Ollivier-Ricci idleness functions of graphs

D. P. Bourne[1], D. Cushing[1], S. Liu[2], F. Münch[3], and N. Peyerimhoff[1]

[1]Department of Mathematical Sciences, Durham University
[2]School of Mathematical Sciences, University of Science and Technology of China
[3]Institute of Mathematics, Universität Potsdam

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Abstract

We study the Ollivier-Ricci curvature of graphs as a function of the chosen idleness. We show that this idleness function is concave and piecewise linear with at most 3 linear parts, with at most 2 linear parts in the case of a regular graph. We then apply our result to show that the idleness function of the Cartesian product of two regular graphs is completely determined by the idleness functions of the factors.

1 Introduction and statement of results

Ricci curvature plays a very important role in the study of Riemannian manifolds. In the discrete setting of graphs, there is very active recent research on various types of Ricci curvature notions and their applications.

In [10] Ollivier developed a notion of Ricci curvature of Markov chains valid on metric spaces including graphs. In this notion an *idleness* parameter, $p \in [0, 1]$, must be set in order to obtain a curvature κ_p. Ollivier considered idleness 0 and $\frac{1}{2}$. Idleness $\frac{1}{2}$ has also been considered by [13]. For graphs, Ollivier's notion for idleness 0 has been studied further in [1, 2, 3, 6, 12]. In [11] Ollivier and Villani considered idleness $\frac{1}{d+1}$, where d is the degree of a regular graph, in order to investigate the curvature of the hypercube. In [5], Lin, Lu, and Yau introduced a modified version of Ollivier-Ricci curvature, which they defined to be the negative of the derivative of κ_p at $p = 1$; see equation (1.1).

We will show that for a regular graph the following holds:

$$\kappa = 2\kappa_{\frac{1}{2}} = \frac{d+1}{d}\kappa_{\frac{1}{d+1}}.$$

Therefore some of these different curvature notions are related to each other by scaling factors.

In [1], Bhattacharya and Mukherjee derive exact expressions of Ollivier-Ricci curvature for bipartite graphs in the special case of idleness $p = 0$ and for graphs of girth at least 5. They use this result to classify all graphs with $\kappa_0 = 0$ for all edges (called 'Ricci flat' in their paper) and girth at least 5. There is a small overlap between some of our methods and theirs in this paper (for example they discuss the existence of integer-valued optimal Kantorovich potentials in the special case of vanishing idleness ($p = 0$)).

To our knowledge, the *global* piecewise linear structure of the function $p \mapsto \kappa_p$ has not yet been established, and the only concrete examples in the literature where the full idleness function is computed are the hypercube and the complete graphs; see [5]. However, some properties of this function have been discussed. In [5], it was shown that the idleness function is concave. It was shown in [8] that κ_p is linear close to idleness $p = 1$, and in [14] it was shown that κ_p is linear if a certain condition is satisfied (see the introductory part of Section 5).

Throughout this article, let $G = (V, E)$ be a locally finite graph with vertex set V, edge set E, and which contains no multiple edges or self loops. Let d_x denote the degree of the vertex $x \in V$ and $d(x, y)$ denote the length of the shortest path between two vertices x and y, that is, the combinatorial distance. We denote the existence of an edge between x and y by $x \sim y$.

We define the following probability measures μ_x for any $x \in V$, $p \in [0, 1]$:

$$\mu_x^p(z) = \begin{cases} p, & \text{if } z = x, \\ \frac{1-p}{d_x}, & \text{if } z \sim x, \\ 0, & \text{otherwise.} \end{cases}$$

Let W_1 denote the 1-Wasserstein distance between two probability measures on V, see [15] page 211. The $p-$Ollivier-Ricci curvature of an edge $x \sim y$ in $G = (V, E)$ is

$$\kappa_p(x, y) = 1 - W_1(\mu_x^p, \mu_y^p).$$

Ollivier-Ricci curvature has also be defined for arbitrary pairs of vertices (not necessarily adjacent vertices) [10] and it is known that

$$\inf_{\substack{x,y \in V \\ x \neq y}} \kappa_p(x, y) \geq \inf_{\substack{x,y \in V \\ x \sim y}} \kappa_p(x, y).$$

In this paper we only consider curvature on edges since edges are the discrete analogues of tangent vectors, i.e., of infinitesimal directions.

Y. Lin, L. Lu, and S.T. Yau introduced in [5] the following Ollivier-Ricci curvature:

$$\kappa(x, y) = \lim_{p \to 1} \frac{\kappa_p(x, y)}{1 - p}.$$

Note that their curvature notion does not have an idleness index, which distinguishes their notion from the idleness function $p \mapsto \kappa_p(x, y)$ in this paper, which we call the *Ollivier-Ricci idleness function*. We will show that

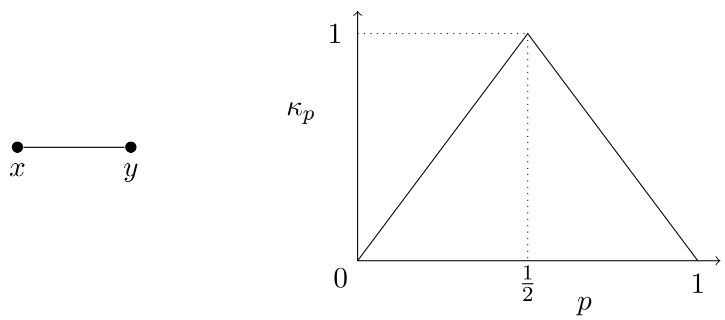
$$\kappa_p(x, y) = (1 - p)\kappa(x, y)$$

for all $p \in \left[\frac{1}{\max\{d_x, d_y\}+1}, 1\right]$ and that $\kappa_0(x, y) \leq \kappa(x, y) \leq \kappa_0(x, y) + \frac{2}{\max\{d_x, d_y\}}$. Observe that

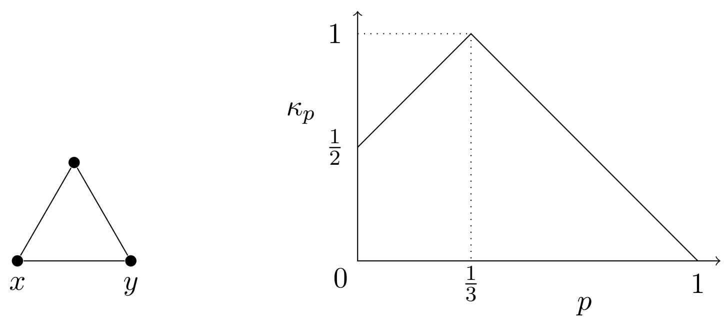
$$\kappa(x, y) = - \frac{\partial}{\partial p}\bigg|_{p=1} \kappa_p(x, y). \tag{1.1}$$

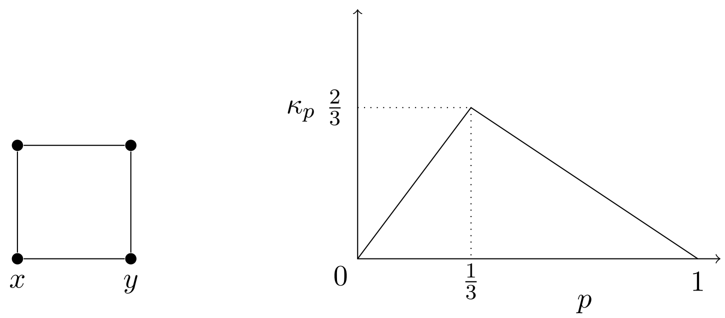
Next, we give some examples of graphs and their Ollivier-Ricci idleness function at a particular edge $x \sim y$.

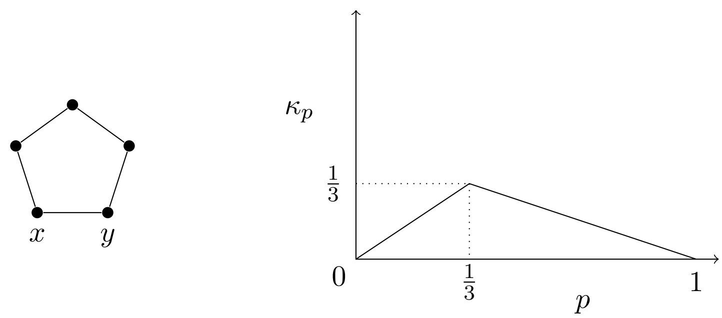
Examples: Below is the one-path and a plot of the corresponding idleness function:



We now present the idleness function for $3-, 4-$ and $5-$ cycles:

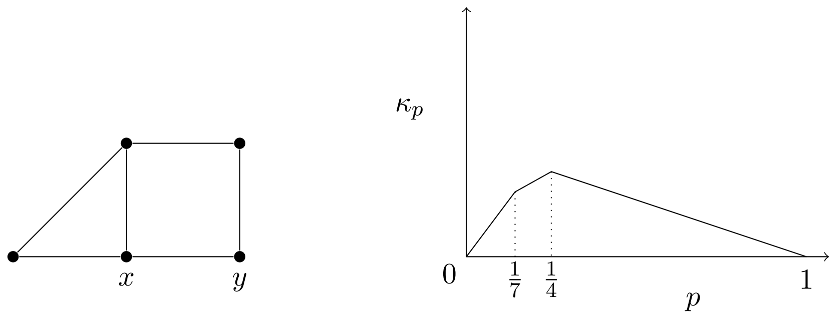






For cycles of length 6 or greater the idleness function at every edge vanishes identically (we call those edges *bone idle*; see Section 7).

So far we have only seen idleness functions with at most 2 linear parts. We will show that if $d_x = d_y$, then this is always the case. However, if $d_x \neq d_y$, then 3 linear parts may occur, as shown in the following example:



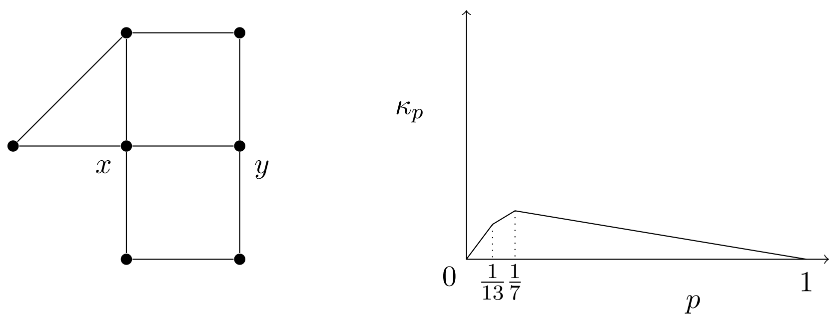
In fact the Ollivier-Ricci idleness function is piecewise linear with *at most* 3 parts always, a fundamental fact which is included in the following theorem (our main result):

Theorem 1.1. *Let $G = (V, E)$ be a locally finite graph. Let $x, y \in V$ with $x \sim y$. Then the function $p \mapsto \kappa_p(x, y)$ is concave and piecewise linear over $[0, 1]$ with at most 3 linear parts. Furthermore $\kappa_p(x, y)$ is linear on the intervals*

$$\left[0, \frac{1}{\mathrm{lcm}(d_x, d_y) + 1}\right] \quad \text{and} \quad \left[\frac{1}{\max(d_x, d_y) + 1}, 1\right].$$

Thus, if we have the further condition $d_x = d_y$, then $\kappa_p(x, y)$ has at most two linear parts.

In our above example of 3 linear parts the changes in slope occurs at $\frac{1}{\mathrm{lcm}(d_x,d_y)+1}$ and $\frac{1}{\max(d_x,d_y)+1}$. However this need not always be the case. Consider the following example:



Here the first change in gradient did occur at $\frac{1}{\mathrm{lcm}(d_x,d_y)+1} = \frac{1}{13}$, but the second change in gradient occurs before $\frac{1}{\max(d_x,d_y)+1} = \frac{1}{5}$.

Remark 1.2. Since $\kappa_p(x, y) = 1 - W_1(\mu_x^p, \mu_y^p)$ and $W_1(\mu_x^p, \mu_y^p)$ is the supremum of affine functions of p (by the Kantorovich Duality Theorem), then $p \mapsto W_1(\mu_x^p, \mu_y^p)$ is convex and so $p \mapsto \kappa_p(x, y)$ is concave. An alternative proof of concavity was given in [5].

A consequence of Theorem 1.1 and the results in [5] is the following Corollary.

Corollary 1.3. *Let $G = (V_G, E_G)$ be a d_G-regular graph and $H = (V_H, E_H)$ be a d_H-regular graph. Let $x_1, x_2 \in V_G$ with $x_1 \sim x_2$ and $y \in V_H$. Then*

$$\kappa_p^{G\times H}((x_1,y),(x_2,y))$$
$$= \begin{cases} \frac{d_G}{d_G+d_H}\kappa_p^G(x_1,x_2) + \frac{d_G d_H}{d_G+d_H}(\kappa^G(x_1,x_2) - \kappa_0^G(x_1,x_2))p, & \text{if } p \in [0, \frac{1}{d_G+d_H+1}], \\ \frac{d_G}{d_G+d_H}\kappa^G(x_1,x_2)(1-p), & \text{if } p \in [\frac{1}{d_G+d_H+1}, 1]. \end{cases}$$

This result shows that the idleness function of the Cartesian product of two regular graphs is completely determined by the idleness functions of the factors.

We finish this introduction with an outline of the rest of this paper. In Section 2 we present the relevant notation and background material. In Section 3 we show that $p \mapsto \kappa_p$ is piecewise linear with at most 3 linear parts. In Sections 4 and 5 we give bounds on the size of the first and last linear part. We prove Corollary 1.3 in Section 6. Finally, in Section 7, we present some open questions. Moreover we discuss the problem of characterising edges with globally linear curvature functions.

2 Definitions and notation

We now introduce the relevant definitions and notation we will need in this paper. First, we recall the Wasserstein distance and the Ollivier-Ricci curvature.

Definition 2.1. Let $G = (V, E)$ be a locally finite graph. Let μ_1, μ_2 be two probability measures on V. The *Wasserstein distance* $W_1(\mu_1, \mu_2)$ between μ_1 and μ_2 is defined as

$$W_1(\mu_1,\mu_2) = \inf_{\pi \in \Pi(\mu_1,\mu_2)} \sum_{y\in V}\sum_{x\in V} d(x,y)\pi(x,y), \tag{2.1}$$

where

$$\Pi(\mu_1,\mu_2) = \left\{ \pi : V \times V \to [0,1] : \mu_1(x) = \sum_{y\in V}\pi(x,y),\ \mu_2(y) = \sum_{x\in V}\pi(x,y) \right\}.$$

The transportation plan π moves a mass distribution given by μ_1 into a mass distribution given by μ_2, and $W_1(\mu_1, \mu_2)$ is a measure for the minimal effort which is required for such a transition. If π attains the infimum in (2.1) we call it an *optimal transport plan* transporting μ_1 to μ_2.

Definition 2.2. The $p-$Ollivier-Ricci curvature of an edge $x \sim y$ in $G = (V, E)$ is

$$\kappa_p(x,y) = 1 - W_1(\mu_x^p, \mu_y^p),$$

where p is called the *idleness.*

A fundamental concept in optimal transport theory and vital to our work is Kantorovich duality. First we recall the notion of 1–Lipschitz functions and then state the Kantorovich Duality Theorem.

Definition 2.3. Let $G = (V, E)$ be a locally finite graph, $\phi : V \to \mathbb{R}$. We say that ϕ is 1-Lipschitz if

$$|\phi(x) - \phi(y)| \leq d(x, y)$$

for all $x, y \in V$. Let 1–Lip denote the set of all 1–Lipschitz functions on V.

Theorem 2.1 (Kantorovich duality [15])**.** *Let $G = (V, E)$ be a locally finite graph. Let μ_1, μ_2 be two probability measures on V. Then*

$$W_1(\mu_1, \mu_2) = \sup_{\substack{\phi: V \to \mathbb{R} \\ \phi \in 1\text{–Lip}}} \sum_{x \in V} \phi(x)(\mu_1(x) - \mu_2(x)).$$

If $\phi \in 1$–Lip attains the supremum we call it an optimal Kantorovich potential *transporting μ_1 to μ_2.*

3 Properties of the idleness function

In this section we prove that the Ollivier-Ricci idleness function has at most 3 linear parts. Two ingredients of this proof are the 'integer-valuedness' of optimal Kantorovich potentials and the Complementary Slackness Theorem, which we state and prove now.

Lemma 3.1. *Let $G = (V, E)$ be a locally finite graph. Let $x, y \in V$ with $x \sim y$. Let $p \in [0, 1]$. Let π and ϕ be an optimal transport plan and an optimal Kantorovich potential transporting μ_x^p to μ_y^p, respectively. Let $u, v \in V$ with $\pi(u, v) \neq 0$. Then*

$$\phi(u) - \phi(v) = d(u, v).$$

This follows from the Complementary Slackness Theorem (see, for example, [9, page 49]) or from standard results in optimal transport theory [15, page 88]. For the sake of completeness we include a short proof here.

Proof. By the definitions of ϕ and π we have

$$W_1(\mu_x^p, \mu_y^p) = \sum_{w \in V} \phi(w)(\mu_x^p - \mu_y^p)(w) = \sum_{w \in V} \sum_{z \in V} d(w, z)\pi(w, z),$$

and

$$\sum_{w \in V} \pi(w, z) = \mu_y^p(z), \quad \sum_{z \in V} \pi(w, z) = \mu_x^p(w).$$

Then

$$\begin{aligned} W_1(\mu_x^p, \mu_y^p) &= \sum_{w \in V} \phi(w)\mu_x^p(w) - \sum_{z \in V} \phi(z)\mu_y^p(z) \\ &= \sum_{w \in V} \phi(w) \sum_{z \in V} \pi(w, z) - \sum_{z \in V} \phi(z) \sum_{w \in V} \pi(w, z) \\ &= \sum_{w \in V} \sum_{z \in V} (\phi(w) - \phi(z))\pi(w, z) \\ &\leq \sum_{w \in V} \sum_{z \in V} d(w, z)\pi(w, z) \\ &= W_1(\mu_x^p, \mu_y^p). \end{aligned}$$

Thus

$$\sum_{w\in V}\sum_{z\in V}(\phi(w)-\phi(z))\pi(w,z)=\sum_{w\in V}\sum_{z\in V}d(w,z)\pi(w,z).$$

Therefore

$$\phi(w)-\phi(z)<d(w,z) \implies \pi(w,z)=0,$$

thus completing the proof. □

As mentioned in the introduction, in [1] the authors discuss the existence of integer-valued optimal Kantorovich potentials in the special case of vanishing idleness ($p=0$). We first introduce the floor and ceiling of functions and then state a corresponding result for the case of arbitrary idleness.

Definition 3.1. Let $G=(V,E)$ be a locally finite graph and let $\phi : V\to\mathbb{R}$. Define the functions $\lfloor\phi\rfloor$ and $\lceil\phi\rceil$ as follows:

$$\begin{aligned}\lfloor\phi\rfloor : V &\to \mathbb{R}\\ v &\mapsto \lfloor\phi(v)\rfloor\,,\\ \lceil\phi\rceil : V &\to \mathbb{R}\\ v &\mapsto \lceil\phi(v)\rceil\,.\end{aligned}$$

Lemma 3.2. *Let $G=(V,E)$ be a locally finite graph. Let $\phi\in 1\text{–Lip}$. Then $\lfloor\phi\rfloor, \lceil\phi\rceil\in$ 1–Lip.*

Proof. For each $v\in V$ set $\delta_v=\phi(v)-\lfloor\phi(v)\rfloor$. Note that $\delta_v\in[0,1)$. Then

$$|\lfloor\phi(v)\rfloor-\lfloor\phi(w)\rfloor|=|\phi(v)-\delta_v-\phi(w)+\delta_w|\le d(v,w)+|\delta_v-\delta_w|.$$

Since $\delta_v-\delta_w\in(-1,1)$ we have $|\lfloor\phi(v)\rfloor-\lfloor\phi(w)\rfloor|<d(v,w)+1$ and so $|\lfloor\phi(v)\rfloor-\lfloor\phi(w)\rfloor|\le d(v,w)$ since $|\lfloor\phi(v)\rfloor-\lfloor\phi(w)\rfloor|$ is integer valued. Thus $\lfloor\phi\rfloor\in$ 1–Lip. The proof that $\lceil\phi\rceil\in$ 1–Lip follows similarly. □

Lemma 3.3 (Integer-Valuedness)**.** *Let $G=(V,E)$ be a locally finite graph. Let $x,y\in V$ with $x\sim y$. Let $p\in[0,1]$. Then there exists $\phi\in$ 1–Lip such that*

$$W_1(\mu_x^p,\mu_y^p)=\sum_{w\in V}\phi(w)(\mu_x^p(w)-\mu_y^p(w)),$$

and $\phi(w)\in\mathbb{Z}$ for all $w\in V$.

Proof. Let Φ be an optimal Kantorovich potential transporting μ_x^p to μ_y^p. Let π be an optimal transport plan transporting μ_x^p to μ_y^p. Consider the following graph H with vertices V and edges given by the following adjacency matrix A:

$$\begin{aligned}A(v,w)&=1 \text{ if } \pi(v,w)>0 \text{ or } \pi(w,v)>0,\\ A(v,w)&=0 \text{ otherwise.}\end{aligned}$$

Let $(W_i)_{i=1}^n$ denote the connected components of H. Fix $u,v\in W_i$ for some $i\in\{1,\dots,n\}$. By Lemma 3.1 we have $|\Phi(u)-\Phi(v)|=d(u,v)$.

Define $\phi : V \to \mathbb{R}$ as follows

$$\phi(v) = \sup\{\psi(v) : \psi : V \to \mathbb{Z},\ \ \psi \in 1\text{–Lip},\ \psi \le \Phi\}.$$

By definition, ϕ is an integer-valued 1–Lipschitz function and $\phi \le \Phi$. Note that $\phi = \lfloor \Phi \rfloor$ since $\lfloor \Phi \rfloor \in 1$–Lip by Lemma 3.2.

Finally we must show that ϕ is optimal. For each $v \in V$ set $\delta_v = \Phi(v) - \lfloor \Phi(v) \rfloor = \Phi(v) - \phi(v)$. Note that $\mu_x^p(W_i) = \mu_y^p(W_i)$ for all i (since no mass is transported between different connected components W_i), and that $\delta_u = \delta_v$ if u, v belong to the same component W_i, for some i. Set $\delta_i = \delta_u$ for any $u \in W_i$. Then

$$\begin{aligned}
\sum_{w\in V} \phi(w)(\mu_x^p(w) - \mu_y^p(w)) &= \sum_{w\in V} (\Phi(w) - \delta_w)(\mu_x^p(w) - \mu_y^p(w)) \\
&= \sum_{i=1}^{n} \sum_{w\in W_i} (\Phi(w) - \delta_w)(\mu_x^p(w) - \mu_y^p(w)) \\
&= \sum_{i=1}^{n} \sum_{w\in W_i} \Phi(w)(\mu_x^p(w) - \mu_y^p(w)) \\
&\quad - \sum_{i=1}^{n} \sum_{w\in W_i} \delta_w(\mu_x^p(w) - \mu_y^p(w)) \\
&= \sum_{w\in V} \Phi(w)(\mu_x^p(w) - \mu_y^p(w)) - \sum_{i=1}^{n} \delta_i \underbrace{(\mu_x^p(W_i) - \mu_y^p(W_i))}_{=0} \\
&= W_1(\mu_x^p, \mu_y^p).
\end{aligned}$$

Therefore ϕ is optimal, as required. □

Now we formulate our main result of this section.

Theorem 3.4. *Let $G = (V, E)$ be a locally finite graph. Let $x, y \in V$ with $x \sim y$. Then $p \mapsto \kappa_p(x, y)$ is piecewise linear over $[0, 1]$ with at most 3 linear parts.*

Proof. For $\phi : V \to \mathbb{R}$, let

$$F(\phi) = d_y \left(\sum_{\substack{z\sim x \\ z\neq y}} \phi(z) \right) - d_x \left(\sum_{\substack{z\sim y \\ z\neq x}} \phi(z) \right). \tag{3.1}$$

For $j \in \{-1, 0, 1\}$, define

$$\mathcal{A}_j = \{\phi : V \to \mathbb{Z} : \phi(x) = j,\ \phi(y) = 0,\ \phi \in 1\text{–Lip}\} \tag{3.2}$$

and define the constants

$$c_j = \sup_{\phi\in\mathcal{A}_j} F(\phi).$$

Finally we define the linear maps

$$f_j(p) = \left(p - \frac{1-p}{d_y} \right) j + \frac{1-p}{d_x d_y} c_j.$$

Then

$$
\begin{aligned}
W_1(\mu_x^p,\mu_y^p) &= \sup_{\phi\in 1\text{–Lip}} \sum_{w\in V} \phi(w)(\mu_x^p(w)-\mu_y^p(w)) \\
&= \sup_{\substack{\phi\in 1\text{–Lip}\\ \phi:V\to\mathbb{Z}\\ \phi(y)=0}} \sum_{w\in V} \phi(w)(\mu_x^p(w)-\mu_y^p(w)) \\
&= \sup_{\substack{\phi\in 1\text{–Lip}\\ \phi:V\to\mathbb{Z}\\ \phi(y)=0}} \left\{ \phi(x)\left(p-\frac{1-p}{d_y}\right) + \frac{1-p}{d_x}\sum_{\substack{w\sim x\\ w\neq y}}\phi(w) - \frac{1-p}{d_y}\sum_{\substack{w\sim y\\ w\neq x}}\phi(w) \right\} \\
&= \sup_{\substack{\phi\in 1\text{–Lip}\\ \phi:V\to\mathbb{Z}\\ \phi(y)=0}} \left\{ \phi(x)\left(p-\frac{1-p}{d_y}\right) + \frac{1-p}{d_xd_y}\left(d_y\sum_{\substack{w\sim x\\ w\neq y}}\phi(w) - d_x\sum_{\substack{w\sim y\\ w\neq x}}\phi(w)\right) \right\} \\
&= \sup_{\substack{\phi\in 1\text{–Lip}\\ \phi:V\to\mathbb{Z}\\ \phi(y)=0}} \left\{ \phi(x)\left(p-\frac{1-p}{d_y}\right) + \frac{1-p}{d_xd_y}F(\phi) \right\} \\
&= \max_{j\in\{-1,0,1\}} \sup_{\phi\in\mathcal{A}_j} \left\{ j\left(p-\frac{1-p}{d_y}\right) + \frac{1-p}{d_xd_y}F(\phi) \right\} \\
&= \max_{j\in\{-1,0,1\}} \left\{ j\left(p-\frac{1-p}{d_y}\right) + \frac{1-p}{d_xd_y}\sup_{\phi\in\mathcal{A}_j}F(\phi) \right\} \\
&= \max_{j\in\{-1,0,1\}} \left\{ j\left(p-\frac{1-p}{d_y}\right) + \frac{1-p}{d_xd_y}c_j \right\} \\
&= \max\{f_{-1}(p), f_0(p), f_1(p)\}. \qquad (3.3)
\end{aligned}
$$

Therefore

$$\kappa_p(x,y) = 1 - \max\{f_{-1}(p), f_0(p), f_1(p)\}.$$

Since $\max\{f_{-1}(p), f_0(p), f_1(p)\}$ is the maximum of three linear functions of p, it is convex and piecewise linear in p with at most 3 linear parts, thus completing the proof. □

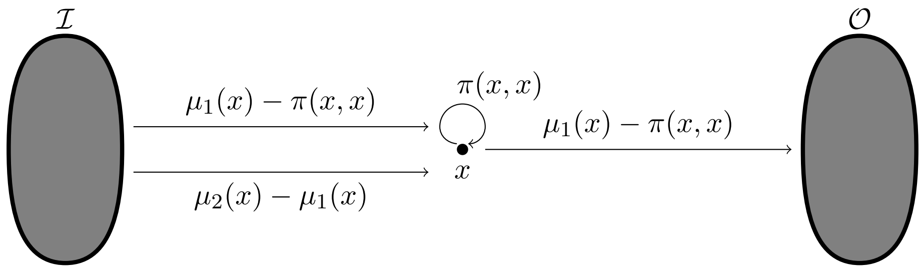
4 Length of the last linear part

Before discussing the size of the last linear part we first need the following lemma about some of the assumptions we can impose on an optimal transport plan. We then show that, if different idlenesses $p_1 < p_2$ share a joint optimal Kantorovich potential, then the Ollivier-Ricci idleness function is linear on the whole interval $[p_1, p_2]$. This was already mentioned in [14] for the special case $p_2 = 1$.

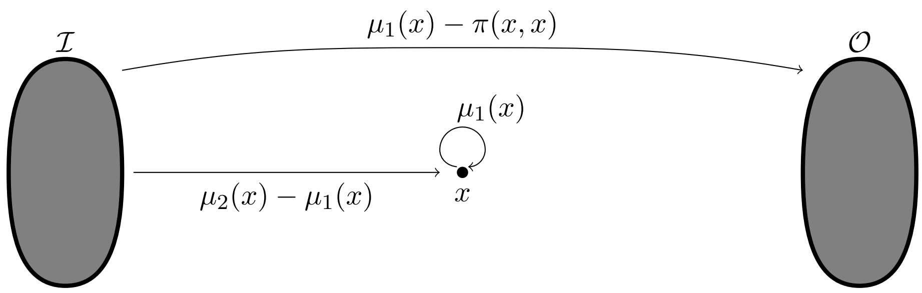
Lemma 4.1. *Let μ_1 and μ_2 be probability measures on V. Then there exists an optimal transport plan π transporting μ_1 to μ_2 with the following property: For all $x \in V$ with $\mu_1(x) \leq \mu_2(x)$ we have $\pi(x,x) = \mu_1(x)$.*

This Lemma could be proved using Corollary 1.16 in [15] (Invariance of Kantorovich-Rubinstein distance under mass subtraction) but we present a proof in our much simpler context, for the reader's convenience.

Proof. Let π be an optimal transport plan transporting μ_1 to μ_2. Assume there exists an $x \in V$ with $\mu_1(x) \leq \mu_2(x)$, but $\pi(x,x) < \mu_1(x)$. Let $\mathcal{I} = \{z \in V \setminus \{x\} : \pi(z,x) > 0\}$ and $\mathcal{O} = \{w \in V \setminus \{x\} : \pi(x,w) > 0\}$. Since π is optimal, we must have $\mathcal{I} \cap \mathcal{O} = \emptyset$. Then the relavent part of π can be depicted as



We now modify π to obtain a new transport plan π' as follows:



This new transport plan π' is still optimal (by the triangle inequality). Note that

$$\pi'(z,z) = \begin{cases} \pi(z,z) & \text{if } z \neq x, \\ \mu_1(x) & \text{if } z = x. \end{cases}$$

Repeating this modification at all other vertices that violate the condition of the lemma successively gives us our required optimal transport plan. □

Lemma 4.2. *Let $G = (V,E)$ be a locally finite graph. Let $x, y \in V$ with $x \sim y$. Let $0 \leq p_1 \leq p_2 \leq 1$. If there exists a 1–Lipschitz function ϕ which is an optimal Kantorovich potential transporting $\mu_x^{p_1}$ to $\mu_y^{p_1}$ and transporting $\mu_x^{p_2}$ to $\mu_y^{p_2}$, then $W_{xy} : [0,1] \to \mathbb{R}$, $W_{xy}(p) = W_1(\mu_x^p, \mu_y^p)$, is linear on $[p_1, p_2]$.*

Proof. Let $\alpha \in [0,1]$. The convexity of W_{xy}, see Remark 1.2, implies that

$$\alpha W_{xy}(p_1) + (1-\alpha) W_{xy}(p_2) \geq W_{xy}(\alpha p_1 + (1-\alpha) p_2).$$

It only remains to show the above inequality is in fact an equality. Observe that

$$\mu_x^{\alpha p_1 + (1-\alpha) p_2} = \alpha \mu_x^{p_1} + (1-\alpha) \mu_x^{p_2},$$
$$\mu_y^{\alpha p_1 + (1-\alpha) p_2} = \alpha \mu_y^{p_1} + (1-\alpha) \mu_y^{p_2}.$$

Then, setting $p = \alpha p_1 + (1-\alpha)p_2$, we have

$$\begin{aligned} W_{xy}(p) &\geq \sum_{w \in V} \phi(w)(\mu_x^{\alpha p_1 + (1-\alpha)p_2}(w) - \mu_y^{\alpha p_1 + (1-\alpha)p_2}(w)) \\ &= \sum_{w \in V} \phi(w) \left(\alpha \mu_x^{p_1}(w) + (1-\alpha)\mu_x^{p_2}(w) - \alpha \mu_y^{p_1}(w) - (1-\alpha)\mu_y^{p_2}(w) \right) \\ &= \alpha \sum_{w \in V} \phi(w) \left(\mu_x^{p_1}(w) - \mu_y^{p_1}(w) \right) + (1-\alpha) \sum_{w \in V} \phi(w) \left(\mu_x^{p_2}(w) - \mu_y^{p_2}(w) \right) \\ &= \alpha W_{xy}(p_1) + (1-\alpha) W_{xy}(p_2). \end{aligned}$$

□

Lemma 4.3. *Let $G = (V, E)$ be a locally finite graph. Let $x, y \in V$ with $x \sim y$ and $d_x \geq d_y$. Let $p \in \left(\frac{1}{1+d_x}, 1 \right]$. Let ϕ be an optimal Kantorovich potential transporting μ_x^p to μ_y^p. Then*

$$\phi(x) - \phi(y) = 1.$$

Proof. Let π be an optimal transport plan transporting μ_x^p to μ_y^p. We may assume that π satisfies the conditions of Lemma 4.1. Since $p > \frac{1}{1+d_x}$, then $\mu_x^p(y) = \frac{1-p}{d_x} < \frac{d_x p}{d_x} = p = \mu_y^p(y)$, therefore there exists $z \in B_1(x) \setminus \{y\}$ such that $\pi(z, y) > 0$. If $z = x$ then $\phi(x) - \phi(y) = 1$, by Lemma 3.1. Suppose $z \sim y$ and $z \neq x$. Then observe that $\mu_x^p(z) = \frac{1-p}{d_x} \leq \frac{1-p}{d_y} = \mu_y^p(z)$. Thus $\pi(z, y) = 0$, by Lemma 4.1, which contradicts our assumption that $\pi(z, y) > 0$.

The only case left to consider is $z \sim x$, $z \nsim y$, $z \neq y$. Then $d(z, y) = 2$, in which case we have $\phi(z) - \phi(y) = 2$, by Lemma 3.1. Then

$$\begin{aligned} 2 &= \phi(z) - \phi(y) \\ &= \phi(z) - \phi(x) + \phi(x) - \phi(y) \\ &\leq 1 + \phi(x) - \phi(y) \\ &\leq 2, \end{aligned}$$

which implies $\phi(x) - \phi(y) = 1$. □

We are now ready to prove the main theorem of this section.

Theorem 4.4. *Let $G = (V, E)$ be a locally finite graph and let $x, y \in V$ with $x \sim y$ and $d_x \geq d_y$. Then $p \mapsto \kappa_p(x, y)$ is linear over $\left[\frac{1}{d_x+1}, 1 \right]$.*

Proof. Let $1 > p_0 > \frac{1}{d_x+1}$ and ϕ be an optimal Kantorovich potential transporting $\mu_x^{p_0}$ to $\mu_y^{p_0}$. Then, by Lemma 4.3, we have $\phi(x) - \phi(y) = 1$. Note that any 1–Lipschitz ψ satisfying $\psi(x) - \psi(y) = 1$ is an optimal Kantorovich potential transporting μ_x^1 to μ_y^1. Thus, by Lemma 4.2, $p \mapsto \kappa_p(x, y)$ is linear over $[p_0, 1]$. By continuity of $p \mapsto \kappa_p(x, y)$, this linearity extends to $\left[\frac{1}{d_x+1}, 1 \right]$. □

Remark 4.5. Note that the above proof shows the existence a 1–Lipschitz function ϕ with $\phi(x)-\phi(y)=1$, which is an optimal Kantorovich potential for all $p \in \left[\frac{1}{d_x+1},1\right]$: We choose ϕ to be an optimal Kantorovich potential transporting $\mu_x^{p_0}$ to $\mu_y^{p_0}$ for some $1>p_0>\frac{1}{d_x+1}$ and satisfying $\phi(x)-\phi(y)=1$, as in the proof of Theorem 4.4. Then both W_{xy} and the function

$$p \mapsto \sum_{w\in V} \phi(w)(\mu_x^p(w)-\mu_y^p(w))$$

are linear over $[\frac{1}{d_x+1},1]$ and agree at $p=p_0$ and $p=1$. Therefore, they agree on the whole interval and, consequently, ϕ is an optimal Kantorovich potential for all $p \in \left[\frac{1}{d_x+1},1\right]$.

5 Length of the first linear part

Lemma 5.1. *Let $G=(V,E)$ be a locally finite graph. Let F be as defined in equation (3.1). Then*

$$\sup_{\substack{\phi\in 1\text{–Lip}\\ \phi:V\to\mathbb{Z}\\ \phi(x)=\phi(y)=0}} F(\phi) = \sup_{\substack{\phi\in 1\text{–Lip}\\ \phi:V\to\mathbb{Z}/2\\ \phi(x)=\phi(y)=0}} F(\phi),$$

where $\mathbb{Z}/2=\{n/2 : n\in\mathbb{Z}\}$.

Proof. Pick $\phi_0 \in 1$–Lip such that $\phi_0 : V\to\mathbb{Z}/2$, $\phi_0(x)=\phi_0(y)=0$ and

$$F(\phi_0) = \sup_{\substack{\phi\in 1\text{–Lip}\\ \phi:V\to\mathbb{Z}/2\\ \phi(x)=\phi(y)=0}} F(\phi).$$

Note that

$$\phi_0(v) = \frac{\lfloor\phi_0(v)\rfloor + \lceil\phi_0(v)\rceil}{2},$$

for all $v\in V$. Thus

$$F(\phi_0) = \frac{F(\lfloor\phi_0\rfloor)+F(\lceil\phi_0\rceil)}{2}.$$

By combining this with $F(\phi_0)\geq F(\lfloor\phi_0\rfloor)$ and $F(\phi_0)\geq F(\lceil\phi_0\rceil)$ we obtain

$$F(\phi_0)=F(\lfloor\phi_0\rfloor)=F(\lceil\phi_0\rceil).$$

Since $\lfloor\phi_0\rfloor : V\to\mathbb{Z}$ this completes the proof. □

The rest of this section is devoted to the proof of the following result.

Theorem 5.2. *Let $G=(V,E)$ be a locally finite graph. Let $x,y\in V$ with $x\sim y$ and $d_x\geq d_y$. Let $\ell=\operatorname{lcm}(d_x,d_y)$. Then $p\mapsto\kappa_p(x,y)$ is linear over $\left[0,\frac{1}{\ell+1}\right]$.*

Proof. Let $F,\mathcal{A}_j,c_j,f_j$ be as defined in the proof of Theorem 3.4. In order to bound the length of the first linear part of κ_p, we look at the intersection points of the functions f_j. First we derive inequalities between the constants c_j. Note that

$$f_j\left(\frac{1}{d_y+1}\right) = \frac{1}{(d_y+1)d_x}c_j,$$

for $j \in \{-1, 0, 1\}$. We claim that $f_1\left(\frac{1}{d_y+1}\right) \geq f_j\left(\frac{1}{d_y+1}\right)$. It then follows that $c_1 \geq c_0$ and $c_1 \geq c_{-1}$. We now prove the claim:

Note that $\frac{1}{d_y+1} \in \left[\frac{1}{d_x+1}, 1\right]$ and that, by Remark 4.5, there exists an optimal Kantorovich potential ϕ at idleness $\frac{1}{d_y+1}$ with $\phi(x) - \phi(y) = 1$. Therefore, by equation (3.3),

$$f_1\left(\frac{1}{d_y+1}\right) = W_{xy}\left(\frac{1}{d_y+1}\right) = \max\left\{f_{-1}\left(\frac{1}{d_y+1}\right), f_0\left(\frac{1}{d_y+1}\right), f_1\left(\frac{1}{d_y+1}\right)\right\},$$

which proves the claim.

Let $\phi_j \in \mathcal{A}_j$ satisfy $F(\phi_j) = c_j = \max_{\mathcal{A}_j} F$. Let $\psi = \frac{\phi_{-1}+\phi_1}{2}$. Note that ψ is 1–Lipschitz and $\psi(x) = \psi(y) = 0$. The function ψ may fail to be integer-valued but we note that $\psi : V \to \mathbb{Z}/2$ and so, by Lemma 5.1, we have

$$c_0 \geq F\left(\frac{\phi_{-1}+\phi_1}{2}\right) = \frac{c_{-1}+c_1}{2} \geq c_{-1}. \tag{5.1}$$

Therefore

$$c_1 \geq c_0 \geq c_{-1}.$$

Let $g = \gcd(d_x, d_y)$. Since the constants c_j are integer linear combinations of d_x and d_y, we have $g|c_j$ for $j \in \{-1, 0, 1\}$. For the computation of the possible intersection points of f_j, we will make use of the following simple observation. Let $b > 0$ and suppose that $0 \leq \frac{a}{a+b} \leq 1$. Then $a > 0$.

Suppose that p' satisfies $f_{-1}(p') = f_0(p')$. Then

$$p' = \frac{d_x - (c_0 - c_{-1})}{d_x d_y + d_x - (c_0 - c_{-1})}.$$

We can write $c_0 - c_{-1} = d_x - Kg$ for some $K \in \mathbb{Z}$. Then

$$p' = \frac{Kg}{d_x d_y + Kg} = \frac{K}{\ell + K},$$

with $\ell = \operatorname{lcm}(d_x, d_y)$. Since $0 \leq p' \leq 1$ we have $K \geq 0$. Thus the smallest strictly positive intersection point is $p' = \frac{1}{\ell+1}$.

Now suppose that p' satisfies $f_1(p') = f_0(p')$. Then

$$p' = \frac{d_x - (c_1 - c_0)}{d_x d_y + d_x - (c_1 - c_0)}.$$

We can write $c_1 - c_0 = d_x - Kg$ for some $K \in \mathbb{Z}$. Then

$$p' = \frac{Kg}{d_x d_y + Kg} = \frac{K}{\ell + K}.$$

Since $0 \leq p' \leq 1$ we have $K \geq 0$. Thus the smallest strictly positive intersection point is again $p' = \frac{1}{\ell+1}$.

Now suppose that p' satisfies $f_{-1}(p') = f_1(p')$. Then

$$f_{-1}(p') = \frac{1}{2}(f_{-1}(p') + f_1(p')) = \frac{1-p'}{d_x d_y}\frac{c_{-1}+c_1}{2} \overset{(5.1)}{\leq} \frac{1-p'}{d_x d_y} c_0 = f_0(p').$$

In particular

$$f_1(p') = f_{-1}(p') = \frac{1}{2}(f_{-1}(p') + f_1(p')) \le f_0(p').$$

Thus either $f_0(p') > f_{-1}(p')$ and $f_0(p') > f_1(p')$, in which case there is no turning point at p', or $f_0(p') = f_{-1}(p') = f_1(p')$, in which case p' is one of the points we have already considered. Thus $p \mapsto \kappa_p(x,y)$ is linear over $[0, \frac{1}{\ell+1}]$. □

Let us finish this section with some observations about relations between various different curvature values. Assume that $d_y|d_x$. Then $\mathrm{lcm}(d_x, d_y) = \max(d_x, d_y)$ and so, by Theorem 1.1, $p \mapsto \kappa_p(x,y)$ has at most two linear parts. We can give a formula for $\kappa_p(x,y)$ in terms of the curvatures $\kappa_0(x,y)$ and $\kappa(x,y)$ by using the fact that $\kappa_p(x,y)$ can change its slope only at $p = \frac{1}{d_x+1}$ and that $\kappa_1 = 0$, $\kappa_1' = -\kappa$. This formula, given in the following theorem, emerges via a straightforward calculation and applies, in particular, to all regular graphs.

Theorem 5.3. *Let $G = (V, E)$ be a locally finite graph. Let $x, y \in V$ with $x \sim y$ and $d_y|d_x$. Then*

$$\kappa_p(x,y) = \begin{cases} (d_x\kappa(x,y) - (d_x+1)\kappa_0(x,y))p + \kappa_0(x,y), & \text{if } p \in [0, \frac{1}{d_x+1}], \\ (1-p)\kappa(x,y), & \text{if } p \in [\frac{1}{d_x+1}, 1]. \end{cases}$$

Remark 5.4. As mentioned earlier $\kappa_{\frac{1}{2}}, \kappa_{\frac{1}{d+1}}$ and κ have been studied in various articles. In fact, the identity

$$\kappa_p(x,y) = (1-p)\kappa(x,y) \tag{5.2}$$

holds true at all edges (even those whose Ollivier-Ricci idleness function has three linear parts) and for all values $p \in \left[\frac{1}{\max\{d_x,d_y\}+1}, 1\right]$. Equation (5.2) follows from Theorem 4.4 and the fact that $\kappa_1 = 0$, $\kappa_1' = -\kappa$. As a consequence, we have

$$\kappa = 2\kappa_{\frac{1}{2}} = \frac{d+1}{d}\kappa_{\frac{1}{d+1}}.$$

We end this section with a connection between κ and κ_0.

Theorem 5.5. *Let $G = (V, E)$ be a locally finite graph. Let $x, y \in V$ with $x \sim y$ and $d_x \ge d_y$. Then*

$$\kappa_0(x,y) \le \kappa(x,y) \le \kappa_0(x,y) + \frac{2}{d_x}.$$

Proof. The first inequality follows from the fact that the graph of a concave function lies below its tangent line at each point and that $\kappa_1 = 0$, $\kappa = -\kappa_1'$:

$$\kappa_0 \le \kappa_1 + \kappa_1'(0-1) = \kappa.$$

Now we prove the second inequality. Let ϕ be a 1–Lipschitz function with $\phi(y) = 0$ such that

$$W_{xy}(0) = \sum_{w\in V} \phi(w)(\mu_x^0(w) - \mu_y^0(w)) = \frac{-1}{d_y}\phi(x) + \frac{1}{d_x}\sum_{\substack{z\sim x\\ z\neq y}}\phi(z) - \frac{1}{d_y}\sum_{\substack{z\sim y\\ z\neq x}}\phi(z).$$

Then

$$
\begin{aligned}
W_{xy}\left(\frac{1}{d_x+1}\right) &\geq \sum_{w\in V} \phi(w)(\mu_x^{\frac{1}{d_x+1}}(w) - \mu_y^{\frac{1}{d_x+1}}(w)) \\
&= \left(\frac{1}{d_x+1} - \frac{d_x}{(d_x+1)d_y}\right)\phi(x) + \frac{1}{d_x+1}\sum_{\substack{z\sim x\\ z\neq y}}\phi(z) - \frac{d_x}{(d_x+1)d_y}\sum_{\substack{z\sim y\\ z\neq x}}\phi(z).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{d_x+1}{d_x}W_{xy}\left(\frac{1}{d_x+1}\right) &\geq \left(\frac{1}{d_x} - \frac{1}{d_y}\right)\phi(x) + \frac{1}{d_x}\sum_{\substack{z\sim x\\ z\neq y}}\phi(z) - \frac{1}{d_y}\sum_{\substack{z\sim y\\ z\neq x}}\phi(z) \\
&= W_{xy}(0) + \frac{1}{d_x}\phi(x) \\
&= W_{xy}(0) + \frac{1}{d_x}(\phi(x) - \phi(y)) \\
&\geq W_{xy}(0) - \frac{1}{d_x}
\end{aligned}
$$

since ϕ is 1–Lipschitz. Therefore

$$
\begin{aligned}
\kappa_{\frac{1}{d_x+1}}(x,y) &\leq 1 + \frac{1}{d_x+1} - \frac{d_x}{d_x+1}W_{xy}(0) \\
&= \frac{2}{d_x+1} + \frac{d_x}{d_x+1}(1 - W_{xy}(0)) \\
&= \frac{2}{d_x+1} + \frac{d_x}{d_x+1}\kappa_0(x,y).
\end{aligned}
$$

Finally, by (5.2),

$$
\kappa(x,y) = \frac{d_x+1}{d_x}\kappa_{\frac{1}{d_x+1}}(x,y) \leq \kappa_0(x,y) + \frac{2}{d_x}.
$$

□

Remark 5.6. Let $G = (V,E)$ be a locally finite d-regular graph. Let $x, y \in V$ with $x \sim y$. Then by the above theorem we have

$$
\kappa_0(x,y) \leq \kappa(x,y) \leq \kappa_0(x,y) + \frac{2}{d}.
$$

These inequalities are sharp. The lower bound on κ is achieved for example by the 6-cycle, and the upper bound on κ is achieved by the 4-cycle. Furthermore, by [8], $\kappa_0(x,y) \in \mathbb{Z}/d$. Similar arguments show that $\kappa(x,y) \in \mathbb{Z}/d$. Thus

$$
\kappa(x,y) = \kappa_0(x,y) + \frac{C}{d},
$$

where $C \in \{0,1,2\}$.

6 Application to the Cartesian product

In [5] the authors proved the following results on the curvature of Cartesian products of graphs:

Theorem 6.1 ([5]). *Let $G = (V_G, E_G)$ be a d_G-regular graph and $H = (V_H, E_H)$ be a d_H-regular graph. Let $x_1, x_2 \in V_G$ with $x_1 \sim x_2$ and $y \in V_H$. Then*

$$\kappa^{G\times H}((x_1,y),(x_2,y)) = \frac{d_G}{d_G+d_H}\kappa^G(x_1,x_2),$$
$$\kappa_0^{G\times H}((x_1,y),(x_2,y)) = \frac{d_G}{d_G+d_H}\kappa_0^G(x_1,x_2).$$

Using our formula from Theorem 5.3, we extend this result and derive relations between the full Ollivier-Ricci idleness functions involved in the Cartesian product.

Corollary 1.3. *Let $G = (V_G, E_G)$ be a d_G-regular graph and $H = (V_H, E_H)$ be a d_H-regular graph. Let $x_1, x_2 \in V_G$ with $x_1 \sim x_2$ and $y \in V_H$. Then*

$$\kappa_p^{G\times H}((x_1,y),(x_2,y)) = \begin{cases} \frac{d_G}{d_G+d_H}\kappa_p^G(x_1,x_2) + \frac{d_Gd_H}{d_G+d_H}(\kappa^G(x_1,x_2) - \kappa_0^G(x_1,x_2))p, & \text{if } p \in [0, \frac{1}{d_G+d_H+1}], \\ \frac{d_G}{d_G+d_H}\kappa^G(x_1,x_2)(1-p), & \text{if } p \in [\frac{1}{d_G+d_H+1}, 1]. \end{cases}$$

Proof. For ease of reading, we define $\kappa_p^{G\times H} := \kappa_p^{G\times H}((x_1,y),(x_2,y))$ and $\kappa_p^G := \kappa_p^G(x_1,x_2)$. Let $p \in [0, \frac{1}{d_G+d_H+1}]$. Then, by Theorem 5.3,

$$\begin{aligned}
\kappa_p^{G\times H} &= ((d_G+d_H)\kappa^{G\times H} - (d_G+d_H+1)\kappa_0^{G\times H})p + \kappa_0^{G\times H} \\
&= \frac{d_G}{d_G+d_H}\left\{((d_G+d_H)\kappa^G - (d_G+d_H+1)\kappa_0^G)p + \kappa_0^G\right\} \\
&= \frac{d_G}{d_G+d_H}\left\{(d_G\kappa^G - (d_G+1)\kappa_0^G)p + \kappa_0^G\right\} + \frac{d_Gd_H}{d_G+d_H}(\kappa^G - \kappa_0^G)p \\
&= \frac{d_G}{d_G+d_H}\kappa_p^G + \frac{d_Gd_H}{d_G+d_H}(\kappa^G - \kappa_0^G)p.
\end{aligned}$$

Now suppose $p \in [\frac{1}{d_G+d_H+1}, 1]$. Then

$$\kappa_p^{G\times H} = \kappa^{G\times H}(1-p) = \frac{d_G}{d_G+d_H}\kappa^G(1-p).$$

□

7 Bone idleness and some open questions

We finish this article with a discussion of when the Ollivier-Ricci idleness function $p \mapsto \kappa_p(x,y)$ is globally linear for all edges. First we introduce the notion *bone idle*.

Definition 7.1. Let $G = (V, E)$ be a locally finite graph. We say an edge $x \sim y$ is *bone idle* if $\kappa_p(x, y) = 0$ for every $p \in [0, 1]$. We say that G is *bone idle* if every edge is bone idle.

Remark 7.1. Note that $\kappa_p(x, y) = 0$ for all $p \in [0, 1]$ if and only if $\kappa_0(x, y) = \kappa(x, y) = 0$. This follows from the concavity of $\kappa_p(x, y)$.

It is an interesting problem to classify the graphs which are bone idle. Due to the above remark this question is closely related to various notions of Ricci flatness. The following result allows us to classify bone idle graphs with girth at least 5. Recall that the girth of a graph is the length of its shortest non-trivial cycle.

Theorem 7.2 ([1]). *Let $G = (V, E)$ be a locally finite graph with girth at least 5. Suppose that $\kappa_0(x, y) = 0$ for all $x, y \in V$ with $x \sim y$. Then G is isomorphic to one of the following graphs:*

(i) The infinite path;

(ii) The cyclic graph C_n for $n \geq 6$;

(iii) The path P_n for $n \geq 2$;

(iv) The Star graph S_n for $n \geq 3$.

Corollary 7.3. *Let $G = (V, E)$ be a locally finite graph with girth at least 5. Suppose that G is bone idle. Then G is isomorphic to one of the following graphs:*

(i) The infinite path;

(ii) The cyclic graph C_n for $n \geq 6$.

Proof. By Remark 7.1, it suffices to check which of the examples given in Theorem 7.2 satisfy $\kappa = 0$, which is straightforward. □

Remark 7.4. Note that the above Corollary shows that there exists no bone idle graph with girth equal to 5.

The full classification of bone idle graphs is still open.

The condition of bone idleness of an edge $x \sim y$ can be weakened to only require that $p \mapsto \kappa_p(x, y)$ is globally linear on $[0, 1]$. This is equivalent to $\kappa_0(x, y) = \kappa(x, y)$. It is a natural desire to understand this weaker condition better.

Recall that $\kappa \geq \kappa_0$. The Petersen graph has $\kappa(x, y) = 0$ and $\kappa_0(x, y) < 0$ for all edges $x \sim y$. In an earlier draft of this paper, we asked whether there exists a regular graph satisfying $\kappa(x, y) > 0$ and $\kappa_0(x, y) < 0$. In the meantime, Riikka Kangaslampi [4] proved that such a graph cannot exist.

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