Double-trace spectrum of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory at strong coupling

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The spectrum of IIB supergravity on $\mathrm{AdS}_5 \times S^5$ contains a number of bound states described by long double-trace multiplets in $\mathcal{N} = 4$ super Yang-Mills theory at large 't Hooft coupling. At large N these states are degenerate and to obtain their anomalous dimensions as expansions in $\frac{1}{N^2}$ one has to solve a mixing problem. We conjecture a formula for the leading anomalous dimensions of all long double-trace operators which exhibits a large residual degeneracy whose structure we describe. Our formula can be related to conformal Casimir operators which arise in the structure of leading discontinuities of supergravity loop corrections to four-point correlators of half-BPS operators.

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I. INTRODUCTION

Recently much progress has been made in understanding the structure of the spectrum of double-trace operators in $\mathcal{N} = 4$ super Yang-Mills theory at large N and large 't Hooft coupling $\lambda = g^2 N$ [1]. Based on these results, operator product expansion (OPE) and bootstrap techniques have been applied in [2,3] to obtain closed form expressions for supergravity loop corrections of certain holographic correlators, uncovering novel and rich structure (see [4,5] for related approaches to such loop corrections). Here we complete the picture for the double-trace spectrum and conjecture a general formula for the leading anomalous dimensions of all long double-trace operators of any twist, spin, and $su(4)$ representation.

In the regime $N \to \infty$ and $\lambda \gg 1$, the theory is in correspondence with classical IIB supergravity on $\mathrm{AdS}_5 \times S^5$ [6]. The graviton and the Kaluza-Klein multiplets are dual to protected half-BPS operators in the $[0, p, 0]$ representation of $su(4)$,

$$\mathcal{O}_p = y^{i_1} \ldots y^{i_p} \mathrm{Tr}(\Phi_{i_1} \ldots \Phi_{i_p}) + \ldots \tag{1}$$

where $\Phi_{i=1,\ldots,6}$ are the elementary scalar fields, the complex vector $\vec{y}$ is null, and the ellipsis stands for $1/N$-suppressed multitrace terms (for $p \geq 4$), whose precise nature will be described in Sec. II.

At leading large N and for any value of λ, we may consider spin l long double-trace superconformal primary operators of the form

$$\mathcal{O}_{pq} = \mathcal{O}_p \partial^l \Box^{\frac{1}{2}(\tau - p - q)} \mathcal{O}_q, \qquad (p \leq q). \tag{2}$$

In the large N limit, the operators $\mathcal{O}_{pq}$ are orthogonal and have dimension $\Delta_0 = \tau + l$, hence τ coincides with the twist in the limit $N \to \infty$. For fixed τ and $su(4)$ labels $[a, b, a]$, there are d allowed values of the pair (p, q). We denote this set by $\mathcal{D}^{\mathrm{long}}_{\tau,l,a,b}$ and we parametrize it as follows:

$$p = i + a + 1 + r, \qquad q = i + a + 1 + b - r,$$
$$i = 1, \ldots, (t-1), \qquad r = 0, \ldots, (\mu - 1), \tag{3}$$

so that $d = \mu(t-1)$ with

$$t \equiv (\tau - b)/2 - a, \qquad \mu \equiv \begin{cases} \lfloor \frac{b+2}{2} \rfloor & a + l \text{ even}, \\ \lfloor \frac{b+1}{2} \rfloor & a + l \text{ odd}. \end{cases} \tag{4}$$

The operators $\mathcal{O}_{pq}$ are in long multiplets, but in the strict large N limit their dimensions are protected. At order $1/N^2$ they acquire anomalous dimensions and mix among themselves and with other long operators. In the supergravity regime $\lambda \gg 1$, operators corresponding to massive string excitations should decouple from the spectrum leaving only those corresponding to supergravity states, e.g., the single-particle states $\mathcal{O}_p$ and the two-particle bound states $\mathcal{O}_{pq}$. At leading order in large N the $\mathcal{O}_{pq}$ just mix among themselves to produce the true scaling eigenstates, which we denote by K_{pq}. Mixing with higher multiparticle states will only occur at higher orders in the $1/N$ expansion. Analysis of the OPE in the tree-level supergravity regime (see Sec. III) leads us to the following conjecture, generalizing results in [1–3].

A. Main conjecture

Up to order $1/N^2$, the dimensions of the operators K_{pq} are given by

$$\boxed{\Delta_{pq} = \Delta_0 - \frac{2}{N^2}\frac{2M_t^{(4)}M_{t+l+1}^{(4)}}{(l+2p-2-a-\frac{1+(-)^{a+l}}{2})_6}} \qquad (5)$$

where $(\ldots)_6$ is the Pochhammer symbol, and

$$M_t^{(4)} \equiv (t-1)(t+a)(t+a+b+1)(t+2a+b+2). \qquad (6)$$

Note that for $\mu > 1$ and $t > 2$ some dimensions exhibit a residual degeneracy because they are independent of q. We display this property with an illustration of $\mathcal{D}^{\text{long}}_{\tau,l,a,b}$ (see Fig. 1). The dots connected by vertical lines in the (p,q) plane represent operators of common anomalous dimension.

II. HOLOGRAPHIC CORRELATORS

The correlators $\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle \equiv \langle p_1p_2p_3p_4\rangle$ may be written as a free part plus an interacting part,

$$\langle p_1p_2p_3p_4\rangle = \langle p_1p_2p_3p_4\rangle_{\text{free}} + \mathcal{P}\times\mathcal{I}\times\mathcal{H}. \qquad (7)$$

The factor $\mathcal{P}$ carries the conformal and $su(4)$ weights and assuming (without loss of generality) $p_{21} \geq 0$, $p_{43} \geq 0$, and $p_{43} \geq p_{21}$, it takes the form

$$\mathcal{P} = N^{\frac{1}{2}\sum p_i} g_{12}^{\frac{p_1+p_2-p_{43}}{2}} g_{14}^{\frac{-p_{21}+p_{43}}{2}} g_{24}^{\frac{p_{21}+p_{43}}{2}} g_{34}^{p_3}, \qquad (8)$$

where $p_{ij} = p_i - p_j$ and $g_{ij} = (y_i\cdot y_j)/x_{ij}^2$. The quantities $\mathcal{I}$ and $\mathcal{H}$ are functions of the variables $x, \bar{x}, y, \bar{y}$, related to the conformal and $su(4)$ cross ratios u, v, σ, τ via

$$u = x\bar{x} = \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}, \qquad v = (1-x)(1-\bar{x}) = \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2},$$
$$\frac{1}{\sigma} = y\bar{y} = \frac{y_{12}^2y_{34}^2}{y_{13}^2y_{24}^2}, \qquad \frac{\tau}{\sigma} = (1-y)(1-\bar{y}) = \frac{y_{14}^2y_{23}^2}{y_{13}^2y_{24}^2}. \qquad (9)$$

In terms of these variables we have

$$\mathcal{I}(x,\bar{x},y,\bar{y}) = (x-y)(x-\bar{y})(\bar{x}-y)(\bar{x}-\bar{y})/(y\bar{y})^2. \qquad (10)$$

The decomposition into free and interacting parts in (7) reflects the property of "partial nonrenormalization" [7], i.e., the statement that all the dependence on the coupling appears in the function $\mathcal{H}$. Here we consider the leading contribution to $\mathcal{H}$ at large λ. In the OPE of $(\mathcal{O}_{p_1}\times\mathcal{O}_{p_2})$ and $(\mathcal{O}_{p_3}\times\mathcal{O}_{p_4})$, the free term contributes both a protected sector and a long sector. Identifying the sectors is nontrivial due to possible semishort multiplet recombination at the unitarity bound [8,9].

At leading order in the $1/N^2$ expansion, a correlator is determined by disconnected contributions to the free part. These only exist for $\langle ppqq\rangle$ and cases related by crossing,

$$\langle ppqq\rangle = pq\mathcal{P}\left[1+\delta_{pq}\left[\left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^p + \left(\frac{g_{14}g_{23}}{g_{12}g_{34}}\right)^p\right]\right]. \qquad (11)$$

At the next order in $1/N^2$ in the supergravity regime, tree-level Witten diagrams contribute both the free theory connected diagrams and the first contribution to $\mathcal{H}$.

A. Supergravity states and free theory

It was noticed in [10] that the connected part of $\langle p_1p_2p_3p_4\rangle_{\text{free}}$, generated via tree-level Witten diagrams, disagrees with free theory four-point functions of single-trace half-BPS operators. The resolution is that single-particle supergravity states are not dual to single-trace half-BPS operators, rather they are uniquely defined as those orthogonal to all multitrace operators. From this property we can identify multitrace contributions to $\text{Tr}\Phi^p$ for $p \geq 4$. The presence of multitrace admixtures was also discussed in [11,12]. Consider e.g., $\mathcal{O}_4$, the condition $\langle\mathcal{O}_4(\mathcal{O}_2)^2\rangle = 0$ determines

$$\mathcal{O}_4 = y^{i_1}\ldots y^{i_4}\text{Tr}(\Phi_{i_1}\ldots\Phi_{i_4}) - \frac{2N^2-3}{N(N^2+1)}(\mathcal{O}_2)^2. \qquad (12)$$

With this identification of $\mathcal{O}_4$ the free theory computation of $\langle 2244\rangle$ agrees with that of supergravity [10]. The correct identification of the operators $\mathcal{O}_p$ is also necessary for the "derivative relation" of [13] to hold, as can be directly observed for the cases $\langle 22nn\rangle$. More generally, connected free theory diagrams where, e.g., O_{p_3} is joined only to $\mathcal{O}_{p_4}$ (see Fig. 2) are absent. To see this note that, at twist p_{43} in the $(\mathcal{O}_{p_3}\times\mathcal{O}_{p_4})$ OPE, only a half-BPS operator $\mathcal{O}_{p_{43}}$ of charge p_{43} could potentially be transferred. By our definition, $\mathcal{O}_{p_4}$ is orthogonal to all multitrace operators and in particular to the double (or higher) trace operator $[\mathcal{O}_{p_{43}}\mathcal{O}_{p_3}]$. But the vanishing two-point function $\langle[\mathcal{O}_{p_{43}}\mathcal{O}_{p_3}]\mathcal{O}_{p_4}\rangle$ is just a nonsingular limit of the three-point function, $\langle\mathcal{O}_{p_{43}}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle$, which therefore also vanishes. Hence no operator $\mathcal{O}_{p_{43}}$ can be exchanged and the

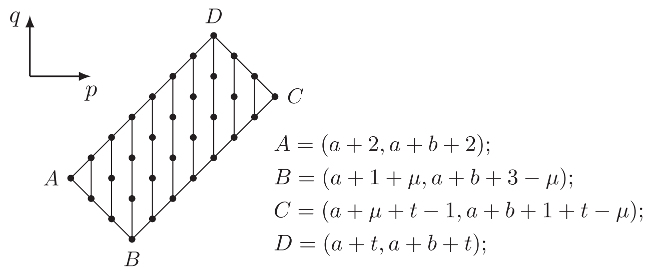

FIG. 1. Illustration of residual degeneracies.

coefficient of the above diagram must vanish. Note that this holds no matter if $\mathcal{O}_{p_{43}}$ is single-trace, multitrace, or a combination thereof. Obviously any topology related by a permutation to Fig. 2 also vanishes.

B. Tree-level dynamics

The conjecture of [14] is a simple Mellin integral for the leading term in $\mathcal{H}$

$$\begin{aligned}
\mathcal{H}_{\rm RZ} &= -\mathcal{N}_{p_1p_2p_3p_4}\oint dz dw\, u^{\frac{z}{2}} v^{\frac{w}{2}} \mathcal{R}\begin{bmatrix} z & w\\ \sigma & \tau\end{bmatrix}\Gamma_{p_1p_2p_3p_4},\\
\Gamma &= \Gamma\left[\frac{p_1+p_2-z}{2}\right]\Gamma\left[\frac{p_3+p_4-z}{2}\right]\Gamma\left[\frac{p_1+p_4-w}{2}\right]\\
&\quad\times\Gamma\left[\frac{p_2+p_3-w}{2}\right]\Gamma\left[\frac{z+w+4-p_1-p_3}{2}\right]\Gamma\\
&\quad\times\left[\frac{z+w+4-p_2-p_4}{2}\right],\\
\mathcal{R} &= \frac{u^{\frac{p_3-p_4}{2}}}{v^{\frac{p_2+p_3}{2}}}\sum_{i,j}\frac{a_{ijk}}{i!j!k!}\frac{\sigma^i\tau^j(\tilde{\mu}-z-w+2i)^{-1}}{(z-\tilde{z}+2k)(w-\tilde{w}+2j)}.
\end{aligned}\tag{13}$$

In the sum $i,\ j,\ k\geq 0$ and we use the notation

$$\begin{aligned}
&\tilde{\mu}=p_2+p_4-2, \qquad \tilde{w}=p_2+p_3-2,\\
&\tilde{z}=\min(p_1+p_2,p_3+p_4)-2, \qquad k=M-1-i-j,\\
&M=p_3-1+\min(0,\Lambda), \qquad \Lambda=\frac{p_1+p_2-p_3-p_4}{2}.
\end{aligned}\tag{14}$$

The coefficients a_{ijk} are given by

$$a_{ijk}=\frac{2^3(M-1)!}{(1+|\Lambda|)_k(1+\frac{p_{43}+p_{21}}{2})_i(1+\frac{p_{43}-p_{21}}{2})_j}.\tag{15}$$

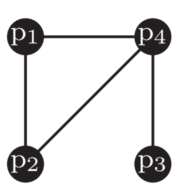

FIG. 2. A free theory diagram absent from $\langle p_1p_2p_3p_4\rangle$.

The conjecture agrees with all known supergravity computations ([15] and refs. therein). The precise integration contour and the assumptions which led to (13) are spelled out in [12].

C. Determining $\mathcal{N}_{p_1p_2p_3p_4}$ from the lightlike limit

The normalization $\mathcal{N}$ is not determined in [14]. Here we fix it using the following nontrivial statement:

$$\boxed{\lim_{u,v\to 0}\frac{\langle p_1p_2p_3p_4\rangle}{\mathcal{P}}\bigg|_{\frac{1}{N^2}}=0, \qquad \frac{u}{v}\ \text{fixed.}}\tag{16}$$

The limit $u,\ v\to 0$ with (u/v) fixed corresponds to taking the points $x_1,\ x_2,\ x_3,\ x_4$ to be sequentially lightlike separated.

Examining both the free theory and interacting contributions to the lhs of (16) above, we find that it takes the form $\sum_{r=1}^{M}A_r(u\tau/v)^r$ where

$$A_r=p_1p_2p_3p_4\frac{p_{21}+p_{43}+2}{2N^2}-\mathcal{N}_{p_1p_2p_3p_4}R^{p_3p_4}_{p_1p_2}.\tag{17}$$

The first term in (17) comes from $\langle p_1p_2p_3p_4\rangle_{\rm free}/\mathcal{P}$ and arises from the diagrams in Fig. 3. The normalization of each of these diagrams in the planar limit can be simply obtained by counting the number of inequivalent planar embeddings. Cyclic rotation on each vertex leaves the diagram unchanged, hence the factor $p_1p_2p_3p_4$. Additionally, the diagonal propagators can be drawn inside or outside the square, giving $\frac{1}{2}(p_{21}+p_{43})+1$ different possibilities. The multitrace terms in $\mathcal{O}_p$ do not affect the leading N result for the diagram. The cases $r=0$ or $r=M+1$ correspond to the diagrams of Fig. 2 which are absent as discussed above.

The second contribution in (17) is obtained from $\mathcal{I}\times\mathcal{H}_{\rm RZ}$. Note that each term in $u^{\frac{z}{2}}v^{\frac{w}{2}}\mathcal{R}$ has the form

$$\frac{u^{\frac{z-p_{43}}{2}}v^{\frac{w-p_2-p_3}{2}}\sigma^i\tau^j}{(z-p_{43}-2-2(i+j))(w-p_2-p_3+2+2j)}\tag{18}$$

and upon residue integration will produce a term proportional to $(u\sigma)^i(u/v)^{1+j}\tau^j$. Since $\mathcal{I}=\tau+O(u,v)$, the contribution to A_r comes from taking the simple poles with $i=0$ in (15). The residue is

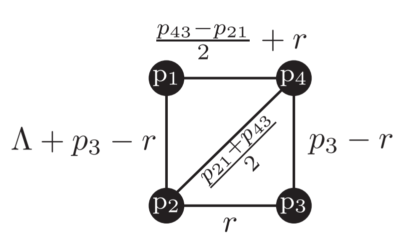

FIG. 3. Free theory diagrams in the lightlike limit.

$$R^{p_3p_4}_{p_1p_2} = |\Lambda|!\left(\frac{p_{43}-p_{21}}{2}\right)!\left(\frac{p_{21}+p_{43}+2}{2}\right)!(M-1)!. \tag{19}$$

Crucially the j dependence cancels between $a_{0jk}/(j!k!)$ and $\Gamma_{p_1p_2p_3p_4}$ and hence A_r is in fact independent of r. Now the statement (16) is clearly equivalent to the statement $A_r = 0$ for all r. Rearranging (17) we thus obtain the result for $\mathcal{N}_{p_1p_2p_3p_4}$,

$$\mathcal{N} = \frac{1}{N^2}\frac{p_1p_2p_3p_4}{|\Lambda|!(\frac{p_{43}-p_{21}}{2})!(\frac{p_{43}+p_{21}}{2})!(M-1)!}. \tag{20}$$

The result combines neatly with the coefficients a_{ijk},

$$\mathcal{N}a_{ijk} = \frac{1}{N^2}\frac{2^3p_1p_2p_3p_4}{(|\Lambda|+k)!(\frac{p_{43}+p_{21}}{2}+i)!(\frac{p_{43}-p_{21}}{2}+j)!}. \tag{21}$$

Note that the expression (20) is consistent with the results for $\mathcal{N}_{ppqq}$ and $\mathcal{N}_{p,p+1,q,q+1}$ obtained in [1,3].

D. Proof of lightlike vanishing

The lightlike limit projects the common OPE of $(\mathcal{O}_{p_1}\times\mathcal{O}_{p_2})$ and $(\mathcal{O}_{p_3}\times\mathcal{O}_{p_4})$ onto operators with large spin and naive twist $\tau \le p_{43}+2M$, i.e., $\tau < \min(p_1+p_2, p_3+p_4)$. To justify the statement (16) let us consider the various contributions to the OPE expected in the supergravity regime. First of all we have single-particle states corresponding to half-BPS superconformal primary operators. Such operators have spin zero and do not contribute in the limit $v\to 0$ which receives contributions from large spin. Next we have (both protected and unprotected) double-trace operators of the form $[\mathcal{O}_p\Box^n\partial^l\mathcal{O}_q]$ or mixtures thereof. The leading large N contribution to three-point functions of the form $\langle\mathcal{O}_p\mathcal{O}_q[\mathcal{O}_{p'}\Box^n\partial^l\mathcal{O}_{q'}]\rangle \sim O(N^{p+q})$ arises when $p=p'$ and $q=q'$ when the three-point function factorizes into a product of two-point functions. The twist τ of the double-trace operator therefore must obey $\tau \ge p+q$, otherwise the three-point function will be suppressed by $1/N^2$. The exchanged operators surviving the lightlike limit (16) all have twist less than both p_1+p_2 and p_3+p_4 and hence the contributions will be suppressed by at least $1/N^4$ and will not contribute to the lhs of (16). Higher multitrace operators are even more suppressed and we conclude that no operators in the supergravity spectrum can contribute in the lightlike limit, justifying (16).

III. UNMIXING EQUATIONS

We now describe how the system of relations implied by the OPE describes an eigenvalue problem which allows us to determine the anomalous dimensions of the true double-trace eigenstates K_{pq}. In particular, we consider the long multiplet superconformal partial wave (SCPW) expansion of the correlators $\langle p_1p_2p_3p_4\rangle$, in which the pairs (p_1,p_2) and (p_3,p_4) both run over the set $\mathcal{D}^{\rm long}_{\tau,l,a,b}$ described in (3). The result is a symmetric $(d\times d)$ matrix whose partial wave expansion reads

$$[\langle p_1p_2p_3p_4\rangle] = \sum_{\tau,l,a,b}\left[\mathcal{A}^{\tau,l}_{a,b}+\frac{1}{N^2}\log u\,\mathcal{M}^{\tau,l}_{a,b}\right]\mathbb{L}^{(\tau|l)}_{[a,b,a]}. \tag{22}$$

Terms of order $1/N^2$ which are analytic at $u=0$, i.e., without a factor of $\log u$, have been dropped on the rhs.

The matrix $\mathcal{A}^{\tau,l}_{a,b}$ in (22) is determined by disconnected free theory and is diagonal due to the form of the disconnected contributions (11). The matrix $\mathcal{M}^{\tau,l}_{a,b}$ is obtained from the discontinuity around $u=0$ of $\mathcal{H}_{\rm RZ}$. For completeness, we recall the explicit expression [16,17] of a long superblock of naive twist τ, spin l, and $su(4)$ rep $\mathfrak{R} = [n-m, 2m+p_{43}, n-m]$,

$$\mathbb{L}^{(\tau|l)}_{\mathfrak{R}} = \mathcal{P}\mathcal{I}(x,\bar{x},y,\bar{y})\frac{\Upsilon_{nm}(y,\bar{y})\mathcal{B}^{2+\frac{\tau}{2}|l}(x,\bar{x})}{u^{2+\frac{p_{43}}{2}}}. \tag{23}$$

This structure is the simplest among the determinantal superconformal blocks [9], since it factorizes into an ordinary conformal block $\mathcal{B}^{s|l}(x,\bar{x})$ [18],

$$\mathcal{B}^{s|l}(x,\bar{x}) = (-)^l\frac{u^sx^{l+1}\mathbf{F}_{s+l}(x)\mathbf{F}_{s-1}(\bar{x})-(x\leftrightarrow\bar{x})}{x-\bar{x}},$$
$$\mathbf{F}_s(x) = {}_2F_1\left[s-\frac{p_{12}}{2}, s+\frac{p_{34}}{2}; 2s\right](x) \tag{24}$$

and an $su(4)$ block $\Upsilon_{nm}(y,\bar{y})$ [19],

$$\Upsilon_{nm}(y,\bar{y}) = -\frac{\mathbf{P}_{n+1}(y)\mathbf{P}_m(\bar{y})-\mathbf{P}_m(y)\mathbf{P}_{n+1}(\bar{y})}{y-\bar{y}},$$
$$\mathbf{P}_n(y) = \frac{n!y}{(n+1+p_{43})_n}\mathrm{JP}_n^{(p_{43}-p_{21}|p_{43}+p_{21})}\left(\frac{2}{y}-1\right), \tag{25}$$

where JP stands for a Jacobi polynomial.

The matrices $\mathcal{A}$ and $\mathcal{M}$ contain conformal field theory data for the operators K_{pq},

$$\mathcal{A}^{\tau,l}_{a,b} = \mathbb{C}_{\tau,l,a,b}\cdot\mathbb{C}^T_{\tau,l,a,b},$$
$$\mathcal{M}^{\tau,l}_{a,b} = \mathbb{C}_{\tau,l,a,b}\cdot\eta\cdot\mathbb{C}^T_{\tau,l,a,b}. \tag{26}$$

Here the $(d\times d)$ matrix $\mathbb{C}$, indexed by pairs (p_1,p_2) and (q_1,q_2) running over $\mathcal{D}^{\rm long}_{\tau,l,a,b}$, is given by

$$\mathbb{C} \equiv [\langle\mathcal{O}_{p_1}\mathcal{O}_{p_2}K_{q_1q_2}\rangle], \tag{27}$$

and $\eta = \mathrm{diag}(\eta_{pq})$ is a $(d\times d)$ diagonal matrix where η_{pq} is (half) the anomalous dimension of the operator K_{pq} for $(p,q)\in\mathcal{D}^{\rm long}_{\tau,l,a,b}$,

$$\Delta_{pq} = \tau + l + \frac{2}{N^2}\eta_{pq} + O(1/N^4). \quad (28)$$

The eigenvalue problem (26) is well defined as a consequence of the equality

$$\left\{ \begin{matrix} \#\text{independent} \\ \text{entries of } \mathcal{A} \,\&\, \mathcal{M} \end{matrix} \right\} = \left\{ \begin{matrix} \#\text{of } \langle \mathcal{O}_{p_i}\mathcal{O}_{p_j}K_{pq}\rangle \\ +\#\text{of } \eta_{pq} \end{matrix} \right\}. \quad (29)$$

Let us comment on the structure of the matrices $\mathcal{A}$ and $\mathcal{M}$. The SCPW expansion of disconnected free theory has the following compact expression:

$$\mathcal{A}^{\tau,l}_{a,b} = \mathrm{diag}(\mathcal{F}_{1+a+i+r,b-2r,r,a,t+a+r})_{\substack{1\le i\le(t-1)\\ 0\le r\le \mu-1}},$$

where the function $\mathcal{F}$ is given by

$$\begin{aligned}
\mathcal{F}_{p,h,m,a,s} &= \frac{p(p+h)(1+\delta_{h0})(1+a)(2m+2+h+a)(l+1)(l+2s+2+h)}{(p-1-m)!(p-2-m-a)!(p+m+h)!(p+m+h+1+a)!} \\
&\quad\times \frac{(m+1+h)_{m+1}}{m!}\frac{(m+2+a+h)_{m+2+a}}{(m+1+a)!}\Pi_s\Pi_{l+s+1}, \\
\Pi_s &\equiv \frac{((s+h)!)^2}{(2s+h)!}(s+1-m)_m(s+1+h)_m(s-m-a)_a(s+2+h+m)_a(s+1-p)_{p-2-m-a} \\
&\quad\times (s+3+h+m+a)_{p-2-m-a}.
\end{aligned} \quad (30)$$

The SCPW of matrix elements in $\mathcal{M}_{\tau,l,a,b}$ has the form

$$\frac{(l+1+t+a+r+\frac{p_{43}-p_{21}}{2})!(l+1+t+a+r+p_{43})!}{(2(l+1+t+r+a)+p_{43})!}\times \mathcal{P}_d(l), \quad (31)$$

where $\mathcal{P}_d(l)$ is a polynomial in l of degree $d=\min(p_1+p_2, p_3+p_4)-(p_{43}-p_{21})-4$, and r labels (p_3,p_4). We determine this polynomial case by case and solve the eigenvalue problem following [1–3]. We have verified that our conjecture (5) holds systematically in the $su(4)$ channels $[a,b,a]$ with $0\le a\le 3$, $0\le b\le 6$ up to twist 24 for both even and odd spins. In particular, we have been able to perform nontrivial tests on the pattern of residual degeneracies. It would be fascinating to understand whether higher order corrections lift the pattern of residual degeneracies observed at order $1/N^2$ or whether they remain due to some as yet unknown symmetry.

IV. CASIMIR OPERATORS

Quadratic and quartic conformal Casimir operators have played a useful role in understanding and simplifying the structure of correlators [3,5,20]. Here we extend the analysis of [3] to all $su(4)$ channels $[a,b,a]$ of any correlator $\langle p_1p_2p_3p_4\rangle$. The quadratic and quartic Casimirs are given by [20,21]

$$\begin{aligned}
\mathcal{D}_2^{\rho_1,\rho_2} &= D_+^{\rho_1,\rho_2} + 2\frac{x\bar{x}}{x-\bar{x}}((1-x)\partial_x - (1-\bar{x})\partial_{\bar{x}}), \\
\mathcal{D}_4^{\rho_1,\rho_2} &= \left(\frac{x\bar{x}}{x-\bar{x}}\right)^2 D_-^{\rho_1,\rho_2}\left(\frac{x\bar{x}}{x-\bar{x}}\right)^{-2} D_-^{\rho_1,\rho_2},
\end{aligned} \quad (32)$$

where $D_\pm^{\rho_1,\rho_2} = D^{\rho_1,\rho_2} \pm \bar{D}^{\rho_1,\rho_2}$ and

$$D^{\rho_1,\rho_2} = x^2\partial_x(1-x)\partial_x - (\rho_1+\rho_2)x^2\partial_x - \rho_1\rho_2 x. \quad (33)$$

The labels ρ_i are given by $\rho_1 = -\frac{1}{2}p_{12}$, $\rho_2 = \frac{1}{2}p_{34}$. The eigenvalues of $\mathcal{D}_2$ and $\mathcal{D}_4$ on $\mathcal{B}^{(2+\frac{\tau}{2}|l)}$ are

$$\begin{aligned}
\lambda_2(\tau,l) &= \frac{1}{2}(l(l+2)+(\tau+l)(\tau+l-4)), \\
\lambda_4(\tau,l) &= l(l+2)(\tau+l-1)(\tau+l-3).
\end{aligned} \quad (34)$$

Consider the combination of Casimirs

$$\begin{aligned}
\Delta^{(8)} &= -\frac{1}{8}(\mathcal{D}_4^{\rho_1,\rho_2} - (\mathcal{D}_2^{\rho_1,\rho_2})^2 + g_1^{a,b}\mathcal{D}_2^{\rho_1,\rho_2} - g_2^{a,b}) \\
&\quad\times(\mathcal{D}_4^{\rho_1,\rho_2} - (\mathcal{D}_2^{\rho_1,\rho_2})^2 + g_3^{a,b}\mathcal{D}_2^{\rho_1,\rho_2} - g_4^{a,b}),
\end{aligned} \quad (35)$$

with the coefficients $g_i^{a,b}$ given by

$$\begin{aligned}
g_1^{a,b} &= (b+2a)^2 + 6(b+2a) + 6, \\
g_2^{a,b} &= \frac{1}{4}(b+2a)(b+2a+2)(b+2a+4)(b+2a+6), \\
g_3^{a,b} &= (b^2+2b-2), \\
g_4^{a,b} &= \frac{1}{4}(b-2)b(b+2)(b+4).
\end{aligned} \quad (36)$$

The operator $\Delta^{(8)}$ has the property that its eigenvalue on the conformal blocks reproduces exactly the numerator of the anomalous dimensions given in Eq. (5), i.e.,

$$\Delta^{(8)}\mathcal{B}^{(2+\frac{\tau}{2}|l)} = -2M_t^{(4)}M_{t+l+1}^{(4)}\mathcal{B}^{(2+\frac{\tau}{2}|l)}. \quad (37)$$

The operator $\Delta^{(8)}$ greatly simplifies the sums which compute the leading discontinuities of a correlator to any loop order. In a large N expansion we have

$$\mathcal{H} = \sum_{k\geq 1}\frac{1}{N^{2k}}\sum_{r=0}^{k}\frac{1}{r!}(\log u)^r\sum_{m\leq n}\Upsilon_{nm}\mathcal{H}^{(k)}_{r,nm}. \quad (38)$$

Then the leading discontinuity $\mathcal{H}^{(k)}_{k,nm}$ in an $su(4)$ channel with $a = n - m$ and $b = 2m - p_{43}$ is given by

$$\mathcal{H}^{(k)}_{k,nm} = \sum_{\tau,l,(q_1,q_2)}(\eta^{\tau,l,a,b}_{q_1q_2})^k C_{q_1q_2}\frac{\mathcal{B}^{(2+\frac{\tau}{2}|l)}}{u^{2+\frac{p_{43}}{2}}}, \quad (39)$$

with $C_{q_1q_2} = \langle\mathcal{O}_{p_1}\mathcal{O}_{p_2}K_{q_1q_2}\rangle\langle\mathcal{O}_{p_3}\mathcal{O}_{p_4}K_{q_1q_2}\rangle$. Since the numerator of the anomalous dimensions does not depend on (q_1, q_2), we may pull out $(k-1)$ factors of $\Delta^{(8)}$ and remove $(k-1)$ powers of the numerator from the anomalous dimension. These reduced sums are considerably simpler. Indeed the resummed result for general k is of a similar complexity as the $k = 1$ case (the $\log u$ coefficient of the tree-level supergravity result). One can then recover the full leading discontinuity by applying $\Delta^{(8)}$ $(k-1)$ times to the resummed expression.

For concreteness, let us consider the simplest example: $p_i = 2$, for which we have $\rho_1, \rho_2 = 0$ and the only $su(4)$ channel for long multiplets is the singlet a, $b = 0$. The $(\log u)^2$ term of the $\langle 2222\rangle$ correlator was computed at one loop in [2] (and recently reproduced using $\Delta^{(8)}$ in [5]). With the aid of $\Delta^{(8)}$ one can produce a closed formula for the highest transcendental weight part (weight k) of the leading $(\log u)^k$ discontinuity for any loop order,

$$\mathcal{H}^{(k)}_k|_{\rm top} = \frac{1}{u^2}(\Delta^{(8)})^{k-1}\left[\frac{G_k(x,\bar{x}) - v^7 G(x',\bar{x}')}{(x-\bar{x})^7}\right],$$
$$G(x,\bar{x}) = a_k(x,\bar{x})\sum_{a_i=0,1}[H_{a_1 0 a_2 0\cdots 1}(x) - (x\leftrightarrow\bar{x})]. \quad (40)$$

Here $x' = \frac{x}{x-1}$ and $H_{c_1\cdots c_n}(x)$ are harmonic polylogarithms of weight k [22]. Finally, the coefficient polynomial for the case $\langle 2222\rangle$ is given by

$$\begin{aligned} a_k(x,\bar{x}) = &-2^{7-3k}3^{1-k}u^4 \\ &\times[2^k(\hat{u}+v)(\hat{u}^2+8\hat{u}v+v(v+6)) \\ &-6(\hat{u}^3+7\hat{u}^2v+3\hat{u}v(v+2)-(v-4)v^2) \\ &+5^{2-k}2(\hat{u}^3-3\hat{u}v^2+3(\hat{u}+2)\hat{u}v+v^3)], \quad (41)\end{aligned}$$

with $\hat{u} = u - 1$. Similar results have been obtained for the correlators $\langle 2233\rangle$, $\langle 2323\rangle$, and $\langle 3333\rangle$, for which the quantum numbers (ρ_1, ρ_2) and (a, b) of $\Delta^{(8)}$ are nontrivial.

We believe that the results on the anomalous dimensions (5) together with the Casimir operators (35) will aid in the construction of one-loop supergravity (i.e., order $1/N^4$) contributions to all correlators $\langle p_1p_2p_3p_4\rangle$. It would be fascinating to see if the methods described in [23] can be used to make contact with such supergravity loop corrections and the spectrum results described here.

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