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On the existence of high-frequency boundary resonances in layered elastic media

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We analyse the asymptotic behaviour of high-frequency vibrations of a three-dimensional layered elastic medium occupying the domain $\Omega=(-a,a)^3$, $a>0$. We show that in both cases of stress-free and zero-displacement boundary conditions on the boundary of Ω a version of the boundary spectrum, introduced in Allaire and Conca (1998 *J. Math. Pures. Appl.* **77**, 153–208. (doi:10.1016/S0021-7824(98)80068-8)), is non-empty and part of it is located below the Bloch spectrum. For zero-displacement boundary conditions, this yields a new type of surface wave, which is absent in the case of a homogeneous medium.

1. Problem formulation and background

There is a growing interest in transport properties of elastic waves in periodic structures. On smaller length scales, new research opportunities emerge in audio filters, nanoscopic phononic lasers, perfect reflectors, phononic integrated circuits. On larger length scales, one may wish to control surface seismic waves in structured soils for civil engineering applications. It may therefore be of importance to consider the asymptotic analysis of spectral problems for the Navier operator of linearized elasticity, with rapidly oscillating coefficients. As we discuss below, the related spectra are shown to accumulate around two sets, the ‘Bloch spectrum’ and the ‘boundary spectrum’, whose eigenfunction sequences behave in essentially different ways that correspond to ‘bulk’ wave propagation and ‘surface’ wave propagation, respectively. The objective of this work is to demonstrate mathematically rigorously that the boundary spectrum can be non-empty, and that it can in fact be located outside the Bloch spectrum. One notable feature of linearized elastic systems in homogeneous media with boundaries is a non-trivial coupling between pressure and shear waves at the boundary, which results in, e.g., the existence of surface waves, known as 'Rayleigh waves', even in the case of constant coefficients. In what follows we exploit some of the extra freedom provided by the general periodic setting in order to give an explicit example of elements of the boundary spectrum. As a by-product, we show the existence of a new kind of surface waves in layered periodic media with a clamped boundary: indeed, it is well known that a clamped boundary does not support surface acoustic waves on any homogeneous half space [1,2].

Consider an ε-periodic linearized elastic medium occupying the domain $\Omega=(-a,a)^3\subset\mathbb{R}^3$, where $\varepsilon\in\Xi:=\{\varepsilon>0:\varepsilon^{-1}a\in\mathbb{N}\}$. We study the following eigenvalue problem, understood in the weak sense:

$$-(\rho^\varepsilon)^{-1}(C^\varepsilon_{ijkl}(u^\varepsilon_k)_{,l})_{,j}=\lambda^\varepsilon u^\varepsilon_i,\quad u^\varepsilon\in[H^1(\Omega)]^3. \tag{1.1}$$

Here, and throughout this article, we sum over repeated indices and use a comma for partial derivatives, for example $u^\varepsilon_{k,j}:=\partial u^\varepsilon_k/\partial x_j$. The functions $\rho^\varepsilon(x)=\rho(x/\varepsilon)$, $C^\varepsilon=(C_{ijkl}(x/\varepsilon))^3_{i,j,k,l=1}$ are the mass density and the elastic tensor of the medium, respectively. We assume that $\rho,\rho^{-1}\in L^\infty(Y)$, $Y:=[0,1)^3$, and that the elastic tensor $C\in[L^\infty(Y)]^{81}$ has the 'major' symmetries $C_{ijkl}=C_{klij}$, $i,j,k,l=1,2,3$, and is uniformly elliptic, i.e. there exists $\nu>0$ such that

$$C_{ijkl}M_{ij}M_{kl}\geq\nu M_{ij}M_{ij}\quad\forall M=(M_{ij})^3_{i,j=1}\in\mathbb{R}^{3\times3}. \tag{1.2}$$

It is well known (e.g. [3]) that, for each ε, the spectrum σ^ε of (1.1) consists of the set of all first elements λ^ε of pairs $(\lambda^\varepsilon,u^\varepsilon)$, $u^\varepsilon\neq0$ that satisfy (1.1), with $\lambda^\varepsilon_m\to\infty$ as $m\to\infty$.

We study the high-frequency spectrum of (1.1); more precisely, we describe the asymptotic behaviour of the set $\varepsilon^2\sigma^\varepsilon$ when the period $\varepsilon\in\Xi$ goes to zero. Without loss of generality, we shall study the set of pairs $(\mu^\varepsilon,u^\varepsilon)\in\mathbb{R}^+\times[H^1(\Omega)]^3$, $u^\varepsilon\neq0$, such that

$$-\varepsilon^2(\rho^\varepsilon)^{-1}(C^\varepsilon_{ijkl}(u^\varepsilon_k)_{,l})_{,j}+u^\varepsilon_i=\mu^\varepsilon u^\varepsilon_i,\quad i=1,2,3. \tag{1.3}$$

Indeed, if μ^ε and u^ε satisfy (1.3) then (1.1) holds for u^ε and $\lambda^\varepsilon=\varepsilon^{-2}(\mu^\varepsilon-1)$. In other words, one has $\lim_{\varepsilon\to0}\varepsilon^2\sigma^\varepsilon=\lim_{\varepsilon\to0}\tilde\sigma^\varepsilon$, where $\tilde\sigma^\varepsilon+1$ is the spectrum of (1.3) for each value of ε. Problem (1.3) defines a compact linear operator $\mathcal{R}^\varepsilon$ on $[L^2(\Omega)]^3$, by the formula $\mathcal{R}^\varepsilon f=u^\varepsilon$ for all $f\in[L^2(\Omega)]^3$, where u^ε is the unique weak solution in $[H^1(\Omega)]^3$ of the problem

$$-\varepsilon^2(\rho^\varepsilon)^{-1}(C^\varepsilon_{ijkl}(u^\varepsilon_k)_{,l})_{,j}+u^\varepsilon_i=f_i,\quad i=1,2,3. \tag{1.4}$$

Note that while $\mathcal{R}^\varepsilon$ are compact they converge in the 'strong two-scale sense' [4–6] to a non-compact operator. In fact, as we recall next, the limit set $\lim_{\varepsilon\to0}\tilde\sigma_\varepsilon$ contains a union of intervals. Indeed, for any $m\in\mathbb{N}^3$ a subsequence of $\mathcal{R}^\varepsilon$ strongly two-scale converges (e.g. [6]) to an operator $\mathcal{R}^{(m)}$ defined on the space $[L^2(\Omega\times mY)]^3$, $mY:=[0,m_1)\times[0,m_2)\times[0,m_3)$, by the formula $\mathcal{R}^{(m)}f=u$, where[1] $u\in[L^2(\Omega;H^1_\#(mY))]^3$ is such that

$$-\rho^{-1}(C_{ijkl}(u_k)_{,y_l})_{,y_j}+u_i=f_i,\quad i=1,2,3. \tag{1.5}$$

(Henceforth, we use the subscripts $_{,y_l},{}_{,y_j},\ldots$ rather than $_{,l},{}_{,j},\ldots$ to refer to the derivatives with respect to the 'cell variable' $y\in mY$ of a function of the pair $(x,y)\in\Omega\times mY$.) In particular, the spectrum of $\mathcal{R}^{(m)}$ is contained in the set $\lim_{\varepsilon\to0}(\tilde\sigma^\varepsilon+1)^{-1}$. The arbitrary choice of m implies that $\sigma_{\text{Bloch}}\subset\lim_{\varepsilon\to0}\tilde\sigma^\varepsilon$, where $(\sigma_{\text{Bloch}}+1)^{-1}$ is the union of the spectra of $\mathcal{R}^{(m)}$ over all $m\in\mathbb{N}^3$. The description of the 'Bloch spectrum' σ_{Bloch} corresponding to the differential expression in (1.5) can be found in a number of standard references, such as e.g. [7]. In the study of $\lim_{\varepsilon\to0}\sigma^\varepsilon$, the question that remains is if this limit set is fully described by the Bloch spectrum, i.e. whether $\lim_{\varepsilon\to0}\tilde\sigma^\varepsilon=\sigma_{\text{Bloch}}$ hold. For problems in the whole space and for problems on a 'supercell' torus (cf. [6]), it can be shown that this is indeed the case. However, introducing a boundary may result in the existence of sequences of eigenvalues of (1.1) that accumulate outside σ_{Bloch},

[1] We denote by $H^1_\#(mY)$ the closure of smooth mY-periodic functions with respect to the H^1-norm, e.g. [3].

as suggested by the numerical analysis of waves propagating in layered elastic media with a stress-free boundary [8–10].

We begin our discussion of boundary phenomena for the equation (1.1) with reference to the work [11] (see also [12,13]), where it is noted that the Bloch spectrum may not be sufficient for the description of $\lim_{\varepsilon\to 0}\tilde{\sigma}^{\varepsilon}$. The paper [11] discusses, among other things, implications of a possible discrepancy between these two sets in the case of scalar elliptic partial differential equations (PDEs). Therein, a subset σ_{boundary} of $\lim_{\varepsilon\to 0}\tilde{\sigma}^{\varepsilon}$ is introduced that was shown to contain any such 'leftover' spectrum. However, it remained to be seen whether such a discrepancy does occur. This leads one to natural questions about technical aspects of σ_{boundary}, namely is it non-empty and if, in general, it is not contained in σ_{Bloch}. The main objective of our present work is to elucidate this matter by providing an example of a PDE system based on the family (1.1) that possesses eigenvalue sequences with accumulation points that belong to σ_{boundary} and lie outside σ_{Bloch}. In our example, we consider a layered medium, where the elastic parameters C_{ijkl} depend on a single spatial variable. We investigate both cases of the 'stress-free' (i.e. Neumann-type) and 'zero-displacement' (i.e. Dirichlet-type) boundary conditions on $\partial\Omega$. We show that in each case a new surface wave is present, in addition to the classical Rayleigh wave in a homogeneous medium. In particular, for a zero-displacement boundary, this demonstrates a surface effect that has not been previously addressed.

The layout of the article is as follows. In §2, we prove that the set $\lim_{\varepsilon\to 0}\tilde{\sigma}^{\varepsilon}$ is the union of σ_{Bloch} and σ_{boundary} for the PDE system (1.1) with Neumann boundary conditions. In §3, we provide a characterization for the set σ_{boundary} in terms of a family of canonical half-space problems. In §4, by analysing the canonical family introduced in §3, we show for a specific example of (1.1) that σ_{boundary} is neither empty nor a strict subset of σ_{Bloch}. In the same section, we also give a new secular equation for the speed of a surface wave that occurs in problems of type (1.1) subject to a zero-displacement condition on part of the boundary of the domain Ω. Appendix A contains the technical details concerning the proof of the main characterization result, presented in §2, of the set $\lim_{\varepsilon\to 0}\tilde{\sigma}^{\varepsilon}$ for the PDE family (1.3) with Neumann boundary conditions.

2. Homogenization and the characterization of the limit spectrum

(a) The inclusion $\sigma_{\text{Bloch}}\subset\lim_{\varepsilon\to 0}\tilde{\sigma}^{\varepsilon}$

The result of this section is demonstrated by a series of standard results that we shall introduce, without proof, and cite where appropriate. To begin with, we shall review the notion of two-scale convergence [4,14].

Definition 2.1. Let u^{ε} be a bounded sequence in $[L^2(\Omega)]^3$ and let $m\in\mathbb{N}^3$.

(i) We say that u^{ε} two-scale converge to $u^0(x,y)\in L^2(\Omega;[L^2(mY)]^3)$, and write $u^{\varepsilon}\overset{2}{\rightharpoonup}u^0$, if

$$\int_{\Omega}u^{\varepsilon}(x)\cdot\varphi(x)\psi\left(\frac{x}{\varepsilon}\right)\mathrm{d}x\longrightarrow|mY|^{-1}\int_{\Omega}\int_{mY}u^0(x,y)\cdot\varphi(x)\psi(y)\,\mathrm{d}y\,\mathrm{d}x\quad\forall\varphi\in C_0^{\infty}(\Omega),\ \forall\psi\in[C_{\#}^{\infty}(mY)]^3.$$

(ii) We say that u^{ε} strongly two-scale converge to u^0, and write $u^{\varepsilon}\overset{2}{\to}u^0$, if for all $v^{\varepsilon}\overset{2}{\rightharpoonup}v^0$ one has

$$\int_{\Omega}u^{\varepsilon}(x)\cdot v^{\varepsilon}(x)\,\mathrm{d}x\longrightarrow|mY|^{-1}\int_{\Omega}\int_{mY}u^0(x,y)\cdot v^0(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

Definition 2.2. Let A^{ε}, A^0 be non-negative self-adjoint operators on $[L^2(\Omega)]^3$ and on a closed linear subspace H of $[L^2(\Omega\times mY)]^3$, respectively. We say that A^{ε} strongly two-scale resolvent converge to A^0 and denote it by $A^\varepsilon \xrightarrow{2} A^0$ if

$$(A^\varepsilon + I)^{-1} f^\varepsilon \xrightarrow{2} (A^0 + I)^{-1} Pf \quad \text{whenever} \quad f^\varepsilon \xrightarrow{2} f \in [L^2(\Omega \times mY)]^3,$$

where $P : L^2(\Omega \times mY) \to H$ is the orthogonal projection onto H.

The main result of this section, corollary 2.4, follows directly from the following standard result (e.g. [6]).

Lemma 2.3. *Suppose that $f^\varepsilon \xrightarrow{2} f^0 \in [L^2(\Omega \times mY)]^3$ for some $m \in \mathbb{N}^3$, and for each $\varepsilon \in \Xi$ denote by u^ε the solution to (1.4) where f is replaced by f^ε. Then $u^\varepsilon \xrightarrow{2} u^0$, where $u^0 \in L^2(\Omega; [H^1_\#(mY)]^3)$ satisfies*

$$\int_\Omega \int_{mY} C_{ijkl} u^0_{k,y_l} \varphi_{i,y_j} + \int_\Omega \int_{mY} \rho u^0_i \varphi_i = \int_\Omega \int_{mY} \rho f^0_i \varphi_i, \quad \forall \varphi \in L^2(\Omega; [C^\infty_\#(mY)]^3). \tag{2.1}$$

Corollary 2.4. *For all $m \in \mathbb{N}^3$, denote by $\sigma^{(m)}$ the set of all values κ for which there exists a function $u \in [H^1_\#(mY)]^3$, $u \neq 0$, such that*

$$\int_{mY} C_{ijkl} u_{k,y_l} \Psi_{i,y_j} = \kappa \int_{mY} \rho u_i \Psi_i \quad \forall \Psi \in [H^1_\#(mY)]^3.$$

Then the inclusion

$$\operatorname{clos}\left(\bigcup_{m \in \mathbb{N}^3} \sigma^{(m)} \right) \subset \lim_{\varepsilon \to 0} \tilde{\sigma}^\varepsilon \tag{2.2}$$

holds.

Proof. For each $\varepsilon > 0$, denote by $\mathcal{H}^\varepsilon$ the closure of $[H^1(\Omega)]^3$ in the weighted space $[L^2_\varepsilon(\Omega)]^3$ with inner product

$$(u, v)_\varepsilon = \int_\Omega \rho^\varepsilon u \cdot v.$$

Note that the bilinear form

$$\mathcal{A}^\varepsilon(u, v) := \varepsilon^2 \int_\Omega C^\varepsilon_{ijkl} u_{k,l} v_{i,j}$$

is closed and non-negative on $\mathcal{H}^\varepsilon$, hence it generates a self-adjoint operator A^ε whose domain $\mathcal{D}(A^\varepsilon)$ is a dense linear subset of $[L^2_\varepsilon(\Omega)]^3$ such that

$$\mathcal{A}^\varepsilon(u, v) = (A^\varepsilon u, v)_\varepsilon, \quad \forall u \in D(A^\varepsilon), v \in \mathcal{H}^\varepsilon.$$

Similarly, for each $m \in \mathbb{N}^3$ we denote by $\mathcal{H}^{(m)}$ the closure of $L^2(\Omega; [H^1_\#(mY)]^3)$ in the space $[L^2_\rho(\Omega \times mY)]^3$ with inner product

$$(u, v)_{(m)} = \int_\Omega \int_{mY} \rho u \cdot v,$$

and note that the bilinear form

$$\mathcal{A}^{(m)}(u, v) := \int_\Omega \int_{mY} C_{ijkl} u_{k,y_l} v_{i,y_j}, \quad u, v \in \mathcal{H}^{(m)}$$

defines a self-adjoint operator $A^{(m)}$ whose domain is a dense linear subset of $[L^2_\rho(\Omega \times mY)]^3$. Clearly, the spectrum of the operator $A^{(m)}$ coincides with the set $\sigma^{(m)}$ defined above. Further, lemma 2.3 implies that $A^\varepsilon \xrightarrow{2} A^{(m)}$, and since the spectrum of A^ε is given by $\tilde{\sigma}^\varepsilon$, the inclusion $\sigma^{(m)} \subset \lim_{\varepsilon\to 0} \tilde{\sigma}^\varepsilon$ follows. Applying lemma 2.3 for all $m \in \mathbb{N}^3$ yields

$$\bigcup_{m \in \mathbb{N}^3} \sigma^{(m)} \subset \lim_{\varepsilon \to 0} \tilde{\sigma}^\varepsilon.$$

The above inclusion, along with the fact that $\lim_{\varepsilon\to 0} \tilde{\sigma}^\varepsilon$ is closed, provides the inclusion (2.2). ■

The inclusion $\sigma_{\text{Bloch}} \subset \lim_{\varepsilon \to 0} \tilde{\sigma}^\varepsilon$ is now demonstrated by the following proposition.

Corollary 2.5. *For a fixed* $\theta \in [0, 2\pi)^3$, *we set* $e^\theta(y) := \exp(i\theta \cdot y)$, $y \in Y$, *and suppose that a pair* $(\kappa, u) \in \mathbb{R}^+ \times [H^1_\#(Y)]^3$ *satisfies*

$$-\rho^{-1}(C_{ijkl}(u_k e^\theta)_{,l})_{,j} = \kappa u_i e^\theta.$$

Then κ *is an element of* $\mathrm{clos}(\bigcup_{m\in\mathbb{N}^3} \sigma^{(m)})$.

Proof. By the min–max variational characterization of eigenvalues, κ can readily be shown to be continuous with respect to θ. The continuity of $\kappa(\theta)$ and the fact that the set of rational numbers is dense in $\mathbb{R}$ imply that

$$\bigcup_{\theta\in[0,2\pi)^3} \kappa(\theta) = \mathrm{clos}\left(\bigcup_{(2\pi)^{-1}\theta\in\mathbb{Q}^3\cap[0,1)^3} \kappa(\theta) \right).$$

Therefore, as $\mathrm{clos}(\bigcup_{m\in\mathbb{N}^3} \sigma^{(m)})$ is closed, it is sufficient to prove the claim of the corollary for the case when $(2\pi)^{-1}\theta \in \mathbb{Q}^3 \cap [0, 1)^3$.

Suppose that $(2\pi)^{-1}\theta \in \mathbb{Q}^3 \cap [0, 1)^3$, i.e. $(2\pi)^{-1}\theta_i = p_i/m_i$ for some $p_i, m_i \in \mathbb{N}$, $i = 1, 2, 3$. For any κ, u as above one has

$$\int_Y C_{ijkl}(u_k e^\theta)_{,y_l} \overline{(\varphi_i e^\theta)}_{,y_j} = \kappa \int_Y \rho u_i \bar{\varphi}_i, \quad \forall \varphi \in [H^1_\#(Y)]^3. \tag{2.3}$$

For each $m \in \mathbb{N}^3$, $\Psi \in [C^\infty_\#(mY)]^3$, we use the 'Gelfand transform of Ψ'

$$\varphi_i(y) = \sum_{z\in\mathbb{Z}^3, 0\le z_i\le m_i-1,\ i=1,2,3.} \Psi_i(y+z)\,\mathrm{e}^{-\mathrm{i}\theta\cdot(y+z)} \tag{2.4}$$

as a test function in (2.3), which is valid since $\varphi \in [H^1_\#(Y)]^3$. Note that since u, C are ρ are Y-periodic, and therefore mY-periodic, one has by direct calculation that

$$\int_{mY} C_{ijkl}(w_k)_{,y_l} \overline{(\Psi_i)}_{,y_j} = \kappa \int_{mY} \rho w_i \overline{\Psi}_i,$$

where $w := \mathrm{e}^\theta u$. In view of the freedom in the choice of Ψ, we infer that κ is an eigenvalue of $A^{(m)}$ with eigenfunction w. ∎

(b) Completeness of spectrum

We shall now introduce, in analogy with [11], the set

$$\sigma_{\mathrm{boundary}} := \{\kappa \mid \exists(\mu^\varepsilon, u^\varepsilon) \text{ solutions of (1.3)}, \|u^\varepsilon\|_{[L^2(\Omega)]^3} = 1,\ \lim_{\varepsilon\to 0} \mu^\varepsilon = \kappa + 1,$$
$$\forall n \in \mathbb{N} \exists C > 0 : \|\mathrm{dist}(x, \partial\Omega)^n u^\varepsilon\|_{[L^2(\Omega)]^3} + \varepsilon \|\mathrm{dist}(x, \partial\Omega)^n \nabla u_\varepsilon\|_{[L^2(\Omega)]^{3\times 3}} \le C\varepsilon^n\}. \tag{2.5}$$

We now present, and prove in this section, the following result.

Theorem 2.6. *Under the above definitions of* $\tilde{\sigma}_\varepsilon$, σ_{Bloch} *and* $\sigma_{\mathrm{boundary}}$, *one has*

$$\lim_{\varepsilon\to 0} \tilde{\sigma}^\varepsilon = \sigma_{\mathrm{Bloch}} \cup \sigma_{\mathrm{boundary}}.$$

Denote by $\mathcal{H}$ the space with inner product $(u, v) = \int_{\mathbb{R}^3} \rho u \cdot v$. To prove theorem 2.6, it suffices to show that all $\kappa \in \lim_{\varepsilon\to 0} \tilde{\sigma}^\varepsilon \setminus \sigma_{\mathrm{boundary}}$ belong to the spectrum of the operator A defined on a dense linear subset of $\mathcal{H}$ by the formula

$$(Au, v) = \mathcal{A}(u, v) := \int_{\mathbb{R}^3} C_{ijkl}(u_k)_{,l}(v_i)_{,j} \quad \forall u \in D(A) \subset [H^1(\mathbb{R}^3)]^3,\ v \in [H^1(\mathbb{R}^3)]^3. \tag{2.6}$$

This is sufficient since, by the Bloch decomposition, one has $\sigma(A) \subset \sigma_{\mathrm{Bloch}}$. We shall prove an equivalent claim, namely that $(\kappa + 1)^{-1}$ is an element of the spectrum of the operator

$B:\mathcal{H}\to[H^1(\mathbb{R}^3)]^3$ given by

$$\mathcal{A}(Bf,v)+(Bf,v)=(f,v)\quad\forall f\in\mathcal{H},\ v\in[H^1(\mathbb{R}^3)]^3, \tag{2.7}$$

i.e. $(\kappa+1)^{-1}$ is an element of the spectrum of $B=(A+1)^{-1}$. Suppose that $\kappa=\lim_{\varepsilon\to 0}\kappa^\varepsilon$, where $\kappa^\varepsilon\in\tilde\sigma^\varepsilon$ for all ε. Our construction of a suitable Weyl sequence for the operator B is based on the following lemma proved in appendix A.

Lemma 2.7. *There exist a sequence $U^\varepsilon\in[H^1(\mathbb{R}^3)]^3$, $\|U^\varepsilon\|_{[L^2(\mathbb{R}^3)]^3}=1$, and a sequence $C_\varepsilon>0$ such that*

$$\left|\int_{\mathbb{R}^3}C_{ijkl}(U^\varepsilon_k)_{,l}(\varphi_i)_{,j}-\kappa^\varepsilon\int_{\mathbb{R}^3}\rho U^\varepsilon\cdot\varphi\right|\le C_\varepsilon\|\varphi\|_{[H^1(\mathbb{R}^3)]^3},\quad\forall\varphi\in[H^1(\mathbb{R}^3)]^3, \tag{2.8}$$

and $C_\varepsilon\to 0$ as $\varepsilon\to 0$.

Proof of theorem 2.6. Let U^ε be one of the sequences provided by lemma 2.7. We will show that U^ε is a Weyl sequence for $(\kappa+1)^{-1}$ and B, i.e. that

$$\|(B-(\kappa+1)^{-1}I)U^\varepsilon\|_{[L^2(\mathbb{R}^3)]^3}\to 0,\quad\text{as }\varepsilon\to 0,$$

where I is the identity operator. First, we note that

$$\begin{aligned}\|(B-(\kappa+1)^{-1}I)U^\varepsilon\|_{[L^2(\mathbb{R}^3)]^3}&\le\|(B-(\kappa+1)^{-1}I)U^\varepsilon\|_{[H^1(\mathbb{R}^3)]^3}\\&=\sup_{\substack{\varphi\in[H^1(\mathbb{R}^3)]^3\\ \|\varphi\|_{H^1}=1}}|((B-(\kappa+1)^{-1}I)U^\varepsilon,\varphi)_{[H^1(\mathbb{R}^3)]^3}|.\end{aligned}$$

Note, that for $\mathcal{A}$ given by (2.6), the function $p:[H^1(\mathbb{R}^3)]^3\mapsto\mathbb{R}$ given by $p(u):=\mathcal{A}(u,u)+(u,u)$ defines an equivalent norm on $[H^1(\mathbb{R}^3)]^3$. Therefore, to prove the theorem it is sufficient to show the convergence

$$\mathcal{A}((B-(\kappa+1)^{-1})U^\varepsilon,\varphi)+((B-(\kappa+1)^{-1})U^\varepsilon,\varphi)\to 0,$$

as $\varepsilon\to 0$ for $\varphi\in[H^1(\mathbb{R}^3)]^3$, $\|\varphi\|_{[H^1(\mathbb{R}^3)]}=1$.

To this end, note that by (2.7), we have

$$\begin{aligned}\mathcal{A}((B-(\kappa+1)^{-1})U^\varepsilon,\varphi)+((B-(\kappa+1)^{-1})U^\varepsilon,\varphi)&=(\kappa+1)^{-1}[(\kappa+1)(U^\varepsilon,\varphi)-\mathcal{A}(U^\varepsilon,\varphi)-(U^\varepsilon,\varphi)]\\&=-(\kappa+1)^{-1}[\mathcal{A}(U^\varepsilon,\varphi)-\kappa(U^\varepsilon,\varphi)]\\&\le|\mathcal{A}(U^\varepsilon,\varphi)-\kappa(U^\varepsilon,\varphi)|,\end{aligned}$$

and, by (2.6), we find that

$$\begin{aligned}\mathcal{A}(U^\varepsilon,\varphi)-\kappa(U^\varepsilon,\varphi)&=\int_{\mathbb{R}^3}C_{ijkl}(U^\varepsilon_k)_{,l}(\varphi_i)_{,j}-\kappa\int_{\mathbb{R}^3}\rho U^\varepsilon\cdot\varphi\\&=\int_{\mathbb{R}^3}C_{ijkl}(U^\varepsilon_k)_{,l}(\varphi_i)_{,j}-\kappa^\varepsilon\int_{\mathbb{R}^3}\rho U^\varepsilon\cdot\varphi+(\kappa^\varepsilon-\kappa)\int_{\mathbb{R}^3}\rho U^\varepsilon\cdot\varphi.\end{aligned}$$

This goes to zero uniformly in φ as $\varepsilon\to 0$ by (2.8), the assumption that $\kappa^\varepsilon\to\kappa$, and the fact that

$$(\kappa^\varepsilon-\kappa)\int_{\mathbb{R}^3}\rho U^\varepsilon\cdot\varphi\le C|\kappa^\varepsilon-\kappa|\|\varphi\|_{[L^2(\mathbb{R}^3)]^3}$$

for some $C>0$. ■

3. A half-space limit problem

Throughout this section, we use the notation $\Pi_{m_1,m_2}:=(0,m_1)\times(0,m_2)\times(0,+\infty)$ for all $(m_1,m_2)\in\mathbb{N}^2$.

We show that the 'boundary' part of the spectrum of the operator generated by the differential expression $-\rho^{-1}(C_{ijkl}(\cdot_k)_{,l})_{,j}$ in the half-space $\mathbb{R}^2\times(0,+\infty)$ is contained in the limit spectrum $\lim_{\varepsilon\to 0}\tilde\sigma^\varepsilon$.

More precisely, for all values of the 'in-plane quasi-momentum' $(\theta_1, \theta_2) \in [0, 2\pi)^2$ consider the operator A_{θ_1,θ_2} in $[L^2_\rho(\Pi_{1,1})]^3$ defined by the bilinear form

$$\mathcal{B}_{\theta_1,\theta_2}(u,v) = \int_{\Pi_{1,1}} C_{ijkl}\hat{\partial}_l u_k \overline{\hat{\partial}_j v_i}, \quad \hat{\partial} := (\partial_1 + \mathrm{i}\theta_1, \partial_2 + \mathrm{i}\theta_2, \partial_3), \quad u, v \in \mathcal{H}^{1,1},$$

where $\partial_l u_k$ denotes the partial derivative of the field component u_k with respect to the lth independent variable. In the above formula, we denote by $[L^2_\rho(\Pi_{m_1,m_2})]^3$ the space with inner product

$$(u,v)_{m_1,m_2} = \int_{\Pi_{m_1,m_2}} \rho u \cdot \bar{v},$$

and $\mathcal{H}^{1,1}$ is the closure of $[H^1_\#(\Pi_{1,1})]^3$ in $[L^2_\rho(\Pi_{1,1})]^3$. We define $[H^1_\#(\Pi_{1,1})]^3$ as the space of 'H^1-functions periodic in the first two variables', or equivalently the closure in $H^1(\Pi_{1,1})$ of the space $\mathcal{S}(\Pi_{1,1})$ defined below.

We claim that for all $(\theta_1, \theta_2) \in [0, 2\pi)^2$ the point spectrum $\sigma^{\text{point}}_{\theta_1,\theta_2}$ of A_{θ_1,θ_2} is contained in $\lim_{\varepsilon\to 0} \tilde{\sigma}^\varepsilon$. The proof of this claim is based on an appropriate modification of the concept of two-scale convergence discussed in §2a, as we describe next.

Denote $\hat{\Omega} := (-a,a)^2 \times (-a,a]$, i.e. the union of Ω and $\partial\Omega \cap \{x_3 = a\}$, and fix $(m_1, m_2) \in \mathbb{N}^2$. We say that a sequence u^ε that is bounded in $[L^2(\Omega)]^3$ boundary two-scale converges ('B2S converges') to a function $u^0 \in [L^2(\Omega \times \Pi_{m_1,m_2})]^3$ (and write $u^\varepsilon \xrightarrow{B2S} u^0$), if

$$\int_\Omega u^\varepsilon(x) \cdot \varphi(x) \psi\left(\frac{x}{\varepsilon}\right) \mathrm{d}x \xrightarrow{\varepsilon\to 0} \frac{1}{m_1 m_2} \int_\Omega \int_{\Pi_{m_1,m_2}} u(x,y) \cdot \varphi(x)\psi(y)\, \mathrm{d}y\, \mathrm{d}x$$
$$\forall \varphi \in C_0^\infty(\hat{\Omega}), \psi \in [\mathcal{S}(\Pi_{m_1,m_2})]^3,$$

where $[\mathcal{S}(\Pi_{m_1,m_2})]^3$ is the set of smooth vector functions on the closure of Π_{m_1,m_2} that are $[0, m_1) \times [0, m_2)$-periodic in the first two arguments and have compact support. By analogy with the second part of definition 2.1, a strong version of the B2S convergence is also defined. The notion of boundary two-scale convergence possesses a number of the properties of the usual two-scale convergence, including compactness of bounded sequences (e.g. [15]).

Following an argument similar to that of lemma 2.3, we prove the following statement (cf. [15]).

Lemma 3.1. *Suppose that* $f^\varepsilon \xrightarrow{B2S} f^0 \in [L^2(\Omega \times \Pi_{m_1,m_2})]^3$ *for some* $(m_1, m_2) \in \mathbb{N}^2$*, and for all values of* $\varepsilon \in \Xi$ *denote by* u^ε *the solution to* (1.4) *where* f *is replaced by* f^ε*. Then the convergence* $u^\varepsilon \xrightarrow{B2S} u^0$ *holds, where* $u^0 \in L^2(\Omega; [H^1_\#(\Pi_{m_1,m_2})]^3)$*. For all* $(\theta_1, \theta_2) \in [0, 2\pi)^2$ *such that* $(2\pi)^{-1}\theta_i = p_i/m_i$ *for some* $p_i \in \mathbb{N}$*,* $i = 1, 2$*, the function* u^0 *satisfies the identity*

$$\int_\Omega \mathcal{B}_{\theta_1,\theta_2}(u^0, \varphi) + \int_\Omega \int_{\Pi_{m_1,m_2}} \rho u^0 \cdot \bar{\varphi} = \int_\Omega \int_{\Pi_{m_1,m_2}} \rho f^0 \cdot \bar{\varphi}, \quad \forall \varphi \in L^2(\Omega; [S(\Pi_{m_1,m_2})]^3). \tag{3.1}$$

Here $[H^1_\#(\Pi_{m_1,m_2})]^3$ *is the closure of* $[S(\Pi_{m_1,m_2})]^3$ *in* $[H^1(\Pi_{m_1,m_2})]^3$*.*

It immediately follows that for all $(2\pi)^{-1}(\theta_1, \theta_2) \in \mathbb{Q}^2 \cap [0, 1)^2$ the operators A^ε strongly BS2 resolvent converge to the operator A^{m_1,m_2} in $[L^2_\rho(\Pi_{m_1,m_2})]^3$ defined by the form $\mathcal{B}_{\theta_1,\theta_2}$ restricted to $\mathcal{H}^{m_1,m_2}$, the closure of $[H^1_\#(\Pi_{m_1,m_2})]^3$ in $[L^2_\rho(\Pi_{m_1,m_2})]^3$. (Note that the identity (3.1), with the integrals over Ω dropped, is a form of the definition of the operator A^{m_1,m_2}.) This, in turn, implies that the point spectrum of A^{m_1,m_2} is contained in $\lim_{\varepsilon\to 0} \tilde{\sigma}^\varepsilon$, and therefore the inclusion $\sigma^{\text{point}}_{\theta_1,\theta_2} \subset \lim_{\varepsilon\to 0} \tilde{\sigma}^\varepsilon$ also holds. Finally, note that the point spectra of the operators A_{θ_1,θ_2} depend on $(\theta_1, \theta_2) \in [0, 2\pi)^2$ in a continuous fashion, which is proved by following a standard argument (e.g. [16]). Therefore, the required inclusion holds for all values of (θ_1, θ_2).

Corollary 3.2. *For all* $(\theta_1, \theta_2) \in [0, 2\pi)^2$*, the point spectrum* $\sigma^{\text{point}}_{\theta_1,\theta_2}$ *of the operator* A_{θ_1,θ_2} *is contained in the limit of the spectra of* A^ε *:*

$$\bigcup_{\theta_1,\theta_2} \sigma^{\text{point}}_{\theta_1,\theta_2} \subset \lim_{\varepsilon\to 0} \tilde{\sigma}^\varepsilon.$$

Remark 3.3. Note that the continuous part of the spectrum of A_{θ_1,θ_2} is also contained in $\lim_{\varepsilon\to 0}\tilde{\sigma}^{\varepsilon}$, although we do not use this fact in what follows.

4. Examples with 'boundary' eigenvalue limits outside the 'bulk' spectrum

Here we analyse the spectrum of a reduced, two-dimensional, version of the half-space set-up discussed in the previous section.

(a) Two-dimensional elasticity for a layered half-space

We consider the problem (1.1) for the case when the coefficients C^{ε}_{ijkl} and ρ^{ε} are independent of the variable x_2 and restrict ourselves to the solutions (eigenfunctions) that are also x_2-independent. In what follows we consider a half-space filled with layers of homogeneous and isotropic materials so that the coefficients C_{ijkl} are given by

$$C_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}),\quad i,j,k,l=1,2,3,$$

where λ, μ are the constant 'Lamé coefficients'. In this case, system (1.1), which is understood in the weak sense, decouples into two systems, one for the component u_2^{ε} and the other for the components u_1^{ε}, u_3^{ε}. Following the argument of §3, we are led to the study of the boundary spectrum of the differential operator $-\rho^{-1}(C_{ijkl}(\cdot_k)_{,l})_{,j}$ in the half-space $\mathbb{R}^2\times(0,+\infty)$. The limit spectral problem is obtained by performing a re-scaling of the independent variable in (1.1) with the factor ε^{-1} and setting $\lim_{\varepsilon\to 0}\varepsilon^2\lambda^{\varepsilon}=:\omega^2$.

In what follows, we consider a particular case when the unit normal $(n_j)_{j=1}^3$ to the layers is given by $(0,0,1)$, in other words the layers are parallel to the (x_1,x_2)-plane (and hence to the boundary of the half-space in question). The decomposition of the problem (1.1) in two independent systems induces a decomposition of the limit boundary spectrum into the 'out-of-plane' part, obtained by solving a problem for the second component $u_2=u_2(x_3)$ of the displacement:

$$-\rho^{-1}(\mu u_2')'=\omega^2 u_2, \tag{4.1}$$

and the 'in-plane' part, obtained by solving a system of equations for the components $u_1=u_1(x_3)$, $u_3=u_3(x_3)$:

$$-\rho^{-1}\sum_{k=1,3}(C_{i3k3}u_k')'=\omega^2 u_i,\quad i=1,3. \tag{4.2}$$

Each of systems (4.1) and (4.2) leads to a classical system in each individual layer subject to the conditions of the continuity across interfaces of the unknown function and of the normal component $\sum_{k=1}^3 C_{i3k3}u_{k,3}$, $i=1,2,3$, of the stress tensor:

$$\begin{aligned}&[u_2]=0,\quad [u_2']=0,\\ &[u_i]=0,\ \ i=1,3,\quad \left[\sum_{k=1,3}C_{i3k3}u_k'\right]=0,\ \ i=1,3,\end{aligned} \tag{4.3}$$

where the square brackets denote the jump across the interface.

In what follows we focus on system (4.2) subject to the interface conditions (4.3).

(b) Rayleigh waves in a layered elastic half space: the case of two homogeneous isotropic layers per period

Suppose that the materials A and B fill two layers of thickness dh and $d(1-h)$ respectively, in the d-periodic fashion, the union of all layers being a half-space $\{(x_1,x_3):x_1\in\mathbb{R},x_3\in(0,+\infty)\}$. We assume that A and B are homogeneous and isotropic with Lamé coefficients λ_{A}, μ_{A} and λ_{B}, μ_{B}, and densities ρ_{A}, ρ_B, and that the material A fills layer nearest to the surface of the half-space. We consider harmonic monochromatic oscillations of frequency ω, propagating along the surface of the half-space as plane waves with wavenumber k and decaying exponentially away from the surface, i.e. solutions to (4.2) and (4.3) of the form

$$u_i = U_i \exp(-\mathrm{i}kx_1), \quad k \in \mathbb{R}, \quad i = 1, 3.$$

Introducing the unknowns

$$W_1 := (\lambda + 2\mu)U_3' - \mathrm{i}\lambda k U_1 \quad \text{and} \quad W_3 := \mu U_3' - \mathrm{i}\mu k U_1,$$

it is a straightforward calculation to check that (4.2) is equivalent to the system

$$\frac{d}{dx_3}\begin{pmatrix} U_1 \\ U_3 \\ W_1 \\ W_3 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{i}k\dfrac{\lambda}{\lambda+2\mu} & \dfrac{1}{\lambda+2\mu} & 0 \\ \mathrm{i}k & 0 & 0 & \dfrac{1}{\mu} \\ -\rho\omega^2 & 0 & 0 & \mathrm{i}k \\ 0 & \dfrac{4\mu(\lambda+\mu)}{\lambda+2\mu}k^2 - \rho\omega^2 & \mathrm{i}k\dfrac{\lambda}{\lambda+2\mu} & 0 \end{pmatrix} \begin{pmatrix} U_1 \\ U_3 \\ W_1 \\ W_3 \end{pmatrix},$$

where λ and μ take indices 'A' and 'B', depending on the layer in which the solution is considered. The above system is considered subject to the condition of continuity across the AB interfaces, which is equivalent to the interface conditions (4.3). In what follows we look for solutions that decay exponentially as $x_3 \to \infty$, which thereby belong to the set $\bigcup \sigma^{\text{point}}_{\theta_1,\theta_2}$.

We use the notation

$$\varkappa_1 := \sqrt{1 - \frac{\rho_A c^2}{\mu_A}}, \quad \varkappa_2 := \sqrt{1 - \frac{\rho_A c^2}{\lambda_A + 2\mu_A}} \tag{4.4}$$

and

$$\tilde{\varkappa}_1 := \sqrt{1 - \frac{\rho_B c^2}{\mu_B}}, \quad \tilde{\varkappa}_2 := \sqrt{1 - \frac{\rho_B c^2}{\lambda_B + 2\mu_B}}, \tag{4.5}$$

where $c := \omega/|k|$ is the absolute value of the phase velocity, which we refer to as the 'wave speed'. In what follows we generally treat the quantities $\varkappa_{1,2}$, $\tilde{\varkappa}_{1,2}$ as functions of c^2.

(i) Zero normal stress boundary conditions

Consider the case when the equations (4.2) and (4.3) are subject to the condition of zero normal stress on the boundary $\{x_3 = 0\}$ of the half-space. The well-known formula (e.g. [17]) for the Rayleigh wave speed $c_{\mathrm{R,N}}$ in a homogeneous half-space filled with material A is

$$4\varkappa_1\varkappa_2 = (\varkappa_1^2 + 1)^2, \tag{4.6}$$

where we set $c = c_{\mathrm{R,N}}$ in the formulae for $\varkappa_1$, $\varkappa_2$. We consider those values of c^2 for which the functions $\varkappa_{1,2}$, $\tilde{\varkappa}_{1,2}$ above take positive real values. This obviously gives one constraint on the values of $c = \omega/k$:

$$c < \min\left\{\sqrt{\frac{\mu_A}{\rho_A}}, \sqrt{\frac{\mu_B}{\rho_B}}\right\}.$$

Under the condition

$$h\varkappa_2 > (1-h)\tilde{\varkappa}_2, \tag{4.7}$$

a surface wave with the Rayleigh wave speed $c_{\mathrm{R,N}}$ given by (4.6) persists, thanks to the decay of the wave amplitude over the period cell implied by (4.7). In the case of the opposite inequality

$$h\varkappa_2 < (1-h)\tilde{\varkappa}_2, \tag{4.8}$$

an analogue of the Rayleigh wave exists, whose wave speed $\hat{c}_{\rm R}$ is found from the equation

$$\begin{aligned}
&-(1-\cosh(h\hat{k}\varkappa_1)\cosh(h\hat{k}\varkappa_2))\left\{\frac{\mu_{\rm B}}{4\mu_{\rm A}}(4\tilde{\varkappa}_1\tilde{\varkappa}_2-(\tilde{\varkappa}_1^2+1)^2)((\varkappa_1^2+1)^2+4)\right.\\
&\quad\left.+(\tilde{\varkappa}_1^2-2\tilde{\varkappa}_1\tilde{\varkappa}_2+1)(\varkappa_1^2+3)(\varkappa_1^2+1)-\frac{2\mu_{\rm A}}{\mu_{\rm B}}(1-\tilde{\varkappa}_1\tilde{\varkappa}_2)(\varkappa_1^2+1)^2\right\}\\
&\quad+\frac{\mu_{\rm B}}{4\mu_{\rm A}}(1-\varkappa_1^2)^2(4\tilde{\varkappa}_1\tilde{\varkappa}_2-(\tilde{\varkappa}_1^2+1)^2)\\
&\quad+\frac{1}{4}\sinh(h\hat{k}\varkappa_1)\sinh(h\hat{k}\varkappa_2)\left\{-\frac{\mu_{\rm B}}{\mu_{\rm A}}(4\tilde{\varkappa}_1\tilde{\varkappa}_2-(\tilde{\varkappa}_1^2+1)^2)\left(4\varkappa_1\varkappa_2+\frac{(\varkappa_1^2+1)^2}{\varkappa_1\varkappa_2}\right)\right.\\
&\quad-(\tilde{\varkappa}_1^2-2\tilde{\varkappa}_1\tilde{\varkappa}_2+1)\left(12\varkappa_1\varkappa_2+(\varkappa_1^2+1)^2+\frac{(\varkappa_1^2+1)^3}{\varkappa_1\varkappa_2}\right)\\
&\quad\left.+\frac{\mu_{\rm A}}{\mu_{\rm B}}(1-\tilde{\varkappa}_1\tilde{\varkappa}_2)\left(16\varkappa_1\varkappa_2+\frac{(\varkappa_1^2+1)^4}{\varkappa_1\varkappa_2}\right)\right\}\\
&\quad+(1-\varkappa_1^2)(1-\tilde{\varkappa}_1^2)\left\{\cosh(h\hat{k}\varkappa_1)\sinh(h\hat{k}\varkappa_2)\left(\tilde{\varkappa}_1\varkappa_2-\frac{1}{4}\tilde{\varkappa}_2\varkappa_2^{-1}(\varkappa_1^2+1)^2\right)\right.\\
&\quad\left.+\sinh(h\hat{k}\varkappa_1)\cosh(h\hat{k}\varkappa_2)\left(\tilde{\varkappa}_2\varkappa_1-\frac{1}{4}\tilde{\varkappa}_1\varkappa_1^{-1}(\varkappa_1^2+1)^2\right)\right\}=0, \qquad (4.9)
\end{aligned}$$

where we set $c=\hat{c}_{\rm R,N}$ in the formulae for $\varkappa_{1,2}$, $\tilde{\varkappa}_{1,2}$ and denote $\hat{k}=d|k|$.

Note that for fixed values of the elastic parameters of the materials A and B, and provided the inequality (4.8) is satisfied, the wave speed $\hat{c}_{\rm R,N}$ depends on the product $h\hat{k}$. In the limit $h\hat{k}\to 0$, the secular equation (4.9) takes the form $4\tilde{\varkappa}_1\tilde{\varkappa}_2=(\tilde{\varkappa}_1^2+1)^2$, i.e. one has $\hat{c}_{\rm R,N}\to\tilde{c}_{\rm R,N}$ as expected, where $\tilde{c}_{\rm R,N}$ is the Rayleigh wave speed for material B. This observation also serves as a proof of the existence of the surface wave in question, at least for sufficiently small values of the parameters h, $\hat{k}$ (note that condition (4.8) is satisfied for small values of h.) For the speed of this wave as a function of the non-dimensional wavenumber $\hat{k}$, see figure 1.

(ii) Zero-displacement boundary condition

It is well known (following an argument of [17]) that when (4.2) and (4.3) are subject to the condition of zero displacement on the boundary $\{x_3=0\}$, no surface waves are found in the case of a homogeneous half-space, i.e. when the materials A and B are identical. Suppose now that the materials A and B are different, in other words the half-space is filled with a medium that is 'genuinely layered', in a periodic manner. Under the same condition (cf. (4.8)) as above,

$$h\varkappa_2<(1-h)\tilde{\varkappa}_2,$$

a surface wave of a new kind with wave speed $\hat{c}_{\rm R,D}$ is found. The corresponding analogue of the above secular equation (4.9) for this case has the form

$$\begin{aligned}
&(1-\cosh(h\hat{k}\varkappa_1)\cosh(h\hat{k}\varkappa_2))\left\{\frac{\mu_{\rm B}}{\mu_{\rm A}}(4\tilde{\varkappa}_1\tilde{\varkappa}_2-(\tilde{\varkappa}_1^2+1)^2)+(\tilde{\varkappa}_1^2-2\tilde{\varkappa}_1\tilde{\varkappa}_2+1)(\varkappa_1^2+3)\right.\\
&\quad\left.-\frac{\mu_{\rm A}}{2\mu_{\rm B}}(1-\tilde{\varkappa}_1\tilde{\varkappa}_2)((\varkappa_1^2+1)^2+4)\right\}+\frac{\mu_{\rm A}}{2\mu_{\rm B}}(1-\varkappa_1^2)^2(1-\tilde{\varkappa}_1\tilde{\varkappa}_2)\\
&\quad+\sinh(h\hat{k}\varkappa_1)\sinh(h\hat{k}\varkappa_2)\left\{\frac{\mu_{\rm B}}{2\mu_{\rm A}}(4\tilde{\varkappa}_1\tilde{\varkappa}_2-(\tilde{\varkappa}_1^2+1)^2)\left(\frac{1}{\varkappa_1\varkappa_2}+\varkappa_1\varkappa_2\right)\right.\\
&\quad\left.+(\tilde{\varkappa}_1^2-2\tilde{\varkappa}_1\tilde{\varkappa}_2+1)\left(2\varkappa_1\varkappa_2+\frac{\varkappa_1^2+1}{\varkappa_1\varkappa_2}\right)-\frac{\mu_{\rm A}}{2\mu_{\rm B}}(1-\tilde{\varkappa}_1\tilde{\varkappa}_2)\left(4\varkappa_1\varkappa_2+\frac{(\varkappa_1^2+1)^2}{\varkappa_1\varkappa_2}\right)\right\}
\end{aligned}$$

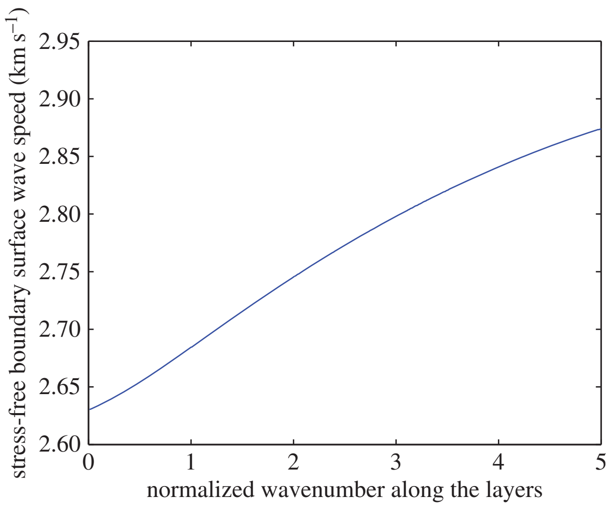

Figure 1. The Rayleigh-type $c^h_{\mathrm{R,N}}$ wave speed (in kilometre per second) for the stress-free boundary, as a function of the parameter $\hat{k}$. Here $\rho_A = 2.8 \times 10^3$, $\rho_B = 2.5 \times 10^3$, $\mu_A = 3.1 \times 10^{10}$, $\mu_B = 2.1 \times 10^{10}$, $\lambda_A = 4.6 \times 10^{10}$, $\lambda_B = 1.4 \times 10^{10}$, $h = 0.1$. (The values chosen for the material densities and Lamé coefficients approximately correspond to the Young moduli $E_A = 8 \times 10^{10}$, $E_B = 5 \times 10^{10}$ and the Poisson ratios $\nu_A = 0.3$, $\nu_B = 0.2$ for materials of layers A and B, respectively, and could represent (in the units of SI) the behaviour of two kinds of rock [18].) (Online version in colour.)

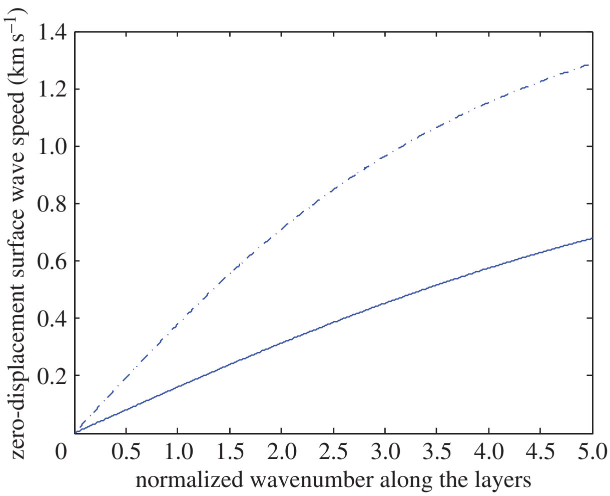

Figure 2. The Rayleigh-type wave speed $c^h_{\mathrm{R,D}}$ for the zero-displacement ('clamped') boundary, as a function of the parameter $\hat{k}$. For the solid line values of the material parameters are the same as in figure 1, for the dashed line the values of the material parameters are swapped between the materials A and B (equivalently, the order of the layers is BABAB. . . starting from the boundary). In both cases we set $h = 0.1$. (Online version in colour.)

$$+ (1 - \varkappa_1^2)^2(1 - \tilde{\varkappa}_1^2)\{\cosh(h\hat{k}\varkappa_1)\sinh(h\hat{k}\varkappa_2)(\tilde{\varkappa}_2\varkappa_2^{-1} - \tilde{\varkappa}_1\varkappa_2)$$
$$+ \sinh(h\hat{k}\varkappa_1)\cosh(h\hat{k}\varkappa_2)(\tilde{\varkappa}_1\varkappa_1^{-1} - \tilde{\varkappa}_2\varkappa_1)\} = 0, \qquad (4.10)$$

where we set $c = \hat{c}_{\mathrm{R,D}}$ in the formulae for $\varkappa_{1,2}$, $\tilde{\varkappa}_{1,2}$ and, as before, denote $\hat{k} = d|k|$. For a plot of the speed of this wave as a function of the parameter $\hat{k}$, see figure 2. Note that in the limit $h\hat{k} \to 0$ one has $\hat{c}_{\mathrm{R,D}} \to 0$, which is consistent with the fact that in the homogeneous half-space this wave is absent.

The analysis of the eigenvalue problem (1.1) subject to the Dirichlet condition on part of the boundary of Ω is similar to that given in §§2 and 3. In particular, appropriate versions of corollaries 2.4 and 3.2 hold. We omit the details of this analysis.

(c) The Bloch spectrum of the layered elastic space

The Bloch spectrum (or 'bulk' spectrum, as it is often referred to) of the two-dimensional problem described in §4 is the union of the two sets: (i) the Bloch spectrum of the 'in-plane displacement' problem (4.2) (subject to boundary conditions at $x_3 = 0$), for which the classical Rayleigh waves are eigenfunctions of the homogeneous half-space problem with stress-free boundary; (ii) the Bloch spectrum of the 'out-of plane displacement' problem (4.1) (subject to corresponding conditions at $x_3 = 0$), for which the Love waves are the eigenfunctions of the homogeneous half-space problem. As in the previous section, we aim at the spectrum of the first of these, using the fact that the two spectra can be analysed independently.

In what follows $\theta = \theta_3$ is the third component quasi-momentum (see §3), $\theta \in [0, 2\pi)$. We use the notation $\varkappa_{1,2}$, $\tilde{\varkappa}_{1,2}$ as above, see (4.4) and (4.5). We also denote

$$S := (1 + \varkappa_1\tilde{\varkappa}_1)(\varkappa_1 - \tilde{\varkappa}_1), \quad T := (\tilde{\varkappa}_1^2 - 1)\varkappa_2 - (\varkappa_1^2 - 1)\tilde{\varkappa}_2 \tag{4.11}$$

and

$$L := (1 - \varkappa_1\tilde{\varkappa}_1)(\varkappa_1 + \tilde{\varkappa}_1), \quad M := (\tilde{\varkappa}_1^2 - 1)\varkappa_2 + (\varkappa_1^2 - 1)\tilde{\varkappa}_2. \tag{4.12}$$

Then, for a given value of k^2 (equivalently, $\hat{k}$), the set of wave speeds c such that $\omega^2 = k^2c^2$ is in the Bloch spectrum for the value θ of the quasi-momentum is given by solutions to the equation

$$\det\begin{pmatrix} L(\varkappa_1^2 - \tilde{\varkappa}_1^2)^{-1}\{\exp(\hat{k}((1-h)\tilde{\varkappa}_1 + h\varkappa_1)) - \exp(\mathrm{i}\theta)\} & -S(\varkappa_1^2 - \tilde{\varkappa}_1^2)^{-1}\{\exp(\hat{k}(-(1-h)\tilde{\varkappa}_1 + h\varkappa_1)) - \exp(\mathrm{i}\theta)\} & \exp(\hat{k}((1-h)\tilde{\varkappa}_2 + h\varkappa_1)) - \exp(\mathrm{i}\theta) & \exp(\hat{k}(-(1-h)\tilde{\varkappa}_2 + h\varkappa_1)) - \exp(\mathrm{i}\theta) \\ S(\varkappa_1^2 - \tilde{\varkappa}_1^2)^{-1}\{\exp(\hat{k}((1-h)\tilde{\varkappa}_1 - h\varkappa_1)) - \exp(\mathrm{i}\theta)\} & -L(\varkappa_1^2 - \tilde{\varkappa}_1^2)^{-1}\{\exp(-\hat{k}((1-h)\tilde{\varkappa}_1 + h\varkappa_1)) - \exp(\mathrm{i}\theta)\} & \exp(\hat{k}((1-h)\tilde{\varkappa}_2 - h\varkappa_1)) - \exp(\mathrm{i}\theta) & \exp(-\hat{k}((1-h)\tilde{\varkappa}_2 + h\varkappa_1)) - \exp(\mathrm{i}\theta) \\ \exp(\hat{k}((1-h)\tilde{\varkappa}_1 + h\varkappa_2)) - \exp(\mathrm{i}\theta) & \exp(\hat{k}(-(1-h)\tilde{\varkappa}_1 + h\varkappa_2)) - \exp(\mathrm{i}\theta) & -M(\varkappa_1^2 - \tilde{\varkappa}_1^2)^{-1}\{\exp(\hat{k}((1-h)\tilde{\varkappa}_2 + h\varkappa_2)) - \exp(\mathrm{i}\theta)\} & T(\varkappa_1^2 - \tilde{\varkappa}_1^2)^{-1}\{\exp(\hat{k}(-(1-h)\tilde{\varkappa}_2 + h\varkappa_2)) - \exp(\mathrm{i}\theta)\} \\ \exp(\hat{k}((1-h)\tilde{\varkappa}_1 - h\varkappa_2)) - \exp(\mathrm{i}\theta) & \exp(-\hat{k}((1-h)\tilde{\varkappa}_1 + h\varkappa_2)) - \exp(\mathrm{i}\theta) & -T(\varkappa_1^2 - \tilde{\varkappa}_1^2)^{-1}\{\exp(\hat{k}((1-h)\tilde{\varkappa}_2 - h\varkappa_2)) - \exp(\mathrm{i}\theta)\} & M(\varkappa_1^2 - \tilde{\varkappa}_1^2)^{-1}\{\exp(-\hat{k}((1-h)\tilde{\varkappa}_2 + h\varkappa_2)) - \exp(\mathrm{i}\theta)\} \end{pmatrix} = 0, \tag{4.13}$$

where as before $\hat{k} = d|k|$. Typical plots of the corresponding dispersion diagrams are shown in figures 3 and 4. For the in-plane problem in the half-space, this is the continuous part of the spectrum while the point part is given by the Rayleigh eigenvalues.

Next we show that the corresponding set of values of ω is situated above the 'boundary frequencies' found in §4. Denote $\alpha = (1-h)\hat{k}\tilde{\varkappa}_1$, $\beta = h\hat{k}\varkappa_1$ $\gamma = (1-h)\hat{k}\tilde{\varkappa}_2$, $\delta = h\hat{k}\varkappa_2$. Equation (4.13)

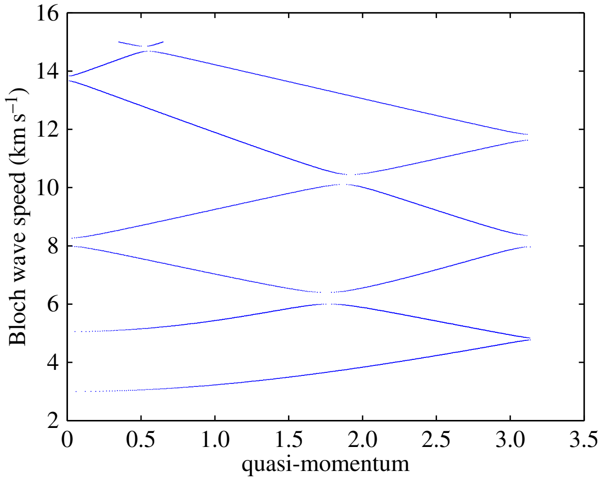

Figure 3. The dispersion relations (wave speed $c = c(\theta)$ as a function of $\theta \in [0, \pi)$) corresponding to the 'in-plane' part of the Bloch spectrum, see (4.2) Here the material parameters and h are the same as in figure 1, $\hat{k} = 2.5$. The values of c for $\theta \in [\pi, 2\pi)$ are given by the formula $c(\theta) = c(2\pi - \theta)$. (Online version in colour.)

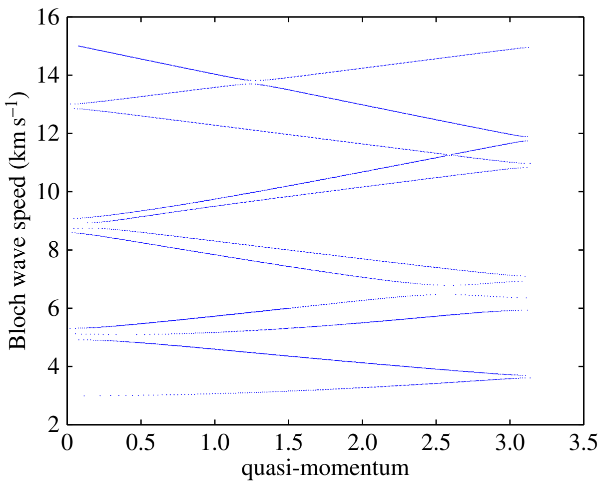

Figure 4. The plot analogous to the figure 3 in the case $\hat{k} = 4.5$, the other parameters being unchanged. (Online version in colour.)

is equivalent to

$$\mathfrak{a}(\varkappa_1^2 - \tilde{\varkappa}_1^2)^4 + \mathfrak{b}(\varkappa_1^2 - \tilde{\varkappa}_1^2)^2 + \mathfrak{c} = 0,$$

where

$$\begin{aligned}
\mathfrak{a} &:= 2(\sinh\alpha)(\sinh\beta)(\sinh\gamma)(\sinh\delta),\\
\mathfrak{b} &:= [(S-L)(M+T)(\sinh\gamma)(\sinh\beta) + (S+L)(T-M)(\sinh\alpha)(\sinh\delta)]\cos\theta\\
&\quad - (S+L)(M+T)(\sinh\alpha)(\sinh\gamma) - (S-L)(T-M)(\sinh\beta)(\sinh\delta)\\
&\quad + (M\sinh(\gamma+\delta) - T\sinh(-\gamma+\delta))(L\sinh(\alpha+\beta) - S\sinh(-\alpha+\beta)),\\
\mathfrak{c} &:= \tfrac{1}{2}((S^2 - L^2)\cos\theta + L^2\cosh(\alpha+\beta) - S^2\cosh(\alpha-\beta))\\
&\quad \times ((T^2 - M^2)\cos\theta + M^2\cosh(\gamma+\delta) - T^2\cosh(\gamma-\delta)).
\end{aligned}$$

Here S, T, L, M are given by formulae (4.11) and (4.12). Solving the above equation for $(\varkappa_1^2 - \tilde{\varkappa}_1^2)^2$ yields

$$(\varkappa_1^2 - \tilde{\varkappa}_1^2)^2 = (2\mathfrak{a})^{-1}\left(-\mathfrak{b} \pm \sqrt{\mathfrak{b}^2 - 4\mathfrak{a}\mathfrak{c}}\right). \tag{4.14}$$

It is verified by a direct calculation of the leading-order behaviour of the right-hand side of (4.14) that for small values of $\hat{k}$ the equation (4.14) implies

$$1 - \cos\theta \sim \frac{(1-h)h\hat{k}^2(\varkappa_1^2 - \tilde{\varkappa}_1^2)^2}{(1-\varkappa_1^2)(1-\tilde{\varkappa}_1^2)} = (1-h)h\hat{k}^2\left(\frac{\mu_{\mathrm{A}}}{\rho_{\mathrm{A}}} - \frac{\mu_{\mathrm{B}}}{\rho_{\mathrm{B}}}\right)^2,$$

which is impossible unless $\mu_{\mathrm{A}}/\rho_{\mathrm{A}} = \mu_{\mathrm{B}}/\rho_{\mathrm{B}}$ and $\theta = 0$. This shows that the wave speeds $\hat{c}_{\mathrm{R,N}}$, $\hat{c}_{\mathrm{R,D}}$, which were discussed in §4b(i),(ii), are situated below Bloch wave speeds. The same relation therefore holds between the corresponding spectra, due to the formula $\omega^2 = c^2k^2$, where k is the wavenumber (see the beginning of §4b) and c is the corresponding wave speed.

5. Conclusion

We have analysed the location of the Bloch spectrum and of part of a boundary spectrum for high-frequency vibrations of an ε-periodic layered elastic medium. We consider the 'scale-interaction' regime, when the frequencies are of the order ε^{-2} as $\varepsilon \to 0$. Formulae (4.9) and (4.10) are the secular equations of two kinds of surface waves that are shown to exist in such a medium: a version of the usual Rayleigh wave in a homogeneous half-space with stress-free boundary (equation (4.9)), and a new kind of wave for the zero-displacement problem (equation (4.10)). The latter does not have an analogue in the case of a homogeneous medium. We prove that the frequencies of the waves of these two kinds are situated below the Bloch frequencies of the full-space formulation. In particular, the 'boundary' part of the original spectra (equation (1.1)) is non-empty for sufficiently small values of ε.

The existence of Rayleigh-type surface waves in a layered elastic half-space with zero displacement on the boundary may have consequences in applications where structures could be affected at frequencies below the Bloch spectrum, e.g. in seismology.

Data accessibility. Data used to generate the graphs in figures 1–4 were automatically deleted by the Matlab software during the analysis. No other data were created during this study.

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Author contributions. K.D.C. and S.C. developed the mathematical formulation for the problem of elastic wave propagation through layered media with a boundary, carried out the proof of the completeness of the union of the boundary and Bloch spectra, derived the secular equations for the boundary modes in the cases of stress-free and zero-displacement boundary conditions and the dispersion relation for the Bloch spectrum, proved the existence of surface waves with frequencies outside the Bloch spectrum, and carried out the numerical simulations for the above secular equations and dispersion relation illustrating the existence of non-empty boundary spectrum.

Conflict of interests. We have no competing interests.

Appendix A. Proof of lemma 2.7

Proof of lemma 2.7. Step 1. Suppose that $(\kappa^\varepsilon + 1, u^\varepsilon)$ is a solution pair for (1.3). The fact that $\lim_{\varepsilon\to 0}\kappa^\varepsilon = \kappa \notin \sigma_{\mathrm{boundary}}$ implies, by definition (2.5) of $\sigma_{\mathrm{boundary}}$, that there exist an integer $n \geq 1$

and a subsequence of ε (denoted by the same letter ε) such that

$$\varepsilon^{-n}(\|d^n u^\varepsilon\|_{[L^2(\Omega)]^3} + \varepsilon\|d^n \nabla u^\varepsilon\|_{[L^2(\Omega)]^{3\times 3}}) \to \infty \quad \text{as } \varepsilon \to 0. \tag{A 1}$$

Henceforth, we denote $\mathrm{dist}(\cdot, \partial\Omega)$ by d. Let us take the smallest of such n, to guarantee that

$$\varepsilon^{-(n-1)}(\|d^{n-1} u^\varepsilon\|_{[L^2(\Omega)]^3} + \varepsilon\|d^{n-1} \nabla u^\varepsilon\|_{[L^2(\Omega)]^{3\times 3}}) \le C, \tag{A 2}$$

for some constant C. Substituting $\varphi = d^{2n} u^\varepsilon$ in the weak form of (1.3) we arrive at

$$\varepsilon^2 \int_\Omega d^{2n} C^\varepsilon_{ijkl} u^\varepsilon_{k,l} u^\varepsilon_{i,j} = \kappa^\varepsilon \int_\Omega \rho^\varepsilon d^{2n} u^\varepsilon_i u^\varepsilon_i - \varepsilon^2 \int_\Omega 2n d^{2n-1} C^\varepsilon_{ijkl} u^\varepsilon_{k,l} d_{,j} u^\varepsilon_i$$

which implies that there exists $C > 0$ independent of ε such that

$$\varepsilon\|d^n \nabla u^\varepsilon\|_{[L^2(\Omega)]^{3\times 3}} \le C(\|d^n u^\varepsilon\|_{[L^2(\Omega)]^3} + \varepsilon^2\|d^{n-1} \nabla u^\varepsilon\|_{[L^2(\Omega)]^{3\times 3}}). \tag{A 3}$$

Inequalities (A 2) and (A 3) imply that

$$\varepsilon\|d^n \nabla u^\varepsilon\|_{[L^2(\Omega)]^{3\times 3}} \le C(\|d^n u^\varepsilon\|_{[L^2(\Omega)]^3} + \varepsilon^n), \tag{A 4}$$

which allows one to conclude that

$$\varepsilon^{-n}\|d^n u^\varepsilon\|_{[L^2(\Omega)]^3} \to \infty \quad \text{as } \varepsilon \to 0, \tag{A 5}$$

for if not, the boundedness of $\varepsilon^{-n}\|d^n u^\varepsilon\|_{[L^2(\Omega)]^3}$ and (A 4) would contradict (A 1).

Step 2. We shall now consider the family w^ε given by $w^\varepsilon = d^n u^\varepsilon$ in Ω and extended by zero to $\mathbb{R}^3 \backslash \Omega$. Since $d = 0$ on $\partial\Omega$ we find that w^ε is continuous across $\partial\Omega$ and therefore belongs to $[H^1(\mathbb{R}^3)]^3$. Introducing

$$\langle \tilde{r}^\varepsilon, \varphi\rangle := \varepsilon^2 \int_{\mathbb{R}^3} C^\varepsilon_{ijkl} w^\varepsilon_{k,l} \varphi_{i,j} - \kappa^\varepsilon \int_{\mathbb{R}^3} \rho^\varepsilon w^\varepsilon_i \varphi_i,$$

we will now show that there exists a constant $C > 0$ independent of ε such that for all $\varphi \in [H^1(\mathbb{R}^3)]^3$ the inequality

$$|\langle \tilde{r}^\varepsilon, \varphi\rangle| \le C\varepsilon^n(\|\varphi\|_{[L^2(\mathbb{R}^3)]^3} + \varepsilon\|\nabla\varphi\|_{[L^2(\mathbb{R}^3)]^{3\times 3}}). \tag{A 6}$$

holds.

By standard calculus arguments and (1.3) we find for fixed $\varphi \in [H^1(\mathbb{R}^3)]^3$ that

$$\begin{aligned}\varepsilon^2 \int_{\mathbb{R}^3} C^\varepsilon_{ijkl} w^\varepsilon_{k,l} \varphi_{i,j} &= \varepsilon^2 \int_\Omega C^\varepsilon_{ijkl} u^\varepsilon_{k,l} (d^n\varphi)_{i,j} + \varepsilon^2 \int_\Omega n d^{n-1} C^\varepsilon_{ijkl}(d_{,l} u^\varepsilon_k \varphi_{i,j} - u^\varepsilon_{k,l} d_{,j} \varphi_i) \\ &= \kappa^\varepsilon \int_\Omega \rho^\varepsilon w^\varepsilon_i \varphi_i + \varepsilon^2 \int_\Omega n d^{n-1} C^\varepsilon_{ijkl}(d_{,l} u^\varepsilon_k \varphi_{i,j} - u^\varepsilon_{k,l} d_{,j} \varphi_i).\end{aligned}$$

This along with (A 2) implies (A 6).

Step 3. Here, we will show that $U^\varepsilon := \|w^\varepsilon(\varepsilon\cdot)\|^{-1}_{[L^2(\mathbb{R}^3)]^3} w^\varepsilon(\varepsilon\cdot)$ satisfies the claim of the lemma. By virtue of the change of variables $x = \varepsilon y$, we find that

$$\int_{\mathbb{R}^3} C_{ijkl} U^\varepsilon_{k,y_l}(y) \varphi_{i,y_j}(y)\,\mathrm{d}y - \kappa^\varepsilon \int_{\mathbb{R}^3} \rho U^\varepsilon_i(y) \varphi_i(y)\,\mathrm{d}y = \|w^\varepsilon(\varepsilon y)\|^{-1}_{[L^2(\mathbb{R}^3)]^3} \varepsilon^{-3} \left\langle \tilde{r}^\varepsilon, \varphi\left(\frac{\cdot}{\varepsilon}\right)\right\rangle.$$

Finally, we use (A 5) and (A 6) to show that

$$\begin{aligned}\|w^\varepsilon(\varepsilon y)\|^{-1}_{[L^2(\mathbb{R}^3)]^3} \varepsilon^{-3} \left\langle \tilde{r}^\varepsilon, \varphi\left(\frac{\cdot}{\varepsilon}\right)\right\rangle &= \varepsilon^{-3/2}\|w^\varepsilon\|^{-1}_{[L^2(\Omega)]^3} \left\langle \tilde{r}^\varepsilon, \varphi\left(\frac{\cdot}{\varepsilon}\right)\right\rangle \\ &\le \varepsilon^{-3/2}\|w^\varepsilon\|^{-1}_{[L^2(\Omega)]^3} C\varepsilon^n \left(\left\|\varphi\left(\frac{\cdot}{\varepsilon}\right)\right\| [L^2(\mathbb{R}^3)]^3 + \varepsilon\left\|\nabla\varphi\left(\frac{\cdot}{\varepsilon}\right)\right\| [L^2(\mathbb{R}^3)]^{3\times 3}\right) \\ &\le C\left(\frac{\varepsilon^n}{\|d^n u^\varepsilon\|_{[L^2(\mathbb{R}^3)]^3}}\right)(\|\varphi\|_{[L^2(\mathbb{R}^3)]^3} + \|\nabla\varphi\|_{[L^2(\mathbb{R}^3)]^{3\times 3}}) \overset{\varepsilon\to 0}{\longrightarrow} 0.\end{aligned}$$

The claim of the lemma now follows. ∎

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