FARVaR: Functional Autoregressive Value-at-Risk[*]

Charlie X. Cai[†], Minjoo Kim[‡], Yongcheol Shin[§] and Qi Zhang[¶]

Abstract

Motivated by the stylized fact that intraday returns can provide additional information on the tail behaviour of daily returns, we propose a functional autoregressive value-at-risk approach which can directly incorporate such informational advantage into the daily value-at-risk forecast. Our approach leads to greater flexibility in modelling the dynamic evolution of the density function of intraday returns and the ability to capture substantial swings in the tails following major events. We comprehensively evaluate our proposed model using intraday transaction data and demonstrate that it can improve coverage ability, reduce economic cost and enhance statistical reliability in market risk management.

Key words: Density Forecasts, Market Risk Management, Functional Autoregressive Model.

JEL classification: C53; G17

In the aftermath of the 2008 global financial crisis, the effective role of value-at-risk (VaR) in risk management has been subjected to intense debate.[1] Although the crisis revealed that the main

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[†]Finance and Accounting, Management School, University of Liverpool, Liverpool L69 3BX, UK. Email: X.Cai7@liverpool.ac.uk.

[‡]**Corresponding author**: Finance and Accounting, Management School, University of Liverpool, Liverpool L69 3BX, UK. Email: Minjoo.Kim@liverpool.ac.uk. Tel: +44 (0)151 795 3000

[§]Economics, University of York, York YO10 5DD, UK. Email: yongcheol.shin@york.ac.uk.

[¶]Economics and Finance, Durham University Business School, Durham University, Durham DH1 3LB, UK. Email: qi.zhang2@durham.ac.uk.

weakness of VaR lies in ignoring the liquidity risk and underestimating the correlation risk, the general consensus is that VaR is still one of the most important risk management tools available (e.g., Croft, 2011). The search for more accurate VaR modelling will therefore continue (e.g., Adams et al., 2014).

Recently, we have been confronted with new challenges. The ever-increasing prominence of computer-based trading in the financial market makes it more important to carefully examine risks arising from the automated intraday activities typically related to high-frequency trading (HFT). While the market microstructure literature is still debating whether and how HFT will influence market quality, intraday dynamics are expected to become an increasingly important input to price discovery.[2] Even for daily risk management, we expect to develop a more accurate VaR model by capturing information generated from the higher frequency data. The usefulness of intraday information has been demonstrated extensively in the realized volatility literature (Andersen et al., 2001; Engle and Giampiero, 2006), though the direct incorporation of intraday return information in risk management modelling is still at an early stage (e.g., Fuertes and Olmo, 2013; Hallam and Olmo, 2014).

It is well-established that intraday returns follow a non-normal and time-varying distribution (e.g., Andersen et al., 2001). Moreover, it would be extremely challenging to model such complex intraday return dynamics, as their random character may be further complicated by market microstructure noise. To address this challenge, we propose a novel semiparametric approach, called functional autoregressive value-at-risk (FARVaR), for forecasting the daily VaR using intraday information.[3] First, we estimate the density of intraday returns nonparametrically for each day and apply the functional autoregressive (FAR) model [4] to the sequence of intraday densities constructed over T days to obtain h-day-ahead forecasts of intraday density. In general, it would be non-trivial to generate the density function of daily returns directly from that of intraday returns, even if the density function of intraday returns were known or given. To address this challenging issue, we suggest two approaches. The first is parametric and relies on the use of the normal inverse Gaussian (NIG) distribution[5] for the returns. The second is based on a nonparametric sampling scheme, which can produce a bootstrap density function of the daily return. Since we are more interested in modelling the VaR of portfolios containing multiple assets in many finance applications, we extend FARVaR for a single asset into one for multiple assets by incorporating copula (mFARVaR).

FARVaR provides a few advances over the existing literature. First, it avoids any (distributional) uncertainty associated with misspecified parametric models, by estimating the intraday density nonparametrically. Furthermore, FAR can easily overcome the shortcomings of nonparametric models by capturing such complex dynamic structure via a functional autoregressive operator, which can represent all contemporaneous and time-dependent associations among all the moments or quantiles. Second, high-frequency financial data is often characterized as extremely dispersed and non-normally distributed (Hasbrouck, 2007). Through a detailed exploratory analysis of the time-varying intraday moments, we establish two stylized facts: (i) volatility and skewness of intraday returns are rather persistent; (ii) there exist complex (potentially nonlinear) and time-varying associations among the moments. In this regard, we expect that neither the parametric nor the nonparametric approach will have the capacity to unravel an exact relationship among intraday moments or quantiles. By contrast, FARVaR is designed to utilize evolutions of intraday return density in directly forecasting daily return density and daily VaR in a flexible manner. Third, while parametric models can reduce the economic cost but tend to underestimate VaR (Netftci, 2000), nonparametric models can provide conservative coverage ability but fail to reduce the economic cost. This fundamental trade-off may reflect their respective forecasting inaccuracies. As confirmed by our empirical evaluation, FARVaR can simultaneously improve the coverage ability and reduce the economic cost. Last, the VaR forecast of FARVaR is less sensitive to the undelying market regime than that of existing semiparametric VaR models. This makes FARVaR superior to other models in terms of coverage ability.

We conduct various evaluation schemes using real data from 30 stocks listed in the Dow Jones Industrial Average (DJIA) index over the period 2000–2008. All the evaluations are based on out-of-sample forecasting. First, we evaluate the performance of intraday return density forecasting for different functional models. We find that FAR is the best predictor for the density of intraday returns as it produces the smallest divergence among the functional models, including a functional martingale process and a functional i.i.d. process.

Next, we employ a wide range of backtesting tools to evaluate and compare the performance of FARVaR against the existing VaR models. A backtest is not only a formal framework for verifying whether the actual loss is in line with the projected loss, but also an essential procedure for selecting suitable internal VaR models for capital requirements as recommended by the Basel

Committee on Banking Supervision (BCBS). We conduct three broad types of test. We examine the coverage ability and the economic cost using conventional quantitative measures such as the empirical coverage probability and the predictive quantile loss (Koenker and Bassett, 1978). We also assess the Basel penalty zone (BCBS, 1996) and the market risk capital requirement (BCBS, 1996, 2005). In addition, we evaluate the statistical adequacy of the VaR models using the conditional coverage test (Christoffersen, 1998) and the dynamic quantile test (Engle and Manganelli, 2004).

These backtesting results demonstrate that FARVaR is the most reliable VaR model. The Basel II Accord was designed to encourage sensible risk-taking using appropriate (internal) models of risk to forecast daily VaR and capital charges. Within the Basel II rules banks may prefer to report high VaRs to avoid the possibility of regulatory intrusion.[6] This conservative risk reporting suggests that efficiency gains may be feasible (McAleer et al., 2013). We demonstrate that FARVaR can achieve such a non-trivial goal by simultaneously improving coverage ability, reducing economic cost and enhancing statistical reliability.

1 FARVaR

The aim of our study is to develop a novel risk-management model which forecasts a daily VaR using intraday information. In this context, an important issue is how best to develop an appropriate econometric methodology for estimating and forecasting the density of daily returns using intraday returns. The most popular approach is to construct realized volatilities from intraday returns and forecast a daily volatility by applying a parametric model to the realized volatilities.[7] However, this approach uses only the realized volatilities, so it is not sufficiently broad to generate the entire density of daily returns. In practice, this approach is bound to suffer from potentially misspecified parametric assumptions imposed on the dynamics of realized volatilities and the distribution of daily returns.

We therefore develop the semiparametric FARVaR, which generates the density forecast of daily returns directly from the density forecast of intraday returns. It is a two-step procedure, as illustrated in Figure 1. First, we estimate the density of intraday returns nonparametrically by a kernel density estimator and forecast an intraday density by means of FAR. Second, we propose two approaches to constructing the density forecast of daily returns directly from the density forecast of intraday returns: the parametric approximation based on the general class of the normal inverse Gaussian (NIG) distribution, and a nonparametric bootstrap scheme. Given the density forecast of daily returns, we can then easily compute a daily VaR.

[FIGURE 1 ABOUT HERE]

1.1 Forecasting Density Function of Intraday Returns by FAR

Bosq (2000) introduces the statistical foundation of FAR modelling. Its asymptotic theory is refined by Cardot et al. (2007) and Mas (2007). Park and Qian (2012) develop the functional regression of a continuous state distribution which is more general than FAR. FAR has been applied to forecasting a climate pattern with regard to temperature (Besse et al., 2000) and ozone (Aneiros-Perez et al., 2004; Damon and Guillas, 2002). Recently, the FAR approach has been adopted in economics and finance. Laukaitis (2008) applies it to forecasting an intraday cash flow and a transaction intensity in the credit card payment system. Bowsher and Meeks (2008) and Kargin and Onatski (2008) employ it to forecast a yield curve as a function of maturity. Chaudhuri et al. (2016) apply it to modelling the cross-sectional distribution of sectoral inflation rates nonparametrically, and develop the flexible framework for forecasting the density of national inflation rates based on FAR.

The foundation of forecasting an intraday density function relies heavily on Park and Qian (2012). Thus we refer readers to the asymptotic theory and its proofs in that paper. In the current paper, we provide detailed descriptions in Appendix Section I for the estimation of FAR and the forecasting of the intraday density function, to facilitate replication.

Using the series of asset prices observed at a fixed time interval (e.g., 5 minutes)[8], we calculate an intraday return by the first difference of logged prices, $r_{t,i} = \ln P_{t,i} - \ln P_{t,i-1}$ for $t = 1, \ldots, T$ and $i = 1, \ldots, m$. The index t denotes the day t and the index i denotes the ith observation over the fixed interval after a market opens on day t. Thus $P_{t,0}$ denotes the opening price and $P_{t,m}$ the closing price on day t.[9] For example, given a 5-minute interval, $r_{2,1}$ and $r_{2,2}$ denote the realized stock returns at 9:35AM and 9:40AM on the second day (see endnote 8). A daily return is then calculated as $r_t = \ln P_{t,m} - \ln P_{t,0}$. As a result, we have the following sequences of time series:

$$X = (X_t)_{t=1}^{T} \text{ and } X_t = (r_{t,i})_{i=1}^{m}, \tag{1}$$

where X_t is an intraday return path on day t and X is the set of intraday return paths over T days.

First, we make an assumption on intraday returns that is crucial for applying FAR to their density function.

Assumption 1 (Piecewise stationarity) *Intraday returns are strictly stationary in short time intervals, so that the density functions of intraday returns can be consistently estimated for each day. However, this local stationarity does not carry over to longer horizons.*

Let the density function of intraday returns on day t be f_t, such that the sequence of intraday densities (f_t) can be well-defined in $\mathcal{H} = L^2(\mathcal{C})$, the Hilbert spaces of square integrable functions defined on the compact subset $\mathcal{C}$ of $\mathbb{R}$. Thus we can treat f_t as a functional-valued random variable. Under **Assumption 1**, it would be very reasonable to model the time-varying density function of intraday returns by the FAR of order 1:

$$w_t = Aw_{t-1} + \epsilon_t, \; t = 1, \ldots, T, \tag{2}$$

where $w_t = f_t - \mathbb{E}[f]$ is the fluctuation of the density function from the well-defined common expectation of the density function, $\mathbb{E}[f]$[10], and A is the so-called autoregressive operator.

Assumption 2 *We assume that:*

(a) A is a compact linear operator in $\mathcal{H}$, satisfying $\|A^\kappa\| < 1$ for some $\kappa > 1$[11];

(b) (ϵ_t) is a sequence of the functional white noise such that $\mathbb{E}[\epsilon_t] = 0$ and $\mathbb{E}\|\epsilon_t\|^4 < \infty$, and is independent of w_0. Further, its distribution is a probability measure on $\mathcal{H}$ such that $\epsilon_t : (\Omega, \mathcal{F}, \mathbb{P}) \mapsto \mathcal{H}$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is the underlying probability space.

The one-step-ahead forecast of a density function can be evaluated by the conditional expectation of the density function on a past information set $(\mathcal{F}_{t-1})$:

$$\mathbb{E}[f_t|\mathcal{F}_{t-1}] = \mathbb{E}[f] + Aw_{t-1}. \tag{3}$$

If A is the zero operator, the best predictor would be the unconditional expectation of density function (AVE). On the other hand, if A is the identity operator, the best predictor would be the last observation (LAST).

Since (f_t) cannot be directly observed in practice, we estimate the density function of intraday returns at each point in time, t. We estimate it nonparametrically using the kernel density estimator[12]:

$$\hat{f}_t(x_j) = \frac{1}{nh_t} \sum_{i=1}^{m} K\left(\frac{x_j - r_{t,i}}{h_t}\right),\ t = 1, \ldots, T;\ j = 1, \ldots, n, \tag{4}$$

where K is a kernel, m is the number of intraday returns, h_t a bandwidth and n the number of discrete grids.[13] One important issue is the selection of kernel and bandwidth value. The optimal value of bandwidth is derived by minimizing a loss function and applying a cross-validation selector. We employ the popular Gaussian kernel and follow Silverman's (1986) rule of thumb such that the optimal bandwidth is given by $1.06\hat{\sigma}_t m^{-1/5}$, with $\hat{\sigma}_t$ being the sample standard deviation of $r_{t,i}$. See also Härdle and Linton (1994) for more detail on the choice of kernel and bandwidth value.

The autoregressive operator in Equation (2) is defined as $A = C_1 C_0^{-1}$, where $C_0 = \mathbb{E}[w_t \otimes w_t]$ and $C_1 = \mathbb{E}[w_t \otimes w_{t-1}]$ are the autocovariance operators of order 0 and 1, respectively. However, since the autocovariance operators are defined in an infinite dimension, the inverse of C_0 is not well-defined; that is, it presents an ill-posed inverse problem. To avoid this, we project C_0 onto a finite L-dimensional subspace in $\mathcal{H}$. Let us denote the inverse of C_0 on the L-dimensional subspace by $C_{0,L}^{+}$. The autoregressive operator restricted to the L-dimensional subspace is $A_L = C_1 C_{0,L}^{+}$. The choice of L is guided by a functional principal component analysis (FPCA) and a cross-validation (CV) (see Ramsey and Silverman, 1997). Then the estimator of A in the L-dimensional subspace is given by $\hat{A}_L = \hat{C}_1 \hat{C}_{0,L}^{+}$. Therefore, the one-step-ahead forecast conditional on the information set $(\mathcal{F}_T)$ is evaluated by

$$\hat{f}_{T+1} = \bar{f} + \hat{A}_L \hat{w}_T, \tag{5}$$

where $\bar{f} = T^{-1} \sum_{t=1}^{T} \hat{f}_t$ and $\hat{w}_T = \hat{f}_T - \bar{f}$. See Theorem 5 of Park and Qian (2012) for the consistency of $\hat{A}_L$ and our Appendix Section I for the estimation of FAR and the forecasting of the intraday density in detail.

In general, the FAR algorithm involves non-negligible computation time. Hence we introduce two popular transforms of $\hat{w}_t$, by the fast Fourier transform and the wavelet transform. Both transforms can reduce the computation time dramatically (making it 30 times faster) by shrinking the dimension of the function. Once we obtain the forecast of the transformed $\hat{w}_{T+1}$, we then recover the original forecast by the inverse Fourier or wavelet transform, and evaluate the density forecast in Equation (5) (see Antoniadis and Sapatinas, 2003; Besse et al., 2000, for details). We call these computationally efficient transforms, respectively, FAR-fft and FAR-wv.

1.2 Forecasting Density Function of Daily Returns and VaR from Density Forecasts of Intraday Returns

We aim to explicitly employ information on intraday returns to improve the forecasting precision of the tail behaviour of daily returns. This requires us to generate the density forecast of the daily returns directly from the density forecast of intraday returns. Let the probability and cumulative distribution functions of the daily returns on day t be g_t and G_t, respectively. In general, this task would be nontrivial even if we successfully obtained the h-day-ahead density forecast of the intraday returns, f_{t+h}, via FAR, as in the previous section. To address this challenging issue, we therefore suggest two approaches. We first consider a parametric approximation on the basis of the normal inverse Gaussian (NIG) distribution, regarded as one of the most flexible distributions for describing the distribution of returns (Barndorff-Nielsen, 1997). We then propose a more general nonparametric bootstrap approximation.

1.2.1 NIG approximation approach (FARVaR-nig)

This approach employs the NIG parametric approximation and constructs the density of daily returns through a flexible match of the first four moments from the density of intraday returns. Let μ_t, v_t, s_t and k_t be, respectively, the mean, variance, skewness and kurtosis of the intraday returns on day t. Then we calculate the four parameters $(\alpha_t, \beta_t, \gamma_t, \delta_t)$ that determine the shape of the NIG distribution as follows:

$$\alpha_t = v_t^{-\frac{1}{2}} \left(3k_t - 4s_t^2 - 9\right)^{\frac{1}{2}} \left(k_t - \frac{5}{3}s_t^2 - 3\right)^{-1}, \beta_t = s_t v_t^{-\frac{1}{2}} \left(k_t - \frac{5}{3}s_t^2 - 3\right)^{-1}, \tag{6}$$
$$\gamma_t = \mu_t - 3s_t v_t^{\frac{1}{2}} \left(3k_t - 4s_t^2 - 9\right)^{-1}, \delta_t = 3^{\frac{3}{2}} \left\{v_t \left(k_t - \frac{5}{3}s_t^2 - 3\right)\right\}^{\frac{1}{2}} \left(3k_t - 4s_t^2 - 9\right)^{-1},$$

where α_t determines the tail heaviness, β_t the asymmetry, γ_t the location and δ_t the scale of the distribution. Note that the kurtosis should satisfy $k_t > 3 + (5/3)\, s_t^2$. The density of the intraday returns is then approximated by the NIG density:

$$f_t\left(x\right) = \left[\frac{\alpha_t \delta_t J_1\left(\alpha_t \sqrt{\delta_t^2 + \left(x - \gamma_t\right)^2}\right)}{\pi \sqrt{\delta_t^2 + \left(x - \gamma_t\right)^2}}\right] e^{\delta_t \lambda_t + \beta_t \left(x - \gamma_t\right)}, \tag{7}$$

where $\lambda_t = \sqrt{\alpha_t^2 - \beta_t^2}$ and J_1 denotes the modified Bessel function of the second kind. The NIG distribution is closed under the convolution of independent random variables X_1 and X_2:

$$X_1 \sim NIG\left(\alpha, \beta, \gamma_1, \delta_1\right) \text{ and } X_2 \sim NIG\left(\alpha, \beta, \gamma_2, \delta_2\right)$$
$$\Rightarrow X_1 + X_2 \sim NIG\left(\alpha, \beta, \gamma_1 + \gamma_2, \delta_1 + \delta_2\right). \tag{8}$$

Hence the density function of the daily returns on day t ($r_t = \sum_{i=1}^{m} r_{t,i}$), denoted $g_t\left(x\right)$, can be approximated by the following NIG density:

$$g_t\left(x\right) = \left[\frac{m \alpha_t \delta_t J_1\left(\alpha_t \sqrt{m^2 \delta_t^2 + \left(x - m\gamma_t\right)^2}\right)}{\pi \sqrt{m^2 \delta_t^2 + \left(x - m\gamma_t\right)^2}}\right] e^{m\delta_t \lambda_t + \beta_t \left(x - m\gamma_t\right)}, \tag{9}$$

where m is the number of intraday observations on day t. Finally, we can evaluate a daily VaR from the cumulative NIG density function of the daily returns. Notice, however, that the analytic form of the cumulative NIG density function does not exist. Hence we approximate it by numerically evaluating the integral of the NIG density in Equation (9) (see also Equation (A.25) in Appendix Section I).

1.2.2 Simulation approach (FARVaR-sim)

The simulation approach is based on repeated random sampling as follows. It is straightforward to construct the cumulative distribution function of the intraday returns from the density forecast of the intraday returns by

$$\hat{F}_{T+1}\left(x\right) = \int_{-\infty}^{x} \hat{f}_{T+1}\left(s\right) ds. \tag{10}$$

As $\hat{f}_{T+1}$ is the discrete approximation of the continuous density function, we also approximate $\hat{F}_{T+1}$ numerically by the following middle Riemann sum:

$$\hat{F}_{T+1}(z_j) = \frac{1}{2}\left[\sum_{i=1}^{j} \hat{f}_{T+1}(x_{i+1})\,\Delta x_{i+1} + \sum_{i=1}^{j} \hat{f}_{T+1}(x_i)\,\Delta x_{i+1}\right],\ j = 1, \ldots, n-1, \tag{11}$$

where $z_j = (x_{j+1} + x_j)/2$ and $\Delta x_{i+1} = x_{i+1} - x_i$.

Next, given **Assumption 1**, we can randomly draw intraday returns from the cumulative distribution function, $\hat{F}_{T+1}$, in Equation (11), in the following way. First, we draw m real numbers randomly from the uniform distribution over $(0, 1)$, and denote them by $\left\{p_1^{(b)}, p_2^{(b)}, \ldots, p_m^{(b)}\right\}$, where $p_i^{(b)} \sim U(0,1)$ for each bootstrap iteration, $b = 1, \ldots, B$. $\hat{F}_{T+1}$ is discretized over the grid set, $\{z_1, z_2, \ldots, z_{n-1}\}$, while $p_i^{(b)}$ is drawn from a continuous domain. In this case, an equality in $\hat{F}_{T+1}(z_j) = p_i^{(b)}$ does not hold for some i . Then we select a z_j such that $\hat{F}_{T+1}(z_j)$ is the nearest to $p_i^{(b)}$. We present this grid selection conveniently by

$$z_i^{(b)} \equiv \underset{z_j \in \{z_1, \ldots, z_{n-1}\}}{\operatorname{argmin}} \left|\hat{F}_{T+1}(z_j) - p_i^{(b)}\right|. \tag{12}$$

We collect a set of m simulated intraday returns in $\left\{z_1^{(b)}, z_2^{(b)}, \ldots, z_m^{(b)}\right\}$ and then construct a daily return by $r^{(b)} = \sum_{i=1}^{m} z_i^{(b)}$ for each iteration, $b = 1, \ldots, B$. Finally, we can approximate the (empirical) cumulative distribution function of the daily returns by

$$\hat{G}_{T+1}(\omega) = \frac{1}{B}\sum_{b=1}^{B} 1\left\{r^{(b)} \leq \omega\right\}. \tag{13}$$

See also Figure 2 for a graphical representation of the simulation approach.

It is then straightforward to evaluate a daily VaR forecast given a nominal probability, α by

$$\widehat{VaR}_{T+1}(\alpha) = \sup\left(\omega | \hat{G}_{T+1}(\omega) \leq \alpha\right). \tag{14}$$

[FIGURE 2 ABOUT HERE]

With regard to our proposed FARVaR approach, we note and address the following issues. First, the randomness of density forecasts is generated by two elements, as shown in Equation (5). The first is associated with the estimation of the autoregressive operator. Since a true density function is unobservable, we have to estimate it. Thus the second is associated with the estimation of the density function. Although it is not straightforward to construct the asymptotic confidence interval of a VaR forecast, we can obtain it by randomly sampling in-sample prediction errors, $\{\hat{\epsilon}_1, \ldots, \hat{\epsilon}_T\}$. See Appendix Section III for the bootstrap-based construction of the confidence interval. Figure 3 presents the empirical distributions of VaR forecasts by FARVaR-nig and FARVaR-sim, along with their bootstrapped 95% confidence intervals.

[FIGURE 3 ABOUT HERE]

Second, the simulation approach is a more general and robust approach than the NIG approximation, because the latter requires strong distributional assumptions: that (i) an intraday return density is characterized by the first four moments such that it follows the NIG distribution, and (ii) intraday returns are independently and identically distributed. In particular, the validity of the second assumption is questionable since many studies have documented evidence that the intraday returns exhibit dependency or autocorrelation (e.g., Andersen and Bollerslev, 1997). However, in the case where such dependency, say at the 5-minute interval, is relatively negligible, we may expect that the NIG approximation works reasonably well.

Third, our proposed FARVaR approach is closely related to the recent study by Hallam and Olmo (2014), who propose a method for estimating the density of daily returns directly from intraday returns by rescaling them through multiplying the scaling factor, under the assumption that the intraday returns are self-affine or unifractal. They estimate a daily density by applying either the location-scale t-distribution or the kernel density estimator to transformed daily returns, and forecast the density of the daily returns by forecasting distributional parameters via a simple autoregressive model or as the weighted sum of past densities. Although their approach is more flexible than the previous studies, it is still restrictive in two ways. First, the higher-order moments such as skewness and kurtosis of both transformed daily and intraday returns are equivalent under the common rescaling. This would violate the market microstructure stylized evidence that intraday returns are more skewed and leptokurtic than daily returns. Second, their proposed density forecasting approach is more restrictive than our proposed FAR modelling in the sense that the dependence structure allowed in their approach is substantially simpler.[14]

2 Multivariate FARVaR

So far we have developed FARVaR for a single asset. However, we are more interested in modelling VaR for a portfolio containing multiple assets in many finance applications, such as a portfolio containing basket options like those for the maximum performance of several stocks. In this case, the dependence structure (or correlation) of the multiple assets is of particular importance for computing a portfolio VaR.

However, there are limitations in directly applying FAR to a multivariate density function. Let us take a bivariate density function as an example. For a univariate density function, $f(x) : \mathbb{R} \to \mathbb{R}_+$, the autoregressive operator is defined in the form $A := \{a(i,j)\}$ for $i, j \in \mathbb{R}$. For the bivariate density function, $f(x,y) : \mathbb{R}^2 \to \mathbb{R}_+$, the autoregressive operator is defined, however, in the form $A := \{a(i,j,k)\}$ for $i, j, k \in \mathbb{R}$. Thus we cannot apply estimations and inferences developed for the univariate density function directly to the multivariate function. For this reason, we develop FARVaR for multiple assets in an indirect way using a copula (hereafter mFARVaR) in this paper.

Let us consider a portfolio containing K assets:

$$r_{p,t,i} = \sum_{k=1}^{K} w_k r_{k,t,i},\ i = 1, \ldots, m; t = 1, \ldots, T. \tag{15}$$

According to Sklar's theorem, every K marginal distribution function $\{F_{1,t}, \ldots, F_{K,t}\}$ can be written as

$$F_t\left(r_{1,t,i}, \ldots, r_{K,t,i}\right) = C_t\left(F_{1,t}\left(r_{1,t,i}\right), \ldots, F_{K,t}\left(r_{K,t,i}\right)\right),\ t = 1, \ldots, T;\ i = 1, \ldots, m \tag{16}$$

for some copula C_t which is uniquely determined on $[0,1]^K$ for the multivariate distribution function with absolutely continuous margins. Conversely, any copula C_t of their joint distribution function may be extracted from Equation (16) by evaluating

$$C_t\left(\boldsymbol{u}_{t,i}\right) := C_t\left(u_{1,t,i}, \ldots, u_{K,t,i}\right) = F_t\left(F_{1,t}^{-1}\left(u_{1,t,i}\right), \ldots, F_{K,t}^{-1}\left(u_{K,t,i}\right)\right), \tag{17}$$

where $F_{k,t}^{-1}$ is the quantile function of the margin. In our exercise, we can compute $\boldsymbol{u}_{t,i} = (u_{1,t,i}, \ldots, u_{K,t,i})'$ from the nonparametric density function, f_k, and estimate the copula C_t given the K marginal density functions for each day.

Next, we assume that C_t follows VAR(1), which is reasonably consistent with the FAR of order 1. For simplicity, we assume that C_t has the form of a multivariate Student's $t\left(\nu_t, \mathbf{0}, \boldsymbol{P}_t\right)$-distribution, where $\boldsymbol{P}_t$ is the correlation matrix and ν_t the degree of freedom. The unique copula is given by

$$C_{\nu_t,\boldsymbol{P}_t}\left(\boldsymbol{u}_{t,i}\right) = \int_{-\infty}^{t_{\nu_t}^{-1}(u_{1,t,i})} \cdots \int_{-\infty}^{t_{\nu_t}^{-1}(u_{K,t,i})} \frac{\Gamma\left(\frac{\nu_t+K}{2}\right)}{\Gamma\left(\frac{\nu_t}{2}\right)\sqrt{\left(\pi\nu_t\right)^K \left|\boldsymbol{P}_t\right|}} \left(1 + \frac{\boldsymbol{x}'\boldsymbol{P}_t^{-1}\boldsymbol{x}}{\nu_t}\right)^{-\frac{\nu_t+K}{2}} dx_K \cdots dx_1, \tag{18}$$

where $t_{\nu_t}^{-1}$ denotes the quantile function of a standard univariate t_{ν_t} distribution. Then we model $vech(\boldsymbol{P}_t)$ and ν_t by VAR(1) and AR(1), as

$$vech(\boldsymbol{P}_t) = \mathbf{c} + \boldsymbol{\Gamma} vech(\boldsymbol{P}_{t-1}) + \mathbf{e}_t, \tag{19}$$

$$\nu_t = \alpha + \varphi\nu_{t-1} + \zeta_t, \tag{20}$$

and forecast both, which are then used to construct a copula forecast, $\hat{C}_{T+1} \equiv C_{\hat{\nu}_{T+1},\hat{\boldsymbol{P}}_{T+1}}$.

Hence we can forecast the K marginal intraday density functions by FAR and their copula by VAR(1). Then we randomly draw $\boldsymbol{u}$ from the copula forecast,

$$\boldsymbol{u}_{T+1,i}^{(b)} = \left(u_{1,T+1,i}^{(b)}, \ldots, u_{K,T+1,i}^{(b)}\right)', \tag{21}$$

and convert these into intraday returns using the K marginal density forecasts, $\hat{f}_{1,T+1}, \ldots, \hat{f}_{K,T+1}$:

$$r_{1,T+1,i}^{(b)} = \hat{F}_{1,T+1}^{-1}\left(u_{1,T+1,i}^{(b)}\right), \ldots, r_K^{(b)} = \hat{F}_{K,T+1}^{-1}\left(u_{K,T+1,i}^{(b)}\right),\ i = 1, \ldots, m. \tag{22}$$

See Equations (10) and (11) for the numerical computation of the cumulative distribution function from the density forecast.

We update the intraday returns of the portfolio using the simulated intraday returns,

$$r_{p,T+1,i}^{(b)} = \sum_{k=1}^{K} w_k r_{k,T+1,i}^{(b)},\ i = 1, \ldots, m, \tag{23}$$

and generate daily portfolio returns by summing the intraday portfolio returns,

$$r_{p,T+1}^{(b)} = \sum_{i=1}^{m} r_{p,T+1,i}^{(b)}, \tag{24}$$

for each iteration $b = 1, 2, \ldots, B$.

Finally, we approximate the (empirical) cumulative distribution function of the daily portfolio returns by

$$\hat{G}_{T+1}(\omega) = \frac{1}{B}\sum_{b=1}^{B} 1\left\{r_{p,T+1}^{(b)} \leq \omega\right\}. \tag{25}$$

We then evaluate the daily VaR forecast for the nominal probability, α, by

$$\widehat{VaR}_{p,T+1}(\alpha) = \sup\left(\omega|\hat{G}_{T+1}(\omega) \leq \alpha\right). \tag{26}$$

Appendix Section II describes the practical algorithm of mFARVaR in detail.

To recap, we have developed the multivariate FARVaR approach together with associated econometric techniques. This clearly shows that the FAR modelling can be extended to analyzing the dependence structure among multiple assets. In practice, we can apply the multivariate FARVaR to the model with the large number of assets (say, 100 or more) so far as we can collect sufficient intraday samples and/or we can employ any factor-based copula.

3 Intraday Data

We consider the constituents of the Dow Jones Industrial Average (DJIA) index over the period 2000–2008.[15] Two market downturns were experienced during the sample period: one caused by the dot-com bubble burst (2001–2002) and the other by the subprime mortgage crisis (2007–2008). In order to guarantee the reliability of backtesting, recommended practice is to include both normal and market downturn periods. The more market events in a sample period, the more scenarios for backtesting; nevertheless, our sample period meets a minimum requirement for the reliability of backtesting.

Three companies experienced crucial financial problems in 2008: bailouts for American International (AIG) and Citigroup were announced by the Federal Reserve Board of Governors; in the case of General Motors (GM), a government bridge loan was given to the auto manufacturers by the US government, which is classed as a *de facto* bailout. It is clearly important for regulators to understand the risk profiles of such companies, so as to monitor and safeguard the US financial system. At the same time, these companies are actively traded and their transactions are thus likely to generate huge amounts of information, especially at the intraday level. Hence we aim to address the important empirical question of whether and how the use of intraday information can improve risk management modelling in practice.

There were five times of change in the components of the DJIA over the sample period: 27 Jan 2003, 8 Apr 2004, 21 Nov 2005, 19 Feb 2008 and 22 Sep 2008.[16] We choose the benchmark list of constituents as at 21 Nov 2005, which corresponds to the middle of our sample period and includes the three financially distressed companies (AIG, Citigroup and GM) which were removed from the list in 2008. See Table 1 for details of the 30 components. Intraday transaction data is collected from the Trade and Quote (TAQ) database. Following existing studies such as Lee and Ready (1991) and Hvidkjaer (2006), we have utilized a filtering procedure to exclude any data likely to be erroneous. Specifically, all the trades (quotes) with condition codes A, C, D, G, L, N, O, R, X, Z, 8, 9 (4, 5, 7–9, 11, 13–17, 19, 20) are eliminated. The trades with a correction code greater than 2 are also removed. (Refer to the TAQ manual for definition of the codes.) Quotes are excluded if the ask is equal to or less than the bid, if the ask spread is above 75% of mid-quote, or if the ask (bid) is more than double or less than half of the previous ask (bid). We only consider trades reported from 9:30 AM to 4:00 PM. According to Lee and Ready (1991), trades occur 5 seconds earlier than the reported time. Thus we calculate trade time as the reported time minus 5 seconds. Trades are deleted if the trade price is more than double or less than half of the previous trade. After filtering, we calculate the closing trade and quote prices in every 5-minute interval (see footnote 8). Closing trade and quote prices are the price of the last trade in each interval and the corresponding quote price when the trade occurs. If there is no trade, the price for the current interval is replaced by the closest trade and quote prices from the previous interval.

[TABLE 1 ABOUT HERE]

3.1 Descriptive Statistics

Table 2 presents the descriptive statistics of the intraday and daily returns of the 30 stocks over the sample period, 2000–2008. Panel A reports results for the intraday returns. The mean returns are very close to zero, a consistent finding in the market microstructure literature. The standard deviation ranges widely between 0.17 (Intel) and 0.54 (AIG). There is significant evidence of asymmetry as skewness is non-zero and mainly negative. As expected, kurtosis is significantly higher than that of the normal distribution, highlighting the typical fat tails of financial data.

Panel B reports the descriptive statistics of the daily returns. The mean returns are small and around a few basis points. The average returns of AIG (-0.17%) and Citigroup (-0.08%) are relatively low, reflecting that they experienced bailouts in 2008. The maximum (minimum) and the standard deviation of these companies are also much larger (smaller). The asymmetry of the return distribution is not as severe as in the intraday returns, though AIG (-6.49) and Procter & Gamble (-5.17) are substantially negatively skewed. Fat tails are observed for all the companies. In particular, the kurtosis of AIG (158.5), Citigroup (47.82) and Procter & Gamble (120.3) are significantly higher. Overall, these findings not only confirm the typical characteristics of financial time series but also display some extreme statistics, especially for companies bailed out during the global financial crisis in 2008.

[TABLE 2 ABOUT HERE]

3.2 Time-Varying Moments of Intraday Returns

To enhance our understanding of complex intraday return dynamics, we provide a time-varying descriptive analysis of the four moments (mean, standard deviation, skewness and kurtosis) of the intraday returns on the value-weighted portfolio of 30 stocks.

In Figure 4 we display the autocorrelation function (ACF) of volatility (standard deviation) and skewness respectively. As expected, volatility exhibits high persistence with its first-order autoregressive (AR(1)) coefficient of around 0.76; this is a consistent finding with those documented in Andersen et al. (2001). Skewness is weakly persistent with its AR(1) coefficient slightly negative at -0.074, but statistically significant. On the other hand, the ACFs of mean and kurtosis are statistically insignificant (and not reported here).

[FIGURE 4 ABOUT HERE]

We turn to analyzing time-varying patterns of contemporaneous correlation among the moments. Figure 5 displays scatter plots between moment pairs.[17] First, from plot (a), we find a U-shaped relationship between return and volatility, suggesting that the correlation is regime-dependent and measured at 0.67 when the mean is positive (bull market) and -0.63 when it is negative (bear market). This finding is qualitatively consistent with the recent literature documenting a non-monotonic relationship between return and volatility (e.g., Rossi and Timmermann, 2010; Chiang and Li, 2012). The U-shaped risk–return relationship observed at the intraday frequency is also in line with the market microstructure model (e.g., Hasbrouck, 2007, chap. 2), where the unconditional mean reverts to zero at high frequency. The high-frequency trading strategy should be a directional bet with information advantage (e.g., knowledge about order flows), since bearing additional volatility risk will not be necessarily compensated by higher returns. This evidence may suggest that the imposition of a time-invariant and linear risk–return trade-off would lead to inaccurate and misleading forecasts.

Second, plot (b) shows that mean and skewness are positively associated, with a correlation coefficient of 0.51. There has been mixed evidence on the mean-skewness relationship, mostly at lower frequencies (weekly or monthly; e.g., Conrad et al., 2013 and Rehmany and Vilkovz, 2012). To the best of our knowledge, there is no documented evidence at the intraday level. Intuitively, our finding suggests that extreme values or outliers may play a significant role in generating intraday returns, as a few large positive or negative movements are likely to render the average return moving in the same direction.

Third, we find from plot (c) that there also exists a U-shaped relationship between volatility and skewness, with the correlation measured at -0.37 when skewness is negative (downside risk) and 0.23 when skewness is positive (upside uncertainty). Given that we obtain the U-shaped risk–return and the linear return-skewness trade-off, this finding is consistent with transitivity. It confirms that extreme large price movements (positive or negative skewness) are associated with higher volatility. The steeper slope observed under downside risk also confirms that the volatility–skewness relationship is stronger when the market is hit by bad news.

Fourth, plot (d) displays a strong quadratic U-shaped relationship between skewness and kurtosis. The correlation is measured at -0.96 when skewness is negative and 0.98 when skewness is positive. This quadratic U-shaped relationship is a pleasing finding mathematically.

[FIGURE 5 ABOUT HERE]

Overall, our time-varying moment-based analysis reveals two stylized facts. First, the second and third moments of the intraday return density are persistent. Second, contemporaneous associations among the four moments are all complex and time-varying. These results provide support for the numerous lower-frequency studies showing that the time-varying characteristics of the higher-order moments should be carefully modelled for the purposes of portfolio allocation and asset pricing (Cenesizoglu and Timmermann, 2012; Harvey and Siddique, 2000). This highlights the challenging issue of modelling intraday return dynamics. A nonparametric approach tends to ignore the time-varying nature of the density function, which leads to an inaccurate forecast. The fully parametric approach, on the other hand, suffering from misspecification, is ill-equipped to unravel an exact relationship among the higher-order moments, resulting in the erroneous measurement of risk. In this regard, we expect FARVaR to take into account the relative advantages of both approaches and thus deal in a robust manner with the complex characteristics of intraday returns when forecasting their density.

3.3 Time-Varying Dependence Structure of Multiple Intraday Returns

The correlation structure of individual assets constituting a portfolio is an essential input to portfolio risk management. Recently, more attention has been focused on dependence structure, which is a broader concept than linear correlation (see McNeil et al., 2005). We too have developed mFARVaR on the basis of a copula by taking into account the dependence structure of the individual assets making up the portfolio. In this section, we estimate the copula correlation coefficients between individual intraday returns and analyze their dynamics.

Given the limited number of intraday returns for each day, we construct 10 portfolios, each consisting of three stocks. To avoid the selection bias, we randomly draw the 10 sets of stocks from 30 stocks without replacement. In each set, we construct the equal-weighted portfolio of the three stocks. We will also use these portfolios for the backtesting of mFARVaR in Section 6.

Table 3 presents the descriptive statistics and autocorrelation tests of copula correlation coefficients between individual intraday returns. We estimate the copula correlation coefficients of Student's t-copula for each day. (See Appendix Section II for estimating a copula in detail.) Panel A reports the minimum, maximum and mean values of the copula correlation coefficients for each pair of individual stocks. The mean value is between 0.2 and 0.4 and the range between –0.4 and 0.9, which shows the large variation of the copula correlation coefficients over time. In Panel B, we compute the sample autocorrelation function of order 1 and conduct the Ljung–Box Q test for the autocorrelations of the copula correlation coefficients at a lag of 5. As the test results show, there are significant autocorrelations in all portfolios.

[TABLE 3 ABOUT HERE.]

We confirm that there is a strong dependence structure of intraday returns constituting a portfolio, beyond their linear correlation, and that it follows the autoregressive process. The overall results therefore support mFARVaR as developed in Section 2.

4 Pseudo Out-of-Sample Forecasting

We conduct pseudo out-of-sample forecasting exercises to evaluate the forecasting performance of a number of functional models as proposed above (FAR, FAR-fft, FAR-wv, AVE and LAST; see Section 1.1 for details). In particular, we choose the continuous version of the fast Fourier transform algorithm, with 40 pairs of coefficients for FAR-fft, and 3-level Daubechies wavelets for FAR-wv, respectively. This practice of holding out a sample is called "pseudo real-time" experiment (e.g., Elliot and Timmermann, 2008). In this way we evaluate the performance in forecasting the density of intraday returns for each functional model.

To examine which of the five functional models described above can produce the most accurate density forecasts of intraday returns, we evaluate the three divergence criteria measuring the distance between the forecast and true density functions: the Hilbert norm (D_H), the uniform norm (D_U) and the generalized entropy (D_E):

$$D_H = \frac{\int \left(\hat{f}\left(x\right) - f\left(x\right)\right)^2 dx}{\int \hat{f}\left(x\right)^2 dx + \int f\left(x\right)^2 dx},\ D_U = \frac{\sup_x \left|\hat{f}_t\left(x\right) - f_t\left(x\right)\right|}{\sup_x f_t\left(x\right)},\ D_E = \int f\left(x\right) \pi \left(\frac{\hat{f}\left(x\right)}{f\left(x\right)}\right) dx, \quad (27)$$

where $\hat{f}$ (f) denotes the forecast (realized) density function of intraday returns and $\pi\left(y\right) = \left(\gamma - 1\right)^{-1}\left(y^{\gamma} - 1\right)$, with $\gamma > 0$ and $\gamma \neq 1$. Since the true density function is unobservable, we proxy it by the kernel density estimator in Equation (4). If π is the natural log, this becomes the Kullback–Leibler divergence measure. We use the generalized version to avoid the log of zero, which would happen when the density estimate or forecast has zero points. We set $\gamma = 1/2$ in our study. All three quantities (called the global error) are non-negative and produce a zero value if $\hat{f} = f$. D_H is useful for evaluating the model's goodness-of-fit, D_U for comparing the function shapes and D_E for assessing the difference in information content.

Throughout the paper, we employ the rolling sample approach, with a window size of 250 business days to accommodate any time-varying patterns. To begin with, we estimate the five functional models using intraday data over the period 3 Jan 2000 – 27 Dec 2000 and compute the one-day-ahead density forecast of intraday returns for 28 Dec 2000. We repeat this procedure by moving forward a day at a time in a rolling manner, ending with the density forecast for 31 Dec 2008. This generates 2013 density forecasts of intraday returns for each firm, except for HP, AT&T and Verizon, with 1428, 2006 and 1887 density forecasts, respectively.

We evaluate the mean divergence for each firm and report the average across 30 firms by

$$\bar{D}_H = \frac{1}{30}\sum_{k=1}^{30}\left(\frac{\sum_{t=1}^{N_k} D_{H,k,t}}{N_k}\right), \bar{D}_U = \frac{1}{30}\sum_{k=1}^{30}\left(\frac{\sum_{t=1}^{N_k} D_{U,k,t}}{N_k}\right), \bar{D}_E = \frac{1}{30}\sum_{k=1}^{30}\left(\frac{\sum_{t=1}^{N_k} D_{E,k,t}}{N_k}\right), \quad (28)$$

where N_k is the number of daily density forecasts for a firm, k. Overall, from Table 4, we find that the FAR models outperform the other functional models, LAST and AVE. In particular, FAR-fft turns out to be the best predictor, producing the smallest divergence. Therefore, we will employ the FAR-fft approach as our FAR model in the further analysis of VaR.

[TABLE 4 ABOUT HERE]

5 Backtesting: Single Asset

We examine the performance of FARVaR against a number of existing VaR models popularly employed by both academics and practitioners: historical simulation (HS), filtered historical simulation (FHS), RiskMetrics (RM; RiskMetrics, 1996), GARCH, the filtered extreme value theory (FEVT) models, conditional autoregressive value-at-risk by regression quantiles (CAViaR), and CAViaR-GARCH. HS is a static nonparametric model and most popular for simplicity (Perignon et al., 2008).[18] RM and GARCH are typical dynamic parametric models; RM assumes the normal distribution of asset returns while GARCH can also allow for the fat-tailed Student's t-distribution. FHS is a hybrid approach, applying HS to returns filtered by GARCH. FEVT is suggested to control for time-varying volatility (Diebold et al., 1998; McNeil and Rudiger, 2000); similarly to FHS, it applies the EVT procedure to returns filtered by GARCH. Here we consider the filtered generalized extreme value (FGEV) distribution and the filtered generalized Pareto distribution (FGPD). Finally, we examine CAViaR and CAViAR–GARCH, which incorporates GARCH in CAViaR (see Engle and Manganelli, 2004). In the literature, FHS has been one of the most successful VaR models (e.g., Barone-Adesi et al., 2002; Kuester et al., 2006; Pritsker, 2006). More details of the models can be found in Appendix Section V. Notice that all the existing VaR models use daily returns as defined in Section 1.1 (i.e., $r_t = \ln P_{t,m} - \ln P_{t,0}$).

We evaluate the performance of the VaR models in terms of coverage ability, economic cost and statistical validity. As strongly recommended by the Basel Committee on Banking Supervision (BCBS), backtesting is a key part of the internal VaR model development for market risk management. We employ a number of backtesting tools: empirical coverage probability (ECP), the Basel penalty zone (BPZ; BCBS, 1996), market risk capital requirement (MRCR; BCBS, 1996), predictive quantile loss (PQL; Koenker and Bassett, 1978), the conditional coverage test (CC test; Christoffersen, 1998) and the dynamic quantile test (DQ test; Engle and Manganelli, 2004).

Backtesting evaluates coverage ability and the economic cost arising from failure to cover the realized extreme event. Hence all backtesting tools have been developed on the basis of the failure of a model. The failure is defined by an indicator function which takes unity when a realized return is not covered by the VaR forecast:

$$H_s = 1\left\{r_s < \widehat{VaR}_s\left(\alpha\right)\right\},\ s = 1, \dots, N, \tag{29}$$

where $\widehat{VaR}_s\left(\alpha\right)$ is the VaR forecast given the information set available at $s-1$ with the nominal coverage probability α.

First, ECP and BPZ evaluate the coverage ability as follows: ECP is calculated by the sample average of H_s, i.e.,

$$ECP = \frac{1}{N}\sum_{s=1}^{N} H_s. \tag{30}$$

BPZ describes the strength of an internal VaR model through evaluating its failure rate, which is the number of daily violations of the 99% VaR over the previous 250 business days; we expect, on average, 2.5 violations under a correctly forecasting model. The Basel Committee rules that up to 4 violations are acceptable, and defines this range as the "Green" zone. If there are 5 or more violations, banks fall into the "Yellow" (5–9) or "Red" (10+) zones. The penalty is cumulatively imposed by the multiplicative factor (κ); this factor is determined according to the number of violations: 3 (for 0–4 violations), 3.4 (5), 3.5 (6), 3.65 (7), 3.75 (8), 3.85 (9) and 4 (10+). In the Yellow zone, the supervisor will decide the penalty according to the reason for the violation.[19] In the Red zone, the penalty will be automatically generated. Hence the regulator prefers the VaR model producing ECP close to the nominal probability and BPZ indicating the Green zone.

Second, MRCR and PQL evaluate the economic costs of the VaR model. Providing that a bank has a sound risk management system, an independent risk-control unit and external audits, MRCR is summarized by the following four factors: (i) the quantitative parameters, (ii) the treatment of correlations, (iii) the market risk charge, and (iv) the plus factor (see BCBS (1996) and BCBS (2005) for details and Jorion (2006) for a compact summary). Then MRCR can be formulated by

$$MRCR_s = \max\left(\kappa \frac{1}{60} \sum_{i=1}^{60} \widehat{VaR}_{s-i}(\alpha), \widehat{VaR}_{s-1}(\alpha)\right) + SRC_s, \; s = 251, \ldots, N, \tag{31}$$

where SRC is the additional capital charge for the specific risk (BCBS, 1996, 2005) and κ is the multiplicative factor from BPZ.[20] We report the average of MRCR as

$$MRCR = \frac{1}{N-250} \sum_{s=251}^{N} MRCR_s. \tag{32}$$

PQL measures the expected economic cost of the VaR model using the "check" function (Koenker and Bassett, 1978). It is consistently estimated by

$$PQL = \frac{1}{N} \sum_{s=1}^{N} (\alpha - H_s)\left[r_s - \widehat{VaR}_s(\alpha)\right]. \tag{33}$$

When the VaR forecast fails to cover a realized return, $|r_s - \widehat{VaR}_s(\alpha)|$ is the economic loss and we impose a harsh penalty on it. Alternatively, when it does cover the realized return, $|r_s - \widehat{VaR}_s(\alpha)|$ is the opportunity cost and we impose a mild penalty on it as compensation for its success. Thus we can rewrite PQL, by following our economic intuition, as

$$PQL = (1-\alpha)\left(\underbrace{\frac{1}{N}\sum_{s=1}^{N}\left|r_s - \widehat{VaR}_s(\alpha)\right| H_s}_{\text{Expected Economic Loss}}\right) + \alpha\left(\underbrace{\frac{1}{N}\sum_{s=1}^{N}\left|r_s - \widehat{VaR}_s(\alpha)\right|(1-H_s)}_{\text{Expected Opportunity Cost}}\right). \tag{34}$$

It is a reasonable measure of the expected economic cost of the VaR model, considering both the economic loss and the opportunity cost.

Finally, the CC and DQ tests evaluate the statistical validity of the VaR model. The CC test verifies whether the conditional expectation of H_s is equal to the coverage probability. Christoffersen (1998) shows that it is equivalent to testing whether H_s is an identically and independently distributed Bernoulli process with probability α. Hence the likelihood ratio statistic simultaneously tests whether the unconditional coverage probability is α (unconditional coverage test) and whether the binary random variable is independent (independence test). It follows the chi-squared distribution with two degrees of freedom under the null hypothesis. The DQ test extends the CC test by allowing for more time-dependent information such as lagged realized violations and the VaR forecast. Specifically, we regress the demeaned binary variable on (constant, lagged demeaned binary variable and the VaR forecast), and test the null, $R^2 = 0$, by the Wald test statistic given by

$$DQ = \frac{\left(\tilde{\mathbf{H}}'\mathbf{Z}\right)\left(\mathbf{Z}'\mathbf{Z}\right)^{-1}\left(\mathbf{Z}'\tilde{\mathbf{H}}\right)}{\alpha(1-\alpha)} = \frac{\hat{\boldsymbol{\beta}}'\mathbf{Z}'\mathbf{Z}\hat{\boldsymbol{\beta}}}{\alpha(1-\alpha)} \overset{a}{\sim} \chi^2_{p+2}, \tag{35}$$

where $\tilde{\mathbf{H}} = \left(\tilde{H}_{p+1}, \ldots, \tilde{H}_N\right)'$, $\tilde{H}_s = H_s - \alpha$, $\mathbf{Z} = (\mathbf{z}_{p+1}, \ldots, \mathbf{z}_N)'$ and $\mathbf{z}_s = \left(1, \tilde{H}_{s-1}, \ldots, \tilde{H}_{s-p}, \widehat{VaR}_s\right)'$. The regulator prefers the VaR model which is not rejected. We use the first four lags, that is, $\mathbf{z}_s = \left(1, \tilde{H}_{s-1}, \ldots, \tilde{H}_{s-4}, \widehat{VaR}_s\right)'$, where the DQ statistic follows the chi-squared distribution with six degrees of freedom under the null.

In the next subsections, we repeat the rolling estimation procedure as described in Section 4 and apply the backtesting to VaR forecasts. We estimate VaR models using the window size of 250 days over the period 3 Jan 2000 – 27 Dec 2000 and compute the one-day-ahead 99% VaR forecasts for 28 Dec 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for 31 Dec 2008. This generates 2013 density forecasts of intraday returns for each firm except for HP, AT&T and Verizon, which generate 1428, 2006 and 1887 forecasts, respectively. This is our main analysis. We also apply the procedure to the value-weighted portfolio of 30 stocks.

We further compare FARVaR with FHS to see how differently these two hybrid approaches use information for the VaR forecast. As a robustness check, we run various exercises in the following subsections. First, we apply the backtesting to subperiods divided by the market regimes: normal and market downturn periods. Second, we compare our probability density function (PDF)-based FARVaR with a cumulative distribution function (CDF)-based one. Third, we check the asymmetry of the return distribution. We investigate how FARVaR performs at the right tail by taking a short position. Finally, we consider a longer window size (500 days) for investigating any effect of the window size on our evaluations. We estimate VaR models using the longer window size of 500 days over the period 3 Jan 2000 – 31 Dec 2001 and compute the one-day-ahead 99% VaR forecast for 2 Jan 2001. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for 31 Dec 2008. This generates 1763 daily density forecasts of intraday returns for each firm except for HP, AT&T and Verizon, which generate 1178, 1756 and 1637 forecasts, respectively.

5.1 Main Analysis

For our main analysis, we apply the VaR models to the left-tail behaviour of return distribution with a window size of 250 days; that is, we hold a long position. We report the average of ECP, MRCR and PQL for the 30 stocks,

$$\overline{ECP} = \frac{1}{30}\sum_{k=1}^{30} ECP_k, \ \overline{MRCR} = \frac{1}{30}\sum_{k=1}^{30} MRCR_k, \ \overline{PQL} = \frac{1}{30}\sum_{k=1}^{30} PQL_k, \tag{36}$$

where k indicates a stock. We report BPZ based on the average violation of the 30 stocks. For the CC and DQ tests, we count the frequency of an individual model being rejected at the 5% significance level out of the 30 asset returns.

Table 5 presents the backtesting results for the long position. In terms of ECP, FARVaR-sim (FARVaR-nig) slightly over-forecasts (under-forecasts) VaR, though both outcomes stay in the BPZ Green zone. Overall, we find that the coverage ability of FARVaR is quite reliable. RM severely under-forecasts VaR, receiving the warning Yellow zone, a finding consistent with Johansson et al. (1999) and Netftci (2000). This implies that the coverage ability of RM is unreliable, rendering it inappropriate for use as an internal VaR model. Despite the fact that GARCH employs the (fat-tailed) Student's t-distribution, its results are not significantly better than those of RM. As expected, the FEVT models substantially over-forecast VaR, though their outcomes stay in the Green zone due to their conservative forecasting. (Under BPZ, over-forecasts of VaR tend to receive better scores, given the Basel Committee's prudential principle.) Finally, the CAViaR models considerably under-forecast VaR, receiving the warning Yellow (CAViaR) and Red (CAViaR-GARCH) zones, respectively. They turn out to be the worst models in terms of coverage ability.

Overall results reveal a number of stylized facts. First, the parametric models tend to under- or over-forecast the tail behaviour of the return distribution. Hence with sufficient sample observations, the nonparametric return distribution could generally improve coverage ability, as demonstrated by the performance of FARVaR and FHS. The coverage ability of the GARCH model can be significantly improved by simulating the tail behaviour from the nonparametric empirical distribution. Second, we can estimate the tail behaviour more precisely when we model the complex dynamic relationships directly among moments or quantiles rather than when we focus on modelling the specific quantile, say at 1% or 5%, like CAViaR. As FARVaR is designed to fully address these two important issues, it can therefore produce more reliable coverage ability than the existing parametric models, RM, GARCH and CAViaR.

Next, we turn to assessing the economic costs accompanying the different VaR models. RM requires the smallest MRCR, followed by GARCH and the FARVaR models. Conversely, the FEVT models incur the highest costs as they considerably over-forecast VaR, and thus necessitate the maintenance of high levels of capital, entailing high opportunity costs.[21] The CAViaR models also incur high economic costs to compensate for their poor coverage ability. FARVaR is able to reduce the economic costs significantly by producing reliable coverage ability as well as by explicitly addressing the dynamics of the return distribution.

Finally, we discuss the statistical adequacy of the different VaR models by applying the CC test and the DQ test to each of the 30 asset returns at the 5% significance level. The relevant columns report the number of individual models rejected out of 30 returns. The null hypothesis of the CC test is rejected for six and four cases respectively in the FARVaR-sim and FARVaR-nig models. These rejection frequencies are well below those of other models, such as GARCH (19), CAViaR (30) and CAViaR-GARCH (30); except for FHS displaying only one rejection. The rejection frequencies for the DQ test are significantly higher for all the models.[22] The lowest rejection frequencies are reported at 11 out of 30 for FARVaR-nig and FGEV, followed by FARVaR-sim and FHS at 12. Again, the null hypothesis of the DQ test is rejected for all stocks when using CAViaR. Combining these results, we may conclude that the semiparametric approach, such as FARVaR and FSH, is statistically more adequate than either parametric or nonparametric models.

In sum, the backtesting performance of VaR modelling can be greatly improved by combining the relative advantages of parametric and nonparametric models. In this regard, we recommend FARVaR as it turns out to be a more reliable approach, substantially improving the coverage ability through avoiding severe misspecification errors, while considerably reducing economic costs through explicitly modelling the dynamics of intraday density in a functional space.

[TABLE 5 ABOUT HERE]

5.2 Portfolio Analysis

The main analysis shows that FARVaR performs well for individual stocks. Furthermore, it is important to analyze how well FARVaR performs relative to other VaR models when applied to portfolios. To this end, we evaluate the performance of the VaR models by applying them to the value-weighted portfolio of 30 stocks. We investigate them to the left-tail behaviour of the return distribution with a window size of 250 days.

Table 6 presents the backtesting results. The FARVaR models slightly over-forecast VaR but maintain relatively low economic cost, which is the highly desirable property for a VaR model. Further, their VaR forecasts are not rejected by the CC and DQ tests. FHS shows performance similar to FARVaR for all backtesting except for ECP. Its ECP is slightly higher than 1%; that is, it slightly under-forecasts VaR.

The nonparametric and parametric models, HS, GARCH and RM, under-forecast VaR. Their ECPs are noticeably higher than 1%. HS stays in a safe zone with relatively low economic cost; further, it is not rejected by the CC and DQ tests. However, both GARCH and RM suffer from weak coverage ability and statistical inadequacy due to their considerable under-forecast; that is, their ECPs are close to 2%. The FEVT models show performance similar to the hybrid semiparametric models but their economic costs are higher than those of the latter models. The CAViaR models perform poorly for all aspects: they largely under-forecast VaR, stay in the Yellow and Red zones, and suffer from poor statistical power.

[TABLE 6 ABOUT HERE]

Both FARVaR and FHS perform reasonably well with the portfolio. We find that FARVaR tends to slightly over-forecast VaR while FHS tends to under-forecast it. However, it is difficult to judge which model performs better from the backtesting only. In order to better understand the differences between the two models, in the next section we further analyze distinctive aspects of how FARVaR and FHS use information and forecast VaR.

5.3 FARVaR vs. FHS

The hybrid approaches, FARVaR and FHS, greatly improve the VaR forecast. However, the two models use information about the distribution of daily stock returns, and forecast the daily VaR, in different ways. FARVaR forecasts VaR relying on information from realized intraday densities, while FHS relies on observed daily returns. So are the two models able to obtain the same information, about the distribution of daily stock returns, in different ways? To answer this question, we investigate how closely the VaR forecasts from the two approaches track each other. This kind of analysis would also help deepen our understanding of whether both FARVaR and FHS capture the same information about stock returns, or whether there are important differences.

Figure 6 plots the VaR forecasts of FARVaR and FHS applied to the value-weighted portfolio of 30 stocks, as described in Section 5.2. The upper panel compares FARVaR-nig with FHS, and the lower one FARVaR-sim with FHS. The time-series patterns of the VaR forecasts by FARVaR and FHS appear to track each other over the sample period, but we find that their responses to the underlying market regime are different. FHS responds sensitively to realized stock returns because it standardizes those by the GARCH filter. It therefore tends to relatively over-forecast VaR in periods of market downturn, such as the dot-com bubble burst (2001 – 2002) or subprime mortgage crisis (2007–2008), while under-forecasting in a normal period, such as (2003–2006).[23] By contrast, FARVaR depends on the realized density rather than the observed stock returns. Thus its VAR forecast is relatively less sensitive to the underlying market regime than that of FHS.

[FIGURE 6 ABOUT HERE]

For a more specific analysis, we investigate the failures of the VaR forecasts in Table 7. Failure is defined as when a realized return is not covered by the VaR forecast. In times of market downturn (dot-com bubble burst and subprime mortgage crisis), the FARVaR models and FHS show a similar number of failures, and the dates of their failures also mostly coincide. In other words, FARVaR relatively under-forecasts VaR to a slightly greater degree than FHS, but there is no notable difference between the two models in terms of coverage ability. However, in normal market periods, FHS under-forecasts VaR to a relatively greater degree and its failures are overwhelmingly more numerous than those of FARVaR.

[TABLE 7 ABOUT HERE]

Overall, the two models offer acceptable VaR forecasts and we cannot find any significant differences. The subtle difference is that the VaR forecast of FARVaR is less sensitive to the underlying market regime than that of FHS. This is due to the way that the two models use information; that is, FARVaR utilizes the entire probability distribution given by the realized density, while FHS relies largely on information provided by the realized returns.

5.4 Subperiod Analysis

As seen in the comparison between FARVaR and FHS, VaR models may have different backtesting results according to the underlying market regime. To investigate this point, we divide the full sample period into three subperiods. First, there are two notable market downturns in our sample: one caused by the dot-com bubble burst (2001–2002) and the other by the subprime mortgage crisis (2007–2008). The remaining subperiod (2003–2006) can be regarded as relatively normal. We apply backtesting to each subperiod in the same way as in our main analysis (long position and 250-day window size).

Table 8 presents the backtesting results for the normal subperiod. The FARVaR models and FHS show reasonable coverage ability and economic cost. They are also statistically reliable. Other models, except for CAViaR, also work reasonably well. However, the FEVT models entail very high economic cost even during the normal subperiod.

[TABLE 8 ABOUT HERE.]

Table 9 presents the backtesting results for the subperiods of market downturn. In the nature of FEVT, it can provide the most prudential VaR forecast during a market downturn. Thus despite the high economic cost of the FEVT models, they successfully reduce the number of failures when the market is in deep recession. In the case of the FARVaR models and FHS, their failures increase but they still remain in the Green zone, at a lower economic cost than for FEVT. In particular, the FARVaR models show statistically reasonable results during the subperiod of the dot-com bubble burst.

[TABLE 9 ABOUT HERE.]

In sum, the difference in how the VaR models use information accounts for the difference in their VaR forecasts as the market regime changes. Although FEVT works more robustly than others during periods of market downturn, FARVaR and FHS also work reasonably well. In particular, FARVaR is statistically more reliable than FHS during periods of market downturn. On balance, FARVaR thus gives the best performance overall across different market conditions.

5.5 PDF vs. CDF

In Section 1.2.2, the numerical approximation of the cumulative distribution function (CDF) in our proposed simulation approach uses the probability density function (PDF) forecast. Since this numerical approximation affects the accuracy of the CDF, we may consider to model CDFs directly instead of PDFs.

The fact that the PDF follows the FAR process means that the PDF tends to revert toward its unconditional mean. We can therefore deduce that the CDF does the same. We model the (centred) CDF, $W(x) = F(x) - \bar{F}(x)$, as the FAR of order 1:

$$W_t = BW_{t-1} + \xi_t, \tag{37}$$

where B denotes the autoregressive operator.

We apply FARVaR-sim in Section 1.2.2 to the CDFs of the value-weighted portfolio of 30 stocks. We use FARVaR-cdf-sim to indicate the CDF-based FARVaR-sim. The empirical CDFs can be consistently estimated by

$$\hat{F}_t\left(x_j\right) = \frac{1}{m}\sum_{i=1}^{m} 1\left(r_{t,i} \le x_j\right),\ t = 1, \ldots, T;\ j = 1, \ldots, n \tag{38}$$

with a large number of intraday observations. However, 78 intraday observations are too small to accurately measure probabilities in tails by Equation (38). Furthermore, we need a smooth empirical CDF to randomly generate intraday returns. We therefore estimate them using the kernel density estimator.

We plot the rolling VaR forecasts for FARVaR-cdf-sim in Figure 7 and show the backtesting results in Table 10. We also present the results of our PDF-based FARVaR-sim for comparison. In Figure 7, FARVaR-cdf-sim under-forecasts VaR relative to FARVaR-sim; this is clearly observable during the normal subperiod. Table 10 shows that FARVaR-cdf-sim under-performs FARVaR-sim for all backtests, given its weak coverage ability due to under-forecasting. Furthermore, the model is rejected for all statistical tests while FARVaR-sim is not rejected for any tests at the 5% significance level.

[FIGURE 7 ABOUT HERE]

[TABLE 10 ABOUT HERE]

Although it is beyond the scope of this paper to investigate the theoretical basis for its under-performance, we do have some doubts about FAR modelling of the CDF: First, with regard to the definition of the error term in Equation (37), based on Equation (2), the error term at a point x is defined by

$$\xi_t(x) = \int_{-\infty}^{x} \epsilon_t(s)ds. \tag{39}$$

Thus it has zero mean and variance,

$$Var\left(\xi_t(x)\right) = Var\left(\int_{-\infty}^{x} \epsilon_t(s)ds\right). \tag{40}$$

It is not clear, however, whether it has a finite variance, because it depends on the covariance of ϵ_t, but it must be a function of x. This heteroscedasticity can affect the estimation of the autoregressive operator. Second, the CDF is more restrictive than the PDF. It is bounded in the range of $[0, 1]$

and is the non-decreasing function. However, the forecast of the CDF can violate these restrictions unless we impose a certain functional transformation. Note that the only restriction on the PDF is that it has a non-negative value. The CDF-based approach is attractive in many respects but it needs more theoretical study with regard to estimations and forecasts. We leave it for future study.

5.6 Short Position

It is not unusual to observe that the return distribution is asymmetric. Hence it is worth investigating how FARVaR performs at the right tail by taking a short position. Table 11 presents the backtesting results for the short position. Overall results are qualitatively similar to those for the long position. More specifically, the coverage ability of FARVaR is somewhat weakened, while its economic cost is slightly reduced. The overall performance of FARVaR remains better than that of other models except for FHS.

Next, we find that the FARVaR models are equally or slightly less rejected than other models by both CC and DQ tests, although, for both FGEV and FGPD, the null hypothesis of the CC test is rejected more while the null hypothesis of the DQ test is less rejected than FARVaR. For CAViaR, the null hypotheses of both tests are rejected for all 30 stocks. Once again, we find that the rejection frequencies of parametric or nonparametric models remain substantially high. This confirms that the hybrid models, FARVaR and FHS, are statistically the most reliable VaR models, irrespective of the asymmetry of the asset return distribution.

[TABLE 11 ABOUT HERE]

5.7 Longer Window Size

In practice, the selection of an optimal window size is a nontrivial issue. As the window size increases, estimation and forecasting precision generally improve. However, there is correspondingly greater uncertainty about the latent market regimes caused by a sequence of rare or extreme shocks hitting the market; for this reason, it would be more desirable to select shorter and homogeneous samples rather than longer and heterogeneous ones. With this caveat, we have also conducted backtesting exercises using the longer window size of 500 days, and find qualitatively similar results to those in Tables 12 and 13.

[TABLE 12 and 13 ABOUT HERE]

6 Backtesting: Multiple Assets

In this section, we examine the performance of mFARVaR against the existing multivariate VaR models: multivariate GARCH-based FHS (mFHS-MGARCH) and copula-based FHS (mFHS-copula).[24] mFHS-MGARCH standardizes individual returns by the multivariate GARCH (MGARCH) filter and applies the Monte Carlo (MC) simulation to those returns as FHS does. For the MGARCH specification, we model the time-varying conditional correlations by the constant conditional correlation (CCC) model of Bollerslev (1990).[25] Analogously, mFHS-copula standardizes individual returns by the copula filter and applies the MC simulation to those returns. We model the dependence structure of the standardized returns by the Student's t-copula.[26] In addition, we evaluate the performance of mFARVaR by imposing no correlations on the multiple intraday returns (mFARVaR-naïve).

We use the 10 equal-weighted portfolios of three stocks constructed in Section 3.3. For each portfolio, we estimate the multivariate VaR models, applied to the left-tail behaviour of the return distribution, with the window size of 250 days, and compute a one-day-ahead 99% VaR forecast. This forecasting exercise generates 2013 daily forecasts for each portfolio except for three: P(DIS,HPQ,MMM), P(KO,MSFT,VS) and P(T,UTX,WMT), which generate 1428, 1887 and 2006 daily forecasts, respectively. (See Table 1 for the details of company tickers.)

Tables 14 – 17 present the backtesting results of mFARVaR, mFARVaR-naïve, mFHS-MGARCH and mFHS-copula, respectively. First, we compare mFARVaR with mFARVaR-naïve. This comparison demonstrates the economic gains obtained by modelling the dependence structure in mFARVaR. For all portfolios, the ECP of mFARVaR-naïve is much higher than 1% and only three portfolios remain in the Green zone. This weak coverage ability is mainly caused by its considerable under-forecast of VaR. For example, Figure 8 plots the VaR forecasts by mFARVaR and mFARVaR-naïve for the portfolio P(T,UTX,WMT). It clearly shows that mFARVaR-naïve systematically under-forecasts VaR compared to mFARVaR.[27] Furthermore, the CC and DQ tests are rejected for all portfolios. By contrast, mFARVaR achieves more accurate VaR forecasts by taking into account the dependence structure of the individual intraday returns. On average, its ECP is slightly higher than 1%, but this is due to more frequent failures in the portfolios that include stocks such as AIG or GM which have experienced serious financial problems. Moreover, all portfolios remain in the Green zone and are less frequently rejected than other models by the CC and DQ tests. From this comparison, we can see how important is the dependence structure of intraday returns constituting a portfolio, and can confirm that the implemented modelling of dependence structure in mFARVaR robustly works well.

[TABLE 14 ABOUT HERE.]

[TABLE 15 ABOUT HERE.]

[FIGURE 8 ABOUT HERE.]

Next, we compare mFARVaR with mFHS, which is the rival model for a single asset. In mFHS-MGARCH, there is no significant difference in economic cost. As seen in the case of a single asset, the CC test is slightly less frequently rejected than for mFARVaR, while the DQ test is slightly more frequently rejected. However, the crucial difference is found in its coverage ability: the ECP for mFHS-MGARCH is higher than 1% for all portfolios. This tendency is not desirable in risk management. For example, Figure 9 plots the VaR forecasts by mFARVaR and mFHS-MGARCH for the portfolio P(T,UTX,WMT). Analogously to the case of a single asset in Figure 6, mFHS-MGARCH tends to over-forecast VaR in a period of market downturn such as the dot-com bubble burst (2001–2002) or the subprime mortgage crisis (2007–2008), while under-forecasting it in a normal period (2003–2006); thereby more failures are observed during the normal period. mFHS-copula shows even worse results: its ECP is much higher than 1% and only three portfolios remain in the Green zone. Moreover, many portfolios are statistically rejected.

[TABLE 16 ABOUT HERE.]

[TABLE 17 ABOUT HERE.]

[FIGURE 9 ABOUT HERE.]

Overall results reveal the following stylized facts. First, the dependence structure of intraday returns constituting a portfolio is essential for enhancing the performance of VaR forecasting in applications considering multiple assets. The copula framework is well equipped in our proposed mFARVaR and contributes to improving its VaR forecasts. Second, there is no dominance between mFARVaR and mFHS-MGARCH in terms of economic cost or statistical adequacy, just as in the case of a single asset. However, we find the crucial differences between the two models in their coverage ability: the ECP of mFHS-MGARCH is higher than that of mFARVaR, but not overwhelmingly so. The problem for mFHS-MGARCH is that it has a tendency to under-forecast VaR, and thus suffers from more failures during a normal market period; this tendency needs to be improved. Consequently, we find that mFARVaR has potential as the most efficient model to utilize information from multiple intraday returns and has competitive advantage over the existing multivariate VaR models.

7 Concluding Remarks

With the growing importance of intraday activity in the financial market, such as high-frequency and algorithmic trading, we have proposed the FARVaR approach in order to improve daily VaR evaluation by explicitly incorporating intraday returns information. FARVaR is a semiparametric approach which applies FAR to forecasting the nonparametric density of intraday returns. Furthermore, we have provided a practical algorithm for constructing the density forecast of daily returns directly from the density forecast of intraday returns, using a parametric approximation based on the NIG distribution and a nonparametric bootstrap approximation. More importantly, we have extended the FARVaR approach to multiple assets for more flexible application.

We have conducted comprehensive evaluation exercises using actual data from 30 stocks listed in the Dow Jones Industrial Average index over the period 2000–2008. First, we find that FAR turns out to be the best functional model predictor for the density of intraday returns for all 30 stocks. Second, we have conducted a number of backtests, which confirm that FARVaR outperforms other nonparametric and parametric VaR models. Third, we find that the overall performance of FARVaR-sim is slightly more favorable than that of FARVaR-nig. Given that the former is a more efficient approach, we recommend the use of FARVaR-sim in practice. Finally, the backtesting for multiple assets demonstrates that mFARVaR has competitive advantage over existing multivariate VaR models.

Overall, this study enhances our understanding of contemporary VaR analysis in several ways. First, as intraday information becomes more helpful in forecasting the daily risk, a robust nonparametric modelling of the density of intraday returns is likely to be a key input to improving daily risk management. Second, FARVaR can accommodate the complex dynamics of intraday return density in a flexible manner because FAR is a generalization of all classes of autoregressive models. Furthermore, FARVaR can be flexibly combined with a framework for multivariate distribution functions like copula. Third, we demonstrate that the hybrid approach can simultaneously improve coverage ability and reduce economic cost. Specifically, the robust nonparametric approach helps to improve coverage ability while the dynamic parametric FAR modelling can reduce economic cost. Finally, Basel III suggests, but does not require, that banks and other financial institutions move away from VaR and towards a coherent measure of tail risk known as "expected shortfall". Once the daily density has been estimated by the FARVaR approach, it is rather straightforward to calculate the expected value of the tail, where the tail is defined as the density beyond the VaR quantile.

Notes

[1]New York Times reporter J. Nocera wrote an extensive piece, "Risk Mismanagement", on 2 Jan 2009, discussing the role VaR played in the financial crisis of 2007–2008. Having interviewed risk managers, she suggests that VaR was very useful to risk experts but nevertheless exacerbated the crisis by giving false security to bank regulators. In 2009 Professor N. N. Taleb testified in Congress asking for the banning of VaR because "tail risks are non-measurable."

[2]Brogaard et al. (2014) show that HFT facilitates price efficiency by trading in the direction of permanent price changes and in the opposite direction to transitory pricing errors, suggesting that HFT is beneficial to price discovery and market quality; see also Hasbrouck and Saarb (2013). On the other hand, Zhang (2010) argues that HFT is harmful to price discovery, as HFT is positively correlated with stock price volatility, and this correlation is stronger during periods of high market uncertainty.

[3]Andersen et al. (2003) document that daily realized volatility estimates based on intraday returns provide volatility forecasts that are superior to forecasts constructed parametrically from daily returns only.

[4]Bosq (2000) introduces the statistical foundation of FAR modelling. See also Park and Qian (2012).

[5]The NIG distribution is one of the most popular parametric distributions for describing the return distribution, and enables matching the first four moments of the observed data (Barndorff-Nielsen, 1997).

[6]Although market risk management is in transition from VaR to expected shortfall (ES) under the Basel III Accord, the Basel II rules are still crucial requirements for the robust ES model.

[7]Giot and Laurent (2004) use an ARCH-type model, Clements et al. (2008) consider a mixed data sampling and a heterogeneous autoregressive model, and Andersen et al. (2003) employ a vector autoregressive model.

[8]In practice, a trading action and its frequency are irregular in time, so we filter the trading data and extract prices at fixed time intervals (e.g., 9:35AM, 9:40AM, 9.45AM, ..., 4:00PM). In the empirical application below, we follow Andersen et al. (2001), and collect intraday returns over 5-minute intervals, as this is short enough to secure the accuracy of continuous record asymptotic, but long enough to weaken cumulative noises from microstructure frictions.

[9]If we include an overnight jump in price, $P_{t,0}$ should be the closing price from day $t-1$. This suggests that the overnight return, $r_{t,1}$, may have a different data-generating process from other intraday returns (e.g. Ahoniemi et al., 2016; Taylor, 2007). The modelling of the overnight jump is beyond the scope of the current study. Notice, however, that we have obtained qualitatively similar evaluation results both with and without including the overnight returns.

[10]The expectation denotes the point-wise functional mean of a density function throughout the paper. See also Equation (A.6) in Appendix Section I

[11]The role of A is to transform one function into another in the infinite dimensional space.

[12]Under the piecewise stationarity of intraday returns, we still allow them to be autocorrelated or dependent. In such a situation, unless we are certain about a true parametric distribution, the nonparametric approach is more reliable in constructing an intraday density (see Hall et al., 1995; Wu, 1997).

[13]The grid set covers the range of sample values for $x_1 < \ldots < x_n$. For simplicity, we use an equal interval, $\delta = x_j - x_{j-1}$, for all $j = 1, \ldots, n$.

[14]For example, Chaudhuri et al. (2016) show that the average of past densities is inferior to FAR in forecasting the density of inflation rates.

[15]Note that the data for HP is collected from 6 May 2002 to 31 Dec 2008 and the data for Verizon Communications is collected from 3 Jul 2000 to 31 Dec 2008.

[16]27 Jan 2003: Name changes only: Allied Signal Incorporated merged with and changed its name to Honeywell International; Exxon Corporation and Mobil merged and changed to Exxon Mobil Corporation; J.P. Morgan & Company changed to JPMorgan Chase & Co.; Minnesota Mining & Manufacturing changed to 3M Company; Philip Morris Companies Inc. changed to Altria Group Incorporated. 8 Apr 2004: AT&T, Eastman Kodak and International Paper were replaced by American International, Pfizer and Verizon. 21 Nov 2005: SBC Communications Inc. was renamed AT&T Inc. after it acquired the original AT&T. 19 Feb 2008: Altria Group and Honeywell were replaced by Bank of America and Chevron. 22 Sep 2008: American International was replaced by Kraft Foods Inc.

[17]These graphs are produced with intraday returns. In particular, while it is tempting to see the relationship between mean and volatility in (a) as a test of the mean–variance relationship, it is not the objective we have here. More formal tests and consideration should be undertaken to draw such a conclusion. These graphs should be considered as revealing empirical regularities which demonstrate the regime-dependent nonlinear relationship between the moments of intraday returns. We thank the referee for pointing out, in particular, the potential for estimation errors and the relevance of intraday returns to the general mean–variance setting.

[18]A McKinsey report in May 2012 estimated that 85% of large banks were using historical simulation.

[19]See Jorion (2006, p.149) for detailed descriptions of the reasons suggested by the Basel Committee: (i) basic integrity of the model, (ii) model accuracy could be improved, (iii) intraday trading, and (iv) bad luck.

[20]It is difficult to identify the specific risk and its capital charge for an individual company. Hence we only include the VaR part of Equation (31) for the purpose of comparison. Since we are comparing alternative VaR models for each stock or portfolio, SRC is fixed across models, and should not therefore affect our overall conclusion.

[21]For example, suppose that the CFO of a bank manages $1bn and reports the economic cost of the internal VaR model using MRCR and PQL on a daily basis. When the bank uses FGPD, then the CFO should allocate $602.1m against the maximum loss over a 10-day horizon, or endure a $78.2m daily loss. If the CFO switches to employ FARVaR-nig, then FARVaR-nig could reduce the capital requirement to $408m and the daily loss to $67.6m, saving $194.1m on the capital requirement and $10.6m on the daily loss by.

[22]Berkowitz et al. (2011) show that the CC test is less powerful against inaccurate VaR models than the DQ test.

[23]Care is needed to select a right filter for a given underlying regime to improve this tendency (see Gurrola-Perez and Murphy, 2015).

[24]Note that our proposed multivariate FARVaR shares the almost same idea with the multivariate FHS.

[25]We also employ the BEKK model (Engle and Kroner, 1995) and the dynamic conditional correlation (DCC) model (Engle, 2002). These models produce qualitatively similar results to those from CCC.

[26]We model the dependence structure relying on the constant Student's t-copula, but the performance of mFHS-copula could be improved by modelling the dependence structure using a time-varying copula such as the generalized

autoregressive score (GAS) model (Creal et al., 2013).

[27] We find near-identical patterns in other portfolios.

Appendix

I. FARVaR

We describe a detailed FARVaR estimation and forecasting procedure. Given a series of equity prices observed at a fixed time interval, we calculate intraday returns by the first difference of the logged prices. Suppose that we observe m intraday returns, $(r_{t,i})_{i=1}^{m}$, at each day t over T days. For simplicity, we consider the following balanced panel data:

$$\mathbf{R} = \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ r_{T,1} & r_{T,2} & \cdots & r_{T,m} \end{bmatrix}_{T\times m}, \tag{A.1}$$

which consists of T row vectors with each row vector containing m intraday returns.

Step 1: Forecasting density of intraday returns by FAR

A. Kernel density estimation of intraday return density

We set an $n \times 1$ vector of discrete grids, $\mathbf{x} = (x_1, \ldots, x_n)'$, which covers the range of intraday samples with $n = 1,024$ as used in the empirical analysis. We then estimate the density of intraday returns by the following kernel density estimator:

$$\tilde{f}_t(x_j) = \frac{1}{nh_t}\sum_{i=1}^{m} K\left(\frac{x_j - r_{t,i}}{h_t}\right),\ t = 1, ..., T,\ j = 1, \ldots, n, \tag{A.2}$$

where K is a kernel and h_t is a bandwidth. Following Silverman (1986), we employ the popular Gaussian kernel and the rule of thumb that the optimal bandwidth is given by $h_t = 1.06\hat{\sigma}_t m^{-1/5}$, where $\hat{\sigma}_t$ is the sample standard deviation of $\{r_{t,1}, \ldots, r_{t,m}\}$.

Since the density function must satisfy $\int f(x)\,dx = 1$, we normalize the estimated density function by

$$\hat{f}_t(x_j) = \frac{\tilde{f}_t(x_j)}{RSUM\left(\tilde{\mathbf{f}}_t\right)}, \tag{A.3}$$

where $RSUM\left(\tilde{\mathbf{f}}_t\right)$ approximates $\int \tilde{f}_t(x)\,dx$ by the following numerical middle Riemann sum of

$\tilde{\mathbf{f}}_t = \left(\tilde{f}_t\left(x_1\right), \ldots, \tilde{f}_t\left(x_n\right)\right)'$:

$$
\begin{aligned}
RSUM\left(\tilde{\mathbf{f}}_t\right) &= \frac{1}{2}\left[\sum_{j=1}^{n-1} \tilde{f}_t\left(x_{j+1}\right)\left(x_{j+1}-x_j\right) + \sum_{j=1}^{n-1} \tilde{f}_t\left(x_j\right)\left(x_{j+1}-x_j\right)\right] \\
&= \frac{1}{2}\left[\sum_{j=1}^{n-1} \tilde{f}_t\left(x_{j+1}\right) + \sum_{j=1}^{n-1} \tilde{f}_t\left(x_j\right)\right]\Delta, \qquad \text{(A.4)}
\end{aligned}
$$

where $\Delta = x_{j+1} - x_j$ is an equally-partitioned interval for all $j = 1, \ldots, n-1$. Next, we construct the matrix of densities by

$$
\hat{\mathbf{F}} = \left[\begin{array}{cccc} \hat{\mathbf{f}}_1 & \hat{\mathbf{f}}_2 & \cdots & \hat{\mathbf{f}}_T \end{array}\right] = \left[\begin{array}{cccc} \hat{f}_1\left(x_1\right) & \hat{f}_2\left(x_1\right) & \cdots & \hat{f}_T\left(x_1\right) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{f}_1\left(x_n\right) & \hat{f}_2\left(x_n\right) & \cdots & \hat{f}_T\left(x_n\right) \end{array}\right]_{n\times T}. \qquad \text{(A.5)}
$$

Then we estimate a point-wise unconditional functional mean by

$$
\bar{\mathbf{f}} = \left[\begin{array}{c} \bar{f}\left(x_1\right) \\ \vdots \\ \bar{f}\left(x_n\right) \end{array}\right] = \left[\begin{array}{c} T^{-1}\sum_{t=1}^{T}\hat{f}_t\left(x_1\right) \\ \vdots \\ T^{-1}\sum_{t=1}^{T}\hat{f}_t\left(x_n\right) \end{array}\right]_{n\times 1}, \qquad \text{(A.6)}
$$

and build the matrix of fluctuations around the unconditional functional mean by

$$
\hat{\mathbf{W}} = \left[\hat{\mathbf{w}}_1, \hat{\mathbf{w}}_2, \ldots, \hat{\mathbf{w}}_T\right] = \left[\begin{array}{cccc} \hat{w}_1\left(x_1\right) & \hat{w}_2\left(x_1\right) & \cdots & \hat{w}_T\left(x_1\right) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{w}_1\left(x_n\right) & \hat{w}_2\left(x_n\right) & \cdots & \hat{w}_T\left(x_n\right) \end{array}\right]_{n\times T}, \qquad \text{(A.7)}
$$

where $\hat{\mathbf{w}}_t = \hat{\mathbf{f}}_t - \bar{\mathbf{f}}$.

B. Estimation of FAR model

We estimate the autocovariance operators of order 0 and 1 by

$$
\hat{C}_0 = \frac{1}{T}\sum_{t=1}^{T}\hat{\mathbf{w}}_t\hat{\mathbf{w}}_t',\ \hat{C}_1 = \frac{1}{T-1}\sum_{t=2}^{T}\hat{\mathbf{w}}_t\hat{\mathbf{w}}_{t-1}', \qquad \text{(A.8)}
$$

and then obtain the eigenvalues (λ_j) of $\hat{C}_0$ and its corresponding eigenfunctions $(\mathbf{v}_j)$ by

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}_{n\times 1}, \mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} v_{11} & v_{21} & \cdots & v_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ v_{1n} & v_{2n} & \cdots & v_{nn} \end{bmatrix}_{n\times n}. \tag{A.9}$$

Next, we choose $L\,(< n)$ eigenvalues, $(\lambda_1, \lambda_2, \ldots, \lambda_L)$, and the corresponding eigenfunctions, $(\mathbf{v}_1, \ldots, \mathbf{v}_L)$. Hence we approximate the inverse of $\hat{C}_0$ in the L-dimensional subspace by

$$\hat{C}^{+}_{0,L} = \sum_{k=1}^{L} \lambda_k^{-1} \mathbf{v}_k \mathbf{v}_k'. \tag{A.10}$$

The choice of L is guided by applying a functional principal component analysis (FPCA) and a cross-validation (CV; see Ramsey and Silverman, 1997). FPCA explains the variation of the fluctuation and CV selects the optimal dimension $L\,(\leq L_{max})$ by minimizing the following criterion:

$$\sum_{i=1}^{N_{cv}} RSUM\left(\left[\hat{\mathbf{w}}^{L}_{T-i+1} - \hat{\mathbf{w}}_{T-i+1}\right]^2\right), \tag{A.11}$$

where N_{cv} is the number of the last observations used in CV and $\hat{\mathbf{w}}^{L}_{T-i+1}$ represents the in-sample forecasts of $\mathbf{w}_{T-i+1}$ on the L-dimensional subspace. We set $L_{max} = 20$ in the empirical analysis, and find that CV selects the optimal value of L ranging between 5 and 10. Finally, we estimate the autoregressive operator A in the L-dimensional subspace, consistently, by

$$\hat{A}_L = \hat{C}_1 \hat{C}^{+}_{0,L}, \tag{A.12}$$

where

$$\hat{A}_L = \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & \cdots & \hat{a}_{1n} \\ \hat{a}_{21} & \hat{a}_{22} & \cdots & \hat{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{a}_{n1} & \hat{a}_{n2} & \cdots & \hat{a}_{nn} \end{bmatrix}. \tag{A.13}$$

C. Density forecast of intraday returns

A one-step-ahead conditional density forecast is evaluated by

$$\hat{\mathbf{f}}_{T+1} = \bar{\mathbf{f}} + \hat{A}_L \hat{\mathbf{w}}_T, \tag{A.14}$$

or equivalently,

$$\begin{bmatrix} \hat{f}_{T+1}(x_1) \\ \hat{f}_{T+1}(x_2) \\ \vdots \\ \hat{f}_{T+1}(x_n) \end{bmatrix} = \begin{bmatrix} \bar{f}(x_1) \\ \bar{f}(x_2) \\ \vdots \\ \bar{f}(x_n) \end{bmatrix} + \begin{bmatrix} \hat{a}_{11}\hat{w}_T(x_1) + \hat{a}_{12}\hat{w}_T(x_2) + \cdots + \hat{a}_{1n}\hat{w}_T(x_n) \\ \hat{a}_{21}\hat{w}_T(x_1) + \hat{a}_{22}\hat{w}_T(x_2) + \cdots + \hat{a}_{2n}\hat{w}_T(x_n) \\ \vdots \\ \hat{a}_{n1}\hat{w}_T(x_1) + \hat{a}_{n2}\hat{w}_T(x_2) + \cdots + \hat{a}_{nn}\hat{w}_T(x_n) \end{bmatrix}. \tag{A.15}$$

Step 2: Forecasting daily VaR

A. NIG approximation approach

We numerically evaluate the first four moments from the density forecast of intraday returns in Equation (A.14) (i.e., mean (μ_{T+1}), variance (v_{T+1}), skewness (s_{T+1}), kurtosis (k_{T+1})), by

$$\hat{\mu}_{T+1} = RSUM\left(\mathbf{x} \odot \hat{\mathbf{f}}_{T+1}\right), \tag{A.16}$$

$$\hat{v}_{T+1} = RSUM\left(\left(\mathbf{x} - \hat{\mu}_{T+1}\right)^2 \odot \hat{\mathbf{f}}_{T+1}\right), \tag{A.17}$$

$$\hat{s}_{T+1} = RSUM\left(\left(\mathbf{x} - \hat{\mu}_{T+1}\right)^3 \odot \hat{\mathbf{f}}_{T+1}\right) / \hat{v}_{T+1}^{3/2}, \tag{A.18}$$

$$\hat{k}_{T+1} = RSUM\left(\left(\mathbf{x} - \hat{\mu}_{T+1}\right)^4 \odot \hat{\mathbf{f}}_{T+1}\right) / \hat{v}_{T+1}^{2}, \tag{A.19}$$

where $\odot$ stands for an element-by-element multiplication operator and $RSUM\left(\mathbf{y}^k \odot \mathbf{f}\right)$ approximates $\int y^k f(y)\,dy$ for $k \geq 1$. We then calculate the four parameters, $\left(\hat{\alpha}_{T+1}, \hat{\beta}_{T+1}, \hat{\gamma}_{T+1}, \hat{\delta}_{T+1}\right)$ from the four moments in Equation (A.16) - (A.19) by

$$\hat{\alpha}_{T+1} = \hat{v}_{T+1}^{-\frac{1}{2}}\left(3\hat{k}_{T+1} - 4\hat{s}_{T+1}^2 - 9\right)^{\frac{1}{2}}\left(\hat{k}_{T+1} - \frac{5}{3}\hat{s}_{T+1}^2 - 3\right)^{-1}, \tag{A.20}$$

$$\hat{\beta}_{T+1} = \hat{s}_{T+1}\hat{v}_{T+1}^{-\frac{1}{2}}\left(\hat{k}_{T+1} - \frac{5}{3}\hat{s}_{T+1}^2 - 3\right)^{-1}, \tag{A.21}$$

$$\hat{\gamma}_{T+1} = \hat{\mu}_{T+1} - 3\hat{s}_{T+1}\hat{v}_{T+1}^{\frac{1}{2}}\left(3\hat{k}_{T+1} - 4\hat{s}_{T+1}^2 - 9\right)^{-1}, \tag{A.22}$$

$$\hat{\delta}_{T+1} = 3^{\frac{3}{2}}\left[\hat{v}_{T+1}\left(\hat{k}_{T+1} - \frac{5}{3}\hat{s}_{T+1}^2 - 3\right)\right]^{\frac{1}{2}}\left(3\hat{k}_{T+1} - 4\hat{s}_{T+1}^2 - 9\right)^{-1}. \tag{A.23}$$

We can approximate the density of daily returns by the following NIG density formula:

$$\hat{g}_{T+1}(x) = \left[\frac{m\hat{\alpha}_{T+1}\hat{\delta}_{T+1} J_1 \left(\hat{\alpha}_{T+1}\sqrt{m^2\hat{\delta}^2_{T+1} + (x - m\hat{\gamma}_{T+1})^2} \right)}{\pi\sqrt{m^2\hat{\delta}^2_{T+1} + (x - m\hat{\gamma}_{T+1})^2}} \right] e^{m\hat{\delta}_{T+1}\hat{\lambda}_{T+1} + \hat{\beta}_{T+1}(x - m\hat{\gamma}_{T+1})}. \tag{A.24}$$

Since the analytic form of the cumulative NIG density function does not exist, we approximate the cumulative NIG density $\hat{G}_{T+1}$ by evaluating the integral of $\hat{g}_{T+1}$:

$$\hat{G}_{T+1}(\omega) = \int_{-\infty}^{\omega} \hat{g}_{T+1}(x)\, dx. \tag{A.25}$$

Since there is no analytic formula for $\hat{G}_{T+1}(\omega)$, the numerical integration method is employed.[1]

B. Simulation approach

To construct the cumulative distribution function of daily returns, $\hat{F}_{T+1}$, from the density forecast of intraday returns, $\hat{f}_{T+1}$, in Equation (A.14), we approximate an empirical cumulative distribution function using the middle Riemann sum:

$$\hat{F}_{T+1}(z_j) = RSUM\left(\hat{\mathbf{f}}_{T+1}[1:j+1]\right),\ j = 1, \ldots, n-1, \tag{A.26}$$

where $\hat{\mathbf{f}}_{T+1}[1:j+1] := \left(\hat{f}_{T+1}(x_1), \ldots, \hat{f}_{T+1}(x_{j+1})\right)'$. We express the cumulative distribution function of intraday returns as an $(n-1) \times 1$ vector:

$$\hat{F}_{T+1} = \begin{bmatrix} \hat{F}_{T+1}(z_1) \\ \hat{F}_{T+1}(z_2) \\ \vdots \\ \hat{F}_{T+1}(z_{n-1}) \end{bmatrix}_{(n-1)\times 1}. \tag{A.27}$$

Then we simulate intraday and daily return forecasts and the daily VaR forecasts from $\hat{F}_{T+1}$, as described in Section 1.2.2 and depicted in Figure 2.

[1] We use the MATLAB package for the NIG distribution provided by Dr. Ralf Werner. The package uses a Gaussian integration rule for computing the CDF. http://uk.mathworks.com/matlabcentral/fileexchange/6050-normal-inverse-gaussion-distribution.

II. Multivariate FARVaR

We describe the detailed estimation and forecasting procedure of mFARVaR. Consider K assets constituting a portfolio in Equation (15).

Step 1. Forecasting K marginal densities of intraday returns by FAR

We forecast K marginal densities individually by FAR (see Equation (A.14)). We then numerically approximate a cumulative distribution function by applying the middle Riemann sum in Equation (A.26) to each density forecast. We express the cumulative distribution function of intraday returns as an $(n-1) \times K$ matrix:

$$\hat{\mathfrak{F}}_{T+1} = \begin{bmatrix} \hat{F}_{1,T+1}(z_1) & \hat{F}_{2,T+1}(z_1) & \cdots & \hat{F}_{K,T+1}(z_1) \\ \hat{F}_{1,T+1}(z_2) & \hat{F}_{2,T+1}(z_2) & \cdots & \hat{F}_{K,T+1}(z_2) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{F}_{1,T+1}(z_{n-1}) & \hat{F}_{2,T+1}(z_{n-1}) & \cdots & \hat{F}_{K,T+1}(z_{n-1}) \end{bmatrix}_{(n-1)\times K} . \tag{A.28}$$

Step 2. Forecasting copula of K intraday return series by VAR(1)

For each day, we construct the matrix of the probability integral transform (PIT) of intraday returns. To this end, we first approximate K cumulative distribution functions by applying the middle Riemann sum in Equation (A.26) to the estimated density functions:

$$\hat{\mathfrak{F}}_{t} = \begin{bmatrix} \hat{F}_{1,t}(z_1) & \hat{F}_{2,t}(z_1) & \cdots & \hat{F}_{K,t}(z_1) \\ \hat{F}_{1,t}(z_2) & \hat{F}_{2,t}(z_2) & \cdots & \hat{F}_{K,t}(z_2) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{F}_{1,t}(z_{n-1}) & \hat{F}_{2,t}(z_{n-1}) & \cdots & \hat{F}_{K,t}(z_{n-1}) \end{bmatrix}_{(n-1)\times K} \quad \text{for } t = 1, \ldots, T. \tag{A.29}$$

Then we compute the PIT of intraday returns by $\hat{u}_{k,t,i} \approx \hat{F}_{k,t}\left(r_{k,t,i}\right)$ and construct a PIT matrix:

$$\hat{\mathbf{U}}_t = \begin{bmatrix} \hat{u}_{1,t,1} & \hat{u}_{2,t,1} & \cdots & \hat{u}_{K,t,1} \\ \hat{u}_{1,t,2} & \hat{u}_{2,t,2} & \cdots & \hat{u}_{K,t,2} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{u}_{1,t,m} & \hat{u}_{2,t,m} & \cdots & \hat{u}_{K,t,m} \end{bmatrix}_{m\times K} \quad \text{for } t = 1,\ldots,T. \tag{A.30}$$

Next, we estimate a copula using the PIT series for each day. Let us assume that the copula has the form of the multivariate Student $t\left(\nu_t, \mathbf{0}, \boldsymbol{P}_t\right)$-distribution, where $\boldsymbol{P}_t$ is the correlation matrix and ν_t the degree of freedom. We estimate the copula parameters, $\hat{\boldsymbol{P}}_t$ and $\hat{\nu}_t$, using the quasi-maximum likelihood estimator (QMLE):

$$\left\{\hat{\nu}_t, \hat{\boldsymbol{P}}_t\right\} = \underset{\{\nu_t, \boldsymbol{P}_t\}\in\boldsymbol{\Theta}}{\operatorname{argmax}} \sum_{i=1}^{m} \ln c_t\left(u_{1,t,i},\ldots,u_{K,t,i};\nu_t,\boldsymbol{P}_t\right). \tag{A.31}$$

Then we model the dynamics of copula parameters by VAR(1) in Equations (19) – (20) and forecast them:

$$\left\{\hat{\nu}_{T+1}, \hat{\boldsymbol{P}}_{T+1}\right\} \Rightarrow C_{\hat{\nu}_{T+1},\hat{\boldsymbol{P}}_{T+1}}. \tag{A.32}$$

Step 3. Simulating daily portfolio VaR

We randomly generate $\mathbf{U}_{T+1}$ from the copula forecast, $C_{\hat{\nu}_{T+1},\hat{\boldsymbol{P}}_{T+1}}$,

$$\mathbf{U}^{(b)}_{T+1} = \begin{bmatrix} u^{(b)}_{1,T+1,1} & u^{(b)}_{2,T+1,1} & \cdots & u^{(b)}_{K,T+1,1} \\ u^{(b)}_{1,T+1,2} & u^{(b)}_{2,T+1,2} & \cdots & u^{(b)}_{K,T+1,2} \\ \vdots & \vdots & \ddots & \vdots \\ u^{(b)}_{1,T+1,m} & u^{(b)}_{2,T+1,m} & \cdots & u^{(b)}_{K,T+1,m} \end{bmatrix}_{m\times K} \quad \text{for } b = 1,\ldots,B, \tag{A.33}$$

and convert those into intraday returns using the inverse cumulative distribution function of intraday returns,

$$\mathbf{R}_{T+1}^{(b)} = \begin{bmatrix} r_{1,T+1,1}^{(b)} & r_{2,T+1,1}^{(b)} & \cdots & r_{K,T+1,1}^{(b)} \\ r_{1,T+1,2}^{(b)} & r_{2,T+1,2}^{(b)} & \cdots & r_{K,T+1,2}^{(b)} \\ \vdots & \vdots & \ddots & \vdots \\ r_{1,T+1,m}^{(b)} & r_{2,T+1,m}^{(b)} & \cdots & r_{K,T+1,m}^{(b)} \end{bmatrix}_{m\times K} \quad \text{for } b = 1, \dots, B, \tag{A.34}$$

where we obtain $r_{k,T+1,i} = \hat{F}_{k,T+1}^{-1}\left(u_{k,T+1,i}^{(b)}\right)$ following the algorithm depicted in Figure 2. Note that we randomly generate $\mathbf{U}_{T+1}$ from the independent multivariate uniform distribution for mFARVaR-naïve.

Then we update the intraday returns of the portfolio, generate the daily portfolio returns and evaluate the daily VAR following Equations (23) – (26).

III. Confidence Interval of VaR Forecast

In order to construct the confidence interval of the VaR forecast, we employ the nonparametric bootstrap approach. First, we draw the prediction errors, $\left\{\epsilon_1^{(b)}, \dots, \epsilon_T^{(b)}\right\}$ for $b = 1, \dots, B$, with replacements from the pool of sample residuals, $\{\hat{\epsilon}_1, \dots, \hat{\epsilon}_1\}$, where $\hat{\epsilon}_t = \hat{w} - \hat{A}_L \hat{w}_{t-1}$. Next, we update the density function based on FAR, using the prediction errors:

$$\left(f_t^{(b)} - \bar{f}\right) = \hat{A}_L\left(f_{t-1}^{(b)} - \bar{f}\right) + \epsilon_t^{(b)},\ t = 1, \dots, T, \tag{A.35}$$

where we assume that $f_1^{(b)}$ is given by the initial density function estimate. We finally forecast a daily VaR using the simulated density functions, $\left\{f_1^{(b)}, \dots, f_T^{(b)}\right\}$. We repeat the bootstrapping many times to construct the empirical distribution of daily VaR forecasts by

$$\hat{D}_{T+1}(\omega) = \frac{1}{B}\sum_{b=1}^{B} 1\left\{\widehat{VaR}_{T+1}^{(b)} \le \omega\right\}. \tag{A.36}$$

Finally, it is straightforward to construct a confidence interval given a probability from the empirical distribution.

IV. Computational Burden

We provide detail on the computational burden that is involved in computing FARVaR for both single and multiple assets in our empirical analysis. As benchmarks we also compare our models with FHS, mFHS-MGARCH and mFHS-copula. We repeat the forecast exercise 100 times and use the average value. The PC used has Intel(R) Core(TM) i7-2640M CPU 2.8 GHz (dual cores and four logical processors) and 8 GB RAM, running 64-bit Windows 10 Pro OS. We use MATLAB 2014b (64-bit). We use Econometrics Toolbox for (AR-GARCH-based) FHS and mFHS-copula, and the Oxford MFE Toolbox for mFHS-MGARCH. We use a 250-day window to estimate models and 1000 iterations for the simulation-based models.

Single Asset			Multiple Assets (3 Assets)		
FARVaR-sim	FARVaR-nig	FHS	mFARVaR	mFHS-MGARCH	mFHS-copula
2.344s	1.368s	0.329s	16.793s	1.537s	1.476s

V. Value-at-Risk Models

Consider the sequence of daily returns (r_t) generated by a probability law

$$\mathbb{P}\left\{r_t \leq x|\mathcal{F}_{t-1}\right\} \equiv F_t\left(r\right) \tag{A.37}$$

conditional on the available information at $t-1$ $(\mathcal{F}_{t-1})$. Then VaR with a given tail probability α is defined as the conditional quantile from F_t such that

$$VaR_t\left(\alpha\right) = F_t^{-1}\left(\alpha\right) \text{ or } F_t\left(VaR_t\left(\alpha\right)\right) = \alpha. \tag{A.38}$$

Daily asset returns are usually represented with mean and volatility,

$$r_t = \mu_t + \sigma_t z_t, \tag{A.39}$$

where $\mu_t = \mathbb{E}\left[r_t|\mathcal{F}_{t-1}\right]$ and $\sigma_t^2 = Var\left(r_t|\mathcal{F}_{t-1}\right)$. The standardized return $z_t = \left(r_t - \mu_t\right)/\sigma_t$ has a conditional distribution function $F_{t*} \equiv \mathbb{P}\left\{z_t \leq z|\mathcal{F}_{t-1}\right\}$. Thus VaR in Equation (A.38) can be alternatively defined as

$$VaR_t\left(\alpha\right) = \mu_t + \sigma_t F_{t*}^{-1}\left(\alpha\right). \tag{A.40}$$

Equations (A.38) and (A.40) are often categorized as unfiltered VaR and filtered VaR, respectively. For practical use, we estimate VaR by

$$\widehat{VaR}_t(\alpha) = \begin{cases} F_t^{-1}(\alpha) & (unfiltered) \\ \hat{\mu}_t + \hat{\sigma}_t F_{t*}^{-1}(\alpha) & (filtered). \end{cases} \tag{A.41}$$

A. Historical Simulation (HS)

HS is a static nonparametric model and is most popular for its simplicity (Perignon et al., 2008). It does not make any strong asumption about the probability distribution of asset returns. It employs a window of observations generally ranging from 6 months to 2 years. Asset returns within this window are sorted in ascending order and the α-quantile of interest is given by the return that leaves α% of the observation on its left side and $(1-\alpha)$% on its right side. Hence it uses the unfiltered VaR, and F_T is an empirical probability distribution function of historical asset returns.

B. RiskMetrics (RM)

RM uses the filtered VaR. It estimates a conditional mean and volatility by $\hat{\mu}_{T+1} = T^{-1}\sum_{t=1}^{T} r_t$ and $\hat{\sigma}_{T+1}^2 = 0.94\hat{\sigma}_T^2 + 0.06\left(r_T - \hat{\mu}_T\right)^2$. Further, it assumes that z_t follows the standard normal distribution.

C. GARCH

GARCH uses the filtered VaR. The conditional mean is estimated by AR(1) and the conditional volatility is estimated by GARCH(1,1). It assumes that z_t follows the Student's t-distribution.

D. Filtered Historical Simulation (FHS)

FHS is the hybrid approach, applying HS to returns filtered by AR(1)-GARCH(1,1). We randomly draw filtered returns $\left\{z_1^{(b)}, \ldots, z_T^{(b)}\right\}$ from the empirical distribution $\{z_1, \ldots, z_T\}$ and simulate a sample path of (r_t) by the AR(1)-GARCH(1,1) process. We iterate this procedure many times and construct the simulated distribution of r_T to get an unfiltered VaR.

E. Filtered Extreme Value Theory (FEVT)

FEVT is suggested to control for time-varying volatility (Diebold et al., 1998; McNeil and Rudiger, 2000). Similarly to FHS, it applies the EVT procedure to returns filtered by AR(1)-GARCH(1,1).

Here we consider the filtered generalized extreme value (FGEV) distribution and the filtered generalized Pareto distribution (FGPD).

In the fixed time interval, consider collection of n filtered returns $\{z_1, \ldots, z_n\}$. Let $z_{(1)} = \min_{1 \leq j \leq n} \{z_j\}$ as the minimum return and $z_{(n)} = \max_{1 \leq j \leq n} \{r_j\}$ as the maximum return. Assume that (z_t) is the sequence of i.i.d random variables. Then the CDF of $z_{(1)}$ is given by

$$GEV: \; F\left(z_{(1)}\right) = \begin{cases} \exp\left[-\left(1+\xi z\right)^{-1/\xi}\right] & \text{if } \xi \neq 0 \\ \exp\left[-\exp\left(z\right)^{-1/\xi}\right] & \text{if } \xi = 0 \end{cases} \tag{A.42}$$

$$GPD: \; F\left(z_{(1)}\right) = \begin{cases} 1-\left(1+\xi z\right)^{-1/\xi} & \text{for } \xi \neq 0 \\ 1-\exp\left(-z\right) & \text{for } \xi = 0, \end{cases} \tag{A.43}$$

where $z_{(1)} \geq 0$ for $\xi \geq 0$ and $0 \leq z_{(1)} \leq -1/\xi$ for $\xi < 0$.

F. Conditional Autoregressive Value-at-Risk by Regression Quantile (CAViaR)

CAViaR and CAViAR–GARCH, which incorporates GARCH in CAViaR (Engle and Manganelli, 2004):

$$CAViaR: \; VaR_t\left(\alpha\right) = \beta_0 + \beta_1 q_{t-1}\left(\alpha\right) + \beta_2 \left|r_{t-1}\right|, \tag{A.44}$$

$$CAViaR - GARCH: \; VaR_t\left(\alpha\right) = \left(\beta_0 + \beta_1 q_{t-1}^2\left(\alpha\right) + \beta_2 r_{t-1}^2\right)^{1/2}. \tag{A.45}$$

G. Multivariate Filtered Historical Simulation (mFHS)

mFHS is the hybrid approach, applying HS to multiple returns filtered by AR(1)-MGARCH(1,1) (mFHS-MGARCH) or AR(1)-copula (mFHS-copula). We randomly draw multiple filtered returns $\left\{\mathbf{z}_1^{(b)}, \ldots, \mathbf{z}_T^{(s)}\right\}$, where $\mathbf{z}_t^{(b)} = \left(z_{1,t}^{(b)}, \ldots, z_{K,t}^{(b)}\right)'$, from its empirical distribution $\{\mathbf{z}_1, \ldots, \mathbf{z}_T\}$ and simulate a sample path of $(r_{1,t}, \ldots, r_{k,t})'$ by the AR(1)-MGARCH(1,1) or AR(1)-copula process. Then we get a simulated portfolio return, $r_{p,T+1}^{(b)} = \sum_{k=1}^{K} w_k r_{k,T+1}^{(b)}$. We iterate this procedure many times and construct the simulated distribution of portfolio returns to get an unfiltered VaR.

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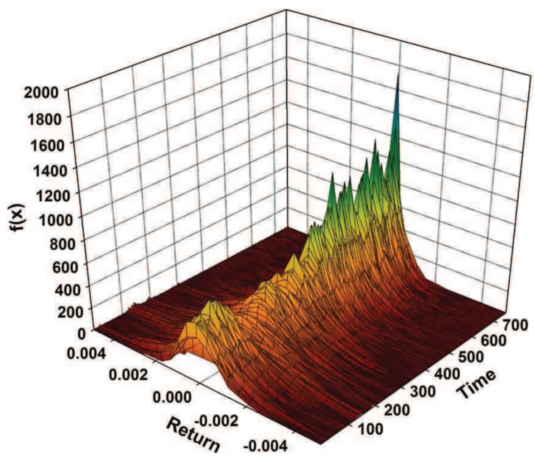
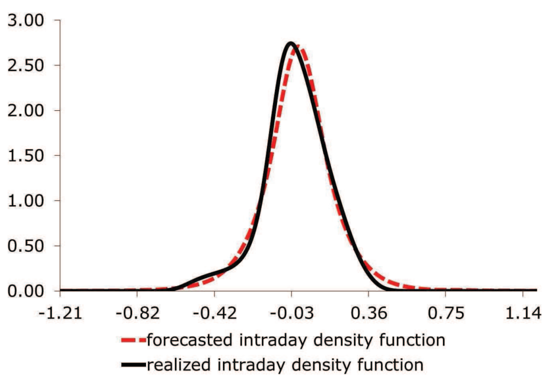
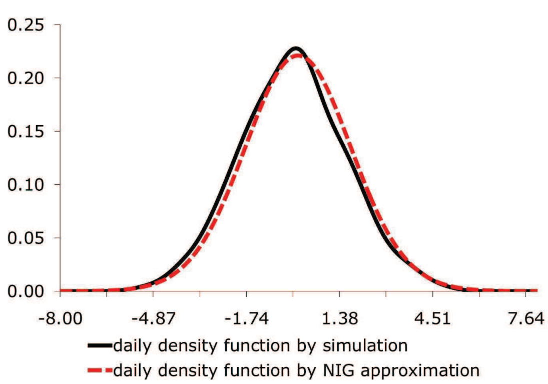
Figure 1: FARVaR algorithm



This figure presents the time-series plot of the density estimates of intraday returns over time. Time denotes days and Return denotes the intraday returns. We estimate the density of intraday returns by Equations (A.2) and (A.3) and plot Equation (A.5): $(\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_T)$.



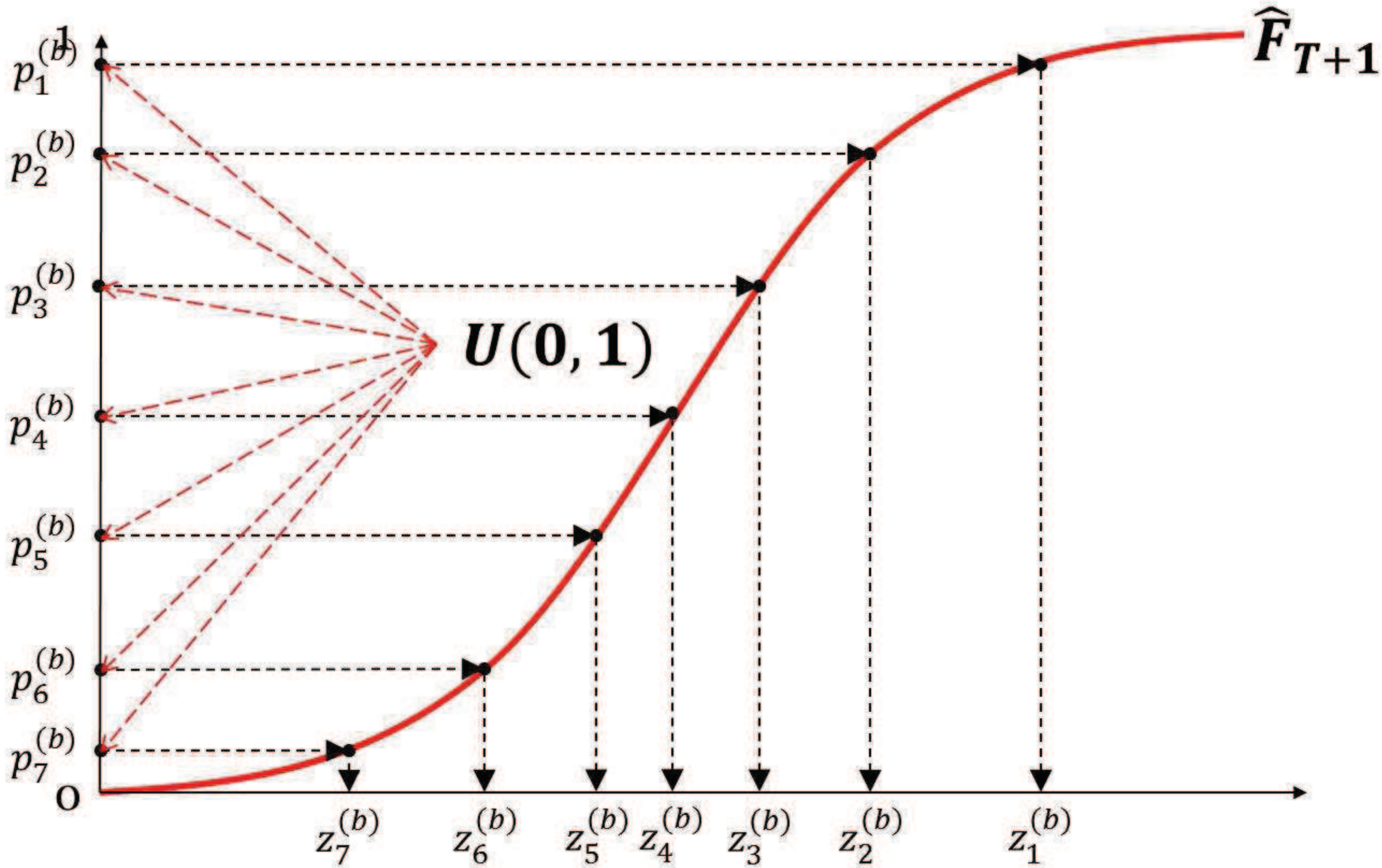
Step 1. This figure presents the one-day-ahead density forecast of intraday returns and the realized density estimate of intraday returns at time $T + 1$. We model the densities of intraday returns by FAR and forecast the density by Equation (10). The realized density of intraday returns is estimated by Equations (A.2) and (A.3).



Step 2. This figure presents the density of daily returns generated from the density forecast of intraday returns in Step 1. There are two densities of daily returns: one generated by the NIG approximation using the first four moments from the density forecast of intraday returns described in Section 1.2.1; the other by simulating intraday returns randomly drawn from the density forecast of intraday returns decribed in Section 1.2.2.

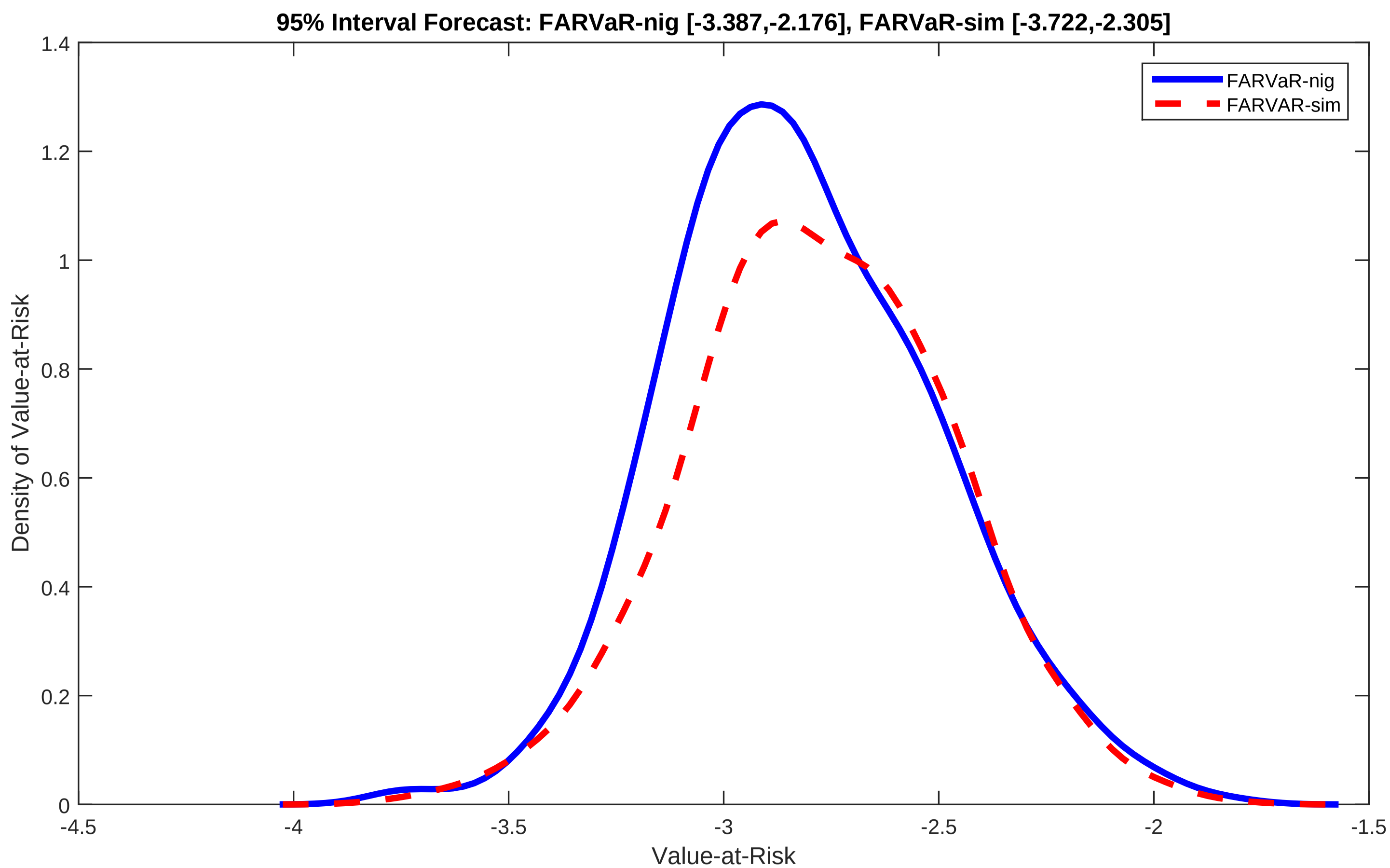
Note: This figure illustrates the two-step algorithm of FARVaR described in Section 1 and Appendix Section I. We use the intraday returns on the equal-weighted portfolio of 30 stocks to estimate the density of intraday returns.

Figure 2: Simulating intraday returns from cumulative distribution function of intraday returns



Note: This figure depicts the idea of simulating intraday returns from the empirical cumulative distribution function of intraday returns in Section 1.2.2. Given the piecewise stationarity assumption of intraday returns, we randomly draw m real numbers from the uniform distribution over $(0,1)$, and denote them by $\left\{p_1^{(b)}, p_2^{(b)}, \ldots, p_m^{(b)}\right\}$, where $p_i^{(b)} \sim U(0,1)$ for each iteration, $b = 1, \ldots, B$. We then select a $z_i^{(b)}$ corresponding to $p_i^{(b)}$, following the algorithm described in Equation (12) and construct a set of m simulated intraday returns, $\left\{z_1^{(b)}, z_2^{(b)}, \ldots, z_m^{(b)}\right\}$.

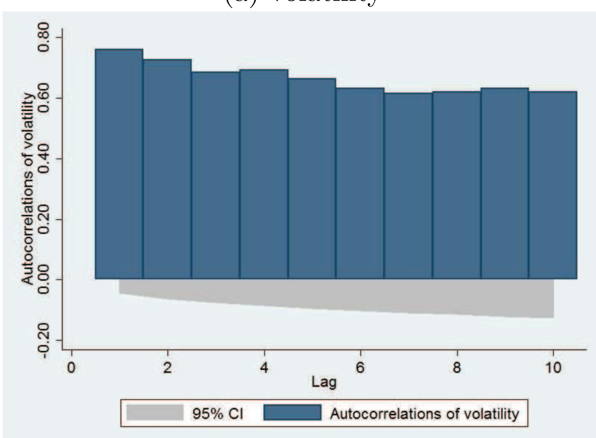
Figure 3: Bootstrap-based confidence interval of VaR forecast



Note: This figure presents the empirical distribution of VaR as forecast by FARVaR-nig and FARVaR-sim, along with their 95% confidence intervals. We obtain the empirical distribution by bootstrapping the intraday density from the in-sample prediction errors. Appendix Section III provides the detailed construction of the empirical distribution.

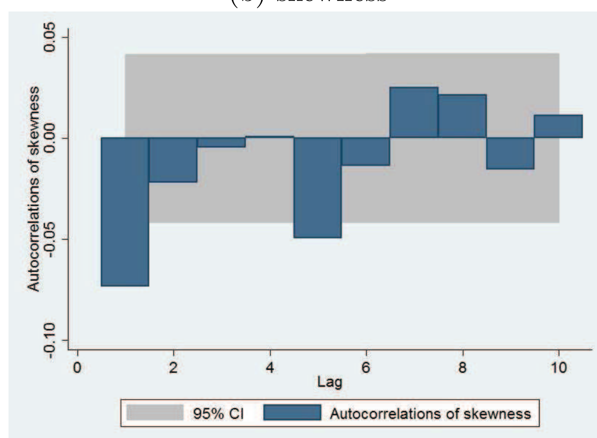
Figure 4: Autocorrelation function of moments of intraday returns

(a) volatility



AR (1) = 0.758 * **

(b) skewness

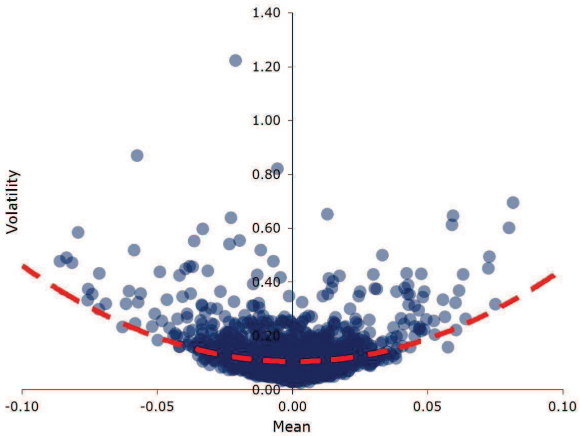


AR (1) = −0.074 * **

Note: This figure displays the autocorrelation functions of the volatility and skewness of intraday returns, constructed by the equal-weighted portfolio of 30 stocks. 95% CI denotes the 95% confidence interval. *, ** and *** indicate that the null of a zero AR(1) coefficient is rejected at the 10%, 5% and 1% significance level, respectively.

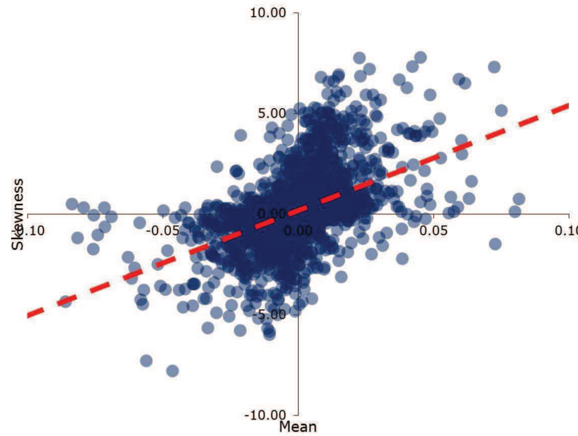
Figure 5: Contemporaneous relationships between moments of intraday returns

(a) mean – volatility



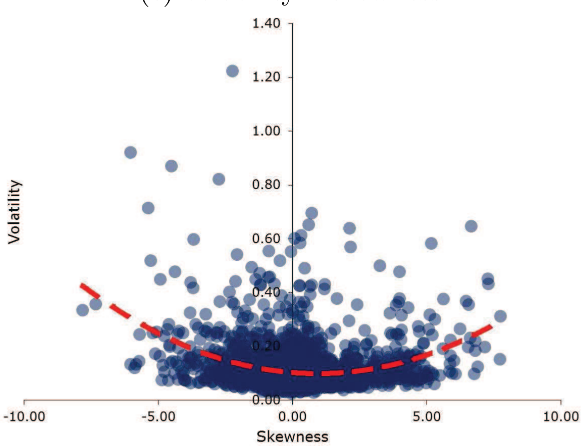
$Corr\,(\mu, \sigma|\mu > 0) = 0.672$
$Corr\,(\mu, \sigma|\mu < 0) = -0.633$

(b) mean – skewness



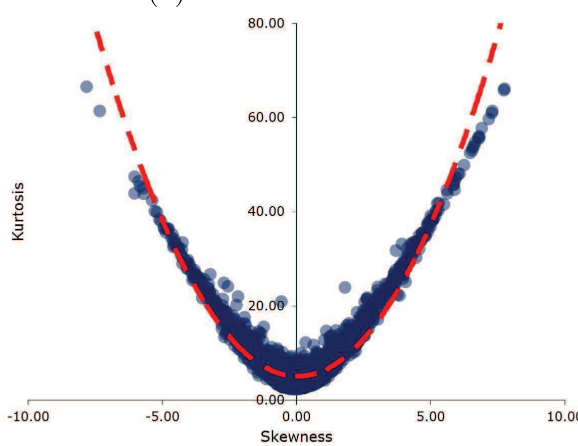
$Corr\,(\mu, s) = 0.512$

(c) volatility – skewness



$Corr\,(\sigma, s|s > 0) = 0.227$
$Corr\,(\sigma, s|s < 0) = -0.366$

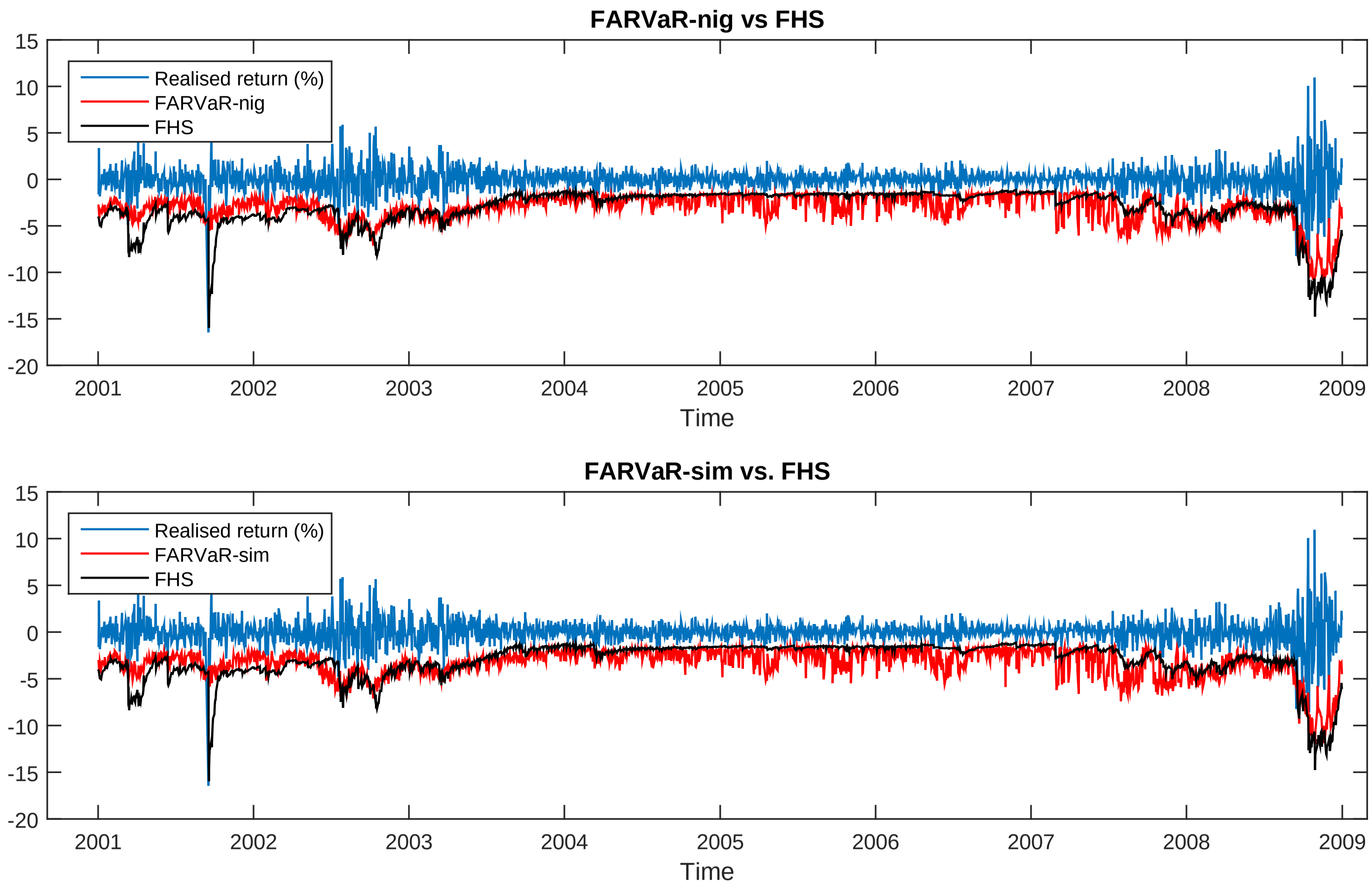
(d) skewness – kurtosis



$Corr\,(s, k|s > 0) = 0.977$
$Corr\,(s, k|s < 0) = -0.963$

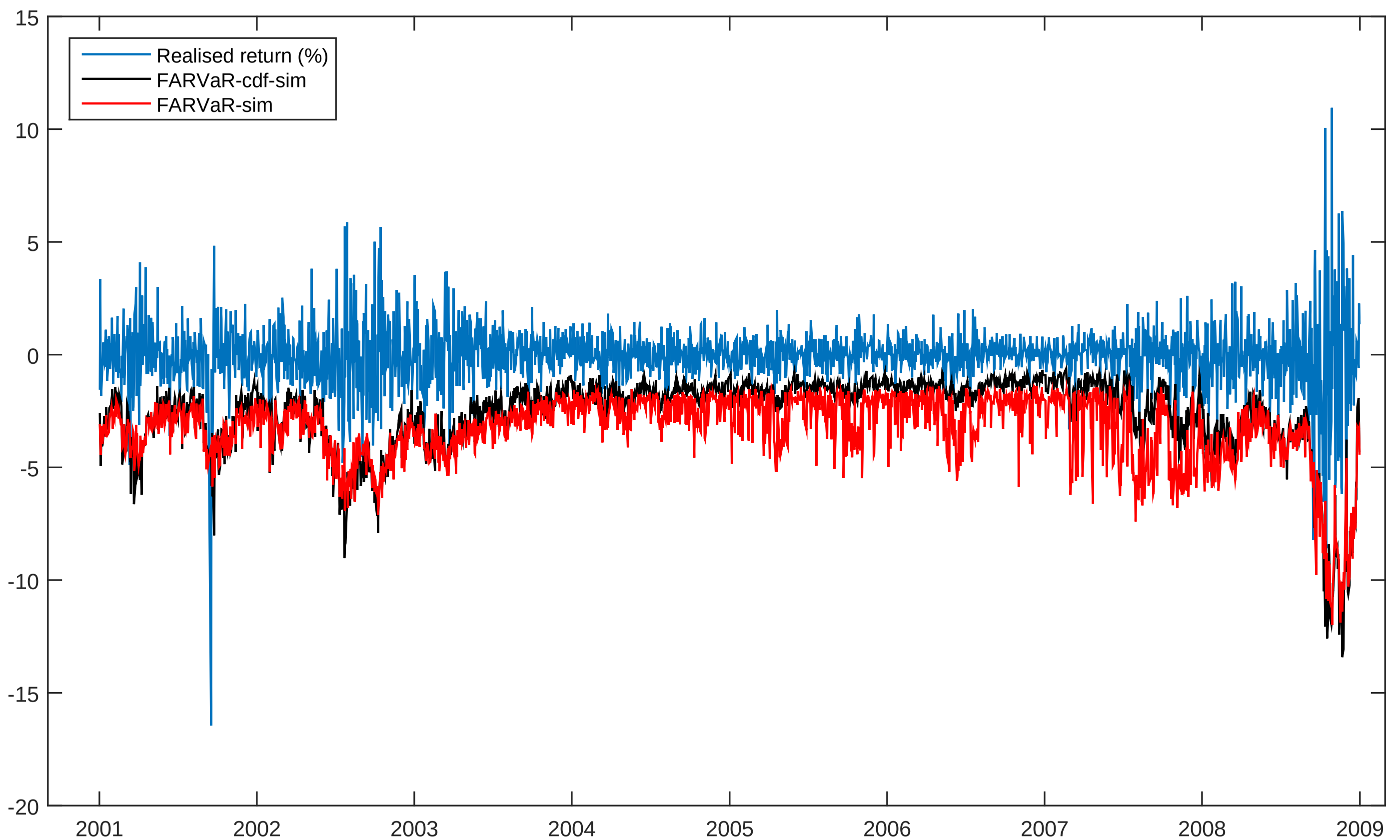
Note: This figure displays scatter plots between moment pairs of intraday returns. $Corr\,(x, y|z)$ denotes a correlation coefficient between x and y, given a condition z. μ, σ and s denote the mean, volatility and skewness of intraday returns constructed by an equal-weighted portfolio of 30 stocks.

Figure 6: VaR forecasts by FARVaR and FHS



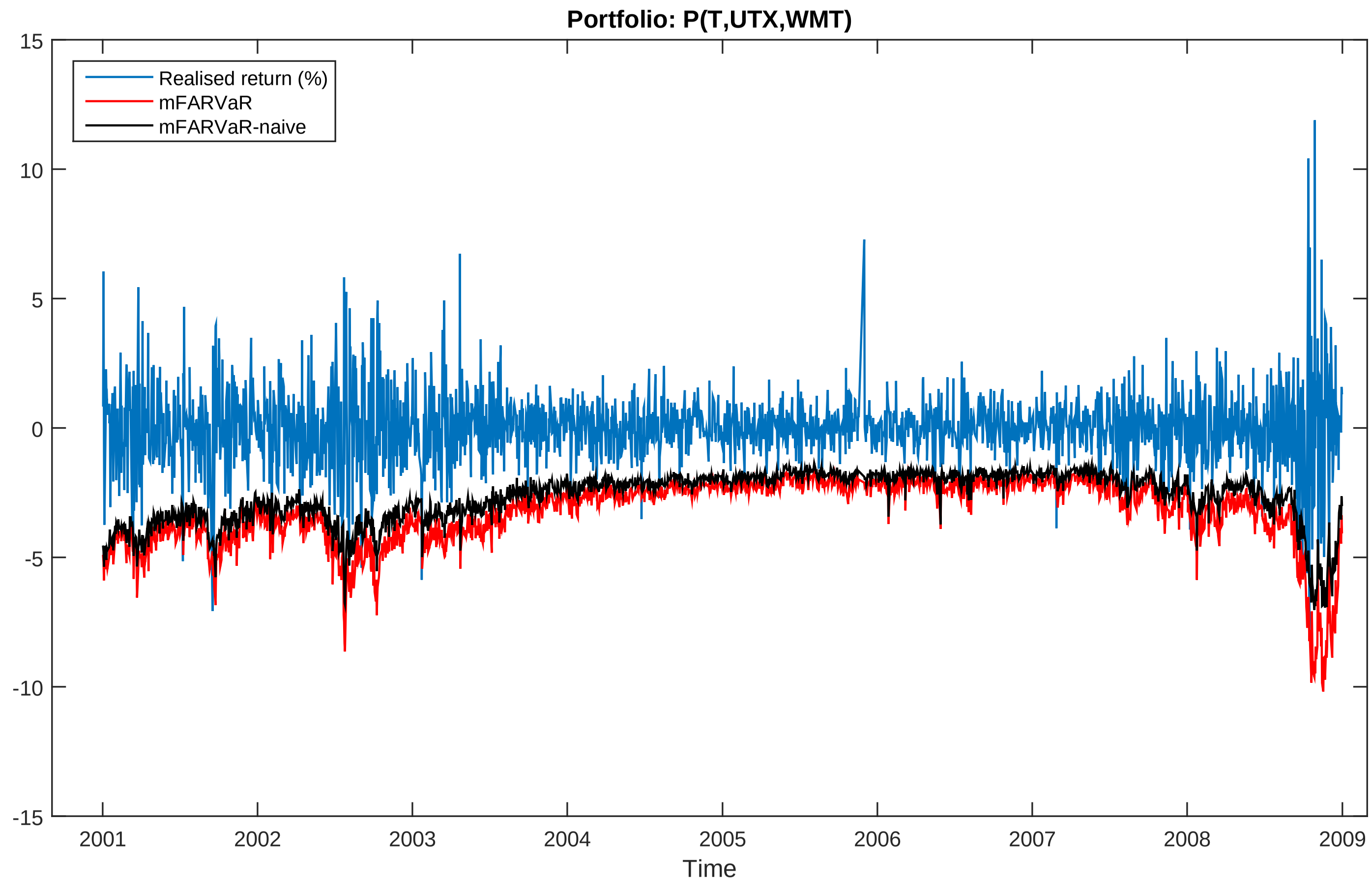
Note: This figure presents the VaR forecasts by FARVaR-nig, FARVaR-sim and FHS applied to the value-weighted portfolio of 30 stocks. The portfolio takes a long position. We estimate the VaR models using the window size of 250 days over the period 3 Jan 2000 – 27 Dec 2000 and compute the one-day-ahead 99% VaR forecast for 28 Dec 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for 31 Dec 2008.

Figure 7: VaR forecasts by FARVaR-cdf-sim and FARVaR-sim



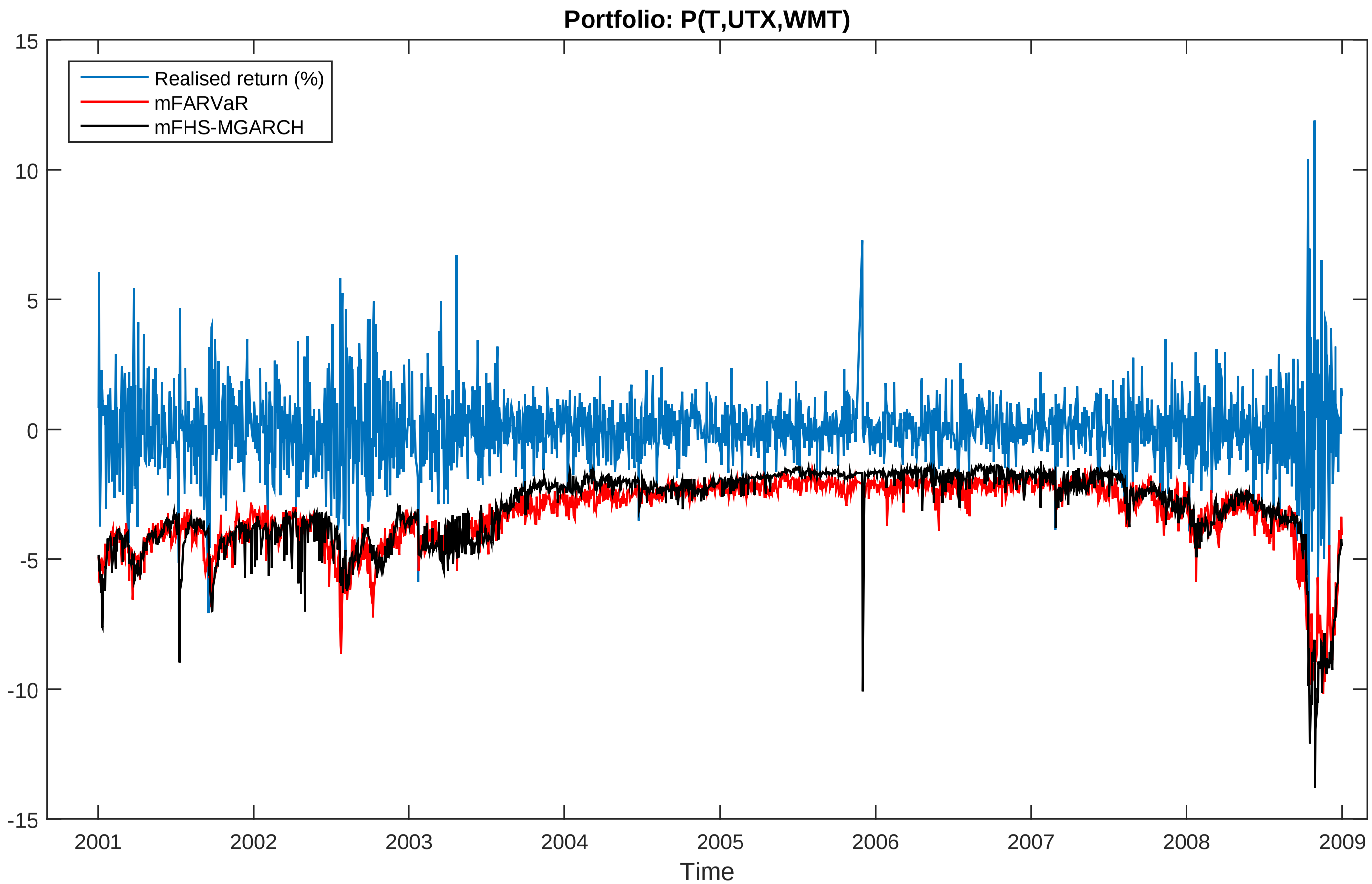
Note: This figure presents the VaR forecasts by FARVaR-cdf-sim applied to the value-weighted portfolio of 30 stocks. The portfolio takes a long position. We estimate the VaR model using the window size of 250 days over the period 3 Jan 2000 – 27 Dec 2000 and compute the one-day-ahead 99% VaR forecast for 28 Dec 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for 31 Dec 2008. For comparison, we also plot the VaR forecasts of our PDF-based FARVaR-sim.

Figure 8: VaR forecasts by mFARVaR and mFARVaR-naïve



Note: This figure presents the VaR forecasts by mFARVaR and mFARVaR-naïve applied to the portfolio P(T,UTX,WMT). The portfolio takes a long position. We estimate the VaR models using the window size of 250 days over the period 3 Jan 2000 – 27 Dec 2000 and compute the one-day-ahead 99% VaR forecast for 28 Dec 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for 31 Dec 2008.

Figure 9: VaR forecasts by mFARVaR and mFHS



Note: This figure presents the VaR forecasts by mFARVaR and mFHS-MGARCH applied to the portfolio P(T,UTX,WMT). The portfolio takes a long position. We estimate the VaR models using the window size of 250 days over the period 3 Jan 2000 – 27 Dec 2000 and compute the one-day-ahead 99% VaR forecast for 28 Dec 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for 31 Dec 2008.

Table 1: Components of the Dow Jones Industrial Average

Ticker: Company	Exchange	Industry	Data period
AA: Alcoa Corporation	NYSE	Aluminum	03/01/00 – 31/12/08
AIG: American International Group, Inc.	NYSE	Multiline insurance & brokers	03/01/00 – 31/12/08
AXP: American Express Company	NYSE	Consumer credit card services	03/01/00 – 31/12/08
BA: The Boeing Company	NYSE	Commercial aircraft manufacturing	03/01/00 – 31/12/08
C: Citigroup Inc.	NYSE	Banking	03/01/00 – 31/12/08
CAT: Caterpillar Inc.	NYSE	Construction machinery	03/01/00 – 31/12/08
DD: E. I. du Pont de Nemours & Company	NYSE	Industrial conglomerate	03/01/00 – 31/12/08
DIS: The Walt Disney Company	NYSE	Broadcasting	03/01/00 – 31/12/08
GE: General Electric	NYSE	Industrial conglomerate	03/01/00 – 31/12/08
GM: General Motors	NYSE	Auto & truck manufacturing	03/01/00 – 31/12/08
HD: The Home Depot, Inc.	NYSE	Home improvement products & services	03/01/00 – 31/12/08
HON: Honeywell International Inc.	NYSE	Industrial conglomerate	06/05/02 – 31/12/08
HPQ: HP Inc.	NYSE	Computer hardware	03/01/00 – 31/12/08
IBM: International Business Machines Corp.	NYSE	Computers & technology	03/01/00 – 31/12/08
INTC: Intel Corporation	NASDAQ	Semiconductors	03/01/00 – 31/12/08
JNJ: Johnson & Johnson	NYSE	Pharmaceuticals	03/01/00 – 31/12/08
JPM: JPMorgan Chase & Co.	NYSE	Banking	03/01/00 – 31/12/08
KO: The Coca-Cola Company	NYSE	Non-alcoholic beverages	03/01/00 – 31/12/08
MCD: McDonald's Corporation	NYSE	Quick service restaurants	03/01/00 – 31/12/08
MMM: 3M Company	NYSE	Industrial conglomerate	03/01/00 – 31/12/08
MO: Altria Group, Inc.	NYSE	Cigars & cigarette manufacturing	03/01/00 – 31/12/08
MRK: Merck & Co., Inc.	NYSE	Pharmaceuticals	03/01/00 – 31/12/08
MSFT: Microsoft Corporation	NASDAQ	Software	03/01/00 – 31/12/08
PFE: Pfizer Inc.	NYSE	Pharmaceuticals	03/01/00 – 31/12/08
PG: Procter & Gamble Co.	NYSE	Personal products	03/01/00 – 31/12/08
T: AT&T Inc.	NYSE	Telecommunications services	03/01/00 – 31/12/08
UTX: United Technologies Corporation	NYSE	Aerospace & defence	03/01/00 – 31/12/08
VZ: Verizon Communications Inc.	NYSE	Telecommunications services	03/07/00 – 31/12/08
WMT: Wal-Mart Stores, Inc.	NYSE	Supermarkets & convenience stores	03/01/00 – 31/12/08
XOM: Exxon Mobil Corporation	NYSE	Oil & gas refining and marketing	03/01/00 – 31/12/08

Note: From 21 Nov 2005, after the close, the Dow Jones Industrial Average consisted of the above 30 major companies.

Table 2: Descriptive statistics of intraday returns and daily returns

	Panel A: Intraday returns				Panel B: Daily returns			
Ticker: Company	Mean	StDev	SK	K	Mean	StDev	SK	K
AA: Alcoa Corporation	−0.0007	0.31	0.18	176.72	−0.0573	2.71	−0.06	10.77
AIG: American International Group, Inc.	−0.0024	0.54	−50.37	10690.99	−0.1698	3.99	−6.49	158.52
AXP: American Express Company	−0.0006	0.29	−2.39	207.30	−0.0423	2.41	−0.37	9.26
BA: The Boeing Company	0.0000	0.25	−3.75	324.89	0.0011	2.10	−0.33	9.37
C: Citigroup Inc.	−0.0010	0.35	15.90	2240.65	−0.0794	2.97	0.58	47.82
CAT: Caterpillar Inc.	0.0003	0.26	−2.11	173.43	0.0267	2.08	−0.21	7.28
DD: E. I. du Pont de Nemours & Company	−0.0005	0.23	−0.80	58.49	−0.0425	1.88	−0.22	8.96
DIS: The Walt Disney Company	−0.0001	0.27	−2.27	434.35	−0.0113	2.22	−0.05	11.87
GE: General Electric	−0.0007	0.24	−1.03	150.30	−0.0508	2.03	−0.22	9.47
GM: General Motors	−0.0006	0.28	−6.38	663.38	0.0300	1.79	0.15	14.63
HD: The Home Depot, Inc.	−0.0003	0.29	0.46	371.78	−0.0485	2.40	−1.11	22.44
HON: Honeywell International Inc.	0.0006	0.28	−0.31	238.78	−0.0253	2.38	−0.48	13.30
HPQ: HP Inc.	−0.0002	0.23	−2.61	477.37	0.0437	2.25	0.13	9.91
IBM: International Business Machines Corp.	−0.0010	0.37	−37.02	6336.19	−0.0127	1.94	−0.05	10.35
INTC: Intel Corporation	0.0001	0.17	−6.78	721.42	−0.0472	2.95	−0.49	9.49
JNJ: Johnson & Johnson	−0.0008	0.44	−148.97	44269.37	0.0110	1.43	−0.84	18.52
JPM: JPMorgan Chase & Co.	−0.0001	0.19	−0.91	76.73	−0.0032	2.64	0.03	11.91
KO: The Coca-Cola Company	0.0003	0.23	−2.18	140.36	−0.0106	1.54	0.08	9.12
MCD: McDonald's Corporation	0.0001	0.20	0.32	80.99	0.0196	1.78	−0.19	7.61
MMM: 3M Company	−0.0003	0.36	−193.24	61862.39	0.0081	1.59	0.19	7.99
MO: Altria Group, Inc.	−0.0005	0.24	−15.69	1810.63	0.0431	1.88	−0.02	12.94
MRK: Merck & Co., Inc.	−0.0006	0.25	−1.39	258.30	−0.0358	2.05	−1.83	31.32
MSFT: Microsoft Corporation	−0.0003	0.23	−3.98	326.59	−0.0488	2.25	−0.08	11.07
PFE: Pfizer Inc.	0.0001	0.20	−44.87	8810.92	−0.0261	1.81	−0.24	7.34
PG: Procter & Gamble Co.	−0.0003	0.32	45.67	9311.84	0.0061	1.66	−5.19	120.31
T: AT&T Inc.	0.0003	0.23	−2.10	229.93	−0.0539	2.44	0.28	14.90
UTX: United Technologies Corporation	−0.0002	0.23	−1.01	68.85	0.0295	1.91	−0.16	9.55
VZ: Verizon Communications Inc.	−0.0001	0.22	−0.14	61.41	−0.0186	1.88	0.17	8.28
WMT: Wal-Mart Stores, Inc.	0.0004	0.20	−0.69	63.75	−0.0089	1.79	0.26	6.88
XOM: Exxon Mobil Corporation	−0.0018	0.39	−1.94	549.39	0.0306	1.79	0.15	14.63

Note: This table presents the descriptive statistics of the intraday and daily returns of the 30 stocks listed in the DJIA index. We calculate the mean, standard deviation (StDev), skewness (SK) and kurtosis (K) of each individual company over the sample period. Panel A reports the descriptive statistics of the intraday returns. Panel B presents the descriptive statistics of the daily returns.

Table 3: Descriptive statistics and autocorrelation of copula correlation coefficients

	Panel A. Min/Mean/Max			Panel B. Autocorrelation			
Portfolio	δ_{12}	δ_{13}	δ_{23}	Statistics	δ_{12}	δ_{13}	δ_{23}
P(AA,CAT,INTC)	–0.30/0.28/0.83	–0.26/0.28/0.80	–0.27/0.33/0.82	ACF(1)	0.484***	0.365***	0.463***
				Q(5)	2161***	1233***	1976***
P(AIG,C,GM)	–0.18/0.36/0.81	–0.30/0.22/0.72	–0.26/0.27/0.72	ACF(1)	0.537***	0.447***	0.455***
				Q(5)	2691***	1874***	2083***
P(AXP,DD,PG)	–0.27/0.32/0.80	–0.50/0.28/0.73	–0.31/0.28/0.80	ACF(1)	0.477***	0.393***	0.435***
				Q(5)	2080***	1338***	1687***
P(BA,GE,XOM)	–0.26/0.30/0.78	–0.31/0.26/0.82	–0.31/0.32/0.81	ACF(1)	0.448***	0.473***	0.567***
				Q(5)	1943***	2233***	3059***
P(DIS,HPQ,MMM)	–0.29/0.28/0.81	–0.22/0.31/0.85	–0.27/0.31/0.79	ACF(1)	0.436***	0.521***	0.402***
				Q(5)	1204***	1915***	957***
P(HD,HON,IBM)	–0.28/0.28/0.86	–0.23/0.31/0.83	–0.39/0.31/0.84	ACF(1)	0.486***	0.452***	0.476***
				Q(5)	2215***	1788***	2043***
P(JNJ,JPM,MCD)	–0.25/0.26/0.75	–0.29/0.22/0.77	–0.33/0.26/0.81	ACF(1)	0.348***	0.376***	0.451***
				Q(5)	1031***	1387***	1773***
P(KO,MSFT,VZ)	–0.34/0.29/0.77	–0.29/0.27/0.79	–0.27/0.30/0.79	ACF(1)	0.422***	0.424***	0.469***
				Q(5)	1420***	1347***	2087***
P(MO,MRK,PFE)	–0.33/0.20/0.74	–0.31/0.21/0.73	–0.45/0.31/0.85	ACF(1)	0.373***	0.382***	0.479***
				Q(5)	1227***	1418***	2257***
P(T,UTX,WMT)	–0.25/0.24/0.83	–0.28/0.25/0.74	–0.27/0.30/0.79	ACF(1)	0.512***	0.417***	0.479***
				Q(5)	2556***	1736***	2046***

Note: This table presents the descriptive statistics and autocorrelation of copula correlation coefficients between the individual intraday returns constituting the equal-weighted portfolio of three stocks. In Panel A, we calculate the minimum/mean/maximum of copula correlation coefficients for each pair of individual stocks. In Panel B, we compute the sample ACF of order 1 and conduct the Ljung –Box Q-test for autocorrelation of copula correlation coefficients at a lag of 5. Note that *, ** and *** indicate that the null of zero autocorrelations is rejected at the 10%, 5% and 1% significance level, respectively.

Table 4: Performance of density forecast of intraday returns

Model	D_H		D_U		D_E	
FAR	0.02852	(9)	0.26648	(13)	0.04124	(4)
FAR-fft	0.02837	(14)	0.26625	(15)	0.04048	(19)
FAR-wv	0.02841	(7)	0.26781	(2)	0.04063	(7)
AVE	0.04568	(0)	0.36825	(0)	0.05762	(0)
LAST	0.03415	(0)	0.27786	(0)	0.05386	(0)

Note: This table presents the performance of the density forecast of intraday returns for different functional models: FAR, FAR-fft, FAR-wv, AVE and LAST. For the 30 stocks, one-step-ahead rolling forecasting is performed based on the 250-day window size. To begin with, we estimate the five functional models using intraday data over the period 3 Jan 2000 – 27 Dec 2000 and compute the one-day-ahead density forecast of intraday returns for 28 Dec 2000. We repeat this procedure by moving forward a day at a time in a rolling manner, ending with the density forecast for 31 Dec 2008. The Hilbert norm (D_H), the uniform norm (D_U) and the generalized entropy (D_E) are employed as divergence criteria (see Equation (27) for the definition), which are evaluated for the mean value of the 2013 intraday return density forecasts (except for HP, AT&T and Verizon, where 1418, 2006 and 1887 forecasts respectively are used for the calculation). The main figure reports the average of each divergence measure for 30 stocks evaluated by Equation (28), and $(\cdot)$ indicates the number of stocks for which a given model achieves the smallest value. The best forecasting performance for each measure is highlighted by a red colour.

Table 5: Backtesting VaR models: 250-day window size and long position

Model	ECP	BPZ	MRCR	PQL	CC	DQ
FARVaR-sim	0.87%	Green	42.65%	6.79%	6	12
FARVaR-nig	1.07%	Green	40.80%	6.76%	4	11
HS	1.19%	Green	45.28%	7.44%	12	25
FHS	1.10%	Green	45.29%	6.87%	1	12
RM	1.82%	Yellow	39.59%	6.89%	25	29
GARCH	1.60%	Green	39.78%	6.80%	19	25
FGEV	0.49%	Green	58.22%	7.61%	19	11
FGPD	0.51%	Green	60.21%	7.82%	20	12
CAViaR	2.83%	Yellow	47.43%	8.25%	30	30
CAViaR-GARCH	4.54%	Red	47.25%	9.74%	30	30

Note: This table presents the backtesting results for the long position. We estimate VaR models using the window size of 250 days over the period 3 Jan 2000 – 27 Dec 2000 and compute the one-day-ahead 99% VaR forecast for 28 Dec 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for 31 Dec 2008. This generates 2013 daily forecasts per company except for HP, AT&T and Verizon, which generate 1428, 2006 and 1887 forecasts, respectively. We report the average of the ECP, MRCR and PQL for the 30 stocks evaluated by Equation (36). The BPZ is evaluated based on the average violation of the 30 stocks. For the CC and DQ tests, we count the frequency of an individual model being rejected at the 5% significance level for the 30 stocks.

Table 6: Backtesting VaR models with value-weighted portfolio of 30 stocks: 250-day window size and long position

Models	ECP	BPZ	MRCR	PQL	CC	DQ
FARVaR-sim	0.65%	Green	31.21%	4.51%	3.065	10.387
FARVaR-nig	0.80%	Green	28.84%	4.44%	1.165	1.379
HS	1.24%	Green	27.35%	4.61%	1.750	8.236
FHS	1.14%	Green	25.89%	4.48%	1.649	9.408
RM	1.79%	Yellow	24.64%	4.38%	10.463***	41.198***
GARCH	1.84%	Yellow	25.00%	4.42%	13.267***	38.099***
FGEV	0.55%	Green	35.47%	5.09%	5.100*	5.490
FGPD	0.65%	Green	35.56%	5.18%	3.065	4.592
CAViaR	2.09%	Yellow	26.66%	4.59%	21.70***	107.515***
CAViaR-GARCH	3.68%	Red	27.56%	5.52%	88.239***	333.923***

Note: This table presents the backtesting results of VaR models applied to the value-weighted portfolio of 30 stocks. The portfolio takes a long position. We estimate VaR models using the window size of 250 days over the period 3 Jan 2000 – 27 Dec 2000 and compute the one-day-ahead 99% VaR forecast for 28 Dec 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for 31 Dec 2008. This generates 2013 daily forecasts per company except for HP, AT&T and Verizon, which generate 1428, 2006 and 1887 forecasts, respectively. We report the ECP, BPZ, MRCR, PQL, CC test statistic and DQ test statistic. *, ** and *** denote that the CC (DQ) test statistic is rejected at the 10%, 5% and 1% significance level, respectively.

Table 7: FARVaR and FHS: 250-day window size and long position

Date	Return	FARVaR-sim	FARVaR-nig	FHS
12 Mar 2001	–4.896	–3.159	–2.985	–3.544
13 Jun 2001	–3.610	–2.807	–2.749	–2.673
17 Sep 2001	–16.453	–3.996	–4.050	–4.257
04 Feb 2002	–2.519	–2.764	–2.471	–4.312
03 Jun 2002	–2.450	–2.772	–2.433	–3.150
10 Jul 2002	–3.365	–4.242	–4.165	–3.269
19 Jul 2002	–4.792	–4.895	–4.788	–3.222
03 Sep 2002	–4.520	–4.414	–4.598	–3.609
24 Mar 2003	–4.363	–4.039	–4.171	–5.116
19 May 2003	–2.703	–2.917	–2.405	–3.176
22 Oct 2003	–1.664	–3.148	–2.867	–1.613
28 Jan 2004	–1.367	–2.014	–1.779	–1.327
10 Mar 2004	–1.701	–2.757	–2.666	–1.366
11 Mar 2004	–1.688	–2.221	–1.993	–1.600
22 Feb 2005	–1.645	–1.943	–1.695	–1.489
15 Apr 2005	–1.738	–3.941	–3.532	–1.621
23 Jun 2005	–1.463	–2.025	–1.565	–1.437
20 Jan 2006	–1.830	–2.006	–2.020	–1.623
17 May 2006	–1.809	–3.398	–3.293	–1.388
30 May 2006	–1.660	–1.904	–1.505	–1.634
27 Nov 2006	–1.364	–1.775	–1.710	–1.134
27 Feb 2007	–3.313	–1.982	–1.547	–1.252
19 Oct 2007	–2.381	–4.274	–4.014	–1.837
11 Dec 2007	–2.007	–1.865	–1.699	–3.651
06 Jun 2008	–3.423	–3.171	–2.800	–2.800
26 Jun 2008	–3.247	–3.085	–2.681	–2.926
15 Sep 2008	–8.228	–5.889	–4.859	–3.168
17 Sep 2008	–6.688	–6.565	–6.844	–4.769
29 Sep 2008	–6.342	–5.701	–5.644	–6.584
01 Dec 2008	–8.625	–4.598	–4.163	–11.463

Note: This table presents the failures of VaR forecasts by FARVaR-sim, FARVaR-nig and FHS applied to the value-weighted portfolio of 30 stocks. The portfolio takes a long position. We estimate VaR models using the window size of 250 days over the period 3 Jan 2000 – 27 Dec 2000 and compute the one-day-ahead 99% VaR forecasts for 28 Dec 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for 31 Dec 2008. Note that a coloured cell indicates the failure of a VaR forecast.

Table 8: Backtesting VaR models excluding periods of market downturn: 250-day window size and long position

Models	ECP	BPZ	MRCR	PQL	CC	DQ
FARVaR-sim	0.57%	Green	32.87%	5.02%	9	5
FARVaR-nig	0.74%	Green	30.81%	4.93%	1	4
HS	0.66%	Green	33.63%	5.39%	2	10
FHS	0.85%	Green	31.98%	5.19%	0	9
RM	1.45%	Green	27.96%	4.95%	5	16
GARCH	1.14%	Green	27.94%	4.92%	3	7
FGEV	0.43%	Green	42.06%	5.94%	11	4
FGPD	0.44%	Green	44.24%	6.14%	9	4
CAViaR	2.59%	Yellow	33.54%	6.22%	26	30
CAViaR-GARCH	4.12%	Red	33.16%	7.26%	29	30

Note: This table presents the backtesting results for the long position excluding the periods of market downturn (the dot-com bubble burst of 2001 – 2002 and the subprime mortgage crisis of 2007 – 2008). We estimate VaR models using the window size of 250 days over the period 4 Jan 2002 – 31 Dec 2002 and compute the one-day-ahead 99% VaR forecast for 2 Jan 2003. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for 31 Dec 2006. This generates 1009 daily forecasts per company except for HP, which generates 924 forecasts. See the note of Table 5 for ECP, BPZ, MRCR, PQL, CC and DQ.

Table 9: Backtesting VaR models during periods of market downturn: 250-day window size and long position

Models	ECP	BPZ	MRCR	PQL	CC	DQ
Panel A. Dot-com bubble burst (2001–2002)						
FARVaR-sim	0.87%	Green	58.86%	8.43%	1	8
FARVaR-nig	1.03%	Green	56.79%	8.34%	1	9
HS	0.97%	Green	63.01%	9.34%	2	16
FHS	1.11%	Green	65.75%	9.19%	1	17
RM	1.86%	Yellow	57.26%	9.17%	7	23
GARCH	1.63%	Yellow	55.88%	9.01%	2	19
FGEV	0.46%	Green	84.98%	10.26%	1	4
FGPD	0.48%	Green	89.70%	10.63%	1	5
CAViaR	2.60%	Yellow	66.91%	10.68%	14	24
CAViaR-GARCH	5.12%	Red	67.88%	13.48%	25	29
Panel B. Subprime mortgage crisis (2007–2008)						
FARVaR-sim	1.47%	Green	59.90%	8.77%	5	15
FARVaR-nig	1.79%	Green	59.20%	8.94%	7	16
HS	2.44%	Yellow	60.04%	9.75%	19	26
FHS	1.60%	Green	69.94%	8.00%	3	14
RM	2.51%	Yellow	65.28%	8.60%	21	27
GARCH	2.51%	Yellow	64.66%	8.43%	20	29
FGEV	0.63%	Green	84.22%	8.38%	0	2
FGPD	0.67%	Green	83.61%	8.49%	0	2
CAViaR	3.48%	Yellow	78.19%	9.90%	27	30
CAViaR-GARCH	4.77%	Red	78.09%	11.05%	29	30

Note: This table presents the backtesting results for the long position during the periods of market downturn (the dot-com bubble burst of 2001 – 2002 and the subprime mortgage crisis of 2007 – 2008). In Panel A (the dot-com bubble burst), we estimate VaR models using the window size of 250 days over the period 5 Jan 2000 – 31 Dec 2000 and compute the one-day-ahead 99% VaR forecast for 3 Jan. 2001. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for 31 Dec 2002. This generates 500 daily forecasts per company except for HP and Verizon, which generate 0 (VaR forecast is available from 2003) and 374 forecasts, respectively. In Panel B (the subprime mortgage crisis), we estimate VaR models using the window size of 250 days over the period 4 Jan 2006 – 29 Dec 2006 and compute the one-day-ahead 99% VaR forecast for 3 Jan 2007. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for 31 Dec 2008. This generates 504 daily forecasts per company except for AT&T, which generate 497 forecasts. See the note of Table 5 for ECP, BPZ, MRCR, PQL, CC and DQ.

Table 10: FARVaR-cdf-sim and FARVaR-sim: 250-day window size and long position

Panel A. Backtesting

Model	ECP	BPZ	MRCR	PQL	CC	DQ
CDF	1.89%	Yellow	26.06%	4.30%	14.228***	33.280***
PDF	0.65%	Green	31.21%	4.51%	3.065	10.387

Panel B. Failures

Date	Return	CDF	PDF	Date	Return	CDF	PDF
12 Mar 2001	–4.896	–3.535	–3.159	17 May 2006	–1.809	–1.578	–3.398
13 Jun 2001	–3.610	–2.727	–2.807	30 May 2006	–1.660	–1.173	–1.904
06 Jul 2001	–1.963	–1.713	–2.420	27 Nov 2006	–1.364	–1.322	–1.775
08 Aug 2001	–1.487	–1.453	–1.849	25 Jan 2007	–1.005	–0.946	–1.861
17 Sep 2001	–16.453	–4.671	–3.996	27 Feb 2007	–3.313	–1.015	–1.982
29 Jan 2002	–2.491	–2.154	–2.706	13 Mar 2007	–1.924	–1.537	–3.386
04 Feb 2002	–2.519	–2.149	–2.764	10 May 2007	–1.048	–0.899	–1.885
03 Jun 2002	–2.450	–1.945	–2.772	10 Jul 2007	–1.313	–1.064	–1.963
03 Sep 2002	–4.520	–4.729	–4.414	24 Jul 2007	–1.508	–1.406	–5.302
24 Mar 2003	–4.363	–4.172	–4.039	09 Aug 2007	–2.636	–2.530	–5.908
19 May 2003	–2.703	–2.217	–2.917	28 Aug 2007	–2.058	–1.537	–3.257
24 Sep 2003	–1.722	–1.666	–3.308	11 Dec 2007	–2.007	–1.406	–1.865
13 Apr 2004	–1.323	–1.173	–1.857	27 Dec 2007	–1.370	–1.003	–2.479
20 Apr 2004	–1.442	–1.406	–1.865	05 Feb 2008	–2.964	–2.884	–3.484
22 Sep 2004	–1.377	–1.249	–2.326	06 Jun 2008	–3.423	–2.567	–3.171
22 Feb 2005	–1.645	–1.388	–1.943	26 Jun 2008	–3.247	–2.706	–3.085
31 May 2005	–0.768	–0.766	–1.916	15 Sep 2008	–8.228	–5.067	–5.889
27 Dec 2005	–0.965	–0.761	–1.912	17 Sep 2008	–6.688	–7.691	–6.565
20 Jan 2006	–1.830	–1.316	–2.006	29 Sep 2008	–6.342	–5.261	–5.701
11 May 2006	–1.172	–1.169	–1.670	01 Dec 2008	–8.625	–3.769	–4.598

Note: This table presents the results of VaR forecasts by FARVaR-cdf-sim and (PDF-based) FARVaR-sim applied to the value-weighted portfolio of 30 stocks. The portfolio takes a long position. We estimate VaR models using the window size of 250 days over the period 3 Jan 2000 – 27 Dec 2000 and compute the one-day-ahead 99% VaR forecast for 28 Dec 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for 31 Dec 2008. Panel A reports the backtesting results. We report the ECP, BPZ, MRCR, PQL, CC test statistic and DQ test statistic. *, ** and *** denote that the CC (DQ) test statistic is rejected at the 10%, 5% and 1% significance level, respectively. Panel B reports the failures of VaR forecasts by FARVaR-cdf-sim (CDF) and FARVaR-sim (PDF). The failure is highlighted by a red colour.

Table 11: Backtesting VaR models: 250-day window size and short position

Model	ECP	BPZ	MRCR	PQL	CC	DQ
FARVaR-sim	1.22%	Green	39.74%	5.98%	5	12
FARVaR-nig	0.96%	Green	41.20%	5.93%	4	7
HS	1.20%	Green	45.13%	7.06%	7	20
FHS	1.03%	Green	42.88%	6.05%	0	3
RM	1.66%	Yellow	38.64%	6.09%	19	20
GARCH	1.46%	Green	39.04%	5.97%	12	11
FGEV	0.48%	Green	52.53%	6.60%	20	6
FGPD	0.48%	Green	53.77%	6.73%	24	4
CAViaR	2.31%	Yellow	44.13%	7.10%	30	30
CAViaR-GARCH	2.27%	Yellow	42.70%	7.30%	30	30

This table presents the backtesting results for the short position. We estimate VaR models using the window size of 250 days over the period 3 Jan 2000 – 27 Dec 2000 and compute the one-day-ahead 99% VaR forecast for 28 Dec 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for 31 Dec 2008. This generates 2013 daily forecasts per company except for HP, AT&T and Verizon, which generate 1428, 2006 and 1887 forecasts, respectively. See the note of Table 5 for ECP, BPZ, MRCR, PQL, CC and DQ.

Table 12: Backtesting VaR models: 500-day window size and long position

Model	ECP	BPZ	MRCR	PQL	CC	DQ
FARVaR-sim	0.89%	Green	40.24%	6.58%	8	10
FARVaR-nig	1.05%	Green	38.77%	6.59%	7	13
HS	1.55%	Green	42.09%	7.85%	22	27
FHS	1.13%	Green	40.88%	6.37%	2	10
RM	1.82%	Yellow	36.84%	6.57%	27	28
GARCH	1.61%	Green	37.25%	6.46%	19	21
FGEV	0.20%	Green	70.93%	8.75%	29	22
FGPD	0.21%	Green	72.42%	8.92%	30	23
CAViaR	1.76%	Yellow	42.12%	6.88%	28	30
CAViaR-GARCH	3.54%	Yellow	45.22%	8.34%	30	30

Note: This table presents the backtesting results for the long position. We estimate VaR models using the window size of 500 days over the period 3 Jan 2000 – 31 Dec 2001 and compute the one-day-ahead 99% VaR forecast for 2 Jan 2002. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for 31 Dec 2008. This generates 1763 daily forecasts per company except for HP, AT&T and Verizon, which generate 1178, 1756 and 1637 forecasts, respectively. See the note of Table 5 for ECP, BPZ, MRCR, PQL, CC and DQ.

Table 13: Backtesting VaR models: 500-day window size and short position

Models	ECP	BPZ	MRCR	PQL	CC	DQ
FARVaR-sim	0.84%	Green	40.12%	5.74%	5	7
FARVaR-nig	0.96%	Green	41.20%	5.93%	3	7
HS	1.20%	Green	45.13%	7.06%	13	29
FHS	1.03%	Green	42.88%	6.05%	2	1
RM	1.66%	Yellow	38.64%	6.09%	24	22
GARCH	1.46%	Green	39.04%	5.97%	13	11
FGEV	0.48%	Green	52.53%	6.60%	30	21
FGPD	0.48%	Green	53.77%	6.73%	30	19
CAViaR	2.31%	Yellow	44.13%	7.10%	19	26
CAViaR-GARCH	1.98%	Yellow	41.02%	6.59%	25	28

Note: This table presents the backtesting results for the short position. We estimate VaR models using the window size of 500 days over the period 3 Jan 2000 – 31 Dec 2001 and compute the one-day-ahead 99% VaR forecast for 2 Jan 2002. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for 31 Dec 2008. This generates 1763 daily forecasts per company except for HP, AT&T and Verizon, which generate 1178, 1756 and 1637 forecasts respectively. See the note of Table 5 for ECP, BPZ, MRCR, PQL, CC and DQ.

Table 14: Backtesting multivariate FARVaR (mFARVaR)

Portfolio	ECP	BPZ	MRCR	PQL	CC	DQ
P(AA,CAT,INTC)	1.54%	Green	38.91%	5.79%	6.094**	16.621**
P(AIG,C,GM)	2.19%	Green	40.47%	8.99%	22.333***	93.375***
P(AXP,DD,PG)	0.99%	Green	28.96%	4.16%	0.403	5.709
P(BA,GE,XOM)	1.24%	Green	31.89%	4.45%	1.776	8.753
P(DIS,HPQ,MMM)	0.91%	Green	26.74%	4.06%	0.353	0.642
P(HD,HON,IBM)	0.60%	Green	35.90%	4.70%	3.998	18.556***
P(JNJ,JPM,MCD)	0.60%	Green	29.29%	3.80%	3.998	5.265
P(KO,MSFT,VZ)	0.74%	Green	27.76%	3.62%	1.588	1.679
P(MO,MRK,PFE)	1.69%	Green	29.73%	4.96%	9.211**	18.443***
P(T,UTX,WMT)	0.85%	Green	29.10%	3.91%	0.779	0.993
Overall	1.14%	Green	31.87%	4.85%	3	4

Note: This table presents the backtesting results of mFARVaR applied to the 10 portfolios. Each portfolio takes a long position and is the equal-weighted portfolio of three stocks randomly drawn without replacement from the 30 stocks. We estimate mFARVaR using the window size of 250 days over the period 3 Jan 2000 – 27 Dec 2000 and compute the one-day-ahead 99% VaR forecast for 28 Dec 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for 31 Dec 2008. This generates 2013 daily forecasts per company except for P(DIS,HPQ,MMM), P(KO,MSFT,VZ) and P(T,UTX,WMT), which generate 1428, 1887 and 2006 forecasts, respectively. Note that we select the t-copula for modelling the tail dependence. We report the ECP, BPZ, MRCR, PQL, CC test statistic and DQ test statistic for each portfolio. *, ** and *** denote that the CC (DQ) test is rejected at the 10%, 5% and 1% significance level, respectively. The 'Overall' row reports the average of the ECP, MRCR and PQL for the 10 portfolios. The BPZ is evaluated based on the average violation of the 10 portfolios. For the CC and DQ tests, we count the number of portfolios being rejected at the 5% significance level.

Table 15: Backtesting naïve multivariate FARVaR (mFARVaR-naïve)

Portfolio	ECP	BPZ	MRCR	PQL	CC	DQ
P(AA,CAT,INTC)	3.43%	Yellow	33.42%	7.32%	74.630***	176.327***
P(AIG,C,GM)	4.03%	Yellow	35.29%	11.83%	113.537***	527.076***
P(AXP,DD,PG)	2.19%	Yellow	24.82%	4.65%	23.403***	60.938***
P(BA,GE,XOM)	2.69%	Yellow	26.80%	5.28%	37.741***	119.653***
P(DIS,HPQ,MMM)	2.66%	Yellow	24.67%	4.55%	29.528***	94.404***
P(HD,HON,IBM)	1.59%	Green	30.53%	4.80%	7.068**	36.896***
P(JNJ,JPM,MCD)	1.59%	Green	26.17%	3.98%	6.416**	35.877***
P(KO,MSFT,VZ)	2.02%	Yellow	25.07%	3.92%	15.267***	27.620***
P(MO,MRK,PFE)	2.88%	Yellow	27.19%	5.79%	47.918***	99.015***
P(T,UTX,WMT)	1.95%	Green	25.64%	4.20%	14.288***	31.111***
Overall	2.50%	Yellow	27.96%	5.63%	10	10

Note: This table presents the backtesting results of naïve mFARVaR applied to the 10 portfolios. Each portfolio takes a long position and is the equal-weighted portfolio of three stocks randomly drawn without replacement from the 30 stocks. We estimate mFARVaR-naïve using the window size of 250 days over the period 3 Jan 2000 – 27 Dec 2000 and compute the one-day-ahead 99% VaR forecasts for 28 Dec 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for 31 Dec 2008. This generates 2013 daily forecasts per company except for P(DIS,HPQ,MMM), P(KO,MSFT,VZ) and P(T,UTX,WMT), which generate 1428, 1887 and 2006 forecasts, respectively. Note that we select the t-copula for modelling the tail dependence. See the note of Table 14 for ECP, BPZ, MRCR, PQL, CC, DQ and Overall.

Table 16: Backtesting MGARCH-based FHS (mFHS-MGARCH)

Portfolio	ECP	BPZ	MRCR	PQL	CC	DQ
P(AA,CAT,INTC)	1.49%	Green	39.66%	5.87%	5.187*	25.394***
P(AIG,C,GM)	1.79%	Green	49.83%	7.33%	11.602***	16.211**
P(AXP,DD,PG)	1.49%	Green	27.30%	4.38%	5.187*	16.864***
P(BA,GE,XOM)	1.44%	Green	30.48%	4.76%	3.614	16.530**
P(DIS,HPQ,MMM)	1.26%	Green	26.94%	4.37%	2.406	11.297*
P(HD,HON,IBM)	1.39%	Green	31.82%	4.51%	3.586	19.192***
P(JNJ,JPM,MCD)	1.39%	Green	29.21%	4.01%	3.586	6.211
P(KO,MSFT,VZ)	1.43%	Green	28.04%	3.84%	3.849	9.003
P(MO,MRK,PFE)	1.34%	Green	34.22%	5.09%	2.896	5.910
P(T,UTX,WMT)	1.55%	Green	28.39%	4.28%	5.646*	12.664**
Overall	1.46%	Green	32.59%	4.84%	1	6

Note: This table presents the backtesting results of mFHS-MGARCH applied to the 10 portfolios. Each portfolio takes a long position and is the equal-weighted portfolio of three stocks randomly drawn without replacement from the 30 stocks. We estimate mFHS-MGARCH using the window size of 250 days over the period 3 Jan 2000 – 27 Dec 2000 and compute the one-day-ahead 99% VaR forecast for 28 Dec 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for 31 Dec 2008. This generates 2013 daily forecasts per company except for P(DIS,HPQ,MMM), P(KO,MSFT,VZ) and P(T,UTX,WMT), which generate 1428, 1887 and 2006 forecasts, respectively. Note that we select the *t*-copula for modelling the tail dependence. See the note of Table 14 for ECP, BPZ, MRCR, PQL, CC, DQ and Overall.

Table 17: Backtesting copula-based mFHS (mFHS-copula)

Portfolio	ECP	BPZ	MRCR	PQL	CC	DQ
P(AA,CAT,INTC)	1.24%	Green	42.85%	5.82%	1.750	17.381***
P(AIG,C,GM)	2.04%	Yellow	50.17%	7.82%	20.451***	53.293***
P(AXP,DD,PG)	1.24%	Green	27.92%	4.21%	1.750	16.142**
P(BA,GE,XOM)	1.34%	Green	31.13%	4.50%	5.494*	25.374***
P(DIS,HPQ,MMM)	1.89%	Yellow	28.25%	4.82%	9.508***	61.765***
P(HD,HON,IBM)	0.94%	Green	34.05%	4.80%	0.425	16.371**
P(JNJ,JPM,MCD)	1.69%	Green	29.41%	4.28%	13.466***	44.084***
P(KO,MSFT,VZ)	1.64%	Green	28.58%	4.14%	12.708***	65.711***
P(MO,MRK,PFE)	2.14%	Yellow	34.68%	5.68%	25.931***	147.051***
P(T,UTX,WMT)	1.90%	Green	30.25%	4.48%	24.731***	106.869***
Overall	1.61%	Green	33.73%	5.06%	6	10

Note: This table presents the backtesting results of mFHS-copula applied to the 10 portfolios. Each portfolio takes a long position and is the equal-weighted portfolio of three stocks randomly drawn without replacement from the 30 stocks. We estimate mFHS-copula using the window size of 250 days over the period 3 Jan 2000 – 27 Dec 2000 and compute the one-day-ahead 99% VaR forecast for 28 Dec 2000. We repeat this procedure moving forward a day at a time in a rolling manner, ending with the forecast for 31 Dec 2008. This generates 2013 daily forecasts per company except for P(DIS,HPQ,MMM), P(KO,MSFT,VZ) and P(T,UTX,WMT), which generate 1428, 1887 and 2006 forecasts, respectively. Note that we select the *t*-copula for modelling the tail dependence. See the note of Table 14 for ECP, BPZ, MRCR, PQL, CC, DQ and Overall.