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OLLIVIER–RICCI IDLENESS FUNCTIONS OF GRAPHS*

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Abstract. We study the Ollivier–Ricci curvature of graphs as a function of the chosen idleness. We show that this idleness function is concave and piecewise linear with at most three linear parts, and at most two linear parts in the case of a regular graph. We then apply our result to show that the idleness function of the Cartesian product of two regular graphs is completely determined by the idleness functions of the factors.

Key words. Ollivier–Ricci, idleness, optimal transport

AMS subject classifications. 05C99, 51K10, 90C08

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1. Introduction and statement of results. Ricci curvature plays a very important role in the study of Riemannian manifolds. In the discrete setting of graphs, there has recently been much active research on various notions of Ricci curvature and its applications.

In [10] Ollivier developed a notion of Ricci curvature of Markov chains valid on metric spaces, including graphs. According to this notion, an *idleness* parameter, $p \in [0, 1]$, must be set in order to obtain a curvature κ_p. Ollivier considered idleness 0 and $\frac{1}{2}$. Idleness $\frac{1}{2}$ has also been considered in [13]. For graphs Ollivier's notion for idleness 0 has been studied further in [1, 2, 3, 6, 12]. In [11] Ollivier and Villani considered idleness $\frac{1}{d+1}$, where d is the degree of a regular graph, in order to investigate the curvature of the hypercube. In [5] Lin, Lu, and Yau introduced a modified version of Ollivier–Ricci curvature, which they defined to be the negative of the derivative of κ_p at $p = 1$; see (1.1).

We will show that for a regular graph the following holds:

$$\kappa = 2\kappa_{\frac{1}{2}} = \frac{d+1}{d}\kappa_{\frac{1}{d+1}}.$$

Therefore some of these different notions of curvature are related to each other by scaling factors.

In [1] Bhattacharya and Mukherjee derived exact expressions for Ollivier–Ricci curvature for bipartite graphs in the special case of idleness $p = 0$ and for graphs of girth at least 5. They used this result to classify all graphs with $\kappa_0 = 0$ for all edges (called "Ricci flat" in their paper) and girth at least 5. There is a small overlap in this paper between some of our methods and theirs (for example, they discussed the existence of integer-valued optimal Kantorovich potentials in the special case of vanishing idleness ($p = 0$)).

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To the best of our knowledge the *global* piecewise linear structure of the function $p \mapsto \kappa_p$ has not yet been established, and the only concrete examples in the literature where the full idleness function is computed are the hypercube and complete graphs [5]. However, some properties of this function have been discussed. In [5] it was shown that the idleness function is concave. It was shown in [8] that κ_p is linear close to idleness $p = 1$, and in [14] that κ_p is linear if a certain condition is satisfied (see the introductory part of section 5).

Throughout this article let $G = (V, E)$ be a locally finite graph with vertex set V and edge set E and containing no multiple edges or self-loops. Let d_x denote the degree of the vertex $x \in V$ and $d(x, y)$ the length of the shortest path between two vertices x and y, that is, the combinatorial distance. We denote the existence of an edge between x and y by $x \sim y$.

We define the following probability measures μ_x for any $x \in V$, $p \in [0, 1]$:

$$\mu_x^p(z) = \begin{cases} p & \text{if } z = x, \\ \dfrac{1-p}{d_x} & \text{if } z \sim x, \\ 0 & \text{otherwise.} \end{cases}$$

Let W_1 denote the 1-Wasserstein distance between two probability measures on V [15, page 211]. The p-Ollivier–Ricci curvature of an edge $x \sim y$ in $G = (V, E)$ is

$$\kappa_p(x, y) = 1 - W_1(\mu_x^p, \mu_y^p).$$

Ollivier–Ricci curvature has also been defined for arbitrary pairs of vertices (not necessarily adjacent vertices) [10], and it is known that

$$\inf_{\substack{x,y \in V \\ x \neq y}} \kappa_p(x, y) \geq \inf_{\substack{x,y \in V \\ x \sim y}} \kappa_p(x, y).$$

In this paper we consider only curvature on edges, since these are the discrete analogues of tangent vectors, i.e., of infinitesimal directions.

Lin, Lu, and Yau [5] introduced the following Ollivier–Ricci curvature:

$$\kappa(x, y) = \lim_{p \to 1} \frac{\kappa_p(x, y)}{1 - p}.$$

Note that their notion of curvature does not have an idleness index, and is thus distinguished from the idleness function $p \mapsto \kappa_p(x, y)$ in this paper, which we call the *Ollivier–Ricci idleness function.* We will show that $\kappa_p(x, y) = (1-p)\kappa(x, y)$ for all $p \in [\frac{1}{\max\{d_x, d_y\}+1}, 1]$ and that

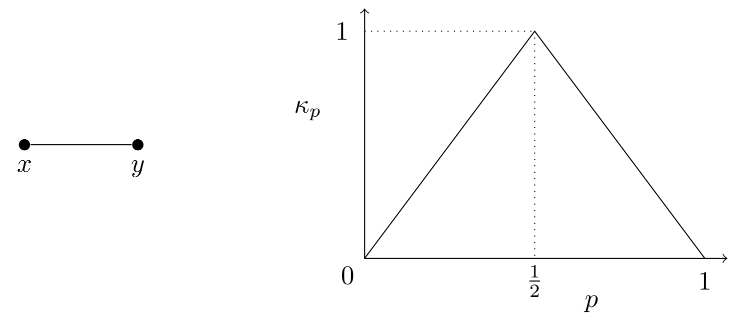
$$\kappa_0(x, y) \leq \kappa(x, y) \leq \kappa_0(x, y) + \frac{2}{\max\{d_x, d_y\}}.$$

Observe that

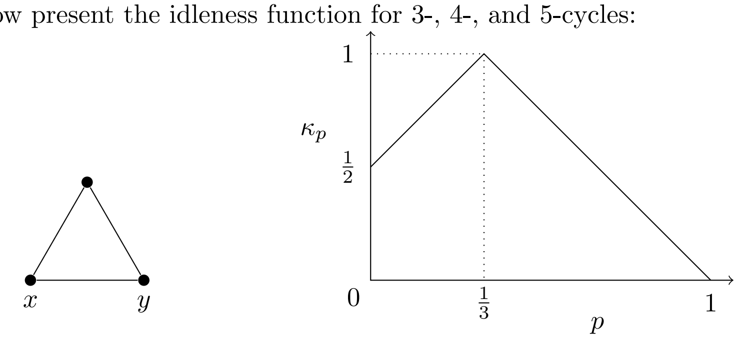
$$\kappa(x, y) = -\left.\frac{\partial}{\partial p}\right|_{p=1} \kappa_p(x, y). \tag{1.1}$$

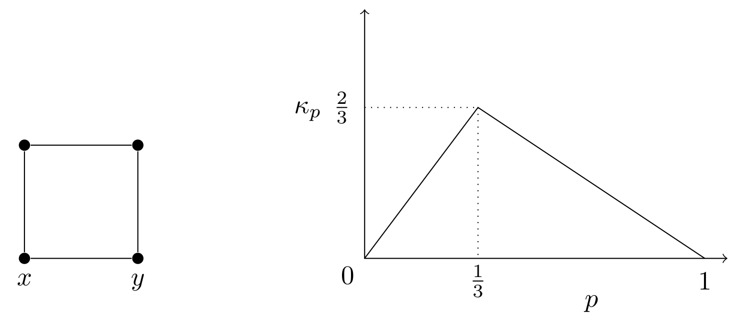
Next we give some examples of graphs and their Ollivier–Ricci idleness functions at a particular edge $x \sim y$.

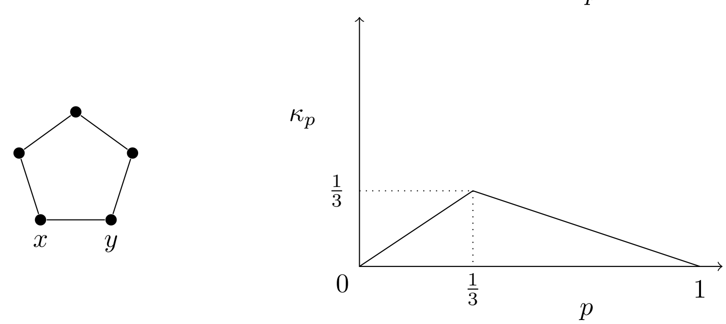
Examples. The following is the 1-path and a plot of the corresponding idleness function:



We now present the idleness function for 3-, 4-, and 5-cycles:

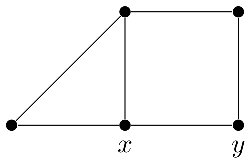


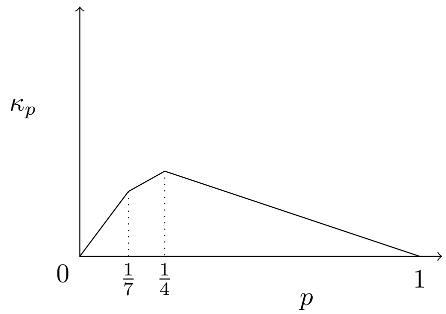




For cycles of length 6 or greater the idleness function at every edge vanishes identically (we call those edges *bone idle*; see section 7).

So far we have only seen idleness functions with at most two linear parts. We will show that if $d_x = d_y$, then this is always the case. However, if $d_x \neq d_y$, then three linear parts may occur, as shown in the following example:





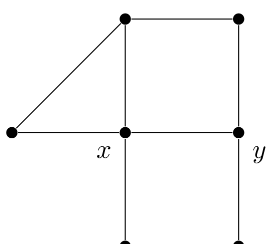
In fact the Ollivier–Ricci idleness function is piecewise linear with *at most* three parts always, a fundamental fact that is included in the following theorem (our main result).

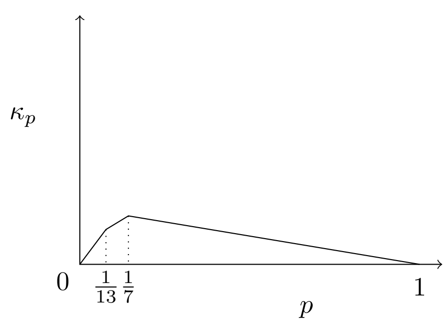
THEOREM 1.1. *Let $G = (V, E)$ be a locally finite graph. Let $x, y \in V$ with $x \sim y$. Then the function $p \mapsto \kappa_p(x, y)$ is concave and piecewise linear over $[0, 1]$ with at most three linear parts. Furthermore, $\kappa_p(x, y)$ is linear on the intervals*

$$\left[0, \frac{1}{\operatorname{lcm}(d_x, d_y) + 1}\right] \quad \textit{and} \quad \left[\frac{1}{\max(d_x, d_y) + 1}, 1\right].$$

Thus, if we have the further condition $d_x = d_y$, then $\kappa_p(x, y)$ has at most two linear parts.

In our example above of three linear parts, the changes in slope occur at $\frac{1}{\operatorname{lcm}(d_x,d_y)+1}$ and $\frac{1}{\max(d_x,d_y)+1}$. However, this need not always be the case. Consider the following example:





Here the first change in gradient does indeed occur at $\frac{1}{\operatorname{lcm}(d_x,d_y)+1} = \frac{1}{13}$, but the second change occurs before $\frac{1}{\max(d_x,d_y)+1} = \frac{1}{5}$.

REMARK 1.2. *Since $\kappa_p(x, y) = 1 - W_1(\mu_x^p, \mu_y^p)$, and $W_1(\mu_x^p, \mu_y^p)$ is the supremum of affine functions of p (by the Kantorovich duality theorem), $p \mapsto W_1(\mu_x^p, \mu_y^p)$ is convex and so $p \mapsto \kappa_p(x, y)$ is concave. An alternative proof of concavity was given in* [5].

A consequence of Theorem 1.1 and the results in [5] is the following corollary.

COROLLARY 1.3. *Let $G = (V_G, E_G)$ be a d_G-regular graph and $H = (V_H, E_H)$ a*

d_H-regular graph. Let $x_1, x_2 \in V_G$ with $x_1 \sim x_2$ and $y \in V_H$. Then

$$\kappa_p^{G\times H}((x_1,y),(x_2,y)) = \begin{cases} \frac{d_G}{d_G+d_H}\kappa_p^G(x_1,x_2) + \frac{d_G d_H}{d_G+d_H}(\kappa^G(x_1,x_2) - \kappa_0^G(x_1,x_2))p & \text{if } p \in \left[0, \frac{1}{d_G+d_H+1}\right], \\ \frac{d_G}{d_G+d_H}\kappa^G(x_1,x_2)(1-p) & \text{if } p \in \left[\frac{1}{d_G+d_H+1}, 1\right]. \end{cases}$$

This result shows that the idleness function of the Cartesian product of two regular graphs is completely determined by the idleness functions of the factors.

We finish this introduction with an outline of the rest of this paper. In section 2 we present the relevant notation and background material. In section 3 we show that $p \mapsto \kappa_p$ is piecewise linear with at most three linear parts. In sections 4 and 5 we give bounds on the sizes of the last and first linear parts, respectively. We prove Corollary 1.3 in section 6. Finally, in section 7 we present some open questions. Moreover, we discuss the problem of characterizing edges with globally linear curvature functions.

2. Definitions and notation.

We now introduce the relevant definitions and notation we will need in this paper. First, we recall the Wasserstein distance and the Ollivier–Ricci curvature.

DEFINITION 2.1. *Let $G = (V, E)$ be a locally finite graph. Let μ_1 and μ_2 be two probability measures on V. The* Wasserstein distance *$W_1(\mu_1, \mu_2)$ between μ_1 and μ_2 is defined as*

$$W_1(\mu_1,\mu_2) = \inf_{\pi\in\Pi(\mu_1,\mu_2)} \sum_{y\in V}\sum_{x\in V} d(x,y)\pi(x,y), \tag{2.1}$$

where

$$\Pi(\mu_1,\mu_2) = \left\{ \pi : V\times V \to [0,1] : \mu_1(x) = \sum_{y\in V}\pi(x,y),\ \mu_2(y) = \sum_{x\in V}\pi(x,y) \right\}.$$

The transportation plan π moves a mass distribution given by μ_1 into a mass distribution given by μ_2, and $W_1(\mu_1, \mu_2)$ is a measure of the minimal effort that is required for such a transition. If π attains its infimum in (2.1), we call it an *optimal transport plan* transporting μ_1 to μ_2.

DEFINITION 2.2. *The* p-Ollivier–Ricci curvature *of an edge $x \sim y$ in $G = (V, E)$ is*

$$\kappa_p(x,y) = 1 - W_1(\mu_x^p, \mu_y^p),$$

where p is called the idleness.

A fundamental concept in optimal transport theory, and vital to our work, is Kantorovich duality. First we recall the notion of 1-Lipschitz functions and then state the Kantorovich duality theorem.

DEFINITION 2.3. *Let $G = (V, E)$ be a locally finite graph, $\phi : V \to \mathbb{R}$. We say that ϕ is* 1-Lipschitz *if*

$$|\phi(x) - \phi(y)| \le d(x,y)$$

for all $x, y \in V$. Let 1-Lip *denote the set of all 1-Lipschitz functions on V.*

THEOREM 2.1 (Kantorovich duality [15]). *Let $G = (V, E)$ be a locally finite graph. Let μ_1, μ_2 be two probability measures on V. Then*

$$W_1(\mu_1, \mu_2) = \sup_{\substack{\phi: V \to \mathbb{R} \\ \phi \in 1\text{–Lip}}} \sum_{x \in V} \phi(x)(\mu_1(x) - \mu_2(x)).$$

If $\phi \in 1$-Lip attains its supremum, we call it an optimal Kantorovich potential *transporting μ_1 to μ_2.*

3. Properties of the idleness function. In this section we prove that the Ollivier–Ricci idleness function has at most three linear parts. Two ingredients of this proof are the "integer-valuedness" of optimal Kantorovich potentials and the complementary slackness theorem, which we now state and prove.

LEMMA 3.1. *Let $G = (V, E)$ be a locally finite graph. Let $x, y \in V$ with $x \sim y$. Let $p \in [0, 1]$. Let π and ϕ be an optimal transport plan and an optimal Kantorovich potential transporting μ_x^p to μ_y^p, respectively. Let $u, v \in V$ with $\pi(u, v) \neq 0$. Then*

$$\phi(u) - \phi(v) = d(u, v).$$

This follows from the complementary slackness theorem (see, for example, [9, page 49]) or from standard results in optimal transport theory [15, page 88]. For the sake of completeness we include a short proof here.

Proof. By the definitions of ϕ and π we have

$$W_1(\mu_x^p, \mu_y^p) = \sum_{w \in V} \phi(w)(\mu_x^p - \mu_y^p)(w) = \sum_{w \in V} \sum_{z \in V} d(w, z)\pi(w, z)$$

and

$$\sum_{w \in V} \pi(w, z) = \mu_y^p(z), \qquad \sum_{z \in V} \pi(w, z) = \mu_x^p(w).$$

Then

$$\begin{aligned} W_1(\mu_x^p, \mu_y^p) &= \sum_{w \in V} \phi(w)\mu_x^p(w) - \sum_{z \in V} \phi(z)\mu_y^p(z) \\ &= \sum_{w \in V} \phi(w) \sum_{z \in V} \pi(w, z) - \sum_{z \in V} \phi(z) \sum_{w \in V} \pi(w, z) \\ &= \sum_{w \in V} \sum_{z \in V} (\phi(w) - \phi(z))\pi(w, z) \\ &\leq \sum_{w \in V} \sum_{z \in V} d(w, z)\pi(w, z) \\ &= W_1(\mu_x^p, \mu_y^p). \end{aligned}$$

Thus

$$\sum_{w \in V} \sum_{z \in V} (\phi(w) - \phi(z))\pi(w, z) = \sum_{w \in V} \sum_{z \in V} d(w, z)\pi(w, z).$$

Therefore

$$\phi(w) - \phi(z) < d(w, z) \implies \pi(w, z) = 0,$$

thus completing the proof. ∎

As mentioned in the introduction, in [1] Bhattacharya and Mukherjee discuss the existence of integer-valued optimal Kantorovich potentials in the special case of vanishing idleness ($p = 0$). We first introduce the floor and ceiling of functions and then state a corresponding result for the case of arbitrary idleness.

Definition 3.1. *Let $G = (V, E)$ be a locally finite graph and let $\phi : V \to \mathbb{R}$. Define the functions $\lfloor \phi \rfloor$ and $\lceil \phi \rceil$ as follows:*

$$\begin{aligned} \lfloor \phi \rfloor &: V \to \mathbb{R} \\ &\quad v \mapsto \lfloor \phi(v) \rfloor, \\ \lceil \phi \rceil &: V \to \mathbb{R} \\ &\quad v \mapsto \lceil \phi(v) \rceil. \end{aligned}$$

Lemma 3.2. *Let $G = (V, E)$ be a locally finite graph. Let $\phi \in$ 1-Lip. Then $\lfloor \phi \rfloor, \lceil \phi \rceil \in$ 1-Lip.*

Proof. For each $v \in V$ set $\delta_v = \phi(v) - \lfloor \phi(v) \rfloor$. Note that $\delta_v \in [0, 1)$. Then

$$|\lfloor \phi(v) \rfloor - \lfloor \phi(w) \rfloor| = |\phi(v) - \delta_v - \phi(w) + \delta_w| \le d(v, w) + |\delta_v - \delta_w|.$$

Since $\delta_v - \delta_w \in (-1, 1)$ we have $|\lfloor \phi(v) \rfloor - \lfloor \phi(w) \rfloor| < d(v, w) + 1$ and so $|\lfloor \phi(v) \rfloor - \lfloor \phi(w) \rfloor| \le d(v, w)$ since $|\lfloor \phi(v) \rfloor - \lfloor \phi(w) \rfloor|$ is integer-valued. Thus $\lfloor \phi \rfloor \in$ 1-Lip. The proof that $\lceil \phi \rceil \in$ 1-Lip follows similarly. ∎

Lemma 3.3 (integer-valuedness). *Let $G = (V, E)$ be a locally finite graph. Let $x, y \in V$ with $x \sim y$. Let $p \in [0, 1]$. Then there exists $\phi \in$ 1-Lip such that*

$$W_1(\mu_x^p, \mu_y^p) = \sum_{w \in V} \phi(w)(\mu_x^p(w) - \mu_y^p(w)),$$

and $\phi(w) \in \mathbb{Z}$ for all $w \in V$.

Proof. Let Φ be an optimal Kantorovich potential transporting μ_x^p to μ_y^p. Let π be an optimal transport plan transporting μ_x^p to μ_y^p. Consider the following graph H with vertices V and edges given by the following adjacency matrix A:

$$\begin{aligned} A(v, w) &= 1 \text{ if } \pi(v, w) > 0 \text{ or } \pi(w, v) > 0, \\ A(v, w) &= 0 \text{ otherwise.} \end{aligned}$$

Let $(W_i)_{i=1}^n$ denote the connected components of H. Fix $u, v \in W_i$ for some $i \in \{1, \ldots, n\}$. By Lemma 3.1 we have $|\Phi(u) - \Phi(v)| = d(u, v)$.

Define $\phi : V \to \mathbb{R}$ by

$$\phi(v) = \sup\{\psi(v) : \psi : V \to \mathbb{Z},\ \psi \in \text{1-Lip},\ \psi \le \Phi\}.$$

By definition, ϕ is an integer-valued 1-Lipschitz function and $\phi \le \Phi$. Note that $\phi = \lfloor \Phi \rfloor$, since $\lfloor \Phi \rfloor \in$ 1-Lip by Lemma 3.2.

Finally we must show that ϕ is optimal. For each $v \in V$ set $\delta_v = \Phi(v) - \lfloor \Phi(v) \rfloor = \Phi(v) - \phi(v)$. Note that $\mu_x^p(W_i) = \mu_y^p(W_i)$ for all i (since no mass is transported between different connected components W_i), and that $\delta_u = \delta_v$ if u and v belong to the same component W_i for some i. Set $\delta_i = \delta_u$ for any $u \in W_i$. Then

$$\begin{aligned}\sum_{w\in V}\phi(w)(\mu_x^p(w)-\mu_y^p(w)) &= \sum_{w\in V}(\Phi(w)-\delta_w)(\mu_x^p(w)-\mu_y^p(w))\\ &= \sum_{i=1}^{n}\sum_{w\in W_i}(\Phi(w)-\delta_w)(\mu_x^p(w)-\mu_y^p(w))\\ &= \sum_{i=1}^{n}\sum_{w\in W_i}\Phi(w)(\mu_x^p(w)-\mu_y^p(w))\\ &\quad -\sum_{i=1}^{n}\sum_{w\in W_i}\delta_w(\mu_x^p(w)-\mu_y^p(w))\\ &= \sum_{w\in V}\Phi(w)(\mu_x^p(w)-\mu_y^p(w))-\sum_{i=1}^{n}\delta_i\underbrace{(\mu_x^p(W_i)-\mu_y^p(W_i))}_{=0}\\ &= W_1(\mu_x^p,\mu_y^p).\end{aligned}$$

Therefore ϕ is optimal, as required. ∎

Now we formulate our main result of this section.

THEOREM 3.4. *Let $G = (V, E)$ be a locally finite graph. Let $x, y \in V$ with $x \sim y$. Then $p \mapsto \kappa_p(x, y)$ is piecewise linear over $[0, 1]$ with at most three linear parts.*

Proof. For $\phi : V \to \mathbb{R}$, let

$$F(\phi) = d_y\left(\sum_{\substack{z\sim x\\ z\neq y}}\phi(z)\right) - d_x\left(\sum_{\substack{z\sim y\\ z\neq x}}\phi(z)\right). \tag{3.1}$$

For $j \in \{-1, 0, 1\}$, define

$$\mathcal{A}_j = \{\phi : V \to \mathbb{Z} : \phi(x) = j,\ \phi(y) = 0,\ \phi \in \text{1-Lip}\} \tag{3.2}$$

and define the constants

$$c_j = \sup_{\phi\in\mathcal{A}_j} F(\phi).$$

Finally define the linear maps

$$f_j(p) = \left(p - \frac{1-p}{d_y}\right) j + \frac{1-p}{d_x d_y} c_j.$$

Then

$$
\begin{aligned}
W_1(\mu_x^p, \mu_y^p) &= \sup_{\phi \in 1\text{-Lip}} \sum_{w \in V} \phi(w)(\mu_x^p(w) - \mu_y^p(w)) \\
&= \sup_{\substack{\phi \in 1\text{-Lip} \\ \phi: V \to \mathbb{Z} \\ \phi(y)=0}} \sum_{w \in V} \phi(w)(\mu_x^p(w) - \mu_y^p(w)) \\
&= \sup_{\substack{\phi \in 1\text{-Lip} \\ \phi: V \to \mathbb{Z} \\ \phi(y)=0}} \left\{ \phi(x) \left(p - \frac{1-p}{d_y} \right) + \frac{1-p}{d_x} \sum_{\substack{w \sim x \\ w \neq y}} \phi(w) - \frac{1-p}{d_y} \sum_{\substack{w \sim y \\ w \neq x}} \phi(w) \right\} \\
&= \sup_{\substack{\phi \in 1\text{-Lip} \\ \phi: V \to \mathbb{Z} \\ \phi(y)=0}} \left\{ \phi(x) \left(p - \frac{1-p}{d_y} \right) + \frac{1-p}{d_x d_y} \left(d_y \sum_{\substack{w \sim x \\ w \neq y}} \phi(w) - d_x \sum_{\substack{w \sim y \\ w \neq x}} \phi(w) \right) \right\} \\
&= \sup_{\substack{\phi \in 1\text{-Lip} \\ \phi: V \to \mathbb{Z} \\ \phi(y)=0}} \left\{ \phi(x) \left(p - \frac{1-p}{d_y} \right) + \frac{1-p}{d_x d_y} F(\phi) \right\} \\
&= \max_{j \in \{-1,0,1\}} \sup_{\phi \in \mathcal{A}_j} \left\{ j \left(p - \frac{1-p}{d_y} \right) + \frac{1-p}{d_x d_y} F(\phi) \right\} \\
&= \max_{j \in \{-1,0,1\}} \left\{ j \left(p - \frac{1-p}{d_y} \right) + \frac{1-p}{d_x d_y} \sup_{\phi \in \mathcal{A}_j} F(\phi) \right\} \\
&= \max_{j \in \{-1,0,1\}} \left\{ j \left(p - \frac{1-p}{d_y} \right) + \frac{1-p}{d_x d_y} c_j \right\} \\
&= \max\{f_{-1}(p), f_0(p), f_1(p)\}. \qquad (3.3)
\end{aligned}
$$

Therefore

$$\kappa_p(x,y) = 1 - \max\{f_{-1}(p), f_0(p), f_1(p)\}.$$

Since $\max\{f_{-1}(p), f_0(p), f_1(p)\}$ is the maximum of three linear functions of p, it is convex and piecewise linear in p with at most three linear parts, and thus the proof is complete. ∎

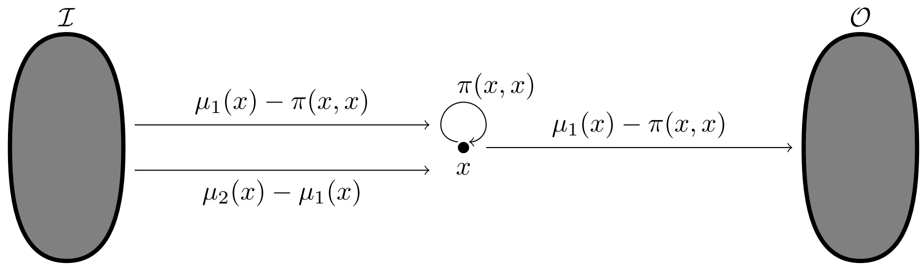
4. Length of the last linear part. Before discussing the size of the last linear part we first need the following lemma about some of the assumptions we can impose on an optimal transport plan. We then show that if different idlenesses $p_1 < p_2$ share a joint optimal Kantorovich potential, then the Ollivier–Ricci idleness function is linear on the whole interval $[p_1, p_2]$. This was already mentioned in [14] for the special case $p_2 = 1$.

LEMMA 4.1. *Let μ_1 and μ_2 be probability measures on V. Then there exists an optimal transport plan π transporting μ_1 to μ_2 with the following property: for all $x \in V$ with $\mu_1(x) \le \mu_2(x)$ we have $\pi(x,x) = \mu_1(x)$.*

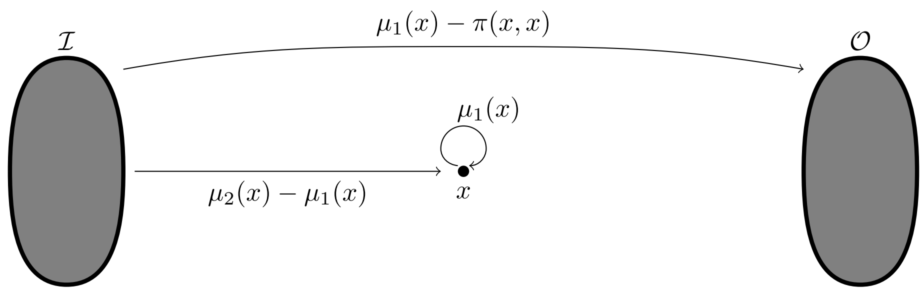
This lemma can be proved using Corollary 1.16 in [15] (invariance of the Kantorovich–Rubinstein distance under mass subtraction), but we present a proof in our much simpler context for the reader's convenience.

Proof. Let π be an optimal transport plan transporting μ_1 to μ_2. Assume there exists an $x \in V$ with $\mu_1(x) \le \mu_2(x)$, but $\pi(x,x) < \mu_1(x)$. Let $\mathcal{I} = \{z \in V \setminus \{x\} :$

$\pi(z,x) > 0\}$ and $\mathcal{O} = \{w \in V \setminus \{x\} : \pi(x,w) > 0\}$. Since π is optimal, we must have $\mathcal{I} \cap \mathcal{O} = \emptyset$. Then the relevant part of π can be depicted as



We now modify π to obtain a new transport plan π' as follows:



This new transport plan π' is still optimal (by the triangle inequality). Note that

$$\pi'(z,z) = \begin{cases} \pi(z,z) & \text{if } z \neq x, \\ \mu_1(x) & \text{if } z = x. \end{cases}$$

Repeating this modification at all other vertices that violate the condition of the lemma successively gives us our required optimal transport plan. ∎

LEMMA 4.2. *Let $G = (V,E)$ be a locally finite graph. Let $x, y \in V$ with $x \sim y$. Let $0 \leq p_1 \leq p_2 \leq 1$. If there exists a 1-Lipschitz function ϕ that is an optimal Kantorovich potential transporting $\mu_x^{p_1}$ to $\mu_y^{p_1}$ and transporting $\mu_x^{p_2}$ to $\mu_y^{p_2}$, then $W_{xy} : [0,1] \to \mathbb{R}$, $W_{xy}(p) = W_1(\mu_x^p, \mu_y^p)$, is linear on $[p_1, p_2]$.*

Proof. Let $\alpha \in [0,1]$. The convexity of W_{xy} (see Remark 1.2) implies that

$$\alpha W_{xy}(p_1) + (1-\alpha) W_{xy}(p_2) \geq W_{xy}(\alpha p_1 + (1-\alpha) p_2).$$

It only remains to show that the above inequality is in fact an equality. Observe that

$$\mu_x^{\alpha p_1 + (1-\alpha) p_2} = \alpha \mu_x^{p_1} + (1-\alpha) \mu_x^{p_2},$$
$$\mu_y^{\alpha p_1 + (1-\alpha) p_2} = \alpha \mu_y^{p_1} + (1-\alpha) \mu_y^{p_2}.$$

Then, setting $p = \alpha p_1 + (1-\alpha) p_2$, we have

$$\begin{aligned} W_{xy}(p) &\geq \sum_{w \in V} \phi(w) (\mu_x^{\alpha p_1 + (1-\alpha) p_2}(w) - \mu_y^{\alpha p_1 + (1-\alpha) p_2}(w)) \\ &= \sum_{w \in V} \phi(w) \left(\alpha \mu_x^{p_1}(w) + (1-\alpha) \mu_x^{p_2}(w) - \alpha \mu_y^{p_1}(w) - (1-\alpha) \mu_y^{p_2}(w) \right) \\ &= \alpha \sum_{w \in V} \phi(w) \left(\mu_x^{p_1}(w) - \mu_y^{p_1}(w) \right) + (1-\alpha) \sum_{w \in V} \phi(w) \left(\mu_x^{p_2}(w) - \mu_y^{p_2}(w) \right) \\ &= \alpha W_{xy}(p_1) + (1-\alpha) W_{xy}(p_2). \end{aligned}$$ ∎

Lemma 4.3. *Let $G = (V, E)$ be a locally finite graph. Let $x, y \in V$ with $x \sim y$ and $d_x \geq d_y$. Let $p \in (\frac{1}{1+d_x}, 1]$. Let ϕ be an optimal Kantorovich potential transporting μ_x^p to μ_y^p. Then*

$$\phi(x) - \phi(y) = 1.$$

Proof. Let π be an optimal transport plan transporting μ_x^p to μ_y^p. We may assume that π satisfies the conditions of Lemma 4.1. Since $p > \frac{1}{1+d_x}$, $\mu_x^p(y) = \frac{1-p}{d_x} < \frac{d_x p}{d_x} = p = \mu_y^p(y)$ and therefore there exists $z \in B_1(x) \setminus \{y\}$ such that $\pi(z, y) > 0$. If $z = x$, then $\phi(x) - \phi(y) = 1$, by Lemma 3.1. Suppose $z \sim y$ and $z \neq x$. Then observe that $\mu_x^p(z) = \frac{1-p}{d_x} \leq \frac{1-p}{d_y} = \mu_y^p(z)$. Thus $\pi(z, y) = 0$, by Lemma 4.1, which contradicts our assumption that $\pi(z, y) > 0$.

The only case left to consider is $z \sim x$, $z \nsim y$, $z \neq y$. Then $d(z, y) = 2$, in which case we have $\phi(z) - \phi(y) = 2$, by Lemma 3.1. Then

$$\begin{aligned} 2 &= \phi(z) - \phi(y) \\ &= \phi(z) - \phi(x) + \phi(x) - \phi(y) \\ &\leq 1 + \phi(x) - \phi(y) \\ &\leq 2, \end{aligned}$$

which implies $\phi(x) - \phi(y) = 1$. ∎

We are now ready to prove the main theorem of this section.

Theorem 4.4. *Let $G = (V, E)$ be a locally finite graph and let $x, y \in V$ with $x \sim y$ and $d_x \geq d_y$. Then $p \mapsto \kappa_p(x, y)$ is linear over $[\frac{1}{d_x+1}, 1]$.*

Proof. Let $1 > p_0 > \frac{1}{d_x+1}$ and let ϕ be an optimal Kantorovich potential transporting $\mu_x^{p_0}$ to $\mu_y^{p_0}$. Then, by Lemma 4.3, we have $\phi(x) - \phi(y) = 1$. Note that any 1-Lipschitz ψ satisfying $\psi(x) - \psi(y) = 1$ is an optimal Kantorovich potential transporting μ_x^1 to μ_y^1. Thus, by Lemma 4.2, $p \mapsto \kappa_p(x, y)$ is linear over $[p_0, 1]$. By continuity of $p \mapsto \kappa_p(x, y)$ this linearity extends to $[\frac{1}{d_x+1}, 1]$. ∎

Remark 4.5. *Note that the above proof shows the existence of a 1-Lipschitz function ϕ with $\phi(x) - \phi(y) = 1$, which is an optimal Kantorovich potential for all $p \in [\frac{1}{d_x+1}, 1]$. We choose ϕ to be an optimal Kantorovich potential transporting $\mu_x^{p_0}$ to $\mu_y^{p_0}$ for some $1 > p_0 > \frac{1}{d_x+1}$ and satisfying $\phi(x) - \phi(y) = 1$, as in the proof of Theorem 4.4. Then both W_{xy} and the function*

$$p \mapsto \sum_{w \in V} \phi(w)(\mu_x^p(w) - \mu_y^p(w))$$

are linear over $[\frac{1}{d_x+1}, 1]$ and agree at $p = p_0$ and $p = 1$. Therefore they agree on the whole interval, and consequently ϕ is an optimal Kantorovich potential for all $p \in [\frac{1}{d_x+1}, 1]$.

5. Length of the first linear part.

Lemma 5.1. *Let $G = (V, E)$ be a locally finite graph. Let F be as defined in (3.1). Then*

$$\sup_{\substack{\phi \in \text{1-Lip} \\ \phi: V \to \mathbb{Z} \\ \phi(x) = \phi(y) = 0}} F(\phi) = \sup_{\substack{\phi \in \text{1-Lip} \\ \phi: V \to \mathbb{Z}/2 \\ \phi(x) = \phi(y) = 0}} F(\phi),$$

where $\mathbb{Z}/2 = \{n/2 : n \in \mathbb{Z}\}$.

Proof. Pick $\phi_0 \in$ 1-Lip such that $\phi_0 : V \to \mathbb{Z}/2$, $\phi_0(x) = \phi_0(y) = 0$, and

$$F(\phi_0) = \sup_{\substack{\phi \in \text{1-Lip} \\ \phi: V \to \mathbb{Z}/2 \\ \phi(x)=\phi(y)=0}} F(\phi).$$

Note that

$$\phi_0(v) = \frac{\lfloor \phi_0(v) \rfloor + \lceil \phi_0(v) \rceil}{2}$$

for all $v \in V$. Thus

$$F(\phi_0) = \frac{F(\lfloor \phi_0 \rfloor) + F(\lceil \phi_0 \rceil)}{2}.$$

By combining this with $F(\phi_0) \geq F(\lfloor \phi_0 \rfloor)$ and $F(\phi_0) \geq F(\lceil \phi_0 \rceil)$ we obtain

$$F(\phi_0) = F(\lfloor \phi_0 \rfloor) = F(\lceil \phi_0 \rceil).$$

Since $\lfloor \phi_0 \rfloor : V \to \mathbb{Z}$, this completes the proof. ∎

The rest of this section is devoted to the proof of the following result.

THEOREM 5.2. *Let $G = (V, E)$ be a locally finite graph. Let $x, y \in V$ with $x \sim y$ and $d_x \geq d_y$. Let $\ell = \operatorname{lcm}(d_x, d_y)$. Then $p \mapsto \kappa_p(x, y)$ is linear over $[0, \frac{1}{\ell+1}]$.*

Proof. Let F, $\mathcal{A}_j$, c_j, and f_j be as defined in the proof of Theorem 3.4. In order to bound the length of the first linear part of κ_p, we look at the intersection points of the functions f_j. First we derive inequalities between the constants c_j. Note that

$$f_j\left(\frac{1}{d_y+1}\right) = \frac{1}{(d_y+1)d_x} c_j$$

for $j \in \{-1, 0, 1\}$. We claim that $f_1(\frac{1}{d_y+1}) \geq f_j(\frac{1}{d_y+1})$. It then follows that $c_1 \geq c_0$ and $c_1 \geq c_{-1}$. We now prove the claim. Note that $\frac{1}{d_y+1} \in [\frac{1}{d_x+1}, 1]$ and that, by Remark 4.5, there exists an optimal Kantorovich potential ϕ at idleness $\frac{1}{d_y+1}$ with $\phi(x) - \phi(y) = 1$. Therefore, by (3.3),

$$f_1\left(\frac{1}{d_y+1}\right) = W_{xy}\left(\frac{1}{d_y+1}\right) = \max\left\{f_{-1}\left(\frac{1}{d_y+1}\right), f_0\left(\frac{1}{d_y+1}\right), f_1\left(\frac{1}{d_y+1}\right)\right\},$$

which proves the claim.

Let $\phi_j \in \mathcal{A}_j$ satisfy $F(\phi_j) = c_j = \max_{\mathcal{A}_j} F$. Let $\psi = \frac{\phi_{-1}+\phi_1}{2}$. Note that ψ is 1-Lipschitz and $\psi(x) = \psi(y) = 0$. The function ψ may fail to be integer-valued, but we note that $\psi : V \to \mathbb{Z}/2$ and so, by Lemma 5.1, we have

$$c_0 \geq F\left(\frac{\phi_{-1}+\phi_1}{2}\right) = \frac{c_{-1}+c_1}{2} \geq c_{-1}. \tag{5.1}$$

Therefore

$$c_1 \geq c_0 \geq c_{-1}.$$

Let $g = \gcd(d_x, d_y)$. Since the constants c_j are integer linear combinations of d_x and d_y, we have $g | c_j$ for $j \in \{-1, 0, 1\}$. For the computation of the possible intersection points of f_j, we will make use of the following simple observation. Let $b > 0$ and suppose that $0 \leq \frac{a}{a+b} \leq 1$. Then $a > 0$.

Suppose that p' satisfies $f_{-1}(p') = f_0(p')$. Then

$$p' = \frac{d_x - (c_0 - c_{-1})}{d_x d_y + d_x - (c_0 - c_{-1})}.$$

We can write $c_0 - c_{-1} = d_x - Kg$ for some $K \in \mathbb{Z}$. Then

$$p' = \frac{Kg}{d_x d_y + Kg} = \frac{K}{\ell + K},$$

with $\ell = \operatorname{lcm}(d_x, d_y)$. Since $0 \le p' \le 1$, we have $K \ge 0$. Thus the smallest strictly positive intersection point is $p' = \frac{1}{\ell+1}$.

Now suppose that p' satisfies $f_1(p') = f_0(p')$. Then

$$p' = \frac{d_x - (c_1 - c_0)}{d_x d_y + d_x - (c_1 - c_0)}.$$

We can write $c_1 - c_0 = d_x - Kg$ for some $K \in \mathbb{Z}$. Then

$$p' = \frac{Kg}{d_x d_y + Kg} = \frac{K}{\ell + K}.$$

Since $0 \le p' \le 1$ we have $K \ge 0$. Thus the smallest strictly positive intersection point is again $p' = \frac{1}{\ell+1}$.

Now suppose that p' satisfies $f_{-1}(p') = f_1(p')$. Then

$$f_{-1}(p') = \frac{1}{2}(f_{-1}(p') + f_1(p')) = \frac{1-p'}{d_x d_y}\,\frac{c_{-1}+c_1}{2} \overset{(5.1)}{\le} \frac{1-p'}{d_x d_y}\,c_0 = f_0(p').$$

In particular

$$f_1(p') = f_{-1}(p') = \frac{1}{2}(f_{-1}(p') + f_1(p')) \le f_0(p').$$

Thus either $f_0(p') > f_{-1}(p')$ and $f_0(p') > f_1(p')$, in which case there is no turning point at p', or $f_0(p') = f_{-1}(p') = f_1(p')$, in which case p' is one of the points we have already considered. Thus $p \mapsto \kappa_p(x,y)$ is linear over $[0, \frac{1}{\ell+1}]$. ∎

Let us finish this section with some observations about relations between various different curvature values. Assume that $d_y | d_x$. Then $\operatorname{lcm}(d_x, d_y) = \max(d_x, d_y)$ and so, by Theorem 1.1, $p \mapsto \kappa_p(x,y)$ has at most two linear parts. We can give a formula for $\kappa_p(x,y)$ in terms of the curvatures $\kappa_0(x,y)$ and $\kappa(x,y)$ by using the fact that $\kappa_p(x,y)$ can change its slope only at $p = \frac{1}{d_x+1}$ and that $\kappa_1 = 0$, $\kappa_1' = -\kappa$. This formula, given in the following theorem, emerges via a straightforward calculation and applies, in particular, to all regular graphs.

THEOREM 5.3. *Let $G = (V,E)$ be a locally finite graph. Let $x, y \in V$ with $x \sim y$ and $d_y | d_x$. Then*

$$\kappa_p(x,y) = \begin{cases} (d_x \kappa(x,y) - (d_x+1)\kappa_0(x,y))p + \kappa_0(x,y) & \text{if } p \in \left[0, \dfrac{1}{d_x+1}\right], \\ (1-p)\kappa(x,y) & \text{if } p \in \left[\dfrac{1}{d_x+1}, 1\right]. \end{cases}$$

REMARK 5.4. *As mentioned earlier $\kappa_{\frac{1}{2}}, \kappa_{\frac{1}{d+1}}$, and κ have been studied in various articles. In fact, the identity*

$$\kappa_p(x,y) = (1-p)\kappa(x,y) \tag{5.2}$$

holds at all edges (even those whose Ollivier–Ricci idleness function has three linear parts) and for all values $p \in [\frac{1}{\max\{d_x,d_y\}+1}, 1]$. Equation (5.2) follows from Theorem 4.4 and the fact that $\kappa_1 = 0$ and $\kappa_1' = -\kappa$. As a consequence, we have

$$\kappa = 2\kappa_{\frac{1}{2}} = \frac{d+1}{d}\kappa_{\frac{1}{d+1}}.$$

We end this section with a connection between κ and κ_0.

THEOREM 5.5. *Let $G = (V,E)$ be a locally finite graph. Let $x, y \in V$ with $x \sim y$ and $d_x \geq d_y$. Then*

$$\kappa_0(x,y) \leq \kappa(x,y) \leq \kappa_0(x,y) + \frac{2}{d_x}.$$

Proof. The first inequality follows from the fact that the graph of a concave function lies below its tangent line at each point and that $\kappa_1 = 0$ and $\kappa = -\kappa_1'$:

$$\kappa_0 \leq \kappa_1 + \kappa_1'(0-1) = \kappa.$$

Now we prove the second inequality. Let ϕ be a 1-Lipschitz function with $\phi(y) = 0$ such that

$$W_{xy}(0) = \sum_{w\in V} \phi(w)(\mu_x^0(w) - \mu_y^0(w)) = \frac{-1}{d_y}\phi(x) + \frac{1}{d_x}\sum_{\substack{z\sim x\\ z\neq y}} \phi(z) - \frac{1}{d_y}\sum_{\substack{z\sim y\\ z\neq x}}\phi(z).$$

Then

$$\begin{aligned} W_{xy}\left(\frac{1}{d_x+1}\right) &\geq \sum_{w\in V}\phi(w)\left(\mu_x^{\frac{1}{d_x+1}}(w) - \mu_y^{\frac{1}{d_x+1}}(w)\right) \\ &= \left(\frac{1}{d_x+1} - \frac{d_x}{(d_x+1)d_y}\right)\phi(x) + \frac{1}{d_x+1}\sum_{\substack{z\sim x\\ z\neq y}}\phi(z) \\ &\quad - \frac{d_x}{(d_x+1)d_y}\sum_{\substack{z\sim y\\ z\neq x}}\phi(z). \end{aligned}$$

Thus

$$\begin{aligned} \frac{d_x+1}{d_x}W_{xy}\left(\frac{1}{d_x+1}\right) &\geq \left(\frac{1}{d_x} - \frac{1}{d_y}\right)\phi(x) + \frac{1}{d_x}\sum_{\substack{z\sim x\\ z\neq y}}\phi(z) - \frac{1}{d_y}\sum_{\substack{z\sim y\\ z\neq x}}\phi(z) \\ &= W_{xy}(0) + \frac{1}{d_x}\phi(x) \\ &= W_{xy}(0) + \frac{1}{d_x}(\phi(x) - \phi(y)) \\ &\geq W_{xy}(0) - \frac{1}{d_x}, \end{aligned}$$

since ϕ is 1-Lipschitz. Therefore

$$\begin{aligned}\kappa_{\frac{1}{d_x+1}}(x,y) &\le 1 + \frac{1}{d_x+1} - \frac{d_x}{d_x+1}W_{xy}(0)\\ &= \frac{2}{d_x+1} + \frac{d_x}{d_x+1}(1 - W_{xy}(0))\\ &= \frac{2}{d_x+1} + \frac{d_x}{d_x+1}\kappa_0(x,y).\end{aligned}$$

Finally, by (5.2),

$$\kappa(x,y) = \frac{d_x+1}{d_x}\kappa_{\frac{1}{d_x+1}}(x,y) \le \kappa_0(x,y) + \frac{2}{d_x}. \qquad \square$$

REMARK 5.6. *Let $G = (V,E)$ be a locally finite d-regular graph. Let $x, y \in V$ with $x \sim y$. Then, by the above theorem, we have*

$$\kappa_0(x,y) \le \kappa(x,y) \le \kappa_0(x,y) + \frac{2}{d}.$$

These inequalities are sharp. The lower bound on κ is achieved, for example, by the 6-cycle, and the upper bound on κ is achieved by the 4-cycle. Furthermore, by [8], $\kappa_0(x,y) \in \mathbb{Z}/d$. Similar arguments show that $\kappa(x,y) \in \mathbb{Z}/d$. Thus

$$\kappa(x,y) = \kappa_0(x,y) + \frac{C}{d},$$

where $C \in \{0,1,2\}$.

6. Application to the Cartesian product. In [5] the authors proved the following results on the curvature of Cartesian products of graphs.

THEOREM 6.1 (see [5]). *Let $G = (V_G, E_G)$ be a d_G-regular graph and $H = (V_H, E_H)$ a d_H-regular graph. Let $x_1, x_2 \in V_G$ with $x_1 \sim x_2$ and $y \in V_H$. Then*

$$\begin{aligned}\kappa^{G\times H}((x_1,y),(x_2,y)) &= \frac{d_G}{d_G+d_H}\kappa^G(x_1,x_2),\\ \kappa_0^{G\times H}((x_1,y),(x_2,y)) &= \frac{d_G}{d_G+d_H}\kappa_0^G(x_1,x_2).\end{aligned}$$

Using our formula from Theorem 5.3, we extend this result and derive relations between the full Ollivier–Ricci idleness functions involved in the Cartesian product.

COROLLARY 1.3. *Let $G = (V_G, E_G)$ be a d_G-regular graph and $H = (V_H, E_H)$ a d_H-regular graph. Let $x_1, x_2 \in V_G$ with $x_1 \sim x_2$ and $y \in V_H$. Then*

$$\begin{aligned}&\kappa_p^{G\times H}((x_1,y),(x_2,y))\\ &= \begin{cases} \frac{d_G}{d_G+d_H}\kappa_p^G(x_1,x_2) + \frac{d_Gd_H}{d_G+d_H}(\kappa^G(x_1,x_2) - \kappa_0^G(x_1,x_2))p & \text{if } p \in \left[0, \frac{1}{d_G+d_H+1}\right],\\ \frac{d_G}{d_G+d_H}\kappa^G(x_1,x_2)(1-p) & \text{if } p \in \left[\frac{1}{d_G+d_H+1}, 1\right].\end{cases}\end{aligned}$$

Proof. For ease of reading, we define $\kappa_p^{G\times H} := \kappa_p^{G\times H}((x_1,y),(x_2,y))$ and $\kappa_p^G := \kappa_p^G(x_1,x_2)$. Let $p \in [0, \frac{1}{d_G+d_H+1}]$. Then, by Theorem 5.3,

$$\begin{aligned}\kappa_p^{G\times H} &= ((d_G+d_H)\kappa^{G\times H} - (d_G+d_H+1)\kappa_0^{G\times H})p + \kappa_0^{G\times H}\\ &= \frac{d_G}{d_G+d_H}\left\{((d_G+d_H)\kappa^G - (d_G+d_H+1)\kappa_0^G)p + \kappa_0^G\right\}\\ &= \frac{d_G}{d_G+d_H}\left\{(d_G\kappa^G - (d_G+1)\kappa_0^G)p + \kappa_0^G\right\} + \frac{d_Gd_H}{d_G+d_H}(\kappa^G - \kappa_0^G)p\\ &= \frac{d_G}{d_G+d_H}\kappa_p^G + \frac{d_Gd_H}{d_G+d_H}(\kappa^G - \kappa_0^G)p.\end{aligned}$$

Now suppose $p \in [\frac{1}{d_G+d_H+1}, 1]$. Then

$$\kappa_p^{G\times H} = \kappa^{G\times H}(1-p) = \frac{d_G}{d_G+d_H}\kappa^G(1-p).$$ ∎

7. Bone idleness and some open questions. We finish this article with a discussion of when the Ollivier–Ricci idleness function $p \mapsto \kappa_p(x,y)$ is globally linear for all edges. First we introduce the notion of *bone idleness*.

DEFINITION 7.1. *Let $G = (V,E)$ be a locally finite graph. We say that an edge $x \sim y$ is* bone idle *if $\kappa_p(x,y) = 0$ for every $p \in [0,1]$. We say that G is* bone idle *if every edge is bone idle.*

REMARK 7.1. *Note that $\kappa_p(x,y) = 0$ for all $p \in [0,1]$ if and only if $\kappa_0(x,y) = \kappa(x,y) = 0$. This follows from the concavity of $\kappa_p(x,y)$.*

It is an interesting problem to classify the graphs that are bone idle. From the above remark this question is closely related to various notions of Ricci flatness. The following result allows us to classify bone idle graphs with girth at least 5. Recall that the girth of a graph is the length of its shortest nontrivial cycle.

THEOREM 7.2 (see [1]). *Let $G = (V,E)$ be a locally finite graph with girth at least 5. Suppose that $\kappa_0(x,y) = 0$ for all $x,y \in V$ with $x \sim y$. Then G is isomorphic to one of the following graphs:*
- (i) *the infinite ray;*
- (ii) *the cyclic graph C_n for $n \geq 5$;*
- (iii) *the path P_n for $n \geq 2$;*
- (iv) *the star graph S_n for $n \geq 3$.*

COROLLARY 7.3. *Let $G = (V,E)$ be a locally finite graph with girth at least 5. Suppose that G is bone idle. Then G is isomorphic to one of the following graphs:*
- (i) *the infinite path;*
- (ii) *the cyclic graph C_n for $n \geq 6$.*

Proof. By Remark 7.1, it suffices to check which of the examples given in Theorem 7.2 satisfy $\kappa = 0$, which is straightforward. ∎

REMARK 7.4. *Note that the above corollary shows that there exists no bone idle graph with girth equal to 5.*

The full classification of bone idle graphs is still open.

The condition of bone idleness of an edge $x \sim y$ can be weakened to require only that $p \mapsto \kappa_p(x,y)$ is globally linear on $[0,1]$. This is equivalent to $\kappa_0(x,y) = \kappa(x,y)$. It is natural to want to understand this weaker condition better.

Recall that $\kappa \geq \kappa_0$. The Petersen graph has $\kappa(x,y) = 0$ and $\kappa_0(x,y) < 0$ for all edges $x \sim y$. In an earlier draft of this paper we asked whether there exists a regular graph satisfying $\kappa(x,y) > 0$ and $\kappa_0(x,y) < 0$. In the meantime, Kangaslampi [4] has proved that such a graph cannot exist.

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