Bidispersive vertical convection

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Abstract

A bidispersive porous material is one which has usual pores but additionally contains a system of micro pores. We consider a fluid saturated bidispersive porous medium in the vertical layer $x \in (-1/2, 1/2)$ with gravity in the $-z$ (downward) direction. The walls of the layer are maintained at different constant temperatures. A suitable Rayleigh number is defined and we derive a global stability threshold below which no instability may arise. We additionally show that the porous layer is stable for all Rayleigh numbers provided the initial temperature gradient is bounded in a precise sense.

1 Introduction

Thermal convection in a vertical porous layer where the vertical sides are maintained at different constant temperatures is currently a major research topic. Gill [1] produced a surprising result by showing that convection will not occur in a vertical porous layer, in the sense that he showed the basic steady solution is linearly stable to two-dimensional disturbances for all Rayleigh numbers. Refinements to Gill's result addressing the three-dimensional nonlinear problem were given by Rees [2] and Straughan [3].

Various generalizations of the porous vertical convection problem have subsequently appeared including mass diffusion and very interesting use of alternative boundary conditions, see e.g. Barletta [4, 5, 6], Barletta and Celli [7], Barletta and Miklavcic [8], Barletta and Storesletten [9], Bera and Khandelwal [10], Celli et al. [11], Rees [12, 13, 14], Shankar and Shivakumara [15], Shankar et al. [16], see also other references in Straughan [17].

This article also addresses the porous vertical convection problem but in a novel context, namely, when the porous medium is one of bidispersive (or double porosity) type. Thermal convection in a bidispersive porous material was introduced in Nield and Kuznetsov [18], and reviews of recent developments may be

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found in Nield and Bejan [19] and in Straughan [20]. These articles concentrate on the case where the different porosity scales allow for different temperatures. Falsaperla et al. [21] adapted the model of Nield and Kuznetsov [18] to allow for a single temperature field, and the horizontal convection problem where the layer is heated from below is analyzed by Gentile and Straughan [22] who show the linear instability and nonlinear stability critical Rayleigh numbers coincide and they determine these numbers.

A bidispersive porous medium is one where there are pores on a macro scale and these are called macro pores. However, the porous skeleton may have cracks or fissures and these give rise to micro pores. Such double porosities may be created in man made materials, see e.g. the picture on p. 3069 of Nield and Kuznetsov. Since there are many real life applications of bidispersive porous materials, see Nield and Bejan [19] and Straughan [20], we deem that a study of the vertical thermal convection problem is timely and useful. We address the three-dimensional fully nonlinear stability problem associated with thermal convection in a fluid saturated bidispersive porous medium and derive two classes of result. The first establishes a Rayleigh number threshold such that stability holds for all initial data when the Rayleigh number is below this threshold. The second demonstrates that the basic solution is nonlinearly stable for all Rayleigh numbers provided the initial value of the temperature gradient does not exceed a precise value.

2   Basic equations

We consider a bidispersive saturated porous medium in which the porosity in the macro pores is ϕ whereas the porosity in the micro pores is ε. The basic equations for non-isothermal flow in a bidispersive porous medium with a single temperature may be derived from the general equations with different temperatures of Nield and Kuznetsov [18] as is done in Falsaperla et al. [21]. Let U_i^f, U_i^p, p^f, p^p denote the velocity fields in the macro and micro pores, and the pressures in the macro and micro pores, and let T denote the temperature. The governing system of equations is

$$\left.\begin{aligned}
\frac{\mu}{K_f}U_i^f + \zeta(U_i^f - U_i^p) &= -p_{,i}^f + \rho_F\, g\, \alpha\, T k_i\,, \\[2mm]
\frac{\mu}{K_p}U_i^p - \zeta(U_i^f - U_i^p) &= -p_{,i}^p + \rho_F\, g\, \alpha\, T k_i\,, \\[2mm]
(\rho c)_m T_{,t} + (\rho c)_f (U_i^f + U_i^p)T_{,i} &= \kappa_m \Delta T\,, \\[2mm]
U_{i,i}^f = 0\,, \quad U_{i,i}^p &= 0\,.
\end{aligned}\right\} \tag{1}$$

In these equations μ, ζ, K_f and K_p denote the dynamic viscosity of the saturating fluid, an interaction coefficient, the permeability attached to the macro pores, and the permeability attached to the micro pores, respectively. In ad-

dition, ρ_F is a reference density, g is gravity, α is the coefficient of expansion of the fluid, $\mathbf{k} = (0,0,1)$, $(\rho c)_m$ is the product of the density and specific heat at constant pressure in the porous medium, $(\rho c)_f$ is the same in the fluid, and κ_m denotes the thermal conductivity of the of the porous medium. Standard indicial notation is employed throughout with $,_i$ denoting $\partial/\partial x_i$.

We suppose the porous medium occupies the vertical layer $x \in (-L/2, L/2)$, $(y,z) \in \mathbb{R}^2$. The temperatures on the boundaries $x = \pm L/2$ are fixed with constant values T_l and T_h, where without loss of generality, we assume $T_h > T_l$. Equations (1) are nondimensionalized with the length, velocity, pressure, temperature and time scales L, $U = \kappa_m/L(\rho c)_f$, $P = L\zeta U$, $T^\sharp = T_h - T_l$, $\mathcal{I} = (\rho c)_m L^2/\kappa_m$. Introduce the non-dimensional parameters

$$\gamma_1 = \frac{\mu}{K_f \zeta}, \quad \gamma_2 = \frac{\mu}{K_p \zeta}, \quad R = \frac{\rho_F\, g\, \alpha\, T^\sharp}{\zeta\, U}, \tag{2}$$

where R is the Rayleigh number. In non-dimensional form (1) may be rewritten

$$\left.\begin{aligned}
&\gamma_1 U_i^f + (U_i^f - U_i^p) = -p_{,i}^f + Rk_i T\,,\\[2mm]
&\gamma_2 U_i^p - (U_i^f - U_i^p) = -p_{,i}^p + Rk_i T\,,\\[2mm]
&T_{,t} + (U_i^f + U_i^p)T_{,i} = \Delta T\,,\\[2mm]
&U_{i,i}^f = 0\,, \quad U_{i,i}^p = 0\,,
\end{aligned}\right\} \tag{3}$$

holding in $\{x \in (-1/2, 1/2)\} \times \{(y,z) \in \mathbb{R}^2)\} \times \{t > 0\}$.
The boundary conditions to be satisfied are that $U_i^f n_i = 0$, $U_i^p n_i = 0$, on $x = \pm 1/2$, where n_i is the unit outward normal to $x = \pm 1/2$.

The steady solution to (3) which corresponds to the analogous solution for a single porosity porous medium as studied by Gill [1], may be found from (3) as $\bar{\mathbf{U}}^f = (\bar{U}^f, \bar{V}^f, \bar{W}^f)$, $\bar{\mathbf{U}}^p = (\bar{U}^p, \bar{V}^p, \bar{W}^p)$, $\bar{U}^f = \bar{U}^p = 0$, $\bar{V}^f = \bar{V}^p = 0$,

$$\bar{T} = x, \quad \bar{W}^f = \Gamma_1 Rx, \quad \bar{W}^p = \Gamma_2 Rx, \tag{4}$$

where

$$\Gamma_1 = \frac{\gamma_2 + 2}{\gamma_1 \gamma_2 + \gamma_1 + \gamma_2}, \quad \Gamma_2 = \frac{\gamma_1 + 2}{\gamma_1 \gamma_2 + \gamma_1 + \gamma_2}\,. \tag{5}$$

Let u_i^f, u_i^p, π^f, π^p and θ be perturbations to the steady solution $(\bar{U}_i^f, \bar{U}_i^p, \bar{p}^f, \bar{p}^p, \bar{T})$ and then from (3)-(5) one may show that the perturbations satisfy the equations

$$\left.\begin{aligned}
&\gamma_1 u_i^f + (u_i^f - u_i^p) = -\pi_{,i}^f + Rk_i \theta\,,\\[2mm]
&\gamma_2 u_i^p - (u_i^f - u_i^p) = -\pi_{,i}^p + Rk_i \theta\,,\\[2mm]
&\theta_{,t} + (u_i^f + u_i^p)\theta_{,i} + u^f + u^p + (\Gamma_1 + \Gamma_2)R\,x\,\theta_{,z} = \Delta\theta\,,\\[2mm]
&u_{i,i}^f = 0\,, \quad u_{i,i}^p = 0\,,
\end{aligned}\right\} \tag{6}$$

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holding in $\{x \in (-1/2, 1/2)\} \times \{(y, z) \in \mathbb{R}^2)\} \times \{t > 0\}$. In (6) $u^f = u_1^f$, $u^p = u_2^p$.

The boundary conditions are

$$u^f = u^p = 0, \quad \theta = 0, \quad \text{on } x = \pm\frac{1}{2}, \tag{7}$$

and in addition $(u_i^f, u_i^p, \pi^f, \pi^p, \theta)$ satisfy a plane tiling periodicity in (y, z) plane. This point is discussed further in Straughan [17], p. 51. For example, the frequently occurring hexagonal shape is discussed there. Whatever the plane tiling shape is, its Cartesian product with $\{x \in [-1/2, 1/2]\}$ defines the periodic cell. The periodic cell which arises is denoted by V.

3 Global stability

We commence by deriving a result of global stability, i.e. for all initial perturbations. By $\| \cdot \|$ and $(\cdot, \cdot)$ denote the norm and inner product on the real Hilbert space $L^2(V)$.

Multiply $(6)_1$ by u_i^f, $(6)_2$ by u_i^p, and integrate over V. Integrate by parts and use $(6)_{4,5}$ and the boundary conditions and add the results to derive the equation

$$0 = -\gamma_1 \parallel \mathbf{u}^f \parallel^2 -\gamma_2 \parallel \mathbf{u}^p \parallel^2 - \parallel \mathbf{u}^f - \mathbf{u}^p \parallel^2 +R(\theta, w^f + w^p). \tag{8}$$

Next, multiply $(6)_3$ by θ and integrate over V to find after using the boundary conditions

$$\frac{d}{dt}\frac{1}{2} \parallel \theta \parallel^2 + (u_i^f + u_i^p, \theta\theta_{,i}) + (\Gamma_1 + \Gamma_2)R(x\,\theta, \theta_{,z}) \tag{9}$$
$$= - \parallel \nabla\theta \parallel^2 -(\theta, u^f + u^p).$$

The cubic term is zero since

$$(u_i^f + u_i^p, \theta\theta_{,i}) = -\frac{1}{2}(u_{i,i}^f + u_{i,i}^p, \theta^2) + \oint_{\partial V} (u_i^f + u_i^p)\, n_i\, \theta^2\, dA = 0, \tag{10}$$

where ∂V denote the boundary of V. In addition

$$(x\,\theta, \theta_{,z}) = \frac{1}{2} \int_V \frac{\partial}{\partial z}(x\,\theta^2)\, dx = 0, \tag{11}$$

once θ satisfies a periodicity condition in y and z. Hence, (9) reduces to

$$\frac{d}{dt}\frac{1}{2} \parallel \theta \parallel^2 = - \parallel \nabla\theta \parallel^2 -(\theta, u^f + u^p). \tag{12}$$

Next, let $\lambda > 0$ be a coupling parameter and form (12)+λ(8) to obtain

$$\frac{d}{dt}\frac{1}{2} \parallel \theta \parallel^2 = - \parallel \nabla\theta \parallel^2 -\lambda\gamma_1 \parallel \mathbf{u}^f \parallel^2 -\lambda\gamma_2 \parallel \mathbf{u}^p \parallel^2 \tag{13}$$
$$- \lambda \parallel \mathbf{u}^f - \mathbf{u}^p \parallel^2 -(\theta, u^f + u^p) + R\lambda(\theta, w^f + w^p).$$

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Use the arithmetic-geometric mean inequality on the u^f, u^p, w^f and w^p terms in the form
$ab \leq a^2/2\sigma + \sigma b^2/2$ for a suitable $\sigma > 0$ to derive from (13)

$$\frac{d}{dt}\frac{1}{2} \parallel \theta \parallel^2 \leq - \parallel \nabla\theta \parallel^2 + \left(\frac{1}{2\lambda} + \frac{\lambda R^2}{2}\right)\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right) \parallel \theta \parallel^2 \\ - \lambda\gamma_1 \parallel v^f \parallel^2 -\lambda\gamma_2 \parallel v^p \parallel^2 -\lambda \parallel \mathbf{u}^f - \mathbf{u}^p \parallel^2 , \tag{14}$$

where $v^f = u_2^f$, $v^p = u_2^p$. From (14) and the Poincaré inequality $\pi \parallel \theta \parallel \leq \parallel \nabla\theta \parallel$, it follows that

$$\frac{d}{dt}\frac{1}{2} \parallel \theta \parallel^2 \leq -k \parallel \theta \parallel^2 -\lambda\gamma_1 \parallel v^f \parallel^2 -\lambda\gamma_2 \parallel v^p \parallel^2 -\lambda \parallel \mathbf{u}^f - \mathbf{u}^p \parallel^2 , \tag{15}$$

where

$$k = \pi^2 - \left(\frac{1}{2\lambda} + \frac{\lambda R^2}{2}\right)\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right) .$$

Pick $\lambda = 1/R$ and then (15) leads to exponential decay of $\parallel \theta(t) \parallel^2$ if $k > 0$, i.e. if

$$R\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right) < \pi^2 . \tag{16}$$

From equation (8) we use the arithmetic-geometric mean inequality to deduce

$$\gamma_1 \parallel \mathbf{u}^f \parallel^2 +\gamma_2 \parallel \mathbf{u}^p \parallel^2 \leq R^2 \left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right) \parallel \theta \parallel^2 . \tag{17}$$

Thus condition (16) also guarantees exponential decay of $\parallel \mathbf{u}^f \parallel$ and $\parallel \mathbf{u}^p \parallel$.

Hence, we have demonstrated that condition (16) leads to global stability.

4   Linear instabilty

In this section we derive the analogue of the Gill [1] result for a bisdispersive porous medium. This then allows us to compare the linear instability result with the global nonlinear stability one. This will also allow us to compare our results to the analogous ones for a single porosity material.

In the next section we address the fully nonlinear problem for stability when the Rayleigh number is not restricted by (16). The linear instability result follows as a by product of equation (28) derived there. In fact, if we employ the linearized equations which arise from (6) then instead of equation (28) one derives, *in the linearized case*,

$$\frac{d}{dt}\frac{1}{2}\|\nabla\theta\|^2 = -\|\Delta\theta\|^2. \tag{18}$$

If we appeal to inequality (35) then we may show

$$\frac{d}{dt}\frac{1}{2}\|\nabla\theta\|^2 \leq -2\pi^2\|\nabla\theta\|^2, \tag{19}$$

and after integration,

$$\|\nabla\theta(t)\|^2 \le \exp(-2\pi^2 t)\,\|\nabla\theta(0)\|^2. \tag{20}$$

We then employ Poincaré's inequality to see that

$$\|\theta(t)\|^2 \le \frac{1}{\pi^2}\exp(-2\pi^2 t)\,\|\nabla\theta(0)\|^2. \tag{21}$$

An appeal to inequality (17) now shows

$$\gamma_1\|\mathbf{u}^f\|^2 + \gamma_2\|\mathbf{u}^p\|^2 \le \frac{R^2}{\pi^2}\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)\|\nabla\theta(0)\|^2\exp(-2\pi^2 t). \tag{22}$$

Observe that $\|\theta\|$, $\|\mathbf{u}^f\|$ and $\|\mathbf{u}^p\|$ then decay exponentially regardless of how large R is. This means that *according to linearized theory* there is never instability in the vertical bidispersive convection problem. This is thus entirely analogous to the single porosity result of Gill [1].

5 Stability for all Rayleigh numbers

In this section our goal is to establish an optimal result. Gill [1] showed that any *two-dimensional* disturbance to the vertical convection problem in a one porosity fluid is stable in the linear sense for *all* values of the Rayleigh number. We have shown this linear result is true for three - dimensional disturbances in the bidispersive problem. We shall now show that one may establish a similar result for the vertical convection problem for a bidispersive porous medium in three-dimensions, in the fully nonlinear case, but, at the expense of a restriction on a suitable measure of the initial data.

The starting point is again the perturbation equations (6). Firstly, take curl curl of $(6)_{1,2}$ to see that with the aid of $(6)_{4,5}$

$$\left.\begin{array}{l} \gamma_1\Delta u_i^f + (\Delta u_i^f - \Delta u_i^p) = R(k_i\Delta\theta - k_j\theta_{,ij})\,, \\[2mm] \gamma_2\Delta u_i^p + (\Delta u_i^p - \Delta u_i^f) = R(k_i\Delta\theta - k_j\theta_{,ij})\,. \end{array}\right\} \tag{23}$$

Next, multiply $(6)_3$ by $-\Delta\theta$ and integrate the result over V. After using the boundary conditions one may see that

$$\frac{d}{dt}\frac{1}{2}\|\nabla\theta\|^2 = -\|\Delta\theta\|^2 + (u^f+u^p,\Delta\theta) + (\Gamma_1+\Gamma_2)R\,(x\,\theta_{,z},\Delta\theta)$$
$$+\int_V (u_i^f+u_i^p)\,\theta_{,i}\,\Delta\theta\,dx\,. \tag{24}$$

Now

$$(x\,\theta_{,z},\Delta\theta) = -\int_V x\,\theta\,\Delta\theta_{,z}\,dx$$
$$= \frac{1}{2}\int_V \frac{\partial}{\partial z}\left(x|\nabla\theta|^2\right)\,dx + (\theta,\theta_{,xz}) = (\theta,\theta_{,xz})\,. \tag{25}$$

Equations (23) for $i = 1$ now yield

$$\left.\begin{array}{l} -\gamma_1 \Delta u^f - (\Delta u^f - \Delta u^p) = R\,\theta_{,xz}\,, \\ -\gamma_2 \Delta u^p + (\Delta u^f - \Delta u^p) = R\,\theta_{,xz}\,. \end{array}\right\} \tag{26}$$

Then using (26) we employ (25) to find

$$\begin{aligned} (\Gamma_1 + \Gamma_2)\,R\,(x\,\theta_{,z}, \Delta\theta) =& \Gamma_1 R(\theta, \theta_{,xz}) + \Gamma_2 R(\theta, \theta_{,xz}) \\ =& \Gamma_1\left(\theta,\, -\gamma_1 \Delta u^f - (\Delta u^f - \Delta u^p)\right) \\ &+ \Gamma_2\left(\theta,\, -\gamma_2 \Delta u^p + (\Delta u^f - \Delta u^p)\right) \\ = [-\Gamma_1(\gamma_1 + 1) + \Gamma_2](u^f, \Delta\theta) &+ [-\Gamma_2(\gamma_2 + 1) + \Gamma_1]\,(u^p, \Delta\theta) \\ =& -(u^f, \Delta\theta) - (u^p, \Delta\theta)\,, \end{aligned} \tag{27}$$

where we have integrated by parts twice and employed the boundary conditions.

Now, employ (27) in (24) to obtain

$$\frac{d}{dt}\frac{1}{2}\parallel \nabla\theta \parallel^2 = -\parallel \Delta\theta \parallel^2 + \int_V (u_i^f + u_i^p)\,\theta_{,i}\,\Delta\theta\,dx\,. \tag{28}$$

The cubic terms are estimated using the inequality

$$\sup_V |\mathbf{u}^\alpha| \le C \parallel \Delta\mathbf{u}^\alpha \parallel\,, \tag{29}$$

for $\alpha = f$ or p, see e.g. Straughan [17], where C is a constant which depends on V. We write

$$\begin{aligned} \int_V (u_i^f + u_i^p)\,\theta_{,i}\,\Delta\theta\,dx \le& \left(\sup_V |\mathbf{u}^f| + \sup_V |\mathbf{u}^p|\right) \parallel \nabla\theta \parallel \parallel \Delta\theta \parallel \\ \le& \, C\left(\parallel \Delta\mathbf{u}^f \parallel + \parallel \Delta\mathbf{u}^p \parallel\right) \parallel \nabla\theta \parallel \parallel \Delta\theta \parallel\,. \end{aligned} \tag{30}$$

One next employs (23) to see that

$$\left.\begin{array}{l} \Delta u_i^f = -\Gamma_1\,R\,(k_j\theta_{,ij} - k_i\Delta\theta) \\ \Delta u_i^p = -\Gamma_2\,R\,(k_j\theta_{,ij} - k_i\Delta\theta) \end{array}\right\} \tag{31}$$

Therefore, for $\alpha = f$ or p,

$$\begin{aligned} \parallel \Delta\mathbf{u}^\alpha \parallel^2 =& \Gamma_\alpha^2\,R^2 \int_V (k_j\,\theta_{,ij} - k_i\,\Delta\theta)\,(k_r\,\theta_{,ir} - k_i\,\Delta\theta)\,dx \\ =& \Gamma_\alpha^2\,R^2(\parallel \Delta\theta \parallel^2 - \parallel \theta_{,zz} \parallel^2 - \parallel \theta_{,xz} \parallel^2 - \parallel \theta_{,yz} \parallel^2) \\ \le& \Gamma_\alpha^2\,R^2 \parallel \Delta\theta \parallel^2\,, \end{aligned} \tag{32}$$

where we have integrated by parts and used the boundary conditions, observing $\theta_{,zz} = 0$ at $x = \pm 1/2$.

Return to (28) and use (30) and (32) to derive

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\parallel \nabla\theta \parallel^2 \le& -\parallel \Delta\theta \parallel^2 + CR\,(\Gamma_1 + \Gamma_2)\parallel \nabla\theta \parallel \parallel \Delta\theta \parallel^2 \\ =& -\parallel \Delta\theta \parallel^2\,[1 - CR\,(\Gamma_1 + \Gamma_2)\parallel \nabla\theta \parallel]\,. \end{aligned} \tag{33}$$

Now, observe

$$\parallel \nabla\theta \parallel^2 = -(\theta, \Delta\theta) \leq \parallel \theta \parallel \parallel \Delta\theta \parallel \leq \frac{1}{\pi} \parallel \nabla\theta \parallel \parallel \Delta\theta \parallel \tag{34}$$

where we have integrated by parts, used the Cauchy-Schwarz inequality and the Poincaré inequality. From (34) it follows that

$$\pi \parallel \nabla\theta \parallel \leq \parallel \Delta\theta \parallel . \tag{35}$$

If now

$$\parallel \nabla\theta(0) \parallel < \frac{1}{CR(\Gamma_1 + \Gamma_2)} \tag{36}$$

then from (33) a continuity argument together with (35) allows one to deduce $\parallel \nabla\theta(t) \parallel$ decays at least exponentially. Further, from Poincaré's inequality we obtain the same decay for $\parallel \theta(t) \parallel$ and then an appeal to (17) ensures similar decay for $\parallel \mathbf{u}^f \parallel$ and $\parallel \mathbf{u}^p \parallel$.

We have, therefore, established that provided (36) holds the vertical bidispersive convection problem is stable for all values of the Rayleigh number R.

Inequality (36) may be perceived as a strong restriction on the H^1 norm of the initial temperature field since the right hand side becomes vanishingly small as $R \to \infty$. However, it is entirely equivalent to the situation in the single porosity case as found by Straughan [3]. To understand this we need to recognize that the Rayleigh number R_s used by Straughan [3] is defined as

$$R_s = \frac{\rho c}{\kappa_m} \frac{K_f \alpha g (T_h - T_L) L}{(\mu/\rho)} .$$

The analogous result of Straughan [3] derives nonlinear stability for all Rayleigh numbers when

$$\|\nabla\theta(0)\| < \frac{1}{R_s C} . \tag{37}$$

We can identify this result with (36) when we make the identification $R = R_s \gamma_1$ between the Rayleigh number used here and R_s as used in Straughan [3]. For the single porosity case we may take $K_p = 0, \zeta = 0$, and then if we take the limit $K_p \to 0, \zeta \to 0$, in (36) we find

$$R(\Gamma_1 + \Gamma_2) = R_s \gamma_1 (\Gamma_1 + \Gamma_2) \to R_s \left(\frac{1}{1 + 1/\gamma_1} \right) \to R_s.$$

Thus, (36) is analogous to (37).

In addition, we may employ inequality (17) and the Poincaré inequality to find

$$\frac{\pi^2}{R^2} \left(\gamma_1 \|\mathbf{u}^f\|^2 + \gamma_2 \|\mathbf{u}^p\|^2 \right) \leq \left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2} \right) \|\nabla\theta\|^2.$$

When we combine this with (36) we obtain the following restriction on the initial velocity fields

$$\pi(\gamma_1 \|\mathbf{u}^f\|^2 + \gamma_2 \|\mathbf{u}^p\|^2)^{1/2} \le R\Big(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\Big)^{1/2} \|\nabla\theta(0)\|$$
$$\le \Big(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\Big)^{1/2} \frac{1}{C(\Gamma_1 + \Gamma_2)}\,.$$

Thus, there is a restriction on the size of the initial velocity field, but it does not depend on the size of R.

Even though inequality (36) does depend on R it represents a restriction only on the initial temperature gradient. In any case, the decay result obtained is a fully nonlinear one.

6   Conclusions

We have studied the problem of convection in a fluid saturated bidispersive porous medium when the porous layer is vertical and the temperatures on the vertical sides are constant but different and thus in the steady state generate a temperature gradient and vertical flow.

We have derived a global stability result which shows that for R less than a critical value one has global stability, i.e. for all initial data. We have further shown that one has nonlinear stability for all values of R if the initial values of the gradient of the temperature field are suitably restricted. Since inequality (36) involves the Rayleigh number R it still leaves open the possibility of a finite amplitude subcritical instability if (36) and (16) are not satisfied. It would be interesting to perform numerical computations in three - dimensions to see if subcritical instabilities can be found in this situation.

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