

# Modelling drinking with information

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## Abstract

In this article we propose a mathematical model that describes the dynamics of a population divided into susceptible drinkers, moderate drinkers, and heavy drinkers subject to an external influence. The external influence is modelled using a supplementary dynamical variable which is not a group of individuals but that enters the equations affecting the choices of the population classes.

The system we define can be investigated using two simplified systems (one of which is a real subsystem) which model the populations of susceptible and moderate drinkers or susceptible and heavy-drinkers independently. The dynamics of these two subsystems can be described exhaustively. The full system is too rich in possible scenarios, but its qualitative behaviour is connected to that of the two simplified systems. We make a complete description only in one particular case by means of numerical simulations.

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## 1 Introduction

Alcohol is well known to cause a multitude of negative effects such as: change of mood, behaviour, and coordination, heart problems, liver problems, cancer [1]. In recent times, particularly in western society, peer pressure drives youngsters towards an extreme behaviour denoted as *binge drinking*. The concept of binge drinking is defined, for instance in [2], and it consists in drinking large amounts of alcohol to become heavily intoxicated. Binge drinking is primarily a practice of youths in the 17-30 age group, and people that indulge in binge drinking are likely to exhibit anti-social behaviour. To binge drinking can be attributed losses of the order of billions of dollars in the U.S. alone, caused by diminished productivity and health issues associated to auto-related accidents, traumas, and crimes (including physical and sexual assault) [3]. Binge drinking can affect well over 25 per cent of the population as may be seen from specific data for Durham City [4] and Stockton-on-Tees [5] in North East England, with some areas being as high as 28.93 per cent, in e.g. Newcastle-upon-Tyne, [4, 5]. In addition the death rate due to drinking alcohol has risen dramatically in the U.K. with, for example, Glasgow recording a death rate of 83.7 per 100,000 for both males and females during 1988-2004, see [6].

Drinking has been investigated using the techniques of epidemical models, and such approach has proven to accurately predict the percentages of binge and moderate drinkers [7, 8, 9, 10, 11, 12, 13, 14].

What we believe has never been modelled in such a field is the influence of external variables that favour or discourage such habits. This external influence can be modelled introducing the concept of an *information variable*. To our knowledge, the idea of expanding an epidemiological system introducing an information variable can be found for the first time in [15, 16, 17, 18, 19, 20]. In such works the authors investigate a vaccinating behaviour where the information variable measures the publicly available information on the state of the disease. This influences the adhesion to vaccination programs which, in turn, influences the state of the disease. An information function approach has been employed in modelling the dynamics of crystal meth (“tik”) abuse by Nyabadza et al. [21]. This is a very interesting article where the information function is effectively the drug supply. This, therefore, yields a way an authority can control access to a drug and assess the availability in the context of treatment. In addition an information function approach has been adopted to study mosquito-borne epidemics like yellow virus, see Avila-Vales et al. [22].

Mathematically, an *information variable* is a dynamical variable  $M(t)$  that represents some type information which influences and is influenced by the system. Different from [19], in our model the information variable models the availability of alcohol, and so is in some ways like the function of Nyabadza et al. [21], albeit the functional form is different. We assume that abundance of alcohol in shops does make it simpler to young adults to try and be involved in binge drinking when exposed to peers that propose that behaviour. To model this fact we assume that the population is split in three classes: the susceptible  $S$ , the moderate drinkers  $A$ , and the binge or heavy drinkers  $B$ . The supply of alcohol  $M$  influences in a monotonically increasing way the magnitude of the *contact rate parameters*  $\beta$  and  $\gamma$  that are connected to the probability that, by peer pressure, a susceptible becomes a moderate drinker or a heavy drinker. On the other hand the existence of moderate drinkers and of heavy drinkers does effect differently the availability of alcohol, so that the number of moderate drinkers  $A$  and

of heavy drinkers  $B$  enter in the evolution equation of the information  $M$  in qualitatively different way (linearly for  $A$  and super-linearly for  $B$ ).

Naturally, more than one information variable can be introduced. For example one could try to model the effects of a stringent legislation on availability of alcohol at certain times of the day or in certain places. This idea could allow the introduction of contact rate functions that are monotonically decreasing functions of the information. We do not pursue this alternative venue. However, we do point out that availability can be linked to cost and a minimum pricing legislation as proposed recently in Scotland may well be modelled by the concept of an information function.

The goal of this work is to investigate the existence and the type of endemic equilibria. We will show that there exist four bifurcation parameters  $R_0, R_1, T_0, T_1$ , two of which  $R_0, T_0$  are the usual basic reproduction numbers associated respectively to the stability of the disease-free equilibrium for a system with susceptible and moderate drinkers only, or susceptible and heavy drinkers only.

The stability of the equilibria is analysed thoroughly in two particular cases: in the first case we consider moderate drinkers alone, in the second we consider only heavy drinkers. The general case is investigated analytically up to a certain degree but, due to the high variety of cases, is analysed completely only for special choices of the parameters.

The plan of the article is the following. In Section 2 we introduce the model and the main dynamical system and find an invariant region. Section 3 is devoted to model moderate drinkers, to study the equilibria and their stability. In Section 4 we study a subsystem of the complete system that model heavy drinkers. In Section 5 we consider an intermediate model to study the transition from the moderate drinkers to the heavy drinkers. Section 6 is devoted to the general model with moderate and heavy drinkers. The stability of the equilibria are presented with some numerical simulations. At the end of the article we draw some conclusions and present some possible further investigations.

## 2 Mathematical formulation

Let a local population be divided into three, time dependent classes,  $S$ ,  $A$  and  $B$ . The class  $S$  represents those individuals who are susceptible to drink, i.e. those who do not drink or drink only very moderately. The class  $A$  denotes the moderate drinkers and  $B$  are the heavy drinkers. According to the *Dietary Guidelines for Americans 2015-2020* [23] moderate drinking is up to 1 drink per day for women and up to 2 drinks per day for men, the National Institute of Alcohol Abuse and Alcoholism defines binge drinking as a pattern of drinking that brings blood alcohol concentration (BAC) levels to 0.08 g/dL. This typically occurs after 4 drinks for women and 5 drinks for men in about 2 hours [24].

The Substance Abuse and Mental Health Services Administration (SAMHSA), which conducts the annual National Survey on Drug Use and Health (NSDUH), defines binge (or heavy) drinking as drinking 5 or more alcoholic drinks on the same occasion on at least 1 day in the past 30 days (see for example <https://www.niaaa.nih.gov/alcohol-health>). Other possible definitions of moderate and heavy drinkers can be found in [25]. Physicians operationally define as *light drinking* the assumption of 1.2 drinks/day, *moderate drinking* the assumption of 2.2 drinks/day, and *heavy drinking* the assumption of 3.5 drinks/day [26, 27].

The total local population  $N = S + A + B$  is constant, and  $M$  is the *information variable*. The model we adopt is governed by the differential equations

$$\begin{cases} \dot{S} = \mu N - \mu S - \frac{\beta(M)}{N} AS - \frac{\gamma(M)}{N} BS + \xi A + \eta B \\ \dot{A} = -\mu A + \frac{\beta(M)}{N} AS - \xi A - \kappa A \\ \dot{B} = -\mu B + \frac{\gamma(M)}{N} BS - \eta B + \kappa A \\ \dot{M} = -\alpha M + \varphi A + \psi B^2, \end{cases} \quad (1)$$

where  $\beta, \gamma$  are called the *information functions* and model the effect of peer pressure on classes. In this work we assume that the information functions have the form

$$\beta(M) = \beta_0 + \beta_1 M, \quad \gamma(M) = \gamma_0 + \gamma_1 M.$$

Other possible choices of information functions can be found in Section 7. The last equation of the vector field (related to the evolution of  $M$ ) models how the information depends on the state of the system. The natural timescale for the model is the year. The information variable  $M$  models the availability of alcohol that is measured in litres of pure alcohol. The parameters appearing in the equations are listed below:

- $\mu$  represents the rate (per unit time) of entry;
- $\xi$  is that fraction of  $A$  (per unit time) who returns into the  $S$  compartment;
- $\eta$  is that fraction of  $B$  (per unit time) who goes into the heavy drinkers compartment;

- $k$  is that fraction of  $A$  (per unit time) who become heavy drinkers;
- $\beta$  and  $\gamma$  are the contact rates which depend on  $M$ ;
- $\beta_0$  and  $\gamma_0$  are the probability (per unit time) that an individual change compartment due to contact with a moderate or a heavy drinker;
- $\beta_1$  and  $\gamma_1$  are the variations of the contact rates due to the information variable;
- $\alpha$  is the decay rate (per unit time) of information;
- $\varphi$  is the growth rate of information (per unit time per unit of population);
- $\psi$  is the growth rate of information (per unit time per unit of squared population).

## 2.1 Invariant region

System (13) has the invariant function  $S + A + B$ , whose value is constant along motions, moreover the only biologically significant set is the set

$$\Omega = \{(S, A, B, M) \in \mathbb{R}^4 \mid S + A + B \equiv N, S, A, B \geq 0\}.$$

Points in  $\Omega$  will be called *biologically admissible*. For the problem to be well posed, the set  $\Omega$  must be positively invariant.

**Lemma 1** *The region  $\Omega$  is positively invariant.*

The proof of this fact is a very standard computation that we will not perform explicitly. As customary, the presence of an invariant of motion allows to reduce the degrees of freedom. Posing  $s = S/N$ ,  $a = A/N$ ,  $b = B/N$ , and  $m = M$ , one has that  $s = 1 - a - b$  and the system can be reduced to

$$\begin{cases} \dot{a} = (\beta(m) - \mu - \xi - \kappa)a - \beta(m)a^2 - \beta(m)ab \\ \dot{b} = (\gamma(m) - \mu - \eta)b - \gamma(m)b^2 - \gamma(m)ab + \kappa a \\ \dot{m} = -\alpha m + \varphi a + \psi b^2. \end{cases} \quad (2)$$

In these new variables, the reduced system admits a biologically significant, positive invariant region

$$\Omega = \{(a, b, m) \in \mathbb{R}^3 \mid a, b \geq 0, a + b \leq 1\}.$$

For simplicity we keep denoting with the letter  $\Omega$  the biologically admissible region of the reduced system. For the same reason such region will also be called with the same name in the various subsystems of this original system.

## 3 The model for moderate drinkers

We consider a model with only two classes: that of non-drinkers  $S$  and that of moderate drinkers  $A$ . The variable  $M$  is the information parameter that, as in the general case, models the supply of alcohol.

We observe that we are not dealing with a subsystem of the original system because the plane  $b = 0$ , is not an invariant submanifold unless  $\kappa = 0$ . Taking into account that plausible values for  $\kappa$  are very small, in this Section we assume  $\kappa = 0$ .

The reduced equations are

$$\begin{cases} \dot{a} = (\beta_0 + \beta_1 m - \mu - \xi)a - (\beta_0 + \beta_1 m)a^2 \\ \dot{m} = -\alpha m + \varphi a. \end{cases} \quad (3)$$

### 3.1 Dimensional form and equilibria

To write the equations in a nondimensional form we pose

$$m = \frac{\varphi}{\alpha} n, \quad s = (\mu + \xi)t, \quad R_0 = \frac{\beta_0}{\mu + \xi}, \quad R_1 = \frac{\varphi}{\alpha} \frac{\beta_1}{\mu + \xi}, \quad \omega = \frac{\varphi}{\mu + \xi}.$$

Denoting with a prime the derivative with respect to the new time  $s$ , one reduces the equations to the system, that we call *moderate drinkers system*,

$$\begin{cases} a' = (R_0 - 1 + R_1 n)a - (R_0 + R_1 n)a^2 \\ n' = \omega(a - n). \end{cases} \quad (4)$$

The equilibria of this system are  $E = (\hat{a}, \hat{n})$  with  $\hat{n} = \hat{a}$  and with  $\hat{a}$  such that  $\hat{a}(R_0 - 1 + (R_1 - R_0)\hat{a} - R_1\hat{a}^2) = 0$ . It follows that the system admits the solution  $\hat{a} = 0$ , that corresponds to the disease-free equilibrium  $E_0 = (0, 0)$  and all biologically admissible solutions associated to zeroes of the quadratic polynomial

$$p_2 = R_0 - 1 + (R_1 - R_0)a - R_1a^2.$$

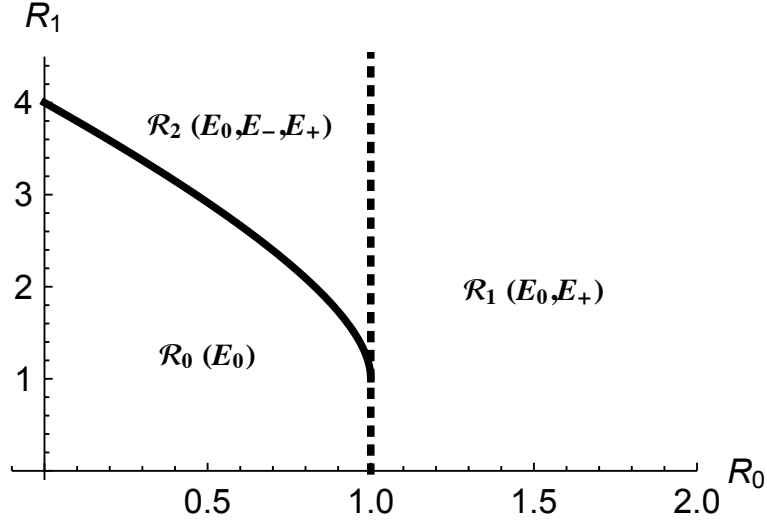


Figure 1: Across the thin-dashed line two negative equilibria collide and become complex. Being negative, such equilibria can be treated as non-existing.

**Theorem 2** *The parameter space  $R_0, R_1$  can be divided into the three regions shown in Fig. 1*

- Region  $\mathcal{R}_0 = \{(R_0, R_1) \mid R_0 < 1, R_1 < 2 - R_0 + 2\sqrt{1 - R_0}\}$  corresponds to systems with no positive equilibria;
- Region  $\mathcal{R}_1 = \{(R_0, R_1) \mid R_0 > 1\}$  corresponds to systems with one positive equilibrium  $E_+$ .
- Region  $\mathcal{R}_2 = \{(R_0, R_1) \mid R_0 < 1, R_1 > 2 - R_0 + 2\sqrt{1 - R_0}\}$  corresponds to systems with two positive equilibria  $E_{\pm}$ ;

**Proof** Other than the equilibrium  $E_0$ , polynomial  $p_2$  admits at most two solutions, which lead to two equilibria  $E_{\pm} = (\hat{a}_{\pm}, \hat{a}_{\pm})$ , where

$$\hat{a}_{\pm} = \frac{R_1 - R_0 \pm \sqrt{(R_1 - R_0)^2 + 4(R_0 - 1)R_1}}{2R_1}.$$

The two equilibria  $E_{\pm}$  are not always biologically admissible (i.e. they do not always belong to  $\Omega$ ). Their existence as real solutions depends on the sign of the discriminant, their positivity depends, according to Descartes's rule, on the changes of sign of the coefficients of the polynomial  $p_2$ . One easily obtains that the polynomial  $p_2$  has

- no positive roots and two or zero negative roots if  $R_0 < 1$  and  $R_1 < R_0$ ;
- two or zero positive roots and no negative roots if  $R_0 < 1$  and  $R_1 > R_0$ ;
- one positive and one negative root if  $R_0 > 1$ .

It is also easy to check that both  $\hat{a}_{\pm}$  are always less than 1. These facts together with the fact that the discriminant of  $p_2$  is  $(R_0 + R_1)^2 - 4R_1$  allow to make Figure 1 and to draw the conclusions.  $\square$

### 3.2 Stability analysis

In order to study the stability of the equilibria, let us first exclude the existence of limiting cycles for (4) using Dulac's theorem [28, pag. 246], [29, pp. 205–210].

**Theorem 3** *System (4) does not admit limiting cycles.*

**Proof** Define a Dulac function by  $F = 1/a$ . If  $X = (X_1, X_2)$  is the vector field associated to (4), a direct calculation leads to

$$\frac{\partial(FX_1)}{\partial a} + \frac{\partial(FX_2)}{\partial n} = -(R_0 + R_1n) - \frac{\omega}{a} < 0$$

for every  $a, n > 0$ . Dulac's theorem allows to prove the theorem.  $\square$

It follows that the phase-portrait can be drawn as soon as the stability of the equilibria becomes clear.

**Theorem 4** *When the parameters belong to the region  $\mathcal{R}_0$  the system admits a unique globally stable equilibrium, that is the disease-free equilibrium  $E_0$ . When the parameters belong to the region  $\mathcal{R}_1$  then the disease free equilibrium is unstable, while the equilibrium  $E_+$  is biologically admissible and globally stable. When the parameters belong to the region  $\mathcal{R}_2$  then there exist two stable equilibria  $E_0, E_+$  and one saddle  $E_-$ . The biologically significant region is the disjoint union of two basins of attraction respectively of the two stable equilibria.*

**Proof** The Jacobian matrix  $J(a, n)$  of the vector field  $X$  is

$$J(a, n) = \begin{pmatrix} R_0 - 1 + R_1n - 2a(R_0 + R_1n) & R_1a(1 - a) \\ \omega & -\omega \end{pmatrix}.$$

It follows that

$$J(E_0) = \begin{pmatrix} R_0 - 1 & 0 \\ \omega & -\omega \end{pmatrix}, \quad J(E_{\pm}) = \begin{pmatrix} -\hat{a}_{\pm}(R_0 + R_1\hat{a}_{\pm}) & R_1\hat{a}_{\pm}(1 - \hat{a}_{\pm}) \\ \omega & -\omega \end{pmatrix}.$$

Computing the determinant and the trace of the Jacobians above, we have

$$\det J(E_0) = -\omega(R_0 - 1), \quad \text{tr } J(E_0) = -\omega + R_0 - 1$$

which implies that  $E_0$  is a stable node if  $R_0 < 1$ , and it is a saddle otherwise. At the other two equilibria

$$\det J(E_{\pm}) = \pm\omega\hat{a}_{\pm}\sqrt{(R_1 - R_0)^2 + 4(R_0 - 1)R_1},$$

$$\text{tr } J(E_{\pm}) = -\omega - \hat{a}_{\pm}(R_0 + R_1\hat{a}_{\pm})$$

which allows to prove that, when they are biologically admissible, are respectively stable  $E_+$  and unstable  $E_-$ .  $\square$

It follows that qualitatively the phase portrait of this system can be summarized in the three panels shown in Figure 2

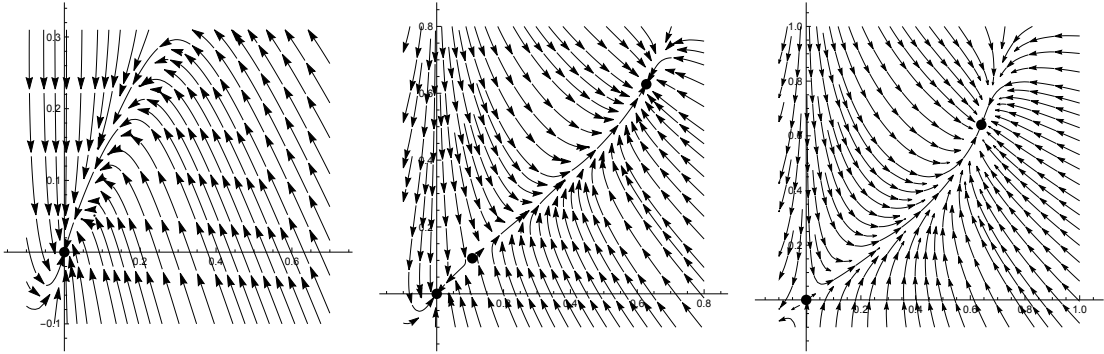


Figure 2: The three possible phase diagrams corresponding to choices of  $R_0, T_0$  in the three regions of Fig 1.

## 4 The model for heavy drinkers

Let us consider a model still possessing two classes, namely non-drinkers and heavy drinkers. This system is a subsystem of (2):

$$\begin{cases} \dot{b} = (\gamma_0 - \mu - \eta + \gamma_1 m)b - (\gamma_0 + \gamma_1 m)b^2 \\ \dot{m} = -\alpha m + \psi b^2. \end{cases}$$

## 4.1 Nondimensionalization and equilibria

To write the equations in a nondimensional form, we pose

$$m = \frac{\psi}{\alpha}n, \quad s = (\mu + \eta)t, \quad T_0 = \frac{\gamma_0}{\mu + \eta}, \quad T_1 = \frac{\psi}{\alpha} \frac{\gamma_1}{\mu + \eta}, \quad \omega = \frac{\psi}{\mu + \eta},$$

and recast the equations in the form, that we call *heavy drinkers system*,

$$\begin{cases} b' = (T_0 - 1 + T_1n)b - (T_0 + T_1n)b^2 \\ n' = \omega(b^2 - n). \end{cases} \quad (5)$$

Also in this case the equilibria are the disease-free solution  $E_0 = (0, 0)$ , the equilibria  $E_i = (\hat{b}_i, \hat{b}_i^2)$  with  $\hat{b}_i$  solution of the cubic polynomial

$$p_3 = -T_1b^3 + T_1b^2 - T_0b + T_0 - 1. \quad (6)$$

The biologically admissible equilibria can hence be from one to four depending on the existence and biological admissible zeroes of  $p_3$ . We summarise the possible situations in a theorem.

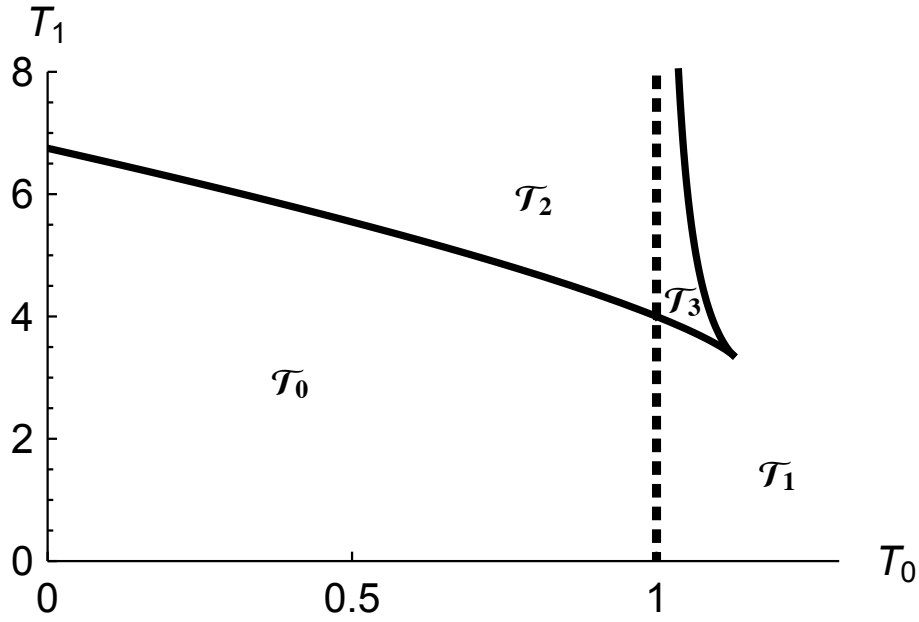


Figure 3: The four regions of the parameter space. The regions  $\mathcal{T}_i$  are labelled according to the number  $i = 0, 1, 2, 3$  that indicates the number of biologically admissible equilibria of the corresponding system, excluding the disease-free equilibrium.

**Theorem 5** *Let*

$$\Delta = -T_1 (4T_0^3 + 8T_1T_0^2 + 4(T_1 - 9)T_1T_0 + (27 - 4T_1)T_1)$$

*be the discriminant of the polynomial  $p_3$ . The parameter space  $T_0, T_1$  can be divided into the four regions, shown in Fig. 3*

- *Region  $\mathcal{T}_0 = \{(T_0, T_1) \mid T_0 < 1, \Delta < 0\}$  corresponds to systems with no positive equilibria;*
- *Region  $\mathcal{T}_1 = \{(T_0, T_1) \mid T_0 > 1, \Delta < 0\}$  corresponds to systems with one positive equilibrium  $E_1$ , the only real root of the polynomial  $p_3$ ;*
- *Region  $\mathcal{T}_2 = \{(T_0, T_1) \mid T_0 < 1, \Delta > 0\}$  corresponds to systems with two positive equilibria  $E_2, E_3$ , the two larger roots among the three real roots of the polynomial  $p_3$ ;*
- *Region  $\mathcal{T}_3 = \{(T_0, T_1) \mid T_0 > 1, \Delta > 0\}$  corresponds to systems with three positive equilibria  $E_1, E_2, E_3$ , the three real roots of the polynomial  $p_3$ .*

**Proof** To investigate the positivity of the equilibria one can use Descartes's rule on the sign of the coefficients of the polynomial  $p_3$  and obtain that

- when  $T_0 < 1$  there are 2 or 0 positive roots;
- when  $T_0 > 1$  there are 3 or 1 positive roots.

The value  $\hat{b}_i$  of the equilibrium  $E_i$  ( $i = 1, 2, 3$ ) is always less than 1 because the cubic polynomial has value  $-1$  in  $b = 1$ , in that point has negative derivative, and has maximum and minimum before  $2/3$ . It follows that it cannot have zeros bigger than 1.

The zeroes of the polynomial  $p_3$  can be either one real and two complex or three real depending on the positivity of its discriminant

$$\Delta = -T_1 (4T_0^3 + 8T_1T_0^2 + 4(T_1 - 9)T_1T_0 + (27 - 4T_1)T_1) .$$

This discriminant can be explicitly solved with respect to  $T_1$  and gives the two curves plotted in Figure 3

$$T_1 = \frac{4T_0(2T_0 - 9) + 27 - \sqrt{(9 - 8T_0)^3}}{8(1 - T_0)}, \quad T_1 = \frac{4T_0(2T_0 - 9) + 27 + \sqrt{(9 - 8T_0)^3}}{8(1 - T_0)} .$$

The graph of the second function is defined and positive for  $T_0 \in [0, 9/8]$  (the apparent singularity at  $T_0 = 1$  is a first kind singularity). The first function is defined and positive for  $T_0 \in (1, 9/8]$ . The result follows from these observations, and can be summarized with Figure 3.  $\square$

## 4.2 Stability analysis

Let us again exclude the possibility of having limit cycles. As done before, we prove this by means of Dulac's theorem [28, pag. 246], [29, pp. 205–210]. We recall that the vector field whose associated differential equations are (5) is given by

$$X = \begin{pmatrix} (T_0 - 1 + T_1m)b - (T_0 + T_1m)b^2 \\ \zeta(-m + b^2) \end{pmatrix} .$$

**Theorem 6** *System (5) does not admit limit cycles.*

**Proof** Let us use the Dulac function  $F = 1/b$ . A direct calculation leads to

$$\frac{\partial(FX_1)}{\partial b} + \frac{\partial(FX_2)}{\partial n} = -(T_0 + T_1n) - \frac{\omega}{b} < 0,$$

for every  $b, n > 0$ .  $\square$

Also in this case, we must discuss the stability of the equilibria.

**Theorem 7** *The parameter space  $T_0, T_1$  can be divided into four regions, shown in Fig. 3*

- Region  $\mathcal{T}_0$  corresponds to systems with only one globally stable equilibrium  $E_0$ ;
- Region  $\mathcal{T}_1$  corresponds to system with two equilibria,  $E_0$  is unstable and  $E_1$  globally stable;
- Region  $\mathcal{T}_2$  corresponds to systems with three equilibria,  $E_0$  and  $E_3$  are stable, while  $E_2$  is unstable (in the unphysical part of phase space there exists another equilibrium  $E_1$  that is unstable). The physical phase space can be divided into two regions that are the basin of attraction of the two stable equilibria, separated by the stable manifold of the equilibrium  $E_2$ ;
- Region  $\mathcal{T}_3$  corresponds to systems with four equilibria,  $E_0$  and  $E_2$  are unstable,  $E_1$  and  $E_3$  are stable. Also in this case the phase space is divided into two regions that are the basin of attraction of the two stable equilibria, separated by the stable manifold of the equilibrium  $E_2$ .

**Proof** The Jacobian matrix of the vector field associated to the system (5) is

$$J(b, n) = \begin{pmatrix} T_0 - 1 + T_1n - 2b(T_0 + T_1n) & T_1b(1 - b) \\ 2\omega b & -\omega \end{pmatrix} .$$

Computing the Jacobian matrix in  $E_0$  one obtains the matrix

$$J(E_0) = \begin{pmatrix} T_0 - 1 & 0 \\ 0 & -\omega \end{pmatrix} .$$

It follows that  $E_0$  is stable if and only if  $T_0 < 1$  and is a saddle otherwise. The other equilibria have the form  $E_i = (\hat{b}_i, \hat{b}_i^2)$  where  $\hat{b}_i$  are the positive real zeroes of the polynomial  $p_3$  given in (6). The condition of being



equilibria immediately implies that  $(T_0 - 1 + T_1 \hat{b}_i^2) = (T_0 + T_1 \hat{b}_i^2) \hat{b}_i$ , hence the Jacobian in the points  $E_i = (\hat{b}_i, \hat{b}_i^2)$  is

$$J(E) = \begin{pmatrix} -\hat{b}_i(T_0 + T_1 \hat{b}_i^2) & T_1 \hat{b}_i(1 - \hat{b}_i) \\ 2\omega \hat{b}_i & -\omega \end{pmatrix},$$

and has negative trace. The determinant of this matrix is  $\omega(\hat{b}_i(T_0 + T_1 \hat{b}_i^2) - T_1 \hat{b}_i(1 - \hat{b}_i)2\hat{b}_i)$ , that is proportional to the derivative  $p_3'(\hat{b}_i)$ . Since in the interior of the regions  $\mathcal{T}_i$  the polynomial  $p_3$  has simple zeroes, the real numbers  $p_3'(\hat{b}_i)$  have alternating signs when ordering the zeroes of  $p_3$ . This allows to conclude that, starting from  $E_0$ , the next equilibrium  $E_1$ , if it is biologically admissible, has opposite stability behavior, and so on.  $\square$

A qualitative plot of the phase portrait for the four cases is given in Figure 4.

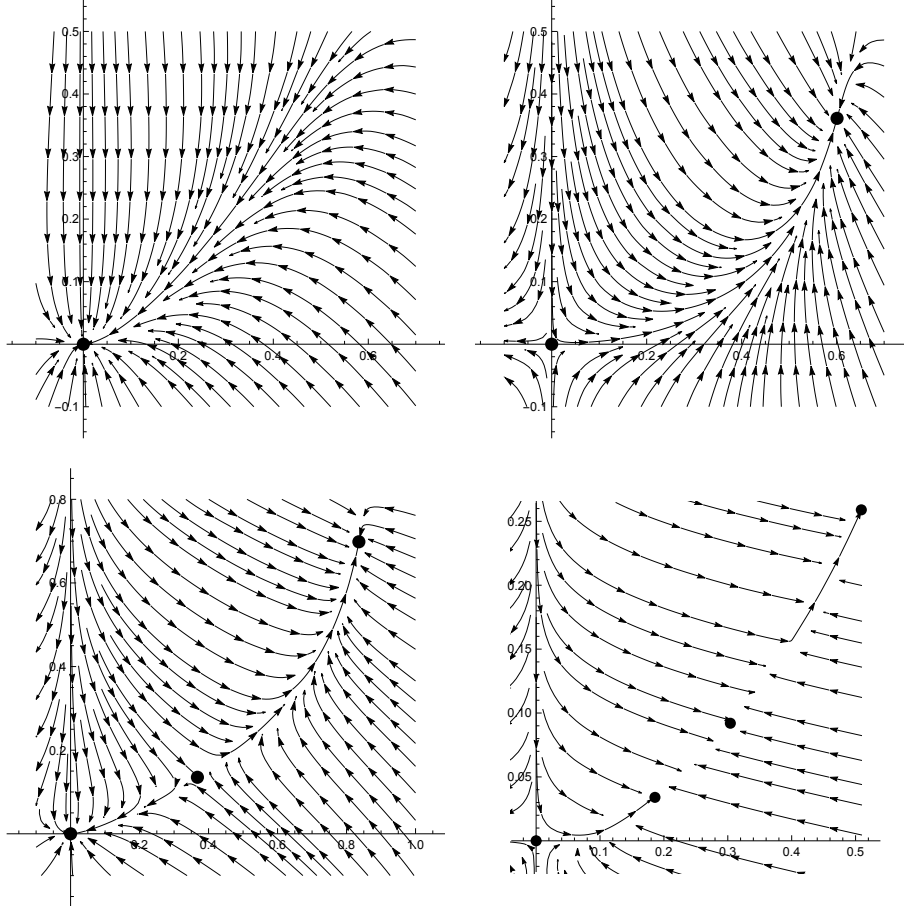


Figure 4: The four possible phase diagrams corresponding to choices of  $R_1, T_1$  in the four regions of Fig. 3.

## 5 An intermediate model

Let us consider now the transition from the moderate drinkers and the heavy drinkers. In this case the equations are

$$\begin{cases} \dot{S} = \mu N - \mu S - \frac{\lambda(M)}{N} CS + \nu C \\ \dot{C} = -\mu C + \frac{\lambda(M)}{N} CS - \nu C \\ \dot{M} = -\alpha M + (1 - \chi)\varphi C + \chi\psi C^2. \end{cases}$$

where  $C$  plays the role of  $A$  when  $\chi = 0$ ,  $\lambda = \beta$ , and  $\nu = \xi$ , it plays the role of  $B$  when  $\chi = 1$ ,  $\lambda = \gamma$ , and  $\nu = \eta$ , or it plays an intermediate role for all other choices of  $\chi \in [0, 1]$ . The same arguments used in the two previous sections allow to consider the normalised populations, to remove the first equation, to substitute  $\lambda(m)$  with  $\lambda_0 + \lambda_1 m$ , and to finally reduce to the ordinary differential equations

$$\begin{cases} \dot{c} = (\lambda_0 + \lambda_1 m - \mu - \nu)c - (\lambda_0 + \lambda_1 m)c^2 \\ \dot{m} = -\alpha m + (1 - \chi)\varphi c + \chi\psi c^2. \end{cases}$$



## 5.1 Nondimensionalization and equilibria

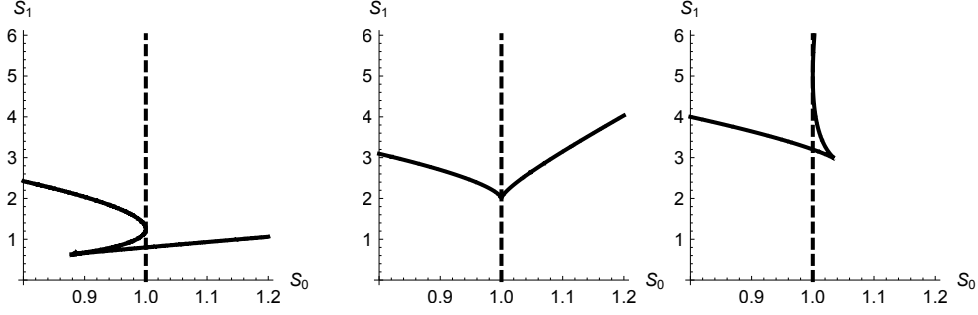


Figure 5: The bifurcation diagrams as  $\rho$  goes from 0 (the moderate drinkers case, first panel) to 1 (the heavy drinkers case, last panel). Not all curves correspond to a change in the number of biologically significant equilibria. The middle pane represents a transition case, at  $\rho = 0.5$ , when the cusp precisely belongs to the vertical line  $S_0 = 1$ .

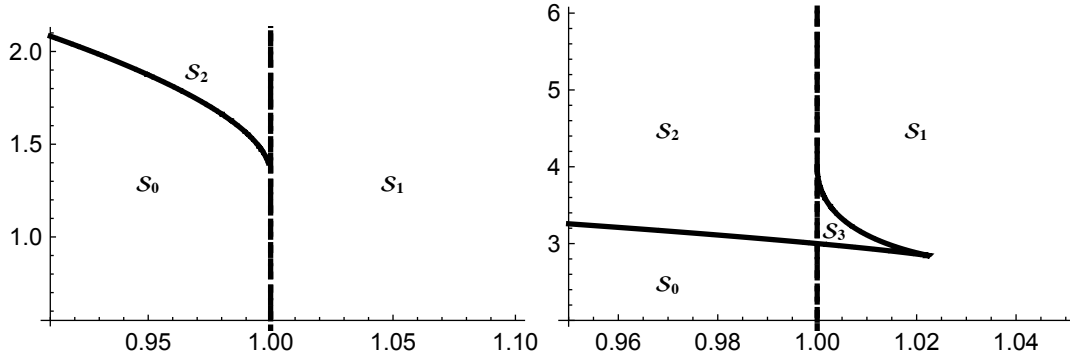


Figure 6: The two intermediate cases with the regions labelled with the same conventions of the previous sections.

The equations can be written in nondimensional form denoting by

$$m = \frac{(1-\chi)\varphi + \chi\psi}{\alpha}n, \quad s = (\mu + \nu)t, \quad \omega = \frac{\varphi(1-\chi) + \chi\psi}{\mu + \nu},$$

$$\rho = \frac{\chi\psi}{(1-\chi)\varphi + \chi\psi}, \quad S_0 = \frac{\lambda_0}{\mu + \nu}, \quad S_1 = \frac{(1-\chi)\varphi + \chi\psi}{\alpha(\mu + \nu)}\lambda_1,$$

and recasting the equations in the form

$$\begin{cases} c' = (S_0 - 1 + S_1n)c - (S_0 + S_1n)c^2 \\ n' = \omega(-n + (1-\rho)c + \rho c^2). \end{cases}$$

The equilibria are points  $(\hat{c}, \hat{n})$  such that  $\hat{n} = \rho\hat{c}^2 + (1-\rho)\hat{c}$  and  $(S_0 - 1 + S_1\hat{n})\hat{c} - (S_0 + S_1\hat{n})\hat{c}^2 = 0$ . Substituting the expression of  $\hat{n}$  in the first component of the system one obtains that  $c$  must be zero of the polynomial

$$p = c(S_0 - 1 + (S_1(1-\rho) - S_0)c + S_1(2\rho - 1)c^2 - S_1\rho c^3). \quad (7)$$

The solutions to this equation are  $\hat{c} = 0$  and the three solutions to the cubic polynomial factor of  $p$  under the condition that they are real and the corresponding equilibria  $E = (\hat{c}, \rho\hat{c}^2 + (1-\rho)\hat{c})$  belong to  $\Omega$ . It follows that the equilibria are  $E_0 = (0, 0)$  and the zeroes of the cubic polynomial that are one real and two complex solutions or three real solutions depending on the positivity of its discriminant  $\Delta$ . A plot of the discriminant-locus  $\Delta = 0$  for various  $\rho$  can be seen in Figure 5 and Figure 6.

The stability can be treated as in the sections above.

## 6 A more realistic model and simulations

Let us finally consider the general model (2). Rescaling time by  $s = (\mu + \xi + \kappa)t$ , posing  $m = \psi/\alpha n$ , and denoting

$$\zeta = \frac{\mu + \eta}{\mu + \xi + \kappa}, \quad \omega = \frac{\alpha}{\mu + \xi + \kappa}, \quad \sigma = \frac{\kappa}{\mu + \xi + \kappa}, \quad \vartheta = \frac{\varphi}{\psi},$$

$$R_0 = \frac{\beta_0}{\mu + \xi + \kappa}, \quad R_1 = \frac{\beta_1 \psi}{\alpha(\mu + \xi + \kappa)}, \quad T_0 = \frac{\gamma_0}{\mu + \eta}, \quad T_1 = \frac{\gamma_1 \psi}{\alpha(\mu + \eta)},$$

the equations can be recast in the nondimensional form

$$\begin{cases} \dot{a} = a \left( -1 + (R_0 + R_1 n)(1 - a - b) \right) \\ \dot{b} = \zeta b \left( -1 + (T_0 + T_1 n)(1 - a - b) \right) + \sigma a \\ \dot{n} = \omega(-n + \vartheta a + b^2). \end{cases} \quad (8)$$

The equilibria of this system are:

1. the disease free equilibrium  $E_0 = (0, 0, 0)$ ;
2. the solutions discussed in Sections 4 relative to the heavy drinkers system, denoted as  $E_{1,2,3} = (0, \hat{b}_{1,2,3}, \hat{b}_{1,2,3}^2)$ ;
3. the solutions with  $a \neq 0$  for which necessarily

$$\hat{n} = \vartheta \hat{a} + \hat{b}^2 \quad (9)$$

and  $\hat{a}, \hat{b}$  are solutions of the system of equations

$$\begin{cases} R_0(1 - a - b) - 1 + R_1(a\vartheta + b^2)(1 - a - b) = 0 \\ b\zeta \left( T_0(1 - a - b) - 1 + T_1(\vartheta a + b^2)(1 - a - b) \right) = -\sigma a. \end{cases} \quad (10)$$

By calling  $f = 1 - a - b$  and  $g = (a\vartheta + b^2)(1 - a - b)$ , equations (10) have the form

$$\begin{cases} R_0 f - 1 + R_1 g = 0 \\ b\zeta(T_0 f - 1 + T_1 g) = -\sigma a. \end{cases} \quad (11)$$

From (11)<sub>1</sub> it follows that  $R_1 g = 1 - R_0 f$ . Multiplying equation (11)<sub>2</sub> by  $R_1$  and using the above identity we have that

$$b\zeta \left( (R_1 T_0 - R_0 T_1)(1 - a - b) + T_1 - R_1 \right) = -\sigma R_1 a.$$

Hence the remaining solutions are given by

$$\hat{a} = \hat{b}\zeta \frac{(R_0 T_1 - R_1 T_0)(1 - \hat{b}) + R_1 - T_1}{\hat{b}\zeta(R_0 T_1 - R_1 T_0) + \sigma R_1}, \quad (12)$$

and  $\hat{b}$  is zero of a polynomial  $p_4(b)$  of degree 4.

The polynomial  $p_4$  can be investigated with the techniques of the previous sections, but its coefficients are complicate expressions of the parameters  $\sigma, \vartheta, R_0, R_1, T_0, T_1$  that make the investigation difficult. For this reason, we resort to a numerical investigation, that indicates a reasonably simple behaviour. In fact

1. for  $\sigma = 0$ , the polynomial  $p_4(b)$  is  $b^2$  times a quadratic polynomial, that can give at most two endemic equilibria interior to  $\Omega$  (and the double root  $b = 0$  indicates the existence of the two equilibria  $E_{\pm}$  found in Section 3);
2. Increasing  $\sigma$  the internal equilibria move, but also the equilibria posed in the boundary  $\{a = 0\}$  of  $\Omega$  move away from the boundary either in the interior of  $\Omega$  or outside of the biologically significant region.

To describe what happens we consider a particular reasonable choice of the parameters. We consider the age group to be all adults, it follows that  $\mu = 1/60$ . From [10] we choose  $\beta_0 = 0.31$  and  $\gamma_0 = 0.29$ . As concerns the other parameters, we choose:  $\xi = \eta = 0.08$ ,  $\alpha = 0.5$ ,  $\varphi = 10$  and  $\psi = 1$ . The parameter  $\kappa$  is chosen to be 0 in the first simulation and  $10^{-3}$  in all other simulations. For the other parameters  $\beta_1 = 0.9 * 10^{-2}$ ,  $\gamma_1 = 1.1 * 10^{-2}$  in the first two simulations, and then they are decreased to  $0.95 * 10^{-2}$  and to  $0.89 * 10^{-2}$  in the third and fourth simulations.

The polynomial  $p_4$ , with this particular choice of parameters, gives a unique, biologically significant, endemic equilibrium. This equilibrium persists under changes of  $\kappa$  from zero to positive values. The following four plots

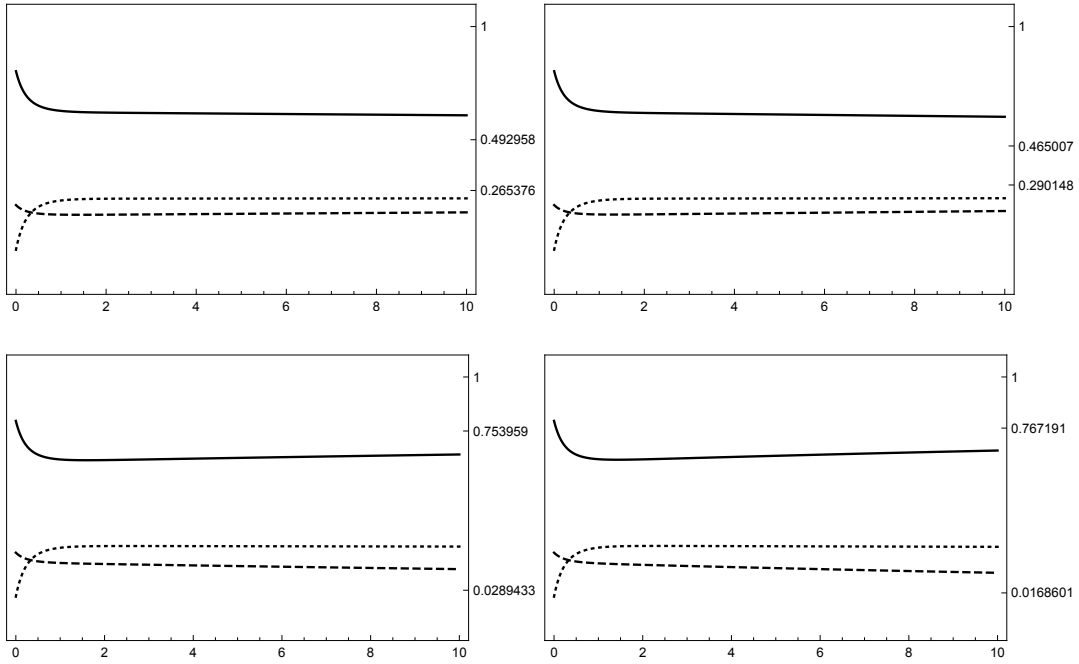


Figure 7: The time-evolution of the solutions. The solid line corresponds to the number of moderate-drinkers  $a$ , the dashed line to the heavy-drinkers  $b$ , the dotted line to the non-drinkers  $s = 1 - a - b$ . The first panel refers to the choice of parameters  $R_0 = 3.20689$ ,  $R_1 = 0.186207$ ,  $T_0 = 3$ ,  $T_1 = 0.22759$ ,  $\zeta = 1$ ,  $\omega = 5.1724138$ ,  $\vartheta = 10$ ,  $\sigma = 0$ . The second refers to a slight increase of the parameter  $\kappa$ , that corresponds to an increase of  $\sigma$  to the value 0.00103341 (and a slight modification of the parameters  $R_0, R_1$ ). The third and fourth panels correspond to the reduction of the  $m$ -derivative of the information function  $\gamma$ , i.e. a decrease of the number  $\gamma_1$ , that in turns corresponds to a decrease of the nondimensional parameter  $T_1$ .

correspond to the time-evolution of the dynamical variables  $a, b, s = 1 - a - b$  for the four possible choices of the parameters indicated above.

From the first panel of Figure 7, we see that the compartments tend asymptotically to equilibria values that are consistent with the data published in the National Survey on Drug Use and Health [30] (relative to statistics made in the U.S. in the year 2014).

The other panels indicate that the number of heavy drinkers is highly sensitive to the parameter  $\gamma_1$  that models how information influences the population of drinkers. This dependence seems to indicate that *an effort to make alcohol less available to the young could have a strong effect on the percentage of heavy drinkers*.

## 7 Conclusions

A mathematical model of drinking with an information function has been proposed. The population has been divided into three compartments: non-drinkers, moderate drinkers, and heavy drinkers. The dynamics of these classes is influenced by an information variable that may indicate the availability of alcohol, and enters the equations by modifying the contact rates  $\beta$  and  $\gamma$  as described in Section 2. Here, as a first hypothesis, the contact rates have been chosen to be linear functions of the information variable. Of course other functional dependences might be used such as functions with a saturation threshold, that is functions of the form

$$\beta(M) = \frac{\beta_u M}{1 + \beta_d M}, \quad \gamma(M) = \frac{\gamma_u M}{1 + \gamma_d M}, \quad \delta(M) = \frac{\gamma_u M}{1 + \gamma_d M}.$$

The saturation makes the functional dependence of the dynamics on information much more realistic, still algebraic, but with polynomials of much higher degree.

The system also admits other possible generalisations. For example, the equations can be extended adding the possible migration by peer pressure from the class  $A$  to the class  $B$ . The associated equations in such case would be

$$\begin{cases} \dot{S} = \mu N - \mu S - \frac{\beta(M)}{N} AS - \frac{\gamma(M)}{N} BS + \xi A + \eta B \\ \dot{A} = -\mu A + \frac{\beta(M)}{N} AS - \frac{\delta(M)}{N} AB - \xi A - \kappa A \\ \dot{B} = -\mu B + \frac{\delta(M)}{N} AB + \frac{\gamma(M)}{N} BS - \eta B + \kappa A \\ \dot{M} = -\alpha M + \varphi A + \psi B^2. \end{cases} \quad (13)$$

This addition creates two additional basic reproduction numbers proportional to  $\delta_0$  and  $\delta_1$ . Finally, the last component of the vector field, which depends on  $B^2$ , should be revised to make the influence of heavy drinkers superlinear but not so overwhelming. This of course would lead to a system that is not algebraic, and it should be treated with completely different techniques. Another use of the information could be related to model laws of the governments to reduce the alcohol consumption.

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