Mean residual life of coherent systems consisting of multiple types of dependent components

Serkan Eryilmaz[*], Frank P.A. Coolen[†] and Tahani Coolen-Maturi[‡]

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Abstract

Mean residual life is a useful dynamic characteristic to study reliability of a system. It has been widely considered in the literature not only for single unit systems but also for coherent systems. This paper is concerned with the study of mean residual life for a coherent system that consists of multiple types of dependent components. In particular, the survival signature based generalized mixture representation is obtained for the survival function of a coherent system and it is used to evaluate the mean residual life function. Furthermore, two mean residual life functions under different conditional events on components' lifetimes are also defined and studied.

Key words. Dependence; Mean residual life; Minimal survival signature; Reliability; Survival signature

1 Introduction

The study of mean residual life of a coherent system has attracted a great deal of attention in reliability theory. Consider a system with components which has two possible states; $\phi(x_1, ..., x_n) = 1$ if the system is functioning and $\phi(x_1, ..., x_n) = 0$ if the system has failed, where $x_i = 1$ if the ith component is functioning and $x_i = 0$ if the ith component has failed. The function $\phi(x_1, ..., x_n)$ is called the structure function. A system with structure function $\phi(x_1, ..., x_n)$ is coherent if it is nondecreasing in each argument, and each component i is relevant to the performance of the system, i.e. $\phi(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_n) = 0$ and $\phi(x_1, ..., x_{i-1}, 1, x_{i+1}, ..., x_n) = 1$ for some states $x_1, ..., x_{i-1}, x_{i+1}, ..., x_n$ of other components $1, 2, ..., i-1, i+1, ..., n$. Besides the classical definition of the mean residual life, different mean residual life functions

[*]Department of Industrial Engineering, Atilim University, 06836, Incek, Ankara, Turkey, e-mail: serkan.eryilmaz@atilim.edu.tr

[†]Department of Mathematical Sciences, Durham University, Durham, United Kingdom

[‡]Durham University Business School, Durham University, Durham, United Kingdom

have been defined and studied in the literature for a coherent system. For a coherent system with lifetime T and components' lifetimes $T_1, ..., T_n$, the usual mean residual life is defined by $E(T - t \mid T > t)$. Navarro and Hernandez (2008) studied the mean residual life function of a system whose reliability function can be written as a generalized mixture. The mean residual life of a coherent system has also been studied under different conditional events, e.g. when all components are functioning at time t. The latter mean residual life can be defined as $E(T - t \mid T_{1:n} > t)$, where $T_{r:n}$ denotes the rth smallest lifetime among $T_1, ..., T_n$ (Asadi and Bayramoglu (2006)). See also Navarro (2016) and Navarro and Durante (2017) for some recent results on the mean residual functions $E(T - t \mid T > t)$ and $E(T - t \mid T_{1:n} > t)$. Asadi and Goliforushani (2008) studied the mean residual life of a system consisting of n components having the property that if it is known that at most r components ($r < n$) have failed, the system is still operating with probability 1, i.e. $E(T - t \mid T_{r:n} > t)$. The concept of signature (see, e.g. Samaniego (2007)) has been used to evaluate the latter mean residual life functions.

For a coherent system that consists of exchangeable components, the survival function can be written as

$$P\{T > t\} = \sum_{i=1}^{n} \alpha_i P\{T_{1:i} > t\}, \tag{1}$$

where the vector of coefficients $(\alpha_1, ..., \alpha_n)$ satisfying $\sum_{i=1}^{n} \alpha_i = 1$ is called minimal signature and only depends on the structure of the system (Navarro et al. (2007)). The equation (1) is a generalized mixture representation for the survival function of a coherent system that consists of a single type of components. With a single type, we mean that all components within the system have a common failure time distribution. The mixture representation given by (1) is useful to study limiting behavior of the mean residual life function $E(T - t \mid T > t)$ (Navarro and Eryilmaz (2007), Navarro and Hernandez (2008)).

Another well-known representation for the survival function of a coherent system that consists of single type of components is given by

$$P\{T > t\} = \sum_{l=0}^{n} \Phi(l) P\{C(t) = l\},$$

where $C(t)$ is the number of working components at time t, and $\Phi(l)$ is the survival signature defined by

$$\Phi(l) = \frac{r_n(l)}{\binom{n}{l}},$$

where $r_n(l)$ denotes the number of path sets of size l (Coolen and Coolen-Maturi (2012)). A path set is a set of components whose simultaneous functioning ensures the functioning of the system.

In this paper, we study mean residual life functions $E(T-t \mid T>t)$, $E(T-t \mid T>t, T_{r:n}>t)$ and $E(T-t \mid T>t, T^{(1)}_{r_1:n_1}>t, ..., T^{(K)}_{r_K:n_K}>t)$ for a coherent system which is composed of $K \geq 2$ types of dependent components, where $T^{(i)}_{r:n_i}$ denotes the rth smallest among the failure times of n_i components of type $i, i=1,...,K$. Under this general setup, the random failure times of components of the same type are exchangeable and dependent and the random failure times of components of different types are dependent. The concept of survival signature has been found to be very useful to study reliability properties of such systems (see, e.g. Coolen and Coolen-Maturi (2012), Samaniego and Navarro (2016)). By utilizing the concept of the survival signature, we obtain a generalized mixture representation for the survival function of a coherent system that consists of K types of dependent components. The obtained mixture representation generalizes the representation given by (1) and is used to study the limiting behavior of $E(T-t \mid T>t)$. The survival signature based representations for $E(T-t \mid T>t, T_{r:n}>t)$ and $E(T-t \mid T>t, T^{(1)}_{r_1:n_1}>t, ..., T^{(K)}_{r_K:n_K}>t)$ are also obtained.

Sadegh (2011) extended the results of Asadi and Goliforushani (2008) when the lifetimes of the system components are independent random variables but not necessarily identically distributed and when the joint distribution of the component lifetimes is exchangeable. Zhang and Meeker (2013) obtained mixture representations of the reliability functions of the residual life and inactivity time of a coherent system with n independent and identically distributed components, given that before time t_1, exactly r $(r<n)$ components have failed and at time t_2, the system is either still working or has failed. Some recent discussions on the mean residual life of systems can be found in Navarro and Gomis (2016), Bayramoglu and Ozkut (2016), Bayramoglu Kavlak (2017).

The paper is organized as follows. In Section 2, we obtain a generalized mixture representation for the survival function of a coherent system consisting of multiple types of dependent component. Section 3 is devoted to study different mean residual life functions.

2 Minimal survival signature

Consider a coherent system with $K \geq 2$ types of n components. Let n_i denote the number of components of type i, $i=1,2,...,K$, where $n=\sum_{i=1}^{K} n_i$. It is assumed that the random failure times of components of the same type are exchangeable and dependent, and that the random failure times of components of different type are dependent. Without loss of generality, the assumption on components' lifetimes can be written as

$$(T_1, ..., T_n) \stackrel{d}{=} (T_{\sigma(1)}, ..., T_{\sigma(n)}),$$

for any permutation σ such that $\sigma(i) \in \{1,...,n_1\}$ for all $i \in \{1,...,n_1\}$, $\sigma(i) \in \{n_1+1,...,n_1+n_2\}$ for all $i \in \{n_1+1,...,n_1+n_2\}$, and so on, where $\stackrel{d}{=}$ denotes equality in distribution. It should be pointed out that this is a quite strong assumption.

If $C_i(t)$ denotes the number of components of type i working at time t, then the survival function of the system can be written as

$$P\{T > t\} = \sum_{l_1=0}^{n_1} \cdots \sum_{l_K=0}^{n_K} \Phi(l_1, ..., l_K) P\{C_1(t) = l_1, ..., C_K(t) = l_K\}, \tag{2}$$

where $\Phi(l_1, ..., l_K)$ represents the survival signature and is defined by

$$\Phi(l_1, ..., l_K) = \frac{r_{n_1,...,n_K}(l_1, ..., l_K)}{\binom{n_1}{l_1} ... \binom{n_K}{l_K}}, \tag{3}$$

(Coolen and Coolen-Maturi (2012, 2015)). In (2), $r_{n_1,...,n_K}(l_1, ..., l_K)$ denotes the number of path sets of the system including exactly l_1 components of type 1, ..., exactly l_K components of type K. The computation of survival signature is a challenging problem. Reed (2017) proposed an efficient algorithm to compute survival signature of a system. Patelli et al. (2017) presented a simulation method for system reliability using the survival signature.

Let $T_j^{(i)}$ denote the failure time of the jth component of type $i, i = 1, 2, ..., K$. Then from Theorem 1 of Eryilmaz (2017), the joint distribution of $C_1(t), ..., C_K(t)$ can be written as

$$P\{C_1(t) = l_1, ..., C_K(t) = l_K\} = \binom{n_1}{l_1} ... \binom{n_K}{l_K} S_{n_1,...,n_K}(t; l_1, ..., l_K), \tag{4}$$

where

$$\begin{aligned} S_{n_1,...,n_K}(t; l_1, ..., l_K) &= \sum_{i_1=0}^{n_1-l_1} \cdots \sum_{i_K=0}^{n_K-l_K} (-1)^{i_1+...+i_K} \binom{n_1 - l_1}{i_1} ... \binom{n_K - l_K}{i_K} \times \\ & \quad P\left\{T_1^{(1)} > t, ..., T_{l_1+i_1}^{(1)} > t, ..., T_1^{(K)} > t, ..., T_{l_K+i_K}^{(K)} > t\right\}. \end{aligned} \tag{5}$$

We first obtain the following generalized mixture representation for the survival function of a coherent system which will be useful in the sequel.

Theorem 1 The survival function of a coherent system consisting of n_i components of type i, $i = 1, 2, ..., K$ can be written as

$$P\{T > t\} = \sum_{m_1=0}^{n_1} \cdots \sum_{m_K=0}^{n_K} \Phi_*(m_1, ..., m_K) P\left\{\min(T_{1:m_1}^{(1)}, ..., T_{1:m_K}^{(K)}) > t\right\}, \tag{6}$$

where $T^{(i)}_{1:m_i} = \min(T^{(i)}_1, ..., T^{(i)}_{m_i}), i = 1, ..., K$, and

$$\begin{aligned}\Phi_*(m_1, ..., m_K) &= \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} (-1)^{m_1-l_1+...+m_K-l_K} \binom{n_1}{l_1} ... \binom{n_K}{l_K} \\ &\times \binom{n_1-l_1}{m_1-l_1} ... \binom{n_K-l_K}{m_K-l_K} \Phi(l_1, ..., l_K), \end{aligned} \tag{7}$$

and for convenience $P\left\{T^{(i)}_{1:0} > t\right\} = 1$.

Proof Let

$$\lambda_{l_1+i_1,...l_K+i_K} = P\left\{T^{(1)}_1 > t, ..., T^{(1)}_{l_1+i_1} > t, ..., T^{(K)}_1 > t, ..., T^{(K)}_{l_K+i_K} > t\right\},$$

then from (2), (4) and (5) we have

$$P\{T > t\} = \sum_{l_1=0}^{n_1} \cdots \sum_{l_K=0}^{n_K} \Phi(l_1, ..., l_K) \binom{n_1}{l_1} ... \binom{n_K}{l_K}$$

$$\times \sum_{i_1=0}^{n_1-l_1} \cdots \sum_{i_K=0}^{n_K-l_K} (-1)^{i_1+...+i_K} \binom{n_1-l_1}{i_1} ... \binom{n_K-l_K}{i_K} \lambda_{l_1+i_1,...i_K+i_K}$$

$$= \sum_{l_1=0}^{n_1} \cdots \sum_{l_K=0}^{n_K} \Phi(l_1, ..., l_K) \binom{n_1}{l_1} ... \binom{n_K}{l_K}$$

$$\times \sum_{m_1=l_1}^{n_1} \cdots \sum_{m_K=l_K}^{n_K} (-1)^{m_1-l_1+...+m_K-l_K} \binom{n_1-l_1}{m_1-l_1} ... \binom{n_K-l_K}{m_K-l_K} \lambda_{m_1,...m_K}$$

$$= \sum_{m_1=0}^{n_1} \cdots \sum_{m_K=0}^{n_K} \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} (-1)^{m_1-l_1+...+m_K-l_K} \binom{n_1}{l_1} ... \binom{n_K}{l_K}$$

$$\times \binom{n_1-l_1}{m_1-l_1} ... \binom{n_K-l_K}{m_K-l_K} \Phi(l_1, ..., l_K) \lambda_{m_1,...m_K}$$

$$= \sum_{m_1=0}^{n_1} \cdots \sum_{m_K=0}^{n_K} \Phi_*(m_1, ..., m_K) \lambda_{m_1,...m_K}$$

$$= \sum_{m_1=0}^{n_1} \cdots \sum_{m_K=0}^{n_K} \Phi_*(m_1, ..., m_K) P\left\{T^{(1)}_1 > t, ..., T^{(1)}_{m_1} > t, ..., T^{(K)}_1 > t, ..., T^{(K)}_{m_K} > t\right\}$$

$$= \sum_{m_1=0}^{n_1} \cdots \sum_{m_K=0}^{n_K} \Phi_*(m_1, ..., m_K) P\left\{T^{(1)}_{1:m_1} > t, ..., T^{(K)}_{1:m_K} > t\right\},$$

where

$$\Phi_*(m_1,...,m_K) = \sum_{l_1=0}^{m_1}\cdots\sum_{l_K=0}^{m_K}(-1)^{m_1-l_1+...+m_K-l_K}\binom{n_1}{l_1}...\binom{n_K}{l_K} \times\binom{n_1-l_1}{m_1-l_1}...\binom{n_K-l_K}{m_K-l_K}\Phi(l_1,...,l_K).\blacksquare$$

Clearly, the coefficients $\Phi_*(m_1,...,m_K)$ in (6) satisfy

$$\sum_{m_1=0}^{n_1}\cdots\sum_{m_K=0}^{n_K}\Phi_*(m_1,...,m_K)=1,$$

but they may take negative values. Therefore equation (5) is a generalized mixture of series systems. Similar to systems with a single type of components, we will call the $\Phi_*(m_1,...,m_K)$ minimal survival signature of the system that consists of multiple types of components.

Corollary 1 If the system consists of independent components such that the common failure time distribution of type i components is $F_i(t), i=1,2,...,K$, then

$$P\{T>t\}=\sum_{m_1=0}^{n_1}\cdots\sum_{m_K=0}^{n_K}\Phi_*(m_1,...,m_K)\bar{F}_1^{m_1}(t)...\bar{F}_K^{m_K}(t).\blacksquare \tag{8}$$

The generalized distorted distribution corresponding to n distribution functions $G_1,G_2,...,G_n$ is represented as

$$F_Q(t)=Q(G_1(t),...,G_n(t)),$$

where the increasing continuous function $Q:[0,1]^n\to[0,1]$ is called multivariate distortion function and satisfies $Q(0,...,0)=0$ and $Q(1,...,1)=1$. For the survival function we have

$$\bar{F}_Q(t)=\bar{Q}(\bar{G}_1(t),...,\bar{G}_n(t)),$$

where $\bar{Q}(u_1,...,u_n)=1-Q(1-u_1,...,1-u_n)$ is called multivariate dual distortion function. The function $\bar{Q}$ is also a multivariate distortion function and it satisfies the same properties as Q (Navarro et al. (2016)).

Proposition 1 Let $\hat{C}$ be a survival copula corresponding to $T_1^{(1)},...,T_{n_1}^{(1)},...,T_1^{(K)},...,T_{n_K}^{(K)}$, i.e.

$$P\left\{T_1^{(1)}>t_1^{(1)},...,T_{n_1}^{(1)}>t_{n_1}^{(1)},...,T_1^{(K)}>t_1^{(K)},...,T_{n_K}^{(K)}>t_{n_K}^{(K)}\right\} = \hat{C}(\bar{F}_1(t_1^{(1)}),...,\bar{F}_1(t_{n_1}^{(1)}),...,\bar{F}_K(t_1^{(K)}),...,\bar{F}_K(t_{n_K}^{(K)})).$$

Then the lifetime T_S of a coherent system that consists of K types of dependent components has a generalized distorted distribution whose survival function is

$$P\left\{T > t\right\} = \bar{Q}(\bar{F}_1(t), ..., \bar{F}_K(t)), \tag{9}$$

where the multivariate distortion function is given by

$$\begin{aligned} \bar{Q}(u_1, ..., u_K) &= \sum_{m_1=0}^{n_1} \cdots \sum_{m_K=0}^{n_K} \Phi_*(m_1, ..., m_K) \\ &\quad \times \hat{C}(\underbrace{u_1, ..., u_1}_{m_1}, \underbrace{1, ..., 1}_{n_1-m_1}, ..., \underbrace{u_K, ..., u_K}_{m_K}, \underbrace{1, ..., 1}_{n_K-m_K}). \end{aligned} \tag{10}$$

Proof The proof is immediate from (6) since

$$\begin{aligned} & P\left\{\min(T^{(1)}_{1:m_1}, ..., T^{(K)}_{1:m_K}) > t\right\} \\ = \; & P\left\{T^{(1)}_1 > t, ..., T^{(1)}_{m_1} > t, ..., T^{(K)}_1 > t, ..., T^{(K)}_{m_K} > t\right\} \\ = \; & \hat{C}(\underbrace{\bar{F}_1(t), ..., \bar{F}_1(t)}_{m_1}, \underbrace{1, ..., 1}_{n_1-m_1}, ..., \underbrace{\bar{F}_K(t), ..., \bar{F}_K(t)}_{m_K}, \underbrace{1, ..., 1}_{n_K-m_K}). \blacksquare \end{aligned}$$

Although Navarro et al. (2016) have represented the system's lifetime distribution as a generalized distorted distribution when components' lifetimes are dependent, their representation was implicit. In particular, they noted that

$$P\left\{T > t\right\} = H(\bar{F}_1(t), ..., \bar{F}_n(t)),$$

where $H = \bar{Q}$ is a function which depends on the minimal path sets of the coherent system structure and on the survival copula $\hat{C}$ (see also Navarro et al. (2017), Miziula and Navarro (2017)). Our representation given by (9)-(10) is explicit as a function of the survival signature which fully characterizes the system structure and can be computed through equation (3). As a direct consequence of Proposition 1, for a coherent system that consists of independent components such that the common failure time distribution of type i components is $F_i(t), i = 1, 2, ..., K$, we have

$$\bar{Q}(u_1, ..., u_K) = \sum_{m_1=0}^{n_1} \cdots \sum_{m_K=0}^{n_K} \Phi_*(m_1, ..., m_K) u_1^{m_1} ... u_K^{m_K}.$$

In the special case, if the system consists of single type of independent components, then

$$\bar{Q}(u) = \sum_{m=0}^{n} \Phi_*(m) u^m$$

which has been called domination function by Navarro and Spizzichino (2015).

It should be noted that (10) can be used jointly with the results in Navarro et al. (2016) to compare different systems.

Example 1 Consider the system in Figure 1 which has been considered in Feng et al. (2016). The system has six components with $K = 2$ types with $n_1 = 3$ and $n_2 = 3$. Type 1 and type 2 components are represented respectively by blank and black boxes. Table 1 displays the minimal survival signature of the system. Note that the minimal survival signature is computed using the relation (7) and the survival signature of the system presented in Table 1 of Feng et al. (2016).

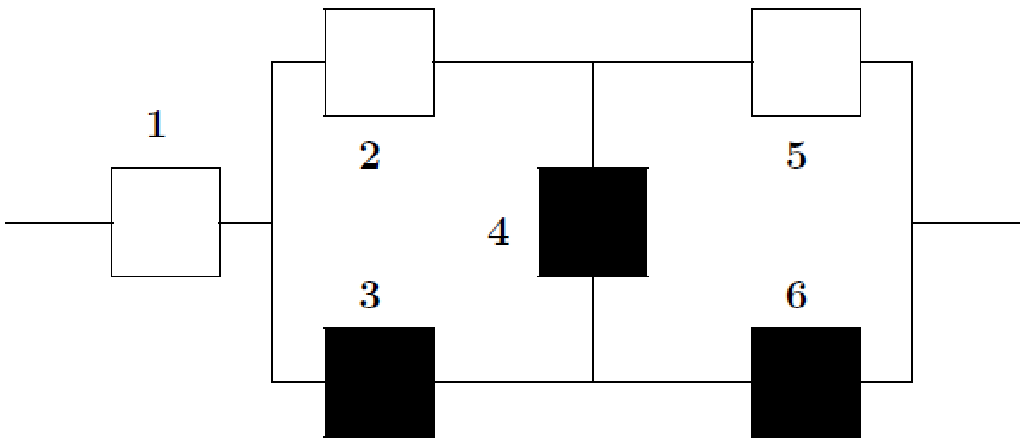

Figure 1. System with two types of components

Using the entries in Table 1 and the equation (5), the survival function of the system can be represented as

$$\begin{aligned} & P\{T > t\} \\ = \ & P\left\{T^{(1)}_{1:1} > t, T^{(2)}_{1:2} > t\right\} + 2P\left\{T^{(1)}_{1:2} > t, T^{(2)}_{1:2} > t\right\} - 2P\left\{T^{(1)}_{1:2} > t, T^{(2)}_{1:3} > t\right\} \\ & + P\left\{T^{(1)}_{1:3} > t\right\} - 3P\left\{T^{(1)}_{1:3} > t, T^{(2)}_{1:2} > t\right\} + 2P\left\{T^{(1)}_{1:3} > t, T^{(2)}_{1:3} > t\right\}. \end{aligned} \tag{11}$$

Using survival copula, the survival function can be represented as

$$P\{T > t\} = \bar{Q}(\bar{F}_1(t), \bar{F}_2(t)),$$

where the distortion function is given by

$$\begin{aligned} \bar{Q}(u_1, u_2) \ = \ & \hat{C}(u_1, u_2, u_2) + 2\hat{C}(u_1, u_1, u_2, u_2) - 2\hat{C}(u_1, u_1, u_2, u_2, u_2) + \hat{C}(u_1, u_1, u_1) \\ & -3\hat{C}(u_1, u_1, u_1, u_2, u_2) + 2\hat{C}(u_1, u_1, u_1, u_2, u_2, u_2). \end{aligned}$$

If the components are independent, then the distortion function becomes

$$\bar{Q}(u_1, u_2) = u_1 u_2^2 + 2u_1^2 u_2^2 - 2u_1^2 u_2^3 + u_1^3 - 3u_1^3 u_2^2 + 2u_1^3 u_2^3.$$

m_1	m_2	$\Phi_*(m_1, m_2)$	m_1	m_2	$\Phi_*(m_1, m_2)$
0	0	0	2	0	0
0	1	0	2	1	0
0	2	0	2	2	2
0	3	0	2	3	-2
1	0	0	3	0	1
1	1	0	3	1	0
1	2	1	3	2	-3
1	3	0	3	3	2

Table 1. Minimal survival signature of the system in Figure 1

3 Mean residual life functions

Using Theorem 1, the MRL of the system that consists of multiple types of components can be computed from

$$\begin{aligned} m(t) &= E(T-t \mid T>t) \\ &= \frac{\sum_{m_1=0}^{n_1}\cdots\sum_{m_K=0}^{n_K}\Phi_*(m_1,...,m_K)\int_0^\infty P\left\{T_{1:m_1}^{(1)}>t+x,...,T_{1:m_K}^{(K)}>t+x\right\}dx}{\sum_{m_1=0}^{n_1}\cdots\sum_{m_K=0}^{n_K}\Phi_*(m_1,...,m_K)P\left\{T_{1:m_1}^{(1)}>t,...,T_{1:m_K}^{(K)}>t\right\}} \end{aligned} \tag{12}$$

The following result of Navarro and Hernandez (2008) is useful to examine the limiting behavior of the MRL function.

Theorem 2 (Navarro and Hernandez (2008)) Let S be a survival function such that

$$S(t)=\sum_{i=1}^{n}\omega_i S_i(t),$$

for all $t \geq 0$, where $S_1(t),...,S_n(t)$ are survival functions, and $\omega_1,...,\omega_n$ are real numbers such that $\sum_{i=1}^{n}\omega_i = 1$. Let $m_i(t)$ be MRL function corresponding to $S_i(t), i=1,...,n$, i.e. $m_i(t) = (S_i(t))^{-1}\int_t^\infty S_i(u)du$. If

$$\lim_{t\to\infty}\inf\frac{m_1(t)}{m_i(t)}>1, \qquad \lim_{t\to\infty}\sup\frac{m_1(t)}{m_i(t)}<\infty,$$

for $i=2,3,...,n$, then the MRL function m of S satisfies

$$\lim_{t\to\infty}\frac{m(t)}{m_1(t)}=1.\blacksquare$$

Because Theorem 1 presents a generalized mixture representation for a coherent system that consists of multiple types of dependent components, Theorem 2 enables us to investigate the limiting behavior of the MRL function for such systems. Application of Theorem 2 needs a multivariate distribution or survival function for modeling lifetimes of components. Suppose that the joint survival function of $T_1^{(1)},...,T_{n_1}^{(1)},...,T_1^{(K)},...,T_{n_K}^{(K)}$ is given by

$$\begin{aligned} & P\left\{T_1^{(1)}>t_1^{(1)},...,T_{n_1}^{(1)}>t_{n_1}^{(1)},...,T_1^{(K)}>t_1^{(K)},...,T_{n_K}^{(K)}>t_{n_K}^{(K)}\right\} \\ = & \left[1+\theta_1\sum_{i=1}^{n_1}t_i^{(1)}+...+\theta_K\sum_{i=1}^{n_K}t_i^{(K)}\right]^{-\alpha}, \end{aligned} \tag{13}$$

for $t_i^{(j)}\geq 0,\ i=1,...,n_j, j=1,...,K;\ \theta_i>0,\alpha>0$. It should be noted that the survival copula corresponding to (13) is

$$\hat{C}(u_1,u_2,...,u_n)=\left[u_1^{-\frac{1}{\alpha}}+u_2^{-\frac{1}{\alpha}}+...+u_n^{-\frac{1}{\alpha}}-(n-1)\right]^{-\alpha},$$

and

$$\begin{aligned} & P\left\{T_1^{(1)}>t_1^{(1)},...,T_{n_1}^{(1)}>t_{n_1}^{(1)},...,T_1^{(K)}>t_1^{(K)},...,T_{n_K}^{(K)}>t_{n_K}^{(K)}\right\} \\ = & \hat{C}(\underbrace{\bar{F}_1(t),...,\bar{F}_1(t)}_{n_1},...,\underbrace{\bar{F}_K(t),...,\bar{F}_K(t)}_{n_K}), \end{aligned}$$

with $\bar{F}_i(t)=(1+\theta_i t)^{-\alpha}$, $i=1,2,...,K$.

In the following, we present the limiting behavior of (12) for the model (13).

Proposition 2 For the multivariate Pareto model given by (13), let $C=\{(i_1,...,i_K): i_1+...+i_K<j_1+...+j_K$ and $\Phi_*(i_1,...,i_K)>0$ for all $j_1=0,1,...,n_1,...,j_K=0,1,...,n_K\}$. If

$$v_1\theta_1+...+v_K\theta_K\leq i_1\theta_1+...+i_K\theta_K, \tag{14}$$

for all $(i_1,...,i_K)\in C$, then

$$\lim_{t\to\infty}\frac{m(t)}{\frac{1}{\alpha-1}\left[t+\frac{1}{v_1\theta_1+...+v_K\theta_K}\right]}=1. \tag{15}$$

Proof The MRL corresponding to $\min(T_{1:i_1}^{(1)},...,T_{1:i_K}^{(K)})$ is

$$\begin{aligned} & \frac{1}{P\left\{T_{1:i_1}^{(1)}>t,...,T_{1:i_K}^{(K)}>t\right\}}\int_0^\infty P\left\{T_{1:i_1}^{(1)}>t+x,...,T_{1:i_K}^{(K)}>t+x\right\}dx \\ = & \frac{1}{[1+\theta_1 i_1 t+...+\theta_K i_K t]^{-\alpha}}\int_0^\infty [1+\theta_1 i_1(t+x)+...+\theta_K i_K(t+x)]^{-\alpha}\,dx \\ = & \frac{1}{\alpha-1}\left[t+\frac{1}{\theta_1 i_1+...+\theta_K i_K}\right], \end{aligned}$$

for $\alpha > 1$. For $(v_1, ..., v_K) \in C$ satisfying (14), the conditions in Theorem 2 hold true for the MRL of $\min(T^{(1)}_{1:v_1}, ..., T^{(K)}_{1:v_K})$. Thus the proof is complete.■

Example 2 For the system in Figure 1, let

$$\begin{aligned} & P\left\{T^{(1)}_1 > t^{(1)}_1, T^{(1)}_2 > t^{(1)}_2, T^{(1)}_3 > t^{(1)}_3, T^{(2)}_1 > t^{(2)}_1, T^{(2)}_2 > t^{(2)}_2, T^{(2)}_3 > t^{(2)}_3\right\} \\ = & \left[1 + \theta_1 \sum_{i=1}^{3} t^{(1)}_i + \theta_2 \sum_{i=1}^{3} t^{(2)}_i\right]^{-\alpha}. \end{aligned}$$

From Table 1, it is easy to see that $C = \{(1, 2), (3, 0)\}$. Thus from Proposition 2, if $\theta_1 \geq \theta_2$, then

$$\lim_{t\to\infty} \frac{m(t)}{\frac{1}{\alpha-1}\left[t + \frac{1}{\theta_1+2\theta_2}\right]} = 1,$$

and if $\theta_2 \geq \theta_1$, then

$$\lim_{t\to\infty} \frac{m(t)}{\frac{1}{\alpha-1}\left[t + \frac{1}{3\theta_1}\right]} = 1.■$$

It should be noted here that the limiting result in (15) depends on determination of the coefficients $v_1, ..., v_K$ defined by (14). As it is clear from Example 2, these coefficients heavily depend on the relation between the parameters θ_1 and θ_2.

Consider a coherent system that has the property that if at most r components ($r < n$) have failed, the system is still operating with probability 1. Then, the conditional expected value $E(T - t \mid T_{r:n} > t)$ represents the mean residual lifetime function of a coherent system given that at least $n - r + 1$ components of the system are working at time t (Asadi and Bayramoglu (2006), Sadegh (2011)). For a coherent system that consists of multiple types of components, define the following mean residual life.

$$E(T - t \mid T > t, T_{r:n} > t) = \int_0^\infty P\{T > t + x \mid T > t, T_{r:n} > t\}\, dx. \qquad (16)$$

For a coherent system consisting of $K \geq 2$ types of components, it is easy to see that

$$\begin{aligned} & P\{T > t, T_{r:n} > t\} \\ = & \sum_{l_1+...+l_K \geq n-r+1} \cdots \sum \Phi(l_1, ..., l_K) P\{C_1(t) = l_1, ..., C_K(t) = l_K\} \\ = & \sum_{l_1+...+l_K \geq n-r+1} \cdots \sum \Phi(l_1, ..., l_K) \binom{n_1}{l_1} ... \binom{n_K}{l_K} S_{n_1,...,n_K}(t; l_1, ..., l_K), \qquad (17) \end{aligned}$$

where $S_{n_1,...,n_K}(t; l_1, ..., l_K)$ is given by (5). In the following Theorem, we present the conditional survival function of T given $\{T > t, T_{r:n} > t\}$.

Theorem 3 For a coherent system consisting of n_i components of type i, $i = 1, 2, ..., K$,

$$
\begin{aligned}
& P\{T > s \mid T > t, T_{r:n} > t\} \\
= & \frac{1}{P\{T_S > t, T_{r:n} > t\}} \sum_{l_1=0}^{n_1} \cdots \sum_{l_K=0}^{n_K} \sum \cdots \sum_{(j_1,\ldots,j_K)\in U} \Phi(l_1, ..., l_K) N(j_1, l_1, n_1; ...; j_K, l_K, n_K) \\
& \times P\left\{n_1 - j_1 \text{ of } T^{(1)}\text{s} \le t, j_1 - l_1 \text{ of } T^{(1)}\text{s} \in (t, s], l_1 \text{ of } T^{(1)}\text{s} > s,\right. \\
& \left. ..., n_K - j_K \text{ of } T^{(K)}\text{s} \le t, j_K - l_K \text{ of } T^{(K)}\text{s} \in (t, s], l_K \text{ of } T^{(K)}\text{s} > s\right\}, \qquad (18)
\end{aligned}
$$

where $U = \{(j_1, ..., j_K) : j_1 + ... + j_K \ge n - r + 1; l_1 \le j_1 \le n_1, ..., l_K \le j_K \le n_K\}$, and

$$
N(j_1, l_1, n_1; ...; j_K, l_K, n_K) = \binom{n_1}{n_1 - j_1, j_1 - l_1, l_1} \cdots \binom{n_K}{n_K - j_K, j_K - l_K, l_K}.
$$

Proof By conditioning on the number of working components of each type at time t and s,

$$
\begin{aligned}
& P\{T > s, T_{r:n} > t\} \\
= & \sum_{l_1=0}^{n_1} \cdots \sum_{l_K=0}^{n_K} \sum \cdots \sum_{(j_1,\ldots,j_K)\in U} \Phi(l_1, ..., l_K) \\
& \times P\{C_1(s) = l_1, ..., C_K(s) = l_K, C_1(t) = j_1, ..., C_K(t) = j_K\}. \qquad (19)
\end{aligned}
$$

Thus the proof follows noting that

$$
\begin{aligned}
& P\{C_1(s) = l_1, ..., C_K(s) = l_K, C_1(t) = j_1, ..., C_K(t) = j_K\} \\
= & \binom{n_1}{n_1 - j_1, j_1 - l_1, l_1} \cdots \binom{n_K}{n_K - j_K, j_K - l_K, l_K} \\
& \times P\left\{n_1 - j_1 \text{ of } T^{(1)}\text{s} \le t, j_1 - l_1 \text{ of } T^{(1)}\text{s} \in (t, s], l_1 \text{ of } T^{(1)}\text{s} > s,\right. \\
& \left. ..., n_K - j_K \text{ of } T^{(K)}\text{s} \le t, j_K - l_K \text{ of } T^{(K)}\text{s} \in (t, s], l_K \text{ of } T^{(K)}\text{s} > s\right\},
\end{aligned}
$$

for $s > t$ and $j_1 \ge l_1, ..., j_K \ge l_K$.■

In equation (19), it is quite interesting to observe that the survival signature depends on only the number of working components of each type at time s (later time point) and independent of $j_1, ..., j_K$ which denote the number of working components of each type at a previous time point t.

Corollary 2 If the system consists of independent components such that the common failure time distribution of type i components is $F_i(t), i = 1, 2, ..., K$, then

$$\begin{aligned} & P\{T > s \mid T > t, T_{r:n} > t\} \\ = & \frac{1}{P\{T > t, T_{r:n} > t\}} \sum_{l_1=0}^{n_1} \cdots \sum_{l_K=0}^{n_K} \sum \cdots \sum_{(j_1,...,j_K)\in U} \Phi(l_1, ..., l_K) \\ & \times \prod_{i=1}^{K} \binom{n_i}{n_i - j_i, j_i - l_i, l_i} F_i^{n_i - j_i}(t)(F_i(s) - F_i(t))^{j_i - l_i}(1 - F_i(s))^{l_i}. \quad (20) \end{aligned}$$

Corollary 3 Let $r = 1$ in Theorem 3. Then the conditional survival function of the system under the condition that all components are working at time t can be represented as

$$\begin{aligned} & P\{T > s \mid T_{1:n} > t\} \\ = & \frac{1}{P\left\{T_1^{(1)} > t, ..., T_{n_1}^{(1)} > t, ..., T_1^{(K)} > t, ..., T_{n_K}^{(K)} > t\right\}} \sum_{l_1=0}^{n_1} \cdots \sum_{l_K=0}^{n_K} \Phi(l_1, ..., l_K) \\ & \times P\{C_1(s) = l_1, ..., C_K(s) = l_K, C_1(t) = n_1, ..., C_K(t) = n_K\}, \quad (21) \end{aligned}$$

for $s > t$.

In the following, we obtain an expression for the joint probability involved in (18) when $K = 2$, i.e. the system consists of two types of components. The following result is useful since it only involves joint survival probabilities.

Proposition 3 For a system that consists of two types of components,

$$\begin{aligned} & P\{C_1(s) = l_1, C_2(s) = l_2, C_1(t) = n_1, C_2(t) = n_2\} \\ = & \binom{n_1}{l_1}\binom{n_2}{l_2} [p_1(s, t, l_1, l_2) - p_2(s, t, l_1, l_2) \\ & - p_3(s, t, l_1, l_2) + p_4(s, t, l_1, l_2)], \quad (22) \end{aligned}$$

where for $s > t$,

$$\begin{aligned} p_1(s, t, l_1, l_2) = & P\left\{T_1^{(1)} > s, ..., T_{l_1}^{(1)} > s, T_{l_1+1}^{(1)} > t, ..., T_{n_1}^{(1)} > t \right. \\ & \left. T_1^{(2)} > s, ..., T_{l_2}^{(2)} > s, T_{l_2+1}^{(2)} > t, ..., T_{n_2}^{(2)} > t\right\}, \quad (23) \end{aligned}$$

$$\begin{aligned} & p_2(s, t, l_1, l_2) \\ = & \sum_{i=1}^{n_1 - l_1} (-1)^{i-1} \binom{n_1 - l_1}{i} P\left\{T_1^{(1)} > s, ..., T_{l_1+i}^{(1)} > s, T_{l_1+i+1}^{(1)} > t, ..., T_{n_1}^{(1)} > t \right. \\ & \left. T_1^{(2)} > s, ..., T_{l_2}^{(2)} > s, T_{l_2+1}^{(2)} > t, ..., T_{n_2}^{(2)} > t\right\}, \quad (24) \end{aligned}$$

$$
\begin{aligned}
& p_3(s,t,l_1,l_2) \\
= & \sum_{i=1}^{n_2-l_2}(-1)^{i-1}\binom{n_2-l_2}{i}P\left\{T_1^{(1)}>s,...,T_{l_1}^{(1)}>s,T_{l_1+1}^{(1)}>t,...,T_{n_1}^{(1)}>t\right. \\
& \left.T_1^{(2)}>s,...,T_{l_2+i}^{(2)}>s,T_{l_2+i+1}^{(2)}>t,...,T_{n_2}^{(2)}>t\right\}, \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
& p_4(s,t,l_1,l_2) \\
= & \sum_{i=1}^{n_1-l_1}\sum_{j=1}^{n_2-l_2}(-1)^{i+j-2}\binom{n_1-l_1}{i}\binom{n_2-l_2}{j}P\left\{T_1^{(1)}>s,...,T_{l_1+i}^{(1)}>s,\right. \\
& T_{l_1+i+1}^{(1)}>t,...,T_{n_1}^{(1)}>t,T_1^{(2)}>s,...,T_{l_2+j}^{(2)}>s, \\
& \left.T_{l_2+j+1}^{(2)}>t,...,T_{n_2}^{(2)}>t\right\} \qquad (26)
\end{aligned}
$$

In equations (24)-(26), $\sum_a^b \equiv 0$ if $a > b$.

Proof Clearly,

$$
\begin{aligned}
& P\left\{C_1(s)=l_1,C_2(s)=l_2,C_1(t)=n_1,C_2(t)=n_2\right\} \\
= & \binom{n_1}{l_1}\binom{n_2}{l_2}P\left\{T_1^{(1)}>s,...,T_{l_1}^{(1)}>s,T_{l_1+1}^{(1)}>t,...,T_{n_1}^{(1)}>t\right. \\
& T_{l_1+1}^{(1)}\le s,...,T_{n_1}^{(1)}\le s,T_1^{(2)}>s,...,T_{l_2}^{(2)}>s,T_{l_2+1}^{(2)}>t,...,T_{n_2}^{(2)}>t \\
& \left.T_{l_2+1}^{(2)}\le s,...,T_{n_2}^{(2)}\le s\right\}.
\end{aligned}
$$

Define the events

$$A_1 \equiv \left\{T_1^{(1)}>s,...,T_{l_1}^{(1)}>s,T_{l_1+1}^{(1)}>t,...,T_{n_1}^{(1)}>t\right\}$$

$$A_2 \equiv \left\{T_1^{(2)}>s,...,T_{l_2}^{(2)}>s,T_{l_2+1}^{(2)}>t,...,T_{n_2}^{(2)}>t\right\}$$

$$B_1 \equiv \bigcup_{i=l_1+1}^{n_1}\left\{T_i^{(1)}>s\right\}\ ,\ B_2 \equiv \bigcup_{i=l_2+1}^{n_2}\left\{T_i^{(2)}>s\right\}.$$

Then

$$
\begin{aligned}
& P\left\{C_1(s)=l_1,C_2(s)=l_2,C_1(t)=n_1,C_2(t)=n_2\right\} \\
= & \binom{n_1}{l_1}\binom{n_2}{l_2}\left[P(A_1\cap A_2)-P(A_1\cap A_2\cap B_1)\right. \\
& \left.-P(A_1\cap A_2\cap B_2)+P(A_1\cap A_2\cap B_1\cap B_2)\right].
\end{aligned}
$$

The proof is now completed using the principle of inclusion-exclusion.■

As it is clear from Proposition 3, to compute $E(T-t \mid T_{1:n}>t)$, it is enough to evaluate the integration in the form

$$\int_0^\infty P\left\{T_1^{(1)}>t+x,...,T_a^{(1)}>t+x,T_{a+1}^{(1)}>t,...,T_b^{(1)}>t,\right.$$
$$\left.T_1^{(2)}>t+x,...,T_c^{(2)}>t+x,T_{c+1}^{(2)}>t,...,T_d^{(2)}>t\right\}dx.$$

For the multivariate Pareto model given by (13), it can be easily seen that the later integral equals to

$$\begin{aligned}&\int_0^\infty [1+\theta_1((t+x)a+(b-a)t)+\theta_2((t+x)c+(d-c)t)]^{-\alpha}\,dx\\ =\;& \frac{1}{\alpha-1}\frac{[1+t(\theta_1 b+\theta_2 d)]^{1-\alpha}}{\theta_1 a+\theta_2 c},\end{aligned}$$

for $\alpha>1$. Thus, using Proposition 3 the MRL of a coherent system when all components are functioning at time t can be computed from

$$\begin{aligned}E(T-t \mid T_{1:n}>t) &= \frac{1}{[1+\theta_1 n_1 t+\theta_2 n_2 t]^{-\alpha}}\frac{1}{(\alpha-1)}\sum_{l_1=0}^{n_1}\sum_{l_2=0}^{n_2}\Phi(l_1,l_2)\\ &\times\binom{n_1}{l_1}\binom{n_2}{l_2}\left[\frac{[1+t(\theta_1(n_1-l_1)+\theta_2(n_2-l_2))]^{1-\alpha}}{\theta_1 l_1+\theta_2 l_2}\right.\\ &-\sum_{i=1}^{n_1-l_1}(-1)^{i-1}\binom{n_1-l_1}{i}\frac{[1+t(\theta_1(n_1-l_1-i)+\theta_2(n_2-l_2))]^{1-\alpha}}{\theta_1(l_1+i)+\theta_2 l_2}\\ &-\sum_{i=1}^{n_2-l_2}(-1)^{i-1}\binom{n_2-l_2}{i}\frac{[1+t(\theta_1(n_1-l_1)+\theta_2(n_2-l_2-i))]^{1-\alpha}}{\theta_1 l_1+\theta_2(l_2+i)}\\ &\sum_{i=1}^{n_1-l_1}\sum_{j=1}^{n_2-l_2}(-1)^{i+j-2}\binom{n_1-l_1}{i}\binom{n_2-l_2}{j}\\ &\left.\times\frac{[1+t(\theta_1(n_1-l_1-i)+\theta_2(n_2-l_2-j))]^{1-\alpha}}{\theta_1(l_1+i)+\theta_2(l_2+j)}\right],\end{aligned} \tag{27}$$

for $\alpha>1$.

Another MRL function that may be of practical interest can be defined as

$$m^{r_1,...,r_K}(t)=E(T-t \mid T>t,T_{r_1:n_1}^{(1)}>t,...,T_{r_K:n_K}^{(K)}>t), \tag{28}$$

for $1\le r_i\le n_i, i=1,...,K$. The function defined by (28) represents the mean residual life of the system given that at least n_i-r_i+1 components of type i are working at time t, $i = 1, ..., K$. Clearly, for $s > t$,

$$\begin{aligned}
& P\left\{T > s, T^{(1)}_{r_1:n_1} > t, ..., T^{(K)}_{r_K:n_K} > t\right\} \\
= & \sum_{l_1=0}^{n_1} \cdots \sum_{l_K=0}^{n_K} \sum \cdots \sum_{(j_1,...,j_K)\in U^*} \Phi(l_1, ..., l_K) \\
& \times P\left\{C_1(s) = l_1, ..., C_K(s) = l_K, C_1(t) = j_1, ..., C_K(t) = j_K\right\}, \qquad (29)
\end{aligned}$$

where $U^* = \{(j_1, ..., j_K) : \max(l_m, n_m - r_m + 1) \leq j_m \leq n_m, m = 1, ..., K\}$. On the other hand,

$$\begin{aligned}
& P\left\{T > t, T^{(1)}_{r_1:n_1} > t, ..., T^{(K)}_{r_K:n_K} > t\right\} \\
= & \sum_{l_1=n_1-r_1+1}^{n_1} \cdots \sum_{l_K=n_K-r_K+1}^{n_K} \Phi(l_1, ..., l_K) P\left\{C_1(t) = l_1, ..., C_K(t) = l_K\right\}. \qquad (30)
\end{aligned}$$

The MRL function defined by (28) can be computed using (29) and (30) in

$$\begin{aligned}
m^{r_1,...,r_K}(t) &= E(T - t \mid T > t, T^{(1)}_{r_1:n_1} > t, ..., T^{(K)}_{r_K:n_K} > t) \\
&= \frac{1}{P\left\{T > t, T^{(1)}_{r_1:n_1} > t, ..., T^{(K)}_{r_K:n_K} > t\right\}} \\
&\quad \times \int_0^{\infty} P\left\{T > t + x, T^{(1)}_{r_1:n_1} > t, ..., T^{(K)}_{r_K:n_K} > t\right\} dx. \qquad (31)
\end{aligned}$$

Equation (31) corresponds to $E(T - t \mid T_{1:n} > t)$ when $r_1 = ... = r_K = 1$.

Example 1 (continued) In Figure 2, we plot $m(t) = E(T - t \mid T > t)$ (MRL), $m^{1,1}(t) = E(T - t \mid T_{1:n} > t)$ (MRL1) and $m^{2,2}(t) = E(T - t \mid T > t, T^{(1)}_{2:n_1} > t, T^{(2)}_{2:n_2} > t)$ (MRL2) for the system in Figure 1 under the model (13) when $\theta_1 = 1, \theta_2 = 2, \alpha = 2$. We have $m(t) \leq m^{2,2}(t) \leq m^{1,1}(t)$ with $m(0) = m^{2,2}(0) =$

$m^{1,1}(0) = E(T) = 0.4103.$

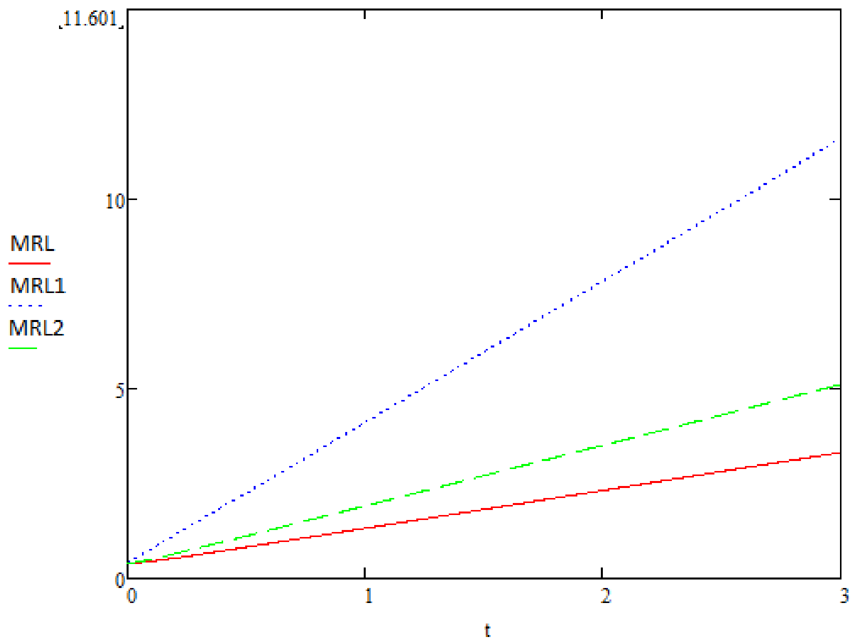

Figure 2. MRL functions of the system in Figure 1.

4 Discussion

This paper has presented general results on survival function and mean residual life for coherent systems, with multiple types of components, with the only assumption that the failure times of components of the same type are exchangeable. Hence, such components can be dependent, and also dependence of components of different types is allowed. The use of the survival signature enabled derivation of a general expression for the mean residual life for such scenarios, in particular through the introduction of the minimal survival signature for such system, generalizing this concept that was introduced by Navarro et al. (2007) for systems with a single type of components. Main future research challenges related to this work include computational issues, in particular for large real world systems, and the use of the mean residual life for decision support, where one can think about aspects like maintenance but also issues of system design.

In addition to the minimal signature, Navarro et al. (2007) also represented the survival function of a coherent system as a generalized mixture of survival functions of parallel systems and called the corresponding set of coefficients as a maximal signature. This concept can be generalized to the maximal survival signature along similar lines as the minimal survival signature presented in this paper, and may be useful for various reliability problems, e.g. stochastic comparison of two different systems.

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