}(\chi^G_{(n020)}\chi^F_{(01)} + \chi^G_{(n101)}\chi^F_{(20)})$$
$$- \sum_{n=0}^{\infty} v^{13+2n}\chi^G_{(n110)}\chi^F_{(10)} + \sum_{n=0}^{\infty} v^{14+2n}\chi^G_{(n200)}. \tag{H.34}$$

As a check, upon specializing $\mathbf{m}_G, \mathbf{m}_F \to 1$ this expression matches with the result in table 91.

- **n = 3, G = E₆, F = SU(3)ₐ × U(1)_b.** In this case we are only able to unambiguously identify the first few coefficients of the Nekrasov partition function:

$$Z_{1-\text{inst}} = v^7\chi^F_{(3)_b \oplus (-3)_b} - v^8(\chi^F_{(30)_a} + \chi^F_{(03)_a}) + v^9\chi^F_{(22)_a \otimes ((3)_b \oplus (-3)_b)}$$
$$- v^{10}(\chi^G_{(100000)}\chi^F_{(12)_a \otimes (-2)_b} + \chi^G_{(000010)}\chi^F_{(21)_a \otimes (2)_b} + \chi^G_{(000001)}\chi^F_{(11)_a}$$
$$- \chi^F_{(30)_a \otimes (\pm 6)_b} - \chi^F_{(03)_a \otimes (\mp 6)_b} - \chi^F_{(33)_a}) + \mathcal{O}(v^{12}), \tag{H.35}$$

where in the last line we do not have sufficient information to fix all U(1) charges.

- **n = 4, G = F₄, F = Sp(1).** We have determined the coefficients of $Z_{1-\text{inst}}$ up to $\mathcal{O}(v^{16})$ and conjecture the following all-order expression:

$$Z_{1-\text{inst}} = v^7 - \sum_{n=0}^{\infty} v^{8+2n}\chi^G_{(n000)}\chi^F_{(3)} + \sum_{n=0}^{\infty} v^{9+2n}\chi^G_{(n001)}\chi^F_{(2)} - \sum_{n=0}^{\infty} v^{10+2n}\chi^G_{(n010)}\chi^F_{(1)}$$
$$+ \sum_{n=0}^{\infty} v^{11+2n}\chi^G_{(n100)}. \tag{H.36}$$

As a check, upon specializing $\mathbf{m}_G, \mathbf{m}_F \to 1$ this matches with the result in table 91.

- **n = 4, G = E₆, F = SU(2)ₐ × U(1)_b.** In this case we are only able to unambiguously identify the first few coefficients of the Nekrasov partition function:

$$Z_{1-\text{inst}} = v^9\chi^F_{(2)_a} + v^{10}\chi^F_{(3)_a \otimes ((3)_b \oplus (-3)_b)} - v^{11}(\chi^G_{(100000)}\chi^F_{(2)_a \otimes (-2)_b} + \chi^G_{(000010)}\chi^F_{(2)_a \otimes (2)_b}$$
$$+ \chi^G_{(000001)} - \chi^F_{(6)_b \oplus (-6)_b} - \chi^F_{(6)_a}) + \mathcal{O}(v^{12}). \tag{H.37}$$

- **n = 5, G = E₆, F = U(1).** We have determined the coefficients of $Z_{1-\text{inst}}$ up to $\mathcal{O}(v^{20})$ and conjecture the following all-order expression:

$$Z_{1-\text{inst}} = v^{10} + \sum_{n=0}^{\infty} v^{11+2n}\chi^G_{(00000n)}\chi^F_{(3)\oplus(-3)} + \sum_{n=0}^{\infty} v^{12+2n}(\chi^G_{(10000n)}\chi^F_{(-2)} + \chi^G_{(00001n)}\chi^F_{(2)})$$
$$+ \sum_{n=0}^{\infty} v^{13+2n}(\chi^G_{(01000n)}\chi^F_{(-1)} + \chi^G_{(00010n)}\chi^F_{(1)}) + \sum_{n=0}^{\infty} v^{14+2n}\chi^G_{(00100n)} \tag{H.38}$$

As a check, upon specializing $\mathbf{m}_G, \mathbf{m}_F \to 1$ this matches with the result in table 92.

- **n = 5, G = E$_7$, F = SO(3).** We write the representations of $G = \mathrm{SO}(3)$ as representations of SU(2). We have determined the coefficients of $Z_{1-\mathrm{inst}}$ up to $\mathcal{O}(v^{30})$ and conjecture the following all-order expression:

$$
\begin{aligned}
Z_{1-\mathrm{inst}} &= v^{14}\chi^F_{(6)} - v^{16}(\chi^G_{(1000000)}\chi^F_{(4)} - \chi^F_{(12)}) + v^{17}(\chi^G_{(0000001)}\chi^F_{(4)} - \chi^G_{(0000010)}\chi^F_{(10)}) \\
&\quad + v^{18}(\chi^G_{(2000000)} + \chi^G_{(0000100)}\chi^F_{(8)}) - v^{19}(\chi^G_{(0001000)}\chi^F_{(6)} + \chi^G_{(1000001)}\chi^F_{(2)}) \\
&\quad + v^{20}(\chi^G_{(0010000)}\chi^F_{(4)} + \chi^G_{(1100000)} + \chi^G_{(0000002)}\chi^F_{(2)}) - v^{21}\chi^G_{(0100001)}\chi^F_{(2)} + v^{22}\chi^G_{(0200000)} \\
&\quad + \sum_{n=0}^{\infty} v^{19+2n}\chi^G_{(n000001)}\chi^F_{(12)} - \sum_{n=0}^{\infty} v^{20+2n}(\chi^G_{(n000011)}\chi^F_{(10)} + \chi^G_{(n100000)}\chi^F_{(12)}) \\
&\quad + \sum_{n=0}^{\infty} v^{21+2n}(\chi^G_{(n000101)}\chi^F_{(8)} + \chi^G_{(n100010)}\chi^F_{(10)}) - \sum_{n=0}^{\infty} v^{22+2n}(\chi^G_{(n001001)}\chi^F_{(6)} + \chi^G_{(n100100)}\chi^F_{(8)}) \\
&\quad + \sum_{n=0}^{\infty} v^{23+2n}(\chi^G_{(n010001)}\chi^F_{(4)} + \chi^G_{(n101000)}\chi^F_{(6)} + \chi^G_{(n000003)}) \qquad (\mathrm{H.39}) \\
&\quad - \sum_{n=0}^{\infty} v^{24+2n}(\chi^G_{(n100002)}\chi^F_{(2)} + \chi^G_{(n110000)}\chi^F_{(4)}) + \sum_{n=0}^{\infty} v^{25+2n}\chi^G_{(n200001)} + \sum_{n=0}^{\infty} v^{26+2n}\chi^G_{(n300000)}.
\end{aligned}
$$

Note that as expected only representations of SO(3), corresponding to representations of SU(2) with labels of the type $(2n)$, appear.

As a check of our result, upon specializing $\mathbf{m}_G, \mathbf{m}_F \to 1$ this expression matches with the result in table 92.

- **n = 6, G = E$_7$, F = SO(2).** We have determined the coefficients of $Z_{1-\mathrm{inst}}$ up to $\mathcal{O}(v^{20})$ and conjecture the following all-order expression:

$$
\begin{aligned}
Z_{1-\mathrm{inst}} &= v^{15}\chi^F_{(2)\oplus(-2)} - v^{17}\chi^G_{(1000000)} + v^{18}\chi^G_{(0000001)}\chi^F_{(1)\oplus(-1)} - v^{19}\chi^G_{(0100000)} \\
&\quad + \sum_{n=0}^{\infty} v^{17+2n}\chi^G_{(n000000)}\chi^F_{(6)\oplus(-6)} - \sum_{n=0}^{\infty} v^{18+2n}\chi^G_{(n000010)}\chi^F_{(5)\oplus(-5)} \\
&\quad + \sum_{n=0}^{\infty} v^{19+2n}\chi^G_{(n000100)}\chi^F_{(4)\oplus(-4)}) - \sum_{n=0}^{\infty} v^{20+2n}\chi^G_{(n001000)}\chi^F_{(3)\oplus(-3)} \\
&\quad + \sum_{n=0}^{\infty} v^{21+2n}(\chi^G_{(n010000)}\chi^F_{(2)\oplus(-2)} + \chi^G_{(n000002)}) - \sum_{n=0}^{\infty} v^{22+2n}\chi^G_{(n100001)}\chi^F_{(1)\oplus(-1)} \\
&\quad + \sum_{n=0}^{\infty} v^{23+2n}\chi^G_{(n200000)}. \qquad (\mathrm{H.40})
\end{aligned}
$$

We label the representations of SO(2) = U(1) in terms of their U(1) charges.

As a check of our result, upon specializing $\mathbf{m}_G, \mathbf{m}_F \to 1$ this expression matches with the result in table 92.

- **n = 7, G = E$_7$.** For this theory the flavor symmetry is trivial. We find the following all-order expression for the one-instanton Nekrasov partition function:

$$
Z_1 = v^{16} + \sum_{n=0}^{\infty} v^{19+2n}\chi^G_{(n00001)} + \sum_{n=0}^{\infty} v^{20+2n}\chi^G_{(n100000)}, \qquad (\mathrm{H.41})
$$

which agrees in the $\mathbf{m}_G, \mathbf{m}_F \to 1$ limit to the result in table 92.

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