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Equivalence of cosmological observables in conformally related scalar tensor theories

François Rondeau* and Baojiu Li†

Institute for Computational Cosmology, Department of Physics, Durham University, Durham DH1 3LE, United Kingdom

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Scalar tensor theories can be expressed in different frames, such as the commonly used Einstein and Jordan frames, and it is generally accepted that cosmological observables are the same in these frames. We revisit this by making a detailed side-by-side comparison of the quantities and equations in two conformally related frames, from the actions and fully covariant field equations to the linearized equations in both real and Fourier spaces. This confirms that the field and conservation equations are equivalent in the two frames, in the sense that we can always re-express equations in one frame using relevant transformations of variables to derive the corresponding equations in the other. We show, with both analytical derivation and a numerical example, that the line-of-sight integration to calculate CMB temperature anisotropies can be done using either Einstein frame or Jordan frame quantities, and the results are identical, provided the correct redshift is used in the Einstein frame ($1 + z \neq 1/a$).

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I. INTRODUCTION

The accelerated expansion of the Universe [1,2] observed about two decades ago is one of the most challenging questions for cosmologists and physicists today. Such an accelerated expansion cannot be explained so far in the standard framework which is built upon the standard model of particle physics and Einstein theory of General Relativity (GR), and therefore hints that new physics beyond our current knowledge might be its driving force. This makes it a very interesting and potentially very important question, and has motivated various ongoing and planned astronomical surveys, such as e BOSS [3], DES [4], HSC [5], DESI [6], LSST [7], EUCLID [8], 4 MOST [9], WFIRST [10] and SKA [11], which are designed to measure various properties of the cosmic acceleration which can in turn be used to shed light on its origin and underlying physics.

From a phenomenological point of view, the simplest possibility to explain the observations is the Λ cold dark matter (ΛCDM) model, where a small positive cosmological constant Λ is assumed to be accelerating the rate of the Hubble expansion. Although this model is compatible with many observations, it has suffered from theoretical difficulties, such as the fine-tuning and coincidence problems. In order to avoid these problems, various theories of dark energy [12] and modified gravity [13,14] have been studied, many of which can be classified as subclasses of the so-called scalar tensor theories [15]. While GR is a tensor theory, in which the mediators of the gravitational interaction (gravitons) are excitations of the metric of the spacetime, in a scalar-tensor theory, a second mediator of gravity is considered—a scalar field ϕ, which couples to the matter or gravitational fields (or both in certain models). The theory can then be studied in different "frames" by suitable field redefinitions. The commonly used frames include the *Einstein frame*, where the matter fields—rather than the gravitational field $g_{\mu\nu}$—are coupled to ϕ and the gravity sector takes its standard form as in GR, and the *Jordan frame*, where the matter fields are uncoupled to ϕ but the gravitational equations are modified due to the coupling to ϕ. The field equations generally look different in the two frames, and usually for certain applications in practice it is advantageous to use one over the other.

Such a freedom of choosing to work in different frames used to be a source for debates in the community, about whether the Einstein and Jordan frames are physically equivalent to each other. The current prevailing opinion is that physics is the same in these frames, but quantities calculated in them need to be interpreted carefully to compare with each other (see, e.g., [16–21] and references therein; for opposite views see, e.g., [22,23]). For example, in [24] it is suggested that calculations can be done in the Einstein frame where the scalar field is a canonical field with minimal coupling to gravity (therefore simplifying the equations) but the interpretation of physical observables should be done in the Jordan frame, which is the "physical frame." In Ref. [17], it is proposed that calculations can be done using frame-independent quantities such that the issue of how to interpret them can be circumvented naturally; [21] similarly demonstrates the equivalence of physics in the two frames by rewriting the action in terms of dimensionless variables which are frame independent; [20] finds the correspondences between variables in the two frames and show that cosmological observables, such as redshift, luminosity distance and temperature anisotropies,

*francois.rondeau@ens-paris-saclay.fr
†baojiu.li@durham.ac.uk

are frame independent. Although these are useful works, it would also be helpful to have a detailed comparison of the linear perturbation variables (not necessarily observables) and equations in the two frames, which will provide further insight about what in a perturbed spacetime are (not) affected by a conformal transformation.

In this paper, we would like to have a closer look at the field equations in the two frames up to first order in linear perturbations. This differs from previous works in that we will write the equations—the fully covariant Einstein and Klein Gordon equations, and their linearized versions in both real space and Fouier space—side by side and demonstrate their equivalence. Using these equations, we will show that physical observables which are gauge invariant, such as the cosmic microwave background (CMB) temperature anisotropies, weak gravitational lensing and the integrated Sachs-Wolfe effect, are the same no matter quantities in which frame are used to calculate them. The derivations will then be supplemented by a numerical example with which we show that the CMB temperature spectra calculated in the two frames are identical provided that care is taken in the Einstein frame so that integrations stop at the correct time (which is not necessarily when the scale factor $a = 1$).

This paper is organized as follows. In Sec. II, we describe general relations between operators and mathematical quantities built from two conformally related metrics. In Sec. III, we present the physical model: a scalar-tensor theory built from the metric tensor $g_{\mu\nu}$ and a scalar field ϕ. The Einstein equations, Klein-Gordon equations and Friedmann equations are derived independently in the two frames, and we check that any equation in one frame can always be obtained directly from its equivalent in the other frame. In Sec. IV, we present the relations of linear perturbation variables in the two frames, and show that the linearized Einstein and matter conservation equations in the two frames are equivalent (in the sense that an equation in one frame can be derived from the same equation in the other frame using the above relations). These will be done in both real space and Fourier space which is more convenient to solve the set of coupled linear equations. In Sec. V, we find expressions of some of the most well-known cosmological observables and explicitly show that they are identical in both frames using the relations between linear perturbation variables in the frames; we explicitly demonstrate this by showing the CMB spectra for a specific scalar field model—the K-mouflage model [25,26], which has a scalar field that is purely kinetic with a non-canonical kinetic term. Finally, we discuss and conclude in Sec. VI.

Throughout this work, we shall use the unit $c = 1$, where c is the speed of light, unless where c is explicitly written. We adopt the metric sign convention $(-,+,+,+)$.

II. THE MATHEMATICAL SETUP

In this section, we will set up the mathematical framework of this paper, by introducing notations and conventions to be used later, as well as some useful relationships between quantities in frames that are conformally related.

Later in the paper we will use untildered and tildered quantities for the Einstein and Jordan frames, respectively. For this section, however, we prefer to keep things more general and so refrain from making connections of the untildered (tildered) frame to the Einstein (Jordan) frame.

A. The 3 + 1 spacetime decomposition

In this paper, we will focus on the analysis and predictions of observable quantities in the linear perturbation regime. The linear perturbation equations will be derived using the 3 + 1 decomposition [see, e.g., [27,28]], which splits the four-dimensional spacetime into a time direction and three-dimensional spatial slices (hypersurfaces) with constant times. The split is done with respect the 4-velocity u^μ of an observer. Note that as per the standard convention Greek indices run over 0,1,2,3. Although later in the paper we will use tildered and untildered quantities for the different frames, the expressions in this subsection are mostly definitions and general for any frame, so we make all quantities un-tildered.

A projection tensor $h_{\mu\nu}$ can be defined as

$$h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu, \tag{1}$$

which is the metric tensor of the 3D hyperspace orthogonal to u^μ. The covariant spatial derivative $\hat{\nabla}^\mu$ of a general tensor field $B^{\sigma\ldots\lambda}_{\nu\ldots\rho}$ can then be expressed using $h_{\mu\nu}$ as

$$\hat{\nabla}^\mu B^{\sigma\ldots\lambda}_{\nu\ldots\rho} \equiv h^\mu_\alpha h^\sigma_\beta \ldots h^\lambda_\gamma h^\delta_\nu \ldots h^\epsilon_\rho \nabla^\alpha B^{\beta\ldots\gamma}_{\delta\ldots\epsilon}, \tag{2}$$

in which ∇ denotes the full covariant derivative compatible with $g_{\mu\nu}$. Similarly, the covariant time derivative can be expressed as

$$\dot{B}^{\sigma\ldots\lambda}_{\nu\ldots\rho} \equiv u^\mu \nabla_\mu B^{\sigma\ldots\lambda}_{\nu\ldots\rho}. \tag{3}$$

The stress-energy tensor of matter and the scalar field can be decomposed as

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} + 2q_{(\mu} u_{\nu)} + \Pi_{\mu\nu}, \tag{4}$$

which gives dynamical quantities including the energy density ρ, isotropic pressure p, energy flux q_μ and the anisotropic stress $\Pi_{\mu\nu}$. The latter two are purely spatial quantities satisfying $u^\mu q_\mu = u^\mu \Pi_{\mu\nu} = 0$, and therefore vanish in an exact Friedman-Robertson-Walker (FRW) universe; in a perturbed FRW spacetime they are linear-order quantities.

Similarly, the covariant derivative of the 4-velocity can be split as

$$\nabla_\mu u_\nu = -u_\mu \dot{u}_\nu + \frac{1}{3}\theta h_{\mu\nu} + \sigma_{\mu\nu} + \varpi_{\mu\nu}, \tag{5}$$

which gives the kinematic quantities including the expansion scalar $\theta \equiv \nabla_\alpha u^\alpha$, the shear tensor $\sigma_{\mu\nu} \equiv \hat{\nabla}_{(\mu}u_{\nu)} - \frac{1}{3}\theta h_{\mu\nu}$, the vorticity $\varpi_{\mu\nu} \equiv \hat{\nabla}_{[\mu}u_{\nu]}$ and the 4-acceleration of the observer $w_\mu \equiv \dot{u}_\mu$. Only θ is nonzero in an exact FRW spacetime, and the other three are purely spatial tensors, which are first order quantities for w_μ and $\sigma_{\mu\nu}$ and second order quantity for $\varpi_{\mu\nu}$ in a perturbed FRW spacetime, satisfying $u^\mu w_\mu = u^\mu \sigma_{\mu\nu} = u^\mu \varpi_{\mu\nu} = 0$.

Note that the metric signature used here is $(-,+,+,+)$, so that $u^\alpha u_\alpha = -1$. The Riemann tensor is defined in terms of the Christoffel symbols as

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\rho\alpha}\Gamma^\alpha{}_{\nu\sigma} - \Gamma^\mu{}_{\sigma\alpha}\Gamma^\alpha{}_{\nu\rho}, \tag{6}$$

and the Ricci tensor and Ricci scalar are given by $R_{\mu\nu} \equiv R^\alpha{}_{\mu\alpha\nu}$ and $R \equiv g^{\mu\nu}R_{\mu\nu} = R^\mu{}_\mu$.

As mentioned in the introduction, we hope that this work will be a useful reference in which linear perturbation equations in both frames are compared side by side and can be found for future work. In the literature, linear perturbation analyses are often done using the synchronous or Newtonian gauges [29]. The equations in the $3+1$ formalism presented here can be re-expressed in general gauges by gauge fixing. For example, the synchronous (Newtonian) gauge corresponds to setting $w = 0$ ($\sigma = 0$) in our equations [see, e.g., [30]] for an explicit mapping to those gauges], where w, σ are, respectively, the Fourier-space expressions of the scalar modes[1] of w_μ and $\sigma_{\mu\nu}$ (see below).

B. Mathematical quantities in conformal transformation

Although the ultimate goal of this paper is to consider the equivalence of physically observable quantities in the Einstein and Jordan frames, it is useful to know how general mathematical quantities are connected in generally conformally-related frames. These relations will also be useful when we compare the equations in the two frames.

The metric tensors in two frames which later will be called the Einstein frame ($g_{\mu\nu}$) and the Jordan frame ($\tilde{g}_{\mu\nu}$) are related by a conformal transformation[2]

$$\tilde{g}_{\mu\nu} = A(\phi) g_{\mu\nu}, \tag{7}$$

[1]In this work, we shall only consider scalar modes of linear perturbations.

[2]Although we will identify these metrics to be the Einstein- and Jordan-frame ones, respectively, later in this section, we shall avoid doing this to keep things general. Instead, if needed, we shall call them untildered and tildered frames.

where A is a function of a scalar field ϕ. If we denote by g and $\tilde{g}$ the determinants of the metrics $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$, respectively, we then have the useful relation:

$$\sqrt{-\tilde{g}} = A^2 \sqrt{-g}. \tag{8}$$

Since the two frames are related by a conformal transformation, if we use conformal time (denoted by η) and comoving spatial coordinates ($\mathbf{x}$), then the coordinates will be the same in the two frames. For example, in the case of exactly FRW spacetimes, the line elements in the two frames can be written as

$$\mathrm{d}s^2 = a^2(-\mathrm{d}\eta^2 + \mathrm{d}\mathbf{x}^2),$$
$$\mathrm{d}\tilde{s}^2 = \tilde{a}^2(-\mathrm{d}\eta^2 + \mathrm{d}\mathbf{x}^2). \tag{9}$$

Therefore, it is convenient to define the covariant derivatives using the conformal and comoving coordinates. In this case, for a general scalar field ψ, the covariant derivatives with lower index (which are equal to the partial derivatives) are the same in the two frames

$$\tilde{\nabla}_\mu \psi = \nabla_\mu \psi. \tag{10}$$

In the meantime, extra care needs to be taken for the covariant derivatives with upper index, as the index is not raised by the same metric in the two frames. We have $\nabla^\mu \psi = g^{\mu\alpha}\nabla_\alpha \psi$ and $\tilde{\nabla}^\mu \psi = \tilde{g}^{\mu\alpha}\tilde{\nabla}_\alpha \psi$, which satisfy

$$\tilde{\nabla}^\mu \psi = \frac{1}{A}\nabla^\mu \psi, \tag{11}$$

where we have used $\tilde{g}^{\mu\nu} = \frac{1}{A}g^{\mu\nu}$, which is the inverse relation of Eq. (7).

The covariant time derivatives in the two frames are, respectively, defined by $\dot{\psi} \equiv u^\alpha \nabla_\alpha \psi$ and $\mathring{\psi} \equiv \tilde{u}^\alpha \tilde{\nabla}_\alpha \psi$. In order to find the link between these two quantities, we have used the relation between the 4-velocities in the two frames:

$$\tilde{u}^\mu = \frac{\mathrm{d}\tilde{x}^\mu}{\mathrm{d}\tilde{s}} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tilde{s}} = \frac{\mathrm{d}x^\mu}{\sqrt{A}\mathrm{d}s} = \frac{1}{\sqrt{A}}u^\mu, \tag{12}$$

where the third equality is because $\mathrm{d}\tilde{s} = \sqrt{A}\mathrm{d}s$. Similarly,

$$\tilde{u}_\mu = \tilde{g}_{\mu\nu}\tilde{u}^\nu = A g_{\mu\nu}\frac{1}{\sqrt{A}}u^\nu = \sqrt{A}u_\mu, \tag{13}$$

and therefore

$$\mathring{\psi} = \frac{1}{\sqrt{A}}\dot{\psi}. \tag{14}$$

Although Eq. (14) is useful in connecting the physical time derivatives in the two frames, it is more convenient to use the conformal time derivative $' \equiv \mathrm{d}/\mathrm{d}\eta$ because it is the same in both frames:

$$\psi' = \tilde{a}\,\overset{\circ}{\psi} = a\dot{\psi}, \tag{15}$$

where a and $\tilde{a}$ are the scale factors in the two frames introduced in Eq. (9) and they satisfy $\tilde{a} = a\sqrt{A}$.

Using the $3+1$ spacetime decomposition introduced in the previous subsection, the covariant derivatives of ψ can be split as

$$\begin{aligned}\nabla_\mu\psi &= -u_\mu\dot{\psi} + \hat{\nabla}_\mu\psi,\\ \tilde{\nabla}_\mu\psi &= -\tilde{u}_\mu\overset{\circ}{\psi} + \hat{\tilde{\nabla}}_\mu\psi,\end{aligned} \tag{16}$$

in the two frames, respectively, from which we obtain the following relations between the spatial derivatives:

$$\hat{\tilde{\nabla}}_\mu\psi = \hat{\nabla}_\mu\psi; \qquad \hat{\tilde{\nabla}}^\mu\psi = \frac{1}{A}\hat{\nabla}^\mu\psi. \tag{17}$$

Finally, starting from the expression of the Christoffel symbols:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}),$$

in the untildered frame, and

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2}\tilde{g}^{\lambda\alpha}(\partial_\mu \tilde{g}_{\alpha\nu} + \partial_\nu \tilde{g}_{\mu\alpha} - \partial_\alpha \tilde{g}_{\mu\nu}),$$

in the tildered frame, and using Eq. (7), one can compute the link between these two quantities

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \frac{1}{2A}(\delta^\lambda_\nu\nabla_\mu A + \delta^\lambda_\mu\nabla_\nu A - g_{\mu\nu}\nabla^\lambda A). \tag{18}$$

This expression is useful to compute the relations between quantities involving covariant derivatives applied to general tensors. An example of its application is the computation of the link between the d'Alembertian operators in the untildered and tildered frames, respectively, defined by $\Box \equiv \nabla_\alpha\nabla^\alpha$ and $\tilde{\Box} \equiv \tilde{\nabla}_\alpha\tilde{\nabla}^\alpha$

$$\tilde{\Box}\psi = \frac{1}{A}\Box\psi + \frac{1}{A^2}\nabla^\alpha\psi\nabla_\alpha A. \tag{19}$$

C. Lagrangian densities and the scalar field

In order to derive and compare the relevant equations in the two frames, we also need to know how the Lagrangian densities and a general scalar field transform in the conformal transformation.

The action built from a Lagrangian density $\mathcal{L}$, expressed in terms of the untildered metric, reads

$$\mathcal{S}[g_{\mu\nu}] = \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}, \tag{20}$$

Using the transformation law for $\sqrt{-g}$ given by Eq. (8), it can be expressed in terms of the tildered metric as

$$S[\tilde{g}_{\mu\nu}] = \int \mathrm{d}^4x\sqrt{-\tilde{g}}\frac{\mathcal{L}}{A^2}, \tag{21}$$

where note that the coordinates are the same in these expressions because they are comoving coordinates.

One can then define the Lagrangian density in the tildered frame, $\tilde{\mathcal{L}}$, as

$$\tilde{\mathcal{L}} = \frac{1}{A^2}\mathcal{L}, \tag{22}$$

which can be used to determine the transformation law of the scalar field between the two frames. To do this, let us have a simple example of the Lagrangian density of a scalar filed, which takes the canonic form in the tildered frame,

$$\tilde{\mathcal{L}}_{\tilde{\phi}} = -\frac{1}{2}\tilde{g}^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\phi}\tilde{\nabla}_\beta\tilde{\phi}. \tag{23}$$

Since the transformed scalar field, $\tilde{\phi}$, is related to the scalar field ϕ in the untildered frame, we can write $\tilde{\nabla}_\mu\tilde{\phi} = (\partial\tilde{\phi}/\partial\phi)\tilde{\nabla}_\mu\phi$. Then, using $\tilde{g}^{\mu\nu} = \frac{1}{A}g^{\mu\nu}$ and $\tilde{\nabla}_\mu\phi = \nabla_\mu\phi$, Eq. (23) can be rewritten as

$$\tilde{\mathcal{L}}_{\tilde{\phi}} = -\left(\frac{\partial\tilde{\phi}}{\partial\phi}\right)^2\frac{1}{2}\frac{g^{\alpha\beta}}{A}\nabla_\alpha\phi\nabla_\beta\phi. \tag{24}$$

Hence, for the scalar field Lagrangian density in the untildered frame, $\mathcal{L}_\phi = -\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi$, the relation in Eq. (22) implies that the scalar fields ϕ and $\tilde{\phi}$ should satisfy

$$\frac{\partial\tilde{\phi}}{\partial\phi} = \frac{1}{\sqrt{A}}. \tag{25}$$

This in turn means that

$$\frac{\partial\tilde{\mathcal{L}}_{\tilde{\phi}}}{\partial(\tilde{\nabla}_\mu\tilde{\phi})} = \frac{1}{A^{\frac{3}{2}}}\frac{\partial\mathcal{L}_\phi}{\partial(\nabla_\mu\phi)}, \tag{26}$$

from which and Eq. (18) we find

$$\tilde{\nabla}_\mu\left[\frac{\partial\tilde{\mathcal{L}}_{\tilde{\phi}}}{\partial(\tilde{\nabla}_\mu\tilde{\phi})}\right] = \frac{1}{A^{\frac{3}{2}}}\nabla_\mu\left[\frac{\partial\mathcal{L}_\phi}{\partial(\nabla_\mu\phi)}\right] + \frac{1}{2A^{\frac{3}{2}}}\frac{\mathrm{d}\ln A}{\mathrm{d}\phi}\frac{\partial\mathcal{L}_\phi}{\partial(\nabla_\mu\phi)}\nabla_\mu\phi. \tag{27}$$

This relation can be used to transform the Klein-Gordon equation between the untildered and the tildered frames.

III. SCALAR-TENSOR THEORIES

We shall consider the classical scalar-tensor theory where gravity is mediated by a scalar field in addition to the metric tensor. Let $\mathcal{M}$ be a four-dimensional manifold representing the spacetime, then $\mathcal{M}$ equipped with a metric $g_{\mu\nu}$, in which the Ricci scalar is not coupled to any scalar field, is denoted by $(\mathcal{M}, g_{\mu\nu})$ and we call this the *Einstein frame*. In this frame, the Einstein equations take their standard form as in GR, but matter is non-minimally coupled to the scalar field and hence free particles do not follow geodesics of the metric $g_{\mu\nu}$ but feel an additional fifth force.

The total action in the Einstein frame reads

$$S[g_{\mu\nu}] = \int d^4x\sqrt{-g}\left\{\frac{1}{2}M_{\rm Pl}^2 R + \mathcal{L}_\phi[\phi, (\nabla\phi)^2]\right\} + S_m[A(\phi), g_{\mu\nu}], \qquad (28)$$

where $M_{\rm Pl}$ is the reduced Planck mass defined by $M_{\rm Pl}^{-2} = 8\pi G$, G is Newton's constant, $\mathcal{L}_\phi$ is the Lagrangian density of the scalar field ϕ, $R = g_{\mu\nu}R^{\mu\nu}$ is the Ricci scalar, $R_{\mu\nu}$ the Ricci tensor, and $S_m[A(\phi), g_{\mu\nu}]$ is the matter action given by

$$S_m[A(\phi), g_{\mu\nu}] = \sum_i \int d^4x\sqrt{-g}\mathcal{L}_m(\psi_m^{(i)}, A(\phi), g_{\mu\nu}), \qquad (29)$$

where $A(\phi)$ is an algebraic function of the scalar field ϕ, $\psi_m^{(i)}$ denotes the ith species of matter fields, and the summation is over all matter species.

We can introduce another, tildered, metric $\tilde{g}_{\mu\nu}$ through $\tilde{g}_{\mu\nu} = A(\phi)g_{\mu\nu}$, and the relation between the Ricci scalars of these two metrics can be straightforwardly found by using the transformations of the Christoffel symbols given in Eq. (18):

$$R = A\tilde{R} + 3\tilde{\Box}A - \frac{9}{2}\frac{(\tilde{\nabla}A)^2}{A}. \qquad (30)$$

Using this relation and the transformations of the metric tensors, we can reexpress the Einstein-frame action Eq. (28) in terms of the tildered metric $\tilde{g}_{\mu\nu}$ and a redefined scalar field $\tilde{\phi}$:

$$S[\tilde{g}_{\mu\nu}] = \int d^4x\sqrt{-\tilde{g}}\left\{\frac{1}{2}M_{\rm Pl}^2\frac{1}{A}\tilde{R} + \frac{3}{4}M_{\rm Pl}^2\frac{(\tilde{\nabla}A)^2}{A^3} + \tilde{\mathcal{L}}_{\tilde{\phi}}(\tilde{\phi}, (\tilde{\nabla}\tilde{\phi})^2)\right\} + S_m[\tilde{g}_{\mu\nu}], \qquad (31)$$

where we have dropped a boundary term which does not contribute to the dynamics of the theory, and the matter action expressed in terms of the tildered metric $\tilde{g}_{\mu\nu}$ is given by

$$S_m[\tilde{g}_{\mu\nu}] = \sum_i \int d^4x\sqrt{-\tilde{g}}\tilde{\mathcal{L}}_m(\tilde{\psi}_m^{(i)}, \tilde{g}_{\mu\nu}). \qquad (32)$$

The manifold $\mathcal{M}$ endowed with the metric $\tilde{g}_{\mu\nu}$, where the metric now is non-minimally coupled to the scalar field, is denoted by $(\mathcal{M}, \tilde{g}_{\mu\nu})$ and called the Jordan frame. In this frame, matter is minimally coupled to the scalar field, and free particles follow geodesics of the metric $\tilde{g}_{\mu\nu}$, just as in general relativity. On the other hand, the Einstein equations are different from those in GR, highlighting the physical result that gravity is now mediated not just by the massless gravitons, but also by the scalar field.

From here on we shall use tildes for Jordan-frame quantities while their Einstein-frame counterparts are untildered.

A. Einstein's equations in the two frames

Varying the action Eq. (28) with respect to the metric $g_{\mu\nu}$ gives the Einstein equations in the Einstein frame:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = M_{\rm Pl}^{-2}(T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(\phi)}), \qquad (33)$$

where the stress energy tensors $T_{\mu\nu}^{(i)}$, $i = m$ or ϕ, are defined by $T_{\mu\nu}^{(i)} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta[\sqrt{-g}\mathcal{L}^{(i)}]}{\delta g^{\mu\nu}}$.

Similarly, varying the action Eq. (31) with respect to the metric $\tilde{g}_{\mu\nu}$ gives the Einstein equations in the Jordan frame:

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} = M_{\rm Pl}^{-2}(A\tilde{T}_{\mu\nu}^{(m)} + A\tilde{T}_{\mu\nu}^{(\tilde{\phi})}) + \tilde{g}_{\mu\nu}\frac{\tilde{\Box}A}{A} - \frac{\tilde{\nabla}_\mu\tilde{\nabla}_\nu A}{A} + \frac{1}{2}\frac{\tilde{\nabla}_\mu A\tilde{\nabla}_\nu A}{A^2} - \frac{5}{4}\tilde{g}_{\mu\nu}\frac{(\tilde{\nabla}A)^2}{A^2}, \qquad (34)$$

where the stress energy tensors $\tilde{T}_{\mu\nu}^{(i)}$, $i = m$ or $\tilde{\phi}$, are defined by $\tilde{T}_{\mu\nu}^{(i)} \equiv -\frac{2}{\sqrt{-\tilde{g}}}\frac{\delta[\sqrt{-\tilde{g}}\tilde{\mathcal{L}}^{(i)}]}{\delta\tilde{g}^{\mu\nu}}$.

Using the links between operators in the two frames given in Sec. II, and the relations between the Ricci tensors and scalars in both frames, which are given by

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{\nabla_\mu\nabla_\nu A}{A} - \frac{g_{\mu\nu}}{2}\frac{\Box A}{A} + \frac{3}{2}\frac{\nabla_\mu A\nabla_\nu A}{A^2}, \qquad (35)$$

and

$$\tilde{R} = \frac{1}{A}\left(R - 3\frac{\Box A}{A} + \frac{3}{2}\frac{(\nabla A)^2}{A^2}\right), \qquad (36)$$

one can check that Eq. (33) is equivalent to Eq. (34).

From Eq. (34), we can define an effective stress-energy tensor $\tilde{\mathbf{T}}^{(\tilde{\phi})}_{\mu\nu}$ for the scalar field in the Jordan frame,

$$\tilde{M}_{\rm Pl}^{-2}\tilde{\mathbf{T}}^{(\tilde{\phi})}_{\mu\nu} \equiv \tilde{M}_{\rm Pl}^{-2}\tilde{T}^{(\tilde{\phi})}_{\mu\nu} + \tilde{g}_{\mu\nu}\frac{\tilde{\Box}A}{A} - \frac{\tilde{\nabla}_\mu\tilde{\nabla}_\nu A}{A} + \frac{1}{2}\frac{\tilde{\nabla}_\mu A\tilde{\nabla}_\nu A}{A^2} - \frac{5}{4}\tilde{g}_{\mu\nu}\frac{(\tilde{\nabla}A)^2}{A^2}, \tag{37}$$

in which we have defined the reduced Planck mass in the Jordan frame as $\tilde{M}_{\rm Pl} = A^{-1/2}M_{\rm Pl}$, and compute the corresponding dynamical quantities using

$$\begin{aligned}\tilde{\boldsymbol{\rho}}^{(\tilde{\phi})} &= \tilde{\mathbf{T}}^{(\tilde{\phi})}_{\mu\nu}\tilde{u}^\mu\tilde{u}^\nu,\\ \tilde{\boldsymbol{p}}^{(\tilde{\phi})} &= \frac{1}{3}\tilde{\mathbf{T}}^{(\tilde{\phi})}_{\mu\nu}\tilde{h}^{\mu\nu},\\ \tilde{\boldsymbol{q}}^{\mu}_{(\tilde{\phi})} &= -\tilde{\mathbf{T}}^{(\tilde{\phi})}_{\alpha\beta}\tilde{u}^\alpha\tilde{h}^{\beta\mu},\\ \tilde{\boldsymbol{\Pi}}^{(\tilde{\phi})}_{\mu\nu} &= \tilde{h}^\alpha_\mu\tilde{h}^\beta_\nu\tilde{\mathbf{T}}^{(\tilde{\phi})}_{\alpha\beta} - \tilde{\mathbf{p}}^{(\tilde{\phi})}\tilde{h}_{\mu\nu}.\end{aligned} \tag{38}$$

Note that the reduced Planck mass is related to Newton's constant and is a fundamental constant in the Einstein frame. In the Jordan frame, however, it depends on the scalar field. Up to linear order, Eqs. (37) and (38) give

$$\tilde{M}_{\rm Pl}^{-2}\tilde{\boldsymbol{\rho}}^{(\tilde{\phi})} = \tilde{M}_{\rm Pl}^{-2}\tilde{\rho}^{(\tilde{\phi})} - \frac{\hat{\tilde{\Box}}A}{A} + \tilde{\theta}\frac{\mathring{A}}{A} - \frac{3}{4}\frac{\mathring{A}^2}{A^2}, \tag{39}$$

$$\tilde{M}_{\rm Pl}^{-2}\tilde{\boldsymbol{p}}^{(\tilde{\phi})} = \tilde{M}_{\rm Pl}^{-2}\tilde{p}^{(\tilde{\phi})} + \frac{2}{3}\frac{\hat{\tilde{\Box}}A}{A} - \frac{\mathring{\mathring{A}}}{A} - \frac{2}{3}\tilde{\theta}\frac{\mathring{A}}{A} + \frac{5}{4}\frac{\mathring{A}^2}{A^2}, \tag{40}$$

$$\tilde{M}_{\rm Pl}^{-2}\tilde{\boldsymbol{q}}^{\mu}_{(\tilde{\phi})} = \tilde{M}_{\rm Pl}^{-2}\tilde{q}^{\mu}_{(\tilde{\phi})} - \frac{\mathring{A}}{2A^2}\hat{\tilde{\nabla}}^\mu A + \frac{\hat{\tilde{\nabla}}^\mu\mathring{A}}{A} - \frac{\tilde{\theta}}{3}\frac{\hat{\tilde{\nabla}}^\mu A}{A}, \tag{41}$$

$$\tilde{M}_{\rm Pl}^{-2}\tilde{\boldsymbol{\Pi}}^{(\tilde{\phi})}_{\mu\nu} = \tilde{M}_{\rm Pl}^{-2}\tilde{\Pi}^{(\tilde{\phi})}_{\mu\nu} - \frac{1}{A}(\hat{\tilde{\nabla}}_{\langle\mu}\hat{\tilde{\nabla}}_{\nu\rangle}A - \mathring{A}\tilde{\sigma}_{\mu\nu}). \tag{42}$$

At the background level, $\hat{\tilde{\Box}}A = 0$ and $\tilde{\theta} = 3\frac{\mathring{\tilde{a}}}{\tilde{a}} = 3\frac{\tilde{a}'}{\tilde{a}^2}$, and using $\mathring{A} = \frac{1}{\tilde{a}}A'$, one can rewrite Eq. (39) as

$$\tilde{M}_{\rm Pl}^{-2}\tilde{\boldsymbol{\rho}}^{(\tilde{\phi})} = \tilde{M}_{\rm Pl}^{-2}\tilde{\rho}^{(\tilde{\phi})} + 3\frac{\tilde{a}'}{\tilde{a}^3}\frac{A'}{A} - \frac{3}{4}\frac{A'^2}{\tilde{a}^2A^2}. \tag{43}$$

Using this equation, it can be easily checked that the Friedmann equations in the Einstein and Jordan frames,

$$3\left(\frac{\dot{a}}{a}\right)^2 = M_{\rm Pl}^{-2}(\rho^{(m)} + \rho^{(\phi)}), \tag{44}$$

$$3\left(\frac{\mathring{\tilde{a}}}{\tilde{a}}\right)^2 = \tilde{M}_{\rm Pl}^{-2}(\tilde{\rho}^{(m)} + \tilde{\boldsymbol{\rho}}^{(\tilde{\phi})}), \tag{45}$$

are equivalent to each other,[3] For this check, we also used the transformation of the density field, $\tilde{\rho} = A^{-2}\rho$, in the two frames, which will be discussed below. This is a consequence of the equivalence between the Einstein equations themselves in the two frames, as we checked above; this equivalence originates from the fact that the equations in the two frames were derived from the same action and thus should contain the same physics despite being expressed differently.

B. The Klein-Gordon equations in the two frames

Varying the Einstein frame action Eq. (28) with respect to the scalar field ϕ leads to the equation of motion for ϕ (the Klein-Gordon equation) in the Einstein frame,

$$\nabla_\mu\left[\frac{\partial\mathcal{L}_\phi(\phi,(\nabla\phi)^2)}{\partial(\nabla_\mu\phi)}\right] = \frac{\partial\mathcal{L}_\phi}{\partial\phi} + \frac{1}{2}\frac{\mathrm{d}\ln A}{\mathrm{d}\phi}T^{(m)}, \tag{46}$$

where $T^{(i)} \equiv T^{(i)}_{\mu\nu}g^{\mu\nu}$ is the trace of the total stress-energy tensor of matter for the ith matter species, and $^{(m)}$ means that the equation only depends on $T^{(i)}$ for all other matter species than the scalar field.

Similarly, varying the Jordan frame action Eq. (31) with respect to the scalar field $\tilde{\phi}$ gives the equation of motion for $\tilde{\phi}$ in the Jordan frame:

$$\begin{aligned}&\frac{3}{2}M_{\rm Pl}^2\left(\frac{\mathrm{d}A}{\mathrm{d}\tilde{\phi}}\right)^2\frac{\tilde{\Box}\tilde{\phi}}{A^3} + \tilde{\nabla}_\mu\left[\frac{\partial\tilde{\mathcal{L}}_{\tilde{\phi}}(\tilde{\phi},(\tilde{\nabla}\tilde{\phi})^2)}{\partial(\tilde{\nabla}_\mu\tilde{\phi})}\right]\\ &= -\frac{1}{2}M_{\rm Pl}^2\frac{1}{A^2}\frac{\mathrm{d}A}{\mathrm{d}\tilde{\phi}}\tilde{R} - \frac{3}{2}M_{\rm Pl}^2\frac{\mathrm{d}A}{\mathrm{d}\tilde{\phi}}\frac{\mathrm{d}^2A}{\mathrm{d}\tilde{\phi}^2}\frac{(\tilde{\nabla}\tilde{\phi})^2}{A^3}\\ &\quad + \frac{9}{4}M_{\rm Pl}^2\left(\frac{\mathrm{d}A}{\mathrm{d}\tilde{\phi}}\right)^3\frac{(\tilde{\nabla}\tilde{\phi})^2}{A^4} + \frac{\partial\tilde{\mathcal{L}}_{\tilde{\phi}}}{\partial\tilde{\phi}}.\end{aligned} \tag{47}$$

This equation can be simplified by using the trace of Eq. (34) to replace $\tilde{R}$ in Eq. (47) with the trace of the stress energy tensor. Doing this leads to

$$\tilde{\nabla}_\mu\left[\frac{\partial\tilde{\mathcal{L}}_{\tilde{\phi}}(\tilde{\phi},(\tilde{\nabla}\tilde{\phi})^2)}{\partial(\tilde{\nabla}_\mu\tilde{\phi})}\right] = \frac{\partial\tilde{\mathcal{L}}_{\tilde{\phi}}}{\partial\tilde{\phi}} + \frac{1}{2}\frac{\mathrm{d}\ln A}{\mathrm{d}\tilde{\phi}}(\tilde{T}^{(m)} + \tilde{T}^{(\tilde{\phi})}). \tag{48}$$

As with the case of the Einstein equations, it can be verified that the Klein-Gordon equations in the two frames, despite having different forms, are mathematically equivalent to

[3]Throughout this paper, when we say that equations in the two frames are equivalent, we mean that we can start from the equation in one frame and derive the corresponding equation in the other frame using transformation laws of variables in these frames.

each other. An explicit check for general scalar field Lagrangians is presented in Appendix B.

IV. PERTURBATION EQUATIONS IN THE TWO FRAMES

In Sec. II, we have given the relations between quantities in two general conformally-related frames. In Sec. III we briefly checked that the fully covariant Einstein and Klein-Gordon equations are equivalent in the two frames, because they are derived from a same action. In order to demonstrate that physical observables are equivalent in the two frames, we next need to have a thorough look at the individual perturbation variables that are relevant to those observables. The aim of this section is to find the transformation laws of these perturbation variables and use them to show that the perturbed field and conservation equations in the two frames contain the same physics.

A. Quantities and equations in real space

We start with the various perturbation quantities and their equations in real space, and move to k (or Fourier) space in the next subsection.

1. Kinematic quantities

First look at the transformations of the kinematic quantities that are related to the curvature of the spacetime—the expansion scalar θ, shear $\sigma_{\mu\nu}$, vorticity $\varpi_{\mu\nu}$ and 4-acceleration $w_\mu = \dot{u}_\mu$.

We already found above that the 4-velocities in the Einstein and Jordan frames are related by $\tilde{u}^\mu = \frac{1}{\sqrt{A}} u^\mu$, $\tilde{u}_\mu = \sqrt{A} u_\mu$. The norm of the 4-velocity is the same in the two frames—$\tilde{u}^\mu \tilde{u}_\mu = u^\mu u_\mu = -1 = -c^2$. Notice that the conformal transformation does not change the speed of light.

Using the definitions for the time derivatives $\dot{\psi} \equiv u^\alpha \nabla_\alpha \psi$ and $\overset{\circ}{\psi} \equiv \tilde{u}^\alpha \tilde{\nabla}_\alpha \psi$, where ψ now is a simplified notation of a generic tensor, and the relation between the Christoffel symbols Eq. (18), one finds the 4-acceleration in the Jordan frame in terms of the one in the Einstein frame:

$$\tilde{w}^\mu = \overset{\circ}{\tilde{u}}{}^\mu = \frac{1}{A} \dot{u}^\mu + \frac{1}{2} \frac{\hat{\nabla}^\mu A}{A^2},$$
$$\tilde{w}_\mu = \overset{\circ}{\tilde{u}}_\mu = \dot{u}_\mu + \frac{1}{2} \frac{\hat{\nabla}_\mu A}{A}. \tag{49}$$

This reflects the fact that the forces felt by particles in the two frames are not the same: in the Einstein frame there is an additional fifth force, which depends on the spatial gradient of A.

Using Eqs. (18) and (12), we can compute the relation between the expansion scalars in the Einstein frame and the Jordan frame, defined by $\theta \equiv \nabla_\alpha u^\alpha$ and $\tilde{\theta} \equiv \tilde{\nabla}_\alpha \tilde{u}^\alpha$, respectively:

$$\tilde{\theta} = \frac{1}{A^{\frac{1}{2}}} \theta + \frac{3}{2} \frac{\dot{A}}{A^{\frac{3}{2}}}. \tag{50}$$

The second term comes from the fact that in the Jordan frame the scale factor is given by $\tilde{a} = \sqrt{A} a$, and an overall factor of $A^{-\frac{1}{2}}$ comes from the fact that the time derivative in $\tilde{\theta}$ (θ) is with respect to $\tilde{a} d\eta$ ($a d\eta$): expressed in conformal time, this becomes

$$\frac{\tilde{a}'}{\tilde{a}} = \frac{a'}{a} + \frac{A'}{2A}. \tag{51}$$

It is straightforward to find the link between the time derivatives of the expansion scalars in both frames by using Eqs. (50) and (14):

$$\overset{\circ}{\tilde{\theta}} = \frac{1}{A} \dot{\theta} - \frac{1}{2} \theta \frac{\dot{A}}{A^2} - \frac{9}{4} \frac{\dot{A}^2}{A^3} + \frac{3}{2} \frac{\ddot{A}}{A^2}. \tag{52}$$

We can similarly find the relations between the spatial gradients of the expansion scalar, $Z_\mu \equiv \hat{\nabla}_\mu \theta$ in the Einstein frame and $\tilde{Z}_\mu \equiv \hat{\tilde{\nabla}}_\mu \tilde{\theta}$ in the Jordan frame. Using Eq. (50) and (52), this is found to be

$$\tilde{Z}_\mu = \frac{1}{A^{\frac{1}{2}}} Z_\mu - \frac{1}{2} \theta \frac{\hat{\nabla}_\mu A}{A^{\frac{3}{2}}} + \frac{3}{2} \frac{\hat{\nabla}_\mu \dot{A}}{A^{\frac{3}{2}}} - \frac{9}{4} \frac{\dot{A}}{A^{\frac{5}{2}}} \hat{\nabla}_\mu A. \tag{53}$$

Z_μ ($\tilde{Z}_\mu$) represents the spatial perturbation of local expansion rate.

The relations between the shear and vorticity tensors in the two frames can be similarly obtained as

$$\tilde{\sigma}_\mu^\nu = \frac{1}{A^{\frac{1}{2}}} \sigma_\mu^\nu; \qquad \tilde{\sigma}_{\mu\nu} = A^{\frac{1}{2}} \sigma_{\mu\nu}, \tag{54}$$

$$\tilde{\varpi}_\mu^\nu = \frac{1}{A^{\frac{1}{2}}} \varpi_\mu^\nu; \qquad \tilde{\varpi}_{\mu\nu} = A^{\frac{1}{2}} \varpi_{\mu\nu}, \tag{55}$$

The relation Eq. (54) between the shear tensors in both frames is linear; this means that if the shear tensor σ_μ^ν in the Einstein frame vanishes, then the same happens for the shear tensor $\tilde{\sigma}_\mu^\nu$ in the Jordan frame. This relation is expected as A is the conformal factor, which affects the volume and therefore the expansion rate, but not the shear.

These transformation laws of the kinematic quantities have a role to play when comparing certain gauges in the two frames. When solving the linear perturbation equations in GR, not all degrees of freedom (dofs) are physical; the nonphysical d.o.f.s do not affect the physical results and can be fixed by choosing to work in some gauge. The choice of gauge is not unique and different choices usually have advantages and disadvantages. Some commonly used gauges are

(i) The synchronous gauge, where the 4-acceleration of the observer $\dot{u}^\mu$ vanishes.
(ii) The Newtonian gauge, where the shear tensor σ_μ^ν vanishes.
(iii) The energy frame, where the energy flux q_μ vanishes.

It is straightforward to see from Eq. (49) that the synchronous gauge is not preserved by a change of frame. For instance, if we choose to work with the synchronous gauge in the Einstein frame by setting $\dot{u}^\mu = 0$, then in the Jordan frame $\overset{\circ}{\tilde{u}}{}^\mu$ generally does not vanish. The Newtonian gauge, on the other hand, is the same in both frames due to Eq. (54). Later we will see that the same applies for the energy frame.

2. Dynamical quantities

Next we turn to the relations between the dynamical quantities in the two frames, starting from the definition of the stress-energy tensor:

$$T_{\mu\nu}^{(i)} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta[\sqrt{-g}\mathcal{L}^{(i)}]}{\delta g^{\mu\nu}}.$$

Using Eqs. (7) and (8) and the fact that the term in brackets is frame independent, one finds the following relations,

$$\tilde{T}_{\mu\nu}^{(i)} = \frac{1}{A}T_{\mu\nu}^{(i)}; \qquad \tilde{T}_\nu^{\mu(i)} = \frac{1}{A^2}T_\nu^{\mu(i)}; \qquad \tilde{T}^{\mu\nu}_{(i)} = \frac{1}{A^3}T^{\mu\nu}_{(i)}, \tag{56}$$

where the superscript $^{(i)}$ indicates that these relations are valid for any species.

Using the decomposition Eq. (4) and the relations Eq. (56), we find that the energy density, isotropic pressure, energy flux and anisotropic stress in the two frames are related by

$$\tilde{\rho}^{(i)} = \frac{1}{A^2}\rho^{(i)}, \tag{57}$$

$$\tilde{p}^{(i)} = \frac{1}{A^2}p^{(i)}, \tag{58}$$

$$\tilde{q}_\mu^{(i)} = \frac{1}{A^{\frac{3}{2}}}q_\mu^{(i)}; \qquad \tilde{q}^\mu_{(i)} = \frac{1}{A^{\frac{5}{2}}}q^\mu_{(i)}, \tag{59}$$

$$\tilde{\Pi}_\mu^{\nu(i)} = \frac{1}{A^2}\Pi_\mu^{\nu(i)}; \qquad \tilde{\Pi}_{\mu\nu}^{(i)} = \frac{1}{A}\Pi_{\mu\nu}^{(i)}. \tag{60}$$

Note that the relation in Eq. (59) between the energy fluxes in the Einstein and Jordan frames is linear, which confirms that the energy gauge is preserved by a conformal transformation.

One can intuitively understand the relation between the energy densities in the two frames given in Eq. (57). Since $\tilde{g}_{\mu\nu} = Ag_{\mu\nu}$, the relation between infinitesimal lengths in the two frames is given as $\mathrm{d}\tilde{s} = \sqrt{A}\mathrm{d}s$. Hence, if one considers a length l in a spatial hypersurface of constant time t, it can be easily found that $\tilde{l} = \sqrt{A}l$. Meanwhile, the relation between masses in the two frames can be found by considering the action of a point particle of mass $\tilde{m}$ expressed in terms of the tildered metric

$$S_m[\tilde{g}_{\mu\nu}] = -\tilde{m}\int\sqrt{-\tilde{g}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu}. \tag{61}$$

Using $\tilde{g}_{\mu\nu} = Ag_{\mu\nu}$, the same action can be expressed in terms of the untildered metric as

$$S_m[g_{\mu\nu}] = -\tilde{m}\int\sqrt{A}\sqrt{-g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu}, \tag{62}$$

which implies that

$$\tilde{m} = \frac{1}{\sqrt{A}}m. \tag{63}$$

Therefore, in this case the density in the Jordan frame is given by $\tilde{\rho} = \tilde{m}\tilde{l}^{-3} = A^{-2}ml^{-3} = A^{-2}\rho$.

From the above equations, we can also find the relations between the time and spatial derivatives of the energy density in the two frames, which is useful for the following sections:

$$\overset{\circ}{\tilde{\rho}} = \frac{1}{A^{\frac{5}{2}}}\dot{\rho} - 2\frac{\dot{A}}{A^{\frac{7}{2}}}\rho, \tag{64}$$

$$\hat{\tilde{\nabla}}_\mu\tilde{\rho} = \frac{1}{A^2}\hat{\nabla}_\mu\rho - 2\rho\frac{\hat{\nabla}_\mu A}{A^3}. \tag{65}$$

We define the perturbation of energy density about the exact zero-order FRW metric by the comoving first order quantity $X_\mu^{(i)} \equiv \hat{\nabla}_\mu\rho^{(i)} \equiv \rho^{(i)}\Delta_\mu^{(i)}$ in the Einstein and $\tilde{X}_\mu^{(i)} \equiv \hat{\tilde{\nabla}}_\mu\tilde{\rho}^{(i)} \equiv \tilde{\rho}^{(i)}\tilde{\Delta}_\mu^{(i)}$ in the Jordan frame. These quantities are related by

$$\tilde{X}_\mu^{(i)} = \frac{1}{A^2}X_\mu^{(i)} - 2\frac{\hat{\nabla}_\mu A}{A^3},$$
$$\tilde{\Delta}_\mu^{(i)} = \Delta_\mu^{(i)} - 2\frac{\hat{\nabla}_\mu A}{A}, \tag{66}$$

and their time derivatives satisfy

$$\overset{\circ}{\tilde{\Delta}}{}_\mu^{(i)} = \frac{1}{A^{\frac{1}{2}}}\dot{\Delta}_\mu^{(i)} - \frac{1}{2}\frac{\dot{A}}{A^{\frac{3}{2}}}\Delta_\mu - \frac{2}{A^{\frac{3}{2}}}\hat{\nabla}_\mu\dot{A} + \frac{2}{3A^{\frac{3}{2}}}\theta\hat{\nabla}_\mu A + 3\frac{\dot{A}}{A^{\frac{5}{2}}}\hat{\nabla}_\mu A - 2\frac{\dot{A}}{A^{\frac{3}{2}}}w_\mu. \tag{67}$$

Finally, it is useful to have the following expressions, which relate the conservations of the stress-energy tensors in the Einstein and the Jordan frames,

$$\tilde{\nabla}_\mu \tilde{T}^{\mu\nu}_{(i)} = \frac{1}{A^3}\nabla_\mu T^{\mu\nu}_{(i)} - \frac{1}{2}\frac{\nabla^\nu A}{A^4}T_{(i)},$$
$$\nabla_\mu \tilde{T}^{\mu(i)}_\nu = \frac{1}{A^2}\nabla_\mu T^{\mu(i)}_\nu - \frac{1}{2}\frac{\nabla_\nu A}{A^3}T^{(i)}. \tag{68}$$

3. Einstein's equations

We can now discuss the perturbed Einstein equations in the two frames. In $3+1$ formalism, the linearized Einstein equations give the relations between the kinematic and dynamical quantities that were introduced above. These include five constraint equations

$$0 = \hat{\nabla}^\alpha(\epsilon^{\mu\nu}_{\alpha\beta}u^\beta \varpi_{\mu\nu}), \tag{69}$$

$$0 = -\frac{2\hat{\nabla}_\mu\theta}{3} + \hat{\nabla}_\nu \ldots + \hat{\nabla}_\nu \varpi^\nu{}_\mu + M_{\rm Pl}^{-2}q_\mu, \tag{70}$$

$$0 = [\hat{\nabla}^\alpha\sigma_{\beta(\mu} - \hat{\nabla}^\alpha\varpi_{\beta(\mu}]\epsilon_{\nu)\gamma\alpha}{}^\beta u^\gamma - H_{\mu\nu}, \tag{71}$$

$$0 = \hat{\nabla}_\nu E^\nu{}_\mu + \frac{1}{2}M_{\rm Pl}^{-2}\left[\hat{\nabla}_\nu\Pi^\nu{}_\mu + \frac{2}{3}\theta q_\mu - \frac{2}{3}\hat{\nabla}_\mu\rho\right], \tag{72}$$

$$0 = \hat{\nabla}^\nu H_{\mu\nu} + \frac{1}{2}M_{\rm Pl}^{-2}[\hat{\nabla}_\alpha q_\beta + (\rho+p)\varpi_{\alpha\beta}]\epsilon_{\mu\nu}{}^{\alpha\beta}u^\nu, \tag{73}$$

and five propagation equations

$$0 = \dot{\theta} + \frac{1}{3}\theta^2 - \hat{\nabla}\cdot w + \frac{1}{2}M_{\rm Pl}^{-2}(\rho+3p), \tag{74}$$

$$0 = \dot{\sigma}^\mu{}_\nu + \frac{2}{3}\theta\sigma^\mu{}_\nu - \hat{\nabla}^{\langle\mu}w_{\nu\rangle} + E^\mu{}_\nu - \frac{1}{2}M_{\rm Pl}^{-2}\Pi^\mu{}_\nu, \tag{75}$$

$$0 = \dot{\varpi}_{\mu\nu} + \frac{2}{3}\theta\varpi_{\mu\nu} - \hat{\nabla}_{[\mu}w_{\nu]}, \tag{76}$$

$$0 = \dot{E}^\mu{}_\nu + \theta E^\mu{}_\nu - \hat{\nabla}^\alpha H^{(\mu}_\beta\epsilon_{\nu)\gamma\alpha}{}^\beta u^\gamma + \frac{1}{2}M_{\rm Pl}^{-2}\left[\dot{\Pi}^\mu{}_\nu + \frac{1}{3}\theta\Pi^\mu{}_\nu\right]$$
$$+ \frac{1}{2}M_{\rm Pl}^{-2}[(\rho+p)\sigma^\mu{}_\nu + \hat{\nabla}^{\langle\mu}q_{\nu\rangle}], \tag{77}$$

$$0 = \dot{H}_{\mu\nu} + \theta H_{\mu\nu} + \hat{\nabla}^\alpha E_{\beta(\mu}\epsilon_{\nu)\gamma\alpha}{}^\beta u^\gamma - \frac{1}{2}M_{\rm Pl}^{-2}\hat{\nabla}^\alpha\Pi_{\beta(\mu}\epsilon_{\nu)\gamma\alpha}{}^\beta u^\gamma. \tag{78}$$

In these equations, $\epsilon_{\mu\nu\alpha\beta}$ is the four-dimensional covariant permutation tensor, $\hat{\nabla}\cdot w \equiv \hat{\nabla}^\alpha w_\alpha$ (the same for general vectors), and $E_{\mu\nu}$ and $H_{\mu\nu}$ are, respectively, the electric and magnetic parts of the Weyl tensor, $C_{\mu\nu\alpha\beta}$, defined by $E_{\mu\nu} \equiv u^\alpha u^\beta C_{\mu\alpha\nu\beta}$ and $H_{\mu\nu} \equiv \frac{1}{2}u^\alpha u^\beta \epsilon^{\gamma\delta}_{\mu\alpha}C_{\gamma\delta\nu\beta}$.

In the Jordan frame, these equations take the same form, but the quantities in them should become tildered. In addition, since there are extra terms in the Jordan-frame *effective* stress-energy tensor from the conformal transformation, as shown in Eq. (37), such terms must be added to the tildered (bold) dynamical quantities in the Einstein equations. For simplicity, we do not repeat all the constraint and propagation equations, but instead only write down those that are directly relevant for linear perturbation evolutions in a spatially-flat perturbed universe:

$$0 = -\frac{2\hat{\tilde{\nabla}}_\mu\tilde{\theta}}{3} + \hat{\tilde{\nabla}}_\nu\tilde{\sigma}^\nu{}_\mu + \hat{\tilde{\nabla}}_\nu\tilde{\varpi}^\nu{}_\mu + \tilde{M}_{\rm Pl}^{-2}\tilde{\mathbf{q}}_\mu, \tag{79}$$

$$0 = \hat{\tilde{\nabla}}_\nu\tilde{E}^\nu{}_\mu + \frac{1}{2}\left[\hat{\tilde{\nabla}}_\nu\frac{\tilde{\mathbf{\Pi}}^\nu{}_\mu}{\tilde{M}_{\rm Pl}^2} + \frac{2}{3}\tilde{\theta}\frac{\tilde{\mathbf{q}}_\mu}{\tilde{M}_{\rm Pl}^2} - \frac{2}{3}\hat{\tilde{\nabla}}_\mu\frac{\tilde{\boldsymbol{\rho}}}{\tilde{M}_{\rm Pl}^2}\right], \tag{80}$$

$$0 = \mathring{\tilde{\theta}} + \frac{1}{3}\tilde{\theta}^2 - \hat{\tilde{\nabla}}\cdot\tilde{w} + \frac{1}{2}\tilde{M}_{\rm Pl}^{-2}(\tilde{\boldsymbol{\rho}} + 3\tilde{\mathbf{p}}), \tag{81}$$

$$0 = \mathring{\tilde{\sigma}}^\mu{}_\nu + \frac{2}{3}\tilde{\theta}\tilde{\sigma}^\mu{}_\nu - \hat{\tilde{\nabla}}^{\langle\mu}\tilde{w}_{\nu\rangle} + \tilde{E}^\mu{}_\nu - \frac{1}{2}\tilde{M}_{\rm Pl}^{-2}\tilde{\mathbf{\Pi}}^\mu{}_\nu, \tag{82}$$

and

$$0 = \mathring{\tilde{E}}^\mu{}_\nu + \tilde{\theta}\tilde{E}^\mu{}_\nu - \hat{\tilde{\nabla}}^\alpha\tilde{H}^{(\mu}_\beta\epsilon_{\nu)\gamma\alpha}{}^\beta\tilde{u}^\gamma + \frac{1}{2}[(\tilde{M}_{\rm Pl}^{-2}\tilde{\mathbf{\Pi}}^\mu{}_\nu)^\circ$$
$$+ \frac{1}{3}\tilde{\theta}\tilde{M}_{\rm Pl}^{-2}\tilde{\mathbf{\Pi}}^\mu{}_\nu] + \frac{1}{2}[\tilde{M}_{\rm Pl}^{-2}(\tilde{\boldsymbol{\rho}} + \tilde{\mathbf{p}})\tilde{\sigma}^\mu{}_\nu + \hat{\tilde{\nabla}}^{\langle\mu}\tilde{M}_{\rm Pl}^{-2}\tilde{\mathbf{q}}_{\nu\rangle}]. \tag{83}$$

We have verified that these Jordan frame equations are equivalent to their Einstein-frame counterparts (for example, one can start from the Jordan frame equations and use the relations of the dynamical and kinematic quantities in these two frames to derive the Einstein-frame equations, and vice versa), for which we have used the following expressions (up to linear order)

$$\tilde{E}_{\mu\nu} = E_{\mu\nu}, \tag{84}$$

$$\hat{\tilde{\Box}}A = \frac{1}{A}\hat{\Box}A, \tag{85}$$

$$\hat{\tilde{\nabla}}^\nu\hat{\tilde{\nabla}}_{\langle\mu}\hat{\tilde{\nabla}}_{\nu\rangle}A = \frac{2}{3A}\hat{\nabla}_\mu\hat{\Box}A - \frac{\dot{A}}{A}\hat{\nabla}_\nu\varpi^\nu{}_\mu, \tag{86}$$

$$\hat{\tilde{\nabla}}_\mu\tilde{w}_\nu = \hat{\nabla}_\mu w_\nu + \frac{1}{2A}\hat{\nabla}_{\langle\mu}\hat{\nabla}_{\nu\rangle}A + \frac{1}{6A}h_{\mu\nu}\hat{\Box}A, \tag{87}$$

and

$$\hat{\tilde{\nabla}}_\mu\left[\frac{\tilde{\boldsymbol{\rho}}^{(\varphi)}}{\tilde{M}_{\rm Pl}^2}\right] = \frac{1}{A}\hat{\nabla}_\mu\left[\frac{\rho^{(\varphi)}}{M_{\rm Pl}^2}\right] - \frac{1}{A^2}\frac{\rho^{(\varphi)}}{M_{\rm Pl}^2}\hat{\nabla}_\mu A + \frac{\dot{A}}{A^2}Z_\mu + \frac{1}{A^2}\left[\theta + \frac{3}{2}\frac{\dot{A}}{A}\right]\hat{\nabla}_\mu\dot{A} - 2\theta\frac{\dot{A}}{A^3}\hat{\nabla}_\mu A - \frac{9}{4}\frac{\dot{A}^2}{A^4}\hat{\nabla}_\mu A - \frac{1}{A^2}\hat{\nabla}_\mu\hat{\Box}A. \quad (88)$$

In addition to the above equations, it is often useful to express the projected Ricci scalar, $\hat{R}$, onto the hypersurfaces orthogonal to u^μ, as

$$\hat{R} = R - 2\dot{\theta} - \frac{4}{3}\theta^2 + 2\hat{\nabla}\cdot w \quad (89)$$

$$= 2M_{\rm Pl}^{-2}\rho - \frac{2}{3}\theta^2. \quad (90)$$

The covariant spatial derivative of the projected Ricci scalar, $\eta_\mu \equiv \hat{\nabla}_\mu\hat{R}/2$, can be derived from the above equation, as

$$\eta_\mu = M_{\rm Pl}^{-2}\hat{\nabla}_\mu\rho - \frac{2}{3}\theta\hat{\nabla}_\mu\theta, \quad (91)$$

and its time evolution is governed by the following propagation equation

$$\dot{\eta}_\mu + \theta\eta_\mu = -\frac{2\theta}{3}\hat{\nabla}_\mu\hat{\nabla}\cdot w - M_{\rm Pl}^{-2}\hat{\nabla}_\mu\hat{\nabla}\cdot q. \quad (92)$$

In the Jordan frame, using the relations Eqs. (89) and (36), it can be found that

$$\hat{\tilde{R}} = \frac{1}{A}\hat{R} - \frac{2}{A^2}\hat{\Box}A, \quad (93)$$

$$\tilde{\eta}_\mu = \frac{1}{A}\eta_\mu - \frac{1}{A^2}\hat{\nabla}_\mu\hat{\Box}A, \quad (94)$$

with which it can be easily shown that the equations

$$\hat{\tilde{R}} = 2\tilde{M}_{\rm Pl}^2\tilde{\boldsymbol{\rho}} - \frac{2}{3}\tilde{\theta}^2, \quad (95)$$

$$\tilde{\eta}_\mu = \tilde{M}_{\rm Pl}^{-2}\hat{\tilde{\nabla}}_\mu\tilde{\boldsymbol{\rho}} - \frac{2}{3}\tilde{\theta}\hat{\tilde{\nabla}}_\mu\tilde{\theta}, \quad (96)$$

$$\dot{\tilde{\eta}}_\mu + \tilde{\theta}\tilde{\eta}_\mu = -\frac{2\tilde{\theta}}{3}\hat{\tilde{\nabla}}_\mu\hat{\tilde{\nabla}}\cdot\tilde{w} - \hat{\tilde{\nabla}}_\mu[\tilde{M}_{\rm Pl}^{-2}\hat{\tilde{\nabla}}\cdot\tilde{\mathbf{q}}]. \quad (97)$$

are equivalent to their Einstein-frame counterparts.

4. Conservation equations

The Jordan frame stress-momentum tensor $\tilde{T}^\mu{}_\nu$ satisfies the conservation equation $\tilde{\nabla}_\nu\tilde{T}^\nu{}_\mu = 0$, which can be decomposed into a component parallel to $\tilde{u}^\mu$ (the continuity equation) plus a component perpendicular (the Euler equation) as:

$$\overset{\circ}{\tilde{\rho}} + (\tilde{\rho} + \tilde{P})\tilde{\theta} + \hat{\tilde{\nabla}}\cdot\tilde{q} = 0, \quad (98)$$

$$\overset{\circ}{\tilde{q}}_\mu + \frac{4}{3}\tilde{\theta}\tilde{q}_\mu + (\tilde{\rho} + \tilde{P})\tilde{w}_\mu + \hat{\tilde{\nabla}}_\mu\tilde{P} + \hat{\tilde{\nabla}}_\nu\tilde{\Pi}^\nu{}_\mu = 0. \quad (99)$$

In the Einstein frame, on the other hand, the stress-energy tensors for individual species do not conserve, but satisfy

$$\nabla_\nu T^\nu{}_\mu = \frac{T}{2A}\nabla_\mu A,$$

according to Eq. (68), where T is the trace of $T^\mu{}_\nu$. Therefore, the continuity and Euler equations can be written as

$$\dot{\rho} + (\rho + P)\theta + \hat{\nabla}\cdot q = -\frac{T}{2A}\dot{A}, \quad (100)$$

$$\dot{q}_\mu + \frac{4}{3}\theta q_\mu + (\rho + P)w_\mu + \hat{\nabla}_\mu P + \hat{\nabla}_\nu\Pi^\nu{}_\mu = \frac{T}{2A}\hat{\nabla}_\mu A. \quad (101)$$

In what follows, we shall explicitly compare the conservation equation for photons and dark matter in the two frames. Other matter species, e.g., massless neutrinos and baryons, are similar. We will not discuss massive neutrinos in this paper.

Photons (and massless particles in general) have zero trace of their stress-energy tensor, and so the conservation equations hold for them even in the Einstein frame. By using Eqs. (50), (57), (64) and $\hat{\tilde{\nabla}}\cdot\tilde{q} = A^{-5/2}\hat{\nabla}\cdot q$, one can straightforwardly check that the continuity equations for photons,

$$\overset{\circ}{\tilde{\rho}}{}^{(\gamma)} + \frac{4}{3}\tilde{\theta}\tilde{\rho}^{(\gamma)} + \hat{\tilde{\nabla}}_\alpha\tilde{q}^\alpha_{(\gamma)} = 0, \quad (102)$$

$$\dot{\rho}^{(\gamma)} + \frac{4}{3}\theta\rho^{(\gamma)} + \hat{\nabla}_\alpha q^\alpha_{(\gamma)} = 0, \quad (103)$$

are equivalent to each other in the two frames. Similarly, it can be checked that the momentum (Euler) equations for photons in the two frames,

$$\overset{\circ}{\tilde{q}}{}^{(\gamma)}_\mu + \frac{4}{3}\tilde{\theta}\tilde{q}^{(\gamma)}_\mu + \hat{\tilde{\nabla}}_\nu\tilde{\Pi}^{\nu(\gamma)}_\mu + \frac{4}{3}\tilde{\rho}^{(\gamma)}\tilde{w}_\mu + \frac{1}{3}\hat{\tilde{\nabla}}_\mu\tilde{\rho}^{(\gamma)} = \tilde{n}_e\tilde{\sigma}_T\left(\frac{4}{3}\tilde{\rho}^{(\gamma)}\tilde{v}^{(b)}_\mu - \tilde{q}^{(\gamma)}_\mu\right), \quad (104)$$

$$\dot{q}^{(\gamma)}_\mu + \frac{4}{3}\theta q^{(\gamma)}_\mu + \hat{\nabla}_\nu \Pi^{\nu(\gamma)}_\mu + \frac{4}{3}\rho^{(\gamma)} w_\mu + \frac{1}{3}\hat{\nabla}_\mu \rho^{(\gamma)} = n_e \sigma_T \left(\frac{4}{3}\rho^{(\gamma)} v^{(b)}_\mu - q^{(\gamma)}_\mu\right), \tag{105}$$

are also equivalent, for which we have used

$$\mathring{\tilde{q}}_\mu = \frac{1}{A^2}\dot{q}_\mu - 2\frac{\dot{A}}{A^3} q_\mu,$$
$$\tilde{v}_\mu = A^{\frac{1}{2}} v_\mu,$$
$$\tilde{n}_e \tilde{\sigma}_\mathrm{T} = A^{-\frac{1}{2}} n_e \sigma_\mathrm{T},$$

where n_e is the electron number density and σ_T is the Thomson scattering cross section. The right-hand sides of the Euler equations are the interactions between electrons and photons. Note that the electron densities in the two frames are related by $\tilde{n}_e = A^{-\frac{3}{2}} n_e$, because the electron numbers are the same while the volumes are related by $\tilde{V} \propto \tilde{l}^3 = (\sqrt{A} l)^3 \propto V$. On the other hand, the Thomson cross sections in the two frames are connected by $\tilde{\sigma}_T = A\sigma_T$, which can be easily checked using the expression $\sigma_T = \frac{8\pi}{3}\left(\frac{q^2}{4\pi\epsilon_0 mc^2}\right)^2$, in which q and m are, respectively, the electron electric charge and the mass; thus $\sigma_\mathrm{T} \propto m^{-2}$ with $\tilde{m} = A^{-\frac{1}{2}} m$.

By taking the spatial gradients of Eqs. (102) and (103), one finds the propagation equations for the photon density contrast $\tilde{\Delta}_\mu$ (Δ_μ) in the Jordan (Einstein) frame (using Eq. (67)

$$\mathring{\tilde{\Delta}}^{(\gamma)}_\mu + \frac{1}{3}\tilde{\theta}\tilde{\Delta}_\mu + \frac{4}{3}\tilde{Z}_\mu + \frac{4}{3}\tilde{\theta}\tilde{w}_\mu + \frac{1}{\tilde{\rho}^{(\gamma)}}\hat{\tilde{\nabla}}_\mu \hat{\tilde{\nabla}}_\alpha \tilde{q}^\alpha_{(\gamma)} = 0, \tag{106}$$

$$\dot{\Delta}^{(\gamma)}_\mu + \frac{1}{3}\theta\Delta_\mu + \frac{4}{3}Z_\mu + \frac{4}{3}\theta w_\mu + \frac{1}{\rho^{(\gamma)}}\hat{\nabla}_\mu \hat{\nabla}_\alpha q^\alpha_{(\gamma)} = 0. \tag{107}$$

Again, it can be checked that these equations in the two frames are equivalent. The fact that all the conservation equations for photons take exactly the same form in the Jordan and Einstein frames is because conformal coupling does not affect the dynamics of photons and, more generally, massless particles.

We next turn to cold dark matter, which is treated as a perfect fluid with zero pressure and anisotropic stress in the perturbation analysis. Using the connection between quantities in the two frames, it can be shown that the continuity equations in the Jordan and Einstein frames:

$$\mathring{\tilde{\rho}}^{(\mathrm{c})} + \tilde{\theta}\tilde{\rho}^{(\mathrm{c})} + \hat{\tilde{\nabla}}_\alpha \tilde{q}^\alpha_{(\mathrm{c})} = 0, \tag{108}$$

$$\dot{\rho}^{(\mathrm{c})} + \left[\theta - \frac{\dot{A}}{2A}\right]\rho^{(\mathrm{c})} + \hat{\nabla}_\alpha q^\alpha_{(\mathrm{c})} = 0, \tag{109}$$

are equivalent to each other. Note that, unlike the case of photons, these equations take slightly different forms in the Jordan and Einstein frames, with the latter having an additional term. While in the Jordan frame $\tilde{\rho}^{(\mathrm{c})}$ satisfies the usual $\tilde{\rho}^{(\mathrm{c})} \propto \tilde{a}^{-3}$ scaling law, in the Einstein frame the mass of dark matter particles evolves in time with $m \propto \sqrt{A}$, and we have a modified scaling law $\rho^{(\mathrm{c})}/\sqrt{A} \propto a^{-3}$ or $\rho^{(\mathrm{c})} \propto \sqrt{A} a^{-3}$, which explains the extra factor in Eq. (109).

Similarly, the momentum equations in the two frames:

$$\mathring{\tilde{q}}^{(\mathrm{c})}_\mu + \frac{4}{3}\tilde{\theta}\tilde{q}^{(\mathrm{c})}_\mu + \tilde{\rho}^{(\mathrm{c})}\tilde{w}_\mu = 0,$$

$$\dot{q}^{(\mathrm{c})}_\mu + \frac{4}{3}\theta q^{(\mathrm{c})}_\mu + \rho^{(\mathrm{c})} w_\mu = -\frac{1}{2A}\rho^{(\mathrm{c})}\hat{\nabla}_\mu A,$$

are equivalent to each other. Indeed, the above equations can be rewritten in terms of the peculiar velocities given by $\tilde{v}^\mathrm{c}_\mu \equiv \tilde{q}^{(\mathrm{c})}_\mu/\tilde{\rho}^{(\mathrm{c})}$ and $v^{(\mathrm{c})}_\mu \equiv q^{(\mathrm{c})}_\mu/\rho^{(\mathrm{c})}$:

$$\mathring{\tilde{v}}^{(\mathrm{c})}_\mu + \frac{\mathring{\tilde{a}}}{\tilde{a}}\tilde{v}^{(\mathrm{c})}_\mu + \tilde{w}_\mu = 0, \tag{110}$$

$$\dot{v}^{(\mathrm{c})}_\mu + \left[\frac{\dot{a}}{a} + \frac{\dot{A}}{2A}\right] v^{(\mathrm{c})}_\mu + w_\mu = -\frac{1}{2A}\hat{\nabla}_\mu A. \tag{111}$$

As expected, in the Jordan frame, the peculiar velocity of dark matter particles is affected by two terms—a "frictional" force caused by the expansion of the Universe [the second, velocity-dependent term in Eq. (110)], and a 4-acceleration $\tilde{w}_\mu$ which encodes the effect of gravity on particle geodesics. The terms are both modified in the Einstein frame: the particles now feel a "fifth" force that is proportional to the gradient[4] of $\ln(A)$ as in the right-hand side of Eq. (111), and an additional frictional force as in the brackets on the left side of Eq. (111).

Taking the spatial gradients of Eqs. (108) and (109), we obtain the propagation equations for the dark matter density contrasts in the two frames [again by using Eq. (67)]:

$$\mathring{\tilde{\Delta}}^{(\mathrm{c})}_\mu + \frac{1}{3}\tilde{\theta}\tilde{\Delta}^{(\mathrm{c})}_\mu + \tilde{Z}_\mu + \tilde{\theta}\tilde{w}_\mu + \frac{1}{\tilde{\rho}^{(\mathrm{c})}}\hat{\tilde{\nabla}}_\mu \hat{\tilde{\nabla}}_\alpha \tilde{q}^\alpha_{(\mathrm{c})} = 0, \tag{112}$$

[4]In this sense, $\ln(A)$ can be considered as the potential of the fifth force.

$$\dot{\Delta}^{(c)}_\mu + \frac{1}{3}\theta\Delta^{(c)}_\mu + Z_\mu + \theta w_\mu + \frac{1}{\rho^{(c)}}\hat{\nabla}_\mu\hat{\nabla}_\alpha q^\alpha_{(c)} = \frac{1}{2A}\hat{\nabla}_\mu\dot{A} - \frac{\dot{A}}{2A^2}\hat{\nabla}_\mu A + \frac{\dot{A}}{2A}w_\mu. \quad (113)$$

Before leaving this section, note that another sanity check of the equations derived above is to verify that the components of the scalar field effective stress-energy tensor: $\tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{q}}_\mu$ and $\tilde{\boldsymbol{\Pi}}^\mu{}_\nu$ satisfy the conservation equations, Eq. (98) and (99). An explicit check of this will require us to know the exact form of $\tilde{\rho}^{(\tilde{\phi})}$, $\tilde{p}^{(\tilde{\phi})}$, $\tilde{q}^{(\tilde{\phi})}_\mu$ and $\tilde{\Pi}^{(\tilde{\phi})}_{\mu\nu}$, as well the stress-energy tensor components for normal matter species (because it is the components of $\tilde{M}^{-2}_{\rm Pl}T^{(\rm matter)}_{\mu\nu}$ that enter the Jordan-frame Einstein equations, and $\tilde{M}_{\rm Pl}$ depends on the scalar field A). A slightly easier check—which still serves the purpose—is to assume that $\tilde{T}^{(\tilde{\phi})}_{\mu\nu} = \tilde{T}^{(\rm matter)}_{\mu\nu} = 0$, which means that there is no matter, including scalar field, in the Universe in the Einstein frame. We have checked that Eqs. (98) and (99) hold for $\tilde{\boldsymbol{\rho}}, \tilde{\boldsymbol{p}}, \tilde{\mathbf{q}}_\mu$ and $\tilde{\boldsymbol{\Pi}}^\mu{}_\nu$ in this case (their Einstein-frame version are simply $0 = 0$ and so hold too trivially).

B. Quantities and equations in k space

The linearized Einstein and conservation equations are usually solved in Fourier (or k) space, where the different Fourier (or k) modes are independent of each other. The spatial derivatives can then be replaced with multiplications by powers of k, so that the equations become ordinary differential equations that can be solved by numerical integration. In this subsection, we'll write the quantities and equations in k space.

For this, we define the zero-order eigenfunctions $Q^{(k)}$ of the comoving spatial d'Alembertian operator $a^2\hat{\Box} \equiv a^2\hat{\nabla}_\alpha\hat{\nabla}^\alpha$ (and $\tilde{a}^2\hat{\tilde{\Box}} \equiv \tilde{a}^2\hat{\tilde{\nabla}}_\alpha\hat{\tilde{\nabla}}^\alpha$) as

$$a^2\hat{\Box}Q^{(k)} = -k^2Q^{(k)}; \qquad \tilde{a}^2\hat{\tilde{\Box}}Q^{(k)} = -k^2Q^{(k)}. \quad (114)$$

$\dot{Q}^{(k)}$ is a zero-order quantity. The multiplication of a^2 not only makes this operator a comoving one, but also means that $Q^{(k)}$ is the same for both the Einstein and the Jordan frames. Vector and (rank-2) tensor perturbation quantities in the Einstein (Jordan) frames can be expanded in terms of $Q^{(k)}_\mu$ ($\tilde{Q}^{(k)}_\mu$) and $Q^{(k)}_{\mu\nu}$ ($\tilde{Q}^{(k)}_{\mu\nu}$), which are defined by $Q^{(k)}_\mu \equiv \frac{a}{k}\hat{\nabla}_\mu Q^{(k)}$ ($\tilde{Q}^{(k)}_\mu \equiv \frac{\tilde{a}}{k}\hat{\tilde{\nabla}}_\mu Q^{(k)}$) and $Q^{(k)}_{\mu\nu} \equiv \frac{a}{k}\hat{\nabla}_{\langle\mu}Q^{(k)}_{\nu\rangle}$ ($\tilde{Q}^{(k)}_{\mu\nu} \equiv \frac{\tilde{a}}{k}\hat{\tilde{\nabla}}_{\langle\mu}\tilde{Q}^{(k)}_{\nu\rangle}$), respectively.

1. Kinematic quantities

Using the notations introduced above, the k-space kinematic quantities (or their gradients) in the two frames can be written as

$$Z_\mu = \sum_k \frac{k^2}{a^2}Z_kQ^{(k)}_\mu, \qquad w_\mu = -\sum_k \frac{k}{a}w_kQ^{(k)}_\mu,$$
$$\sigma_{\mu\nu} = -\sum_k \frac{k}{a}\sigma_kQ^{(k)}_{\mu\nu}, \qquad h_\mu = \sum_k kh_kQ^{(k)}_\mu,$$
$$\eta_\mu = \sum_k \frac{k^3}{a^3}\eta_kQ^{(k)}_\mu, \qquad \tilde{Z}_\mu = \sum_k \frac{k^2}{\tilde{a}^2}\tilde{Z}_k\tilde{Q}^{(k)}_\mu,$$
$$\tilde{w}_\mu = -\sum_k \frac{k}{\tilde{a}}\tilde{w}_k\tilde{Q}^{(k)}_\mu, \qquad \tilde{\sigma}_{\mu\nu} = -\sum_k \frac{k}{\tilde{a}}\tilde{\sigma}_k\tilde{Q}^{(k)}_{\mu\nu},$$
$$\tilde{h}_\mu = \sum_k k\tilde{h}_k\tilde{Q}^{(k)}_\mu, \qquad \tilde{\eta}_\mu = \sum_k \frac{k^3}{\tilde{a}^3}\tilde{\eta}_k\tilde{Q}^{(k)}_\mu, \quad (115)$$

where $h_\mu \equiv \hat{\nabla}_\mu a$ and $\tilde{h}_\mu \equiv \hat{\tilde{\nabla}}_\mu\tilde{a}$. From these relations, we find

$$\tilde{w}_k = w_k - \frac{1}{2A}\xi_k, \quad (116)$$

$$k\tilde{Z}_k = kZ_k + \frac{3}{2}\frac{1}{A}\xi'_k - \frac{3}{2}\frac{1}{A}\left[\frac{a'}{a} + \frac{3A'}{2A}\right]\xi_k + \frac{3}{2}\frac{A'}{A}w_k, \quad (117)$$

$$\tilde{\sigma}_k = \sigma_k, \quad (118)$$

$$\tilde{h}_k = h_k + \frac{1}{2A}\xi_k, \quad (119)$$

$$\tilde{\eta}_k = \eta_k + \frac{1}{A}\xi_k, \quad (120)$$

in which $'$ means the derivative with respect to the conformal time η (not to be confused with η_k with a subscript $_k$, which is the Fourier coefficient of η_μ), and where we have used the Fourier expansion of $\hat{\nabla}_\mu A$ given as

$$\hat{\nabla}_\mu A = \sum_k \frac{k}{a}\xi_kQ^{(k)}_\mu, \quad (121)$$

and the relations (which hold to zero order)

$$\frac{1}{\tilde{a}}\tilde{Q}^{(k)}_\mu = \frac{1}{a}Q^{(k)}_\mu, \qquad \frac{1}{\tilde{a}^2}\tilde{Q}^{(k)}_{\mu\nu} = \frac{1}{a^2}Q^{(k)}_{\mu\nu}. \quad (122)$$

2. Dynamical quantities

Similarly, for the dynamical quantities or their gradients, we can write

$$\Delta_\mu = \sum_k \frac{k}{a}\Delta_k Q_\mu^{(k)}, \qquad q_\mu = -\sum_k q_k Q_\mu^{(k)},$$
$$\Pi_{\mu\nu} = \sum_k \Pi_k Q_{\mu\nu}^{(k)}, \qquad \tilde{\Delta}_\mu = \sum_k \frac{k}{\tilde{a}}\tilde{\Delta}_k \tilde{Q}_\mu^{(k)},$$
$$\tilde{q}_\mu = -\sum_k \tilde{q}_k \tilde{Q}_\mu^{(k)}, \qquad \tilde{\Pi}_{\mu\nu} = \sum_k \tilde{\Pi}_k \tilde{Q}_{\mu\nu}^{(k)}. \tag{123}$$

From these expressions, we can find the following relations between the two frames,

$$\tilde{\Delta}_k = \Delta_k - \frac{2}{A}\xi_k, \tag{124}$$

$$\tilde{\Delta}_k^p = \Delta_k^p - \frac{2}{A}\xi_k, \tag{125}$$

$$\tilde{q}_k = A^{-2} q_k, \tag{126}$$

$$\tilde{\Pi}_k = A^{-2}\Pi_k, \tag{127}$$

where Δ_k^p is the expansion coefficient for $\Delta_\mu^p \equiv \hat{\nabla}_\mu p/\rho$. Note that because $\tilde{\rho} = A^{-2}\rho$, if we define $v_k = q_k/\rho$ and $\pi_k = \Pi_k/\rho$ (and similarly for their Jordan-frame counterparts), we will have

$$\tilde{v}_k = v_k, \qquad \tilde{\pi}_k = \pi_k. \tag{128}$$

For the relevant effective stress-energy tensor components of the scalar field given in Eqs. (39) to (42), the Fourier expansion coefficient can be written as

$$\begin{aligned}\frac{\tilde{\mathbf{X}}_k \tilde{a}^2}{\tilde{M}_{\rm Pl}^2} =& \frac{X_k^{(\varphi)} a^2}{M_{\rm Pl}^2} - \frac{1}{A}\frac{\rho a^2}{M_{\rm Pl}^2}\xi_k + \frac{k^2}{A}\xi_k + \frac{A'}{A}kZ_k \\ &+ \frac{3}{A}\left[\frac{a'}{a} + \frac{A'}{2A}\right]\xi_k' - \frac{6A'}{A^2}\left[\frac{a'}{a} + \frac{3}{8}\frac{A'}{A}\right]\xi_k \\ &+ \frac{3A'}{A}\left[\frac{a'}{a} + \frac{A'}{2A}\right]w_k,\end{aligned} \tag{129}$$

$$\frac{\tilde{\mathbf{q}}_k \tilde{a}^2}{\tilde{M}_{\rm Pl}^2} = \frac{q_k^{(\varphi)} a^2}{M_{\rm Pl}^2} - \frac{1}{A}k\xi_k' + \frac{1}{A}\left[\frac{a'}{a} + \frac{3}{2}\frac{A'}{A}\right]k\xi_k - \frac{A'}{A}kw_k, \tag{130}$$

$$\frac{\tilde{\Pi}_k \tilde{a}^2}{\tilde{M}_{\rm Pl}^2} = \frac{\Pi_k^{(\varphi)} a^2}{M_{\rm Pl}^2} - \frac{1}{A}k^2\xi_k - \frac{A'}{A}k\sigma_k, \tag{131}$$

where in Eq. (129) ρ is the total energy density of all matter species in the Einstein frame.

3. Einstein's equations

With the above results, we can now write down the Fourier-space versions of the linearized Einstein equations in the Jordan and Einstein frames, and check their equivalence. For the constraint equation, Eqs. (79) and (70), we have

$$\frac{2}{3}k^2(\tilde{\sigma}_k - \tilde{Z}_k) = \tilde{M}_{\rm Pl}^{-2}\tilde{\mathbf{q}}_k \tilde{a}^2,$$
$$\frac{2}{3}k^2(\sigma_k - Z_k) = M_{\rm Pl}^{-2} q_k a^2, \tag{132}$$

and using Eqs. (117), (118) and (130), it can be shown that they are equivalent. Using Eqs. (116), (117), (120), (129) and (130), it can also be found that the Fourier-space versions of Eqs. (96) and (91),

$$k^2\tilde{\eta}_k = \tilde{M}_{\rm Pl}^{-2}\tilde{\mathbf{X}}_k \tilde{a}^2 - 2k\frac{\tilde{a}'}{\tilde{a}}\tilde{Z}_k,$$
$$k^2\eta_k = M_{\rm Pl}^{-2} X_k a^2 - 2k\frac{a'}{a}Z_k, \tag{133}$$

and of Eqs. (97) and (92),

$$k\tilde{\eta}_k' = -\tilde{M}_{\rm Pl}^{-2}\tilde{\mathbf{q}}_k \tilde{a}^2 - 2k\frac{\tilde{a}'}{\tilde{a}}\tilde{w}_k,$$
$$k\eta_k' = -M_{\rm Pl}^{-2} q_k a^2 - 2k\frac{a'}{a}w_k, \tag{134}$$

are equivalent.

The constraint equations Eqs. (80) and (72) in the Fourier space become

$$k^3\tilde{\Phi}_k = -\frac{1}{2}\left[k\tilde{M}_{\rm Pl}^{-2}(\tilde{\Pi}_k + \tilde{\mathbf{X}}_k) + 3\frac{\tilde{a}'}{\tilde{a}}\tilde{M}_{\rm Pl}^{-2}\mathbf{q}_k\right]\tilde{a}^2,$$
$$k^3\Phi_k = -\frac{1}{2}\left[kM_{\rm Pl}^{-2}(\Pi_k + X_k) + 3\frac{a'}{a}M_{\rm Pl}^{-2}q_k\right]a^2, \tag{135}$$

the equivalence of which can be checked by using Eqs. (129), (130), (131) and (132). In the above, Φ_k and $\tilde{\Phi}_k$ are, respectively, the Fourier coefficients of $E_{\mu\nu}$ and $\tilde{E}_{\mu\nu}$:

$$E_{\mu\nu} = \sum_k \frac{k^2}{a^2}\Phi_k Q_{\mu\nu}, \qquad \tilde{E}_{\mu\nu} = \sum_k \frac{k^2}{\tilde{a}^2}\tilde{\Phi}_k \tilde{Q}_{\mu\nu}, \tag{136}$$

so that the Weyl potentials in the two frames satisfy $\tilde{\Phi}_k = \Phi_k$ following $\tilde{E}_{\mu\nu} = E_{\mu\nu}$.

The propagation equations for the shear, Eqs. (82) and (75), become

$$k\left(\tilde{\sigma}_k' + \frac{\tilde{a}'}{\tilde{a}}\tilde{\sigma}_k\right) - k^2(\tilde{\Phi}_k + \tilde{w}_k) + \frac{1}{2}\tilde{M}_{\rm Pl}^{-2}\tilde{\Pi}_k\tilde{a}^2 = 0,$$
$$k\left(\sigma_k' + \frac{a'}{a}\sigma_k\right) - k^2(\Phi_k + w_k) + \frac{1}{2}M_{\rm Pl}^{-2}\Pi_k a^2 = 0, \tag{137}$$

and the propagation equations for the Weyl potential, Eqs. (83) and (77), can be written in k-space as

$$k^2\left(\tilde{\Phi}'_k + \frac{\tilde{a}'}{\tilde{a}}\tilde{\Phi}_k\right) = \frac{1}{2}\left[k\tilde{M}_{\rm Pl}^{-2}[(\tilde{\boldsymbol{\rho}} + \tilde{\mathbf{P}})\tilde{\sigma}_k + \tilde{\mathbf{q}}_k] - (\tilde{M}_{\rm Pl}^{-2}\tilde{\Pi}_k)' - \frac{\tilde{a}'}{\tilde{a}}\tilde{M}_{\rm Pl}^{-2}\tilde{\Pi}_k\right]\tilde{a}^2,$$

$$k^2\left(\Phi'_k + \frac{a'}{a}\Phi_k\right) = \frac{1}{2}\left[kM_{\rm Pl}^{-2}[(\rho + P)\sigma_k + q] - M_{\rm Pl}^{-2}\Pi'_k - \frac{a'}{a}M_{\rm Pl}^{-2}\Pi\right]a^2. \tag{138}$$

These again can be shown to be equivalent to each other.

Therefore, for all components of the linearized Einstein equations that are relevant here, the two frames are physically identical—not only do the equations take the same forms, but also they have the same physical content.

Note that, in this subsubsection, we used bold symbols to denote total quantities, including contributions from normal matter *and* the effective stress-energy tensor of the scalar field in the Jordan frame.

4. Conservation equations

Expressed in k space, the perturbed continuity equations for photons in the Jordan and Einstein frames can be written as

$$(\tilde{\Delta}_k^{(\gamma)})' + \frac{4}{3}k\tilde{Z}_k - 4\frac{\tilde{a}'}{\tilde{a}}\tilde{w}_k + k\tilde{v}_k^{(\gamma)} = 0,$$

$$(\Delta_k^{(\gamma)})' + \frac{4}{3}kZ_k - 4\frac{a'}{a}w_k + kv_k^{(\gamma)} = 0. \tag{139}$$

The Euler equations for photons can be written as

$$(\tilde{v}_k^{(\gamma)})' - \frac{1}{3}k\tilde{\Delta}_k^{(\gamma)} + \frac{2}{3}k\tilde{\pi}_k^{(\gamma)} + \frac{4}{3}k\tilde{w}_k - \tilde{n}_e\tilde{\sigma}_{\rm T}\tilde{a}\left(\frac{4}{3}\tilde{v}_k^{(b)} - \tilde{v}_k^{(\gamma)}\right) = 0,$$

$$(v_k^{(\gamma)})' - \frac{1}{3}k\Delta_k^{(\gamma)} + \frac{2}{3}k\pi_k^{(\gamma)} + \frac{4}{3}kw_k - n_e\sigma_{\rm T}a\left(\frac{4}{3}v_k^{(b)} - v_k^{(\gamma)}\right) = 0. \tag{140}$$

The perturbed continuity equations for cold dark matter become

$$(\tilde{\Delta}_k^{(c)})' + k\tilde{Z}_k - 3\frac{\tilde{a}'}{\tilde{a}}\tilde{w}_k + k\tilde{v}_k^{(c)} = 0,$$

$$(\Delta_k^{(c)})' + kZ_k - 3\frac{a'}{a}w_k + kv_k^{(c)} = \frac{1}{2A}\xi'_k - \frac{A'}{2A^2}\xi_k, \tag{141}$$

and the Euler equations are

$$(\tilde{v}_k^{(c)})' + \frac{\tilde{a}'}{\tilde{a}}\tilde{v}_k^{(c)} + k\tilde{w}_k = 0,$$

$$(v_k^{(c)})' + \left[\frac{a'}{a} + \frac{A'}{2A}\right]v_k^{(c)} + kw_k = \frac{1}{2A}k\xi_k. \tag{142}$$

All these equations are equivalent between their Jordan frame and Einstein frame versions, as can be straightforwardly checked using the relations between the k-space quantities in the two frames given above. As in the real space case, we shall not present the relevant equations for baryons and massless neutrinos as they are similar.

Note that Eq. (139) can be rewritten, in a form that more directly shows that they are "continuity" equations, as

$$[\tilde{\Delta}_k^{(\gamma)} + 4\tilde{h}_k]' + k\tilde{v}_k^{(\gamma)} = 0,$$

$$[\Delta_k^{(\gamma)} + 4h_k]' + kv_k^{(\gamma)} = 0, \tag{143}$$

where we have used $h'_k = \frac{1}{3}kZ_k - \frac{a'}{a}w_k$ and a similar relation in the Jordan-frame. Therefore, even though both $\Delta_k^{(\gamma)}$ and h_k are frame-dependent, the value of their combination in the brackets are not because $\tilde{v}_k^{(\gamma)} = v_k^{(\gamma)}$. This is because $(\Delta_k^{(\gamma)} + 4h_k)$ is the fractional perturbation of $\rho^{(\gamma)}a^4$, where $\rho^{(\gamma)}$ is the *local* photon energy density and a is the *local* scale factor: the conformal transformation changes the size of a volume element and therefore the density in it, but it does not change the total energy inside the volume element. Doing the same for cold dark matter, we obtain

$$[\tilde{\Delta}_k^{(c)} + 3\tilde{h}_k]' + k\tilde{v}_k^{(c)} = 0,$$

$$\left[\Delta_k^{(\gamma)} + 3h_k - \frac{1}{2A}\xi_k\right]' + kv_k^{(\gamma)} = 0. \tag{144}$$

The equation in the Jordan frame can be understood as the mass conservation as in the case of photons, while the Einstein-frame equation looks different because of the variation of mass $m \propto A^{\frac{1}{2}}$—here what is conserved is not the mass in a volume element, $\rho^{(c)}a^3$, but the number of particles in it, which is proportional to $\rho^{(c)}A^{-\frac{1}{2}}a^3$.

V. FRAME-INDEPENDENCE OF COSMOLOGICAL OBSERVABLES

In the previous sections, we have explicitly checked the mathematical equivalence of the Einstein and conservation equations in the Jordan and Einstein frames. We have seen that although quantities such as the matter density perturbation, spatial curvature and gradient of the expansion scalar are different in these frames, and the matter contents in them are also not exactly the same, certain combinations of quantities are frame independent. To demonstrate the physical equivalence between the frames, we need to show that the quantities that are directly related to observables are frame independent.

A. The CMB power spectrum

The CMB temperature map, whose anisotropy information is often presented in the form of its angular power spectrum $C(\ell)$, has been a primary cosmological observable, and can be used to simultaneously constrain all six cosmological parameters in the simple ΛCDM model.

The CMB temperature anisotropies are primarily due to the inhomogeneities of photon densities at the time of last scattering, plus late-time secondary temperature fluctuations induced by the CMB photons falling in and climbing out of the potential wells created by the large-scale structures on their way to the observer. From the distribution function of photons, $f(E, e)$, where E is photon energy and e is the direction vector, the mean energy density can be written as

$$\rho^{(\gamma)} = \int \mathrm{d}E\mathrm{d}\Omega E^3 f(E, e) \propto T^4_{(\gamma)}, \tag{145}$$

where Ω is the solid angle and $T_{(\gamma)}$ is the mean photon temperature. Hence, the direction-dependent CMB temperature fluctuation around the mean value is given by

$$[1+\delta_{\mathrm{T}}(e)]^4 = \frac{4\pi}{\rho^{(\gamma)}}\int \mathrm{d}EE^3 f(E, e). \tag{146}$$

The e dependence of the distribution function can be expanded using projected symmetric trace-free tensors as

$$f = \sum_{\ell=0}^{\infty} F_{A_\ell} e^{A_\ell} = F + F_\mu e^\mu + F_{\mu\nu} e^\mu e^\nu + \cdots, \tag{147}$$

where F is the unperturbed distribution function and F_μ, $F_{\mu\nu}$ are first order quantities characterizing the direction dependence. To linear order, this gives the following expansion [28,31]

$$\delta_{\mathrm{T}}(e) = \frac{1}{4}\sum_{\ell=1}^{\infty}\frac{(2\ell+1)!}{(-2)^\ell(\ell!)^2} I_{A_\ell} e^{A_\ell}, \tag{148}$$

in which I_{A_ℓ} are projected symmetric trace-free energy-integrated multipoles of the distribution function

$$I_{A_\ell} \equiv \frac{4\pi}{\rho^{(\gamma)}}\frac{(-2)^\ell(\ell!)^2}{(2\ell+1)!}\int_0^\infty \mathrm{d}EE^3 F_{A_\ell}. \tag{149}$$

The collisional Boltzmann equation for photons can then be written order by order in ℓ, which results in a hierarchy of coupled equations

$$\begin{aligned} &I'_{\ell,k} + k\left[\frac{\ell+1}{2\ell+1} I_{\ell+1,k} - \frac{\ell}{2\ell+1} I_{\ell-1,k}\right] + 4h'_k\delta^0_\ell + \frac{4}{3}kw_k\delta^1_\ell \\ &\quad - \frac{8}{15}k\sigma_k\delta^2_\ell \\ &= -an_e\sigma_{\mathrm{T}}\left[I_{\ell,k} - \delta^0_\ell I_{0,k} - \frac{4}{3}\delta^1_\ell v^{(b)}_k - \frac{1}{10}\delta^2_\ell I_{2,k}\right], \end{aligned} \tag{150}$$

where $I_{\ell,k}$ is the k-space counterpart of I_{A_ℓ}: $I_{A_\ell} = \sum_k I_{\ell,k} Q^{(k)}_{A_\ell}$, and δ^0_ℓ etc. are Kronecker deltas. The right-hand side of Eq. (150) are collisional terms coming from Thomson scattering and we have neglected polarization for the discussion here. The lowest three multipoles of $I_{\ell,k}$ ($\ell = 0, 1, 2$) are, respectively, $\Delta^{(\gamma)}_k$, $v^{(\gamma)}_k$, $\pi^{(\gamma)}_k$, and one can check that the $\ell = 0$, 1 components of Eq. (150) are, respectively, Eqs. (139) and (140). From the discussion in previous sections, it follows that $\tilde{I}_{\ell,k} = I_{\ell,k}$ for all $\ell > 0$, and hence the CMB temperature anisotropies should be the same in the two frames.

One can check this more explicitly. The solution to Eq. (150) can be written in the line-of-sight integral formula as [31]:

$$\begin{aligned} I_{\ell,k}(\eta_0) = 4\int^{\eta_0} \mathrm{d}\eta e^{-\tau}\Bigg(&\left[k\sigma_k + \frac{3}{16}an_e\sigma_{\mathrm{T}}\pi^{(\gamma)}_k\right] \\ &\times\left[\frac{1}{3}j_\ell + j^{**}_\ell\right] + [an_e\sigma_{\mathrm{T}}v^{(b)}_k - kw_k]j^*_\ell \\ &+ \left[\frac{1}{4}an_e\sigma_{\mathrm{T}}\Delta^{(\gamma)}_k - h'_k\right]j_\ell\Bigg), \end{aligned} \tag{151}$$

in which $j_\ell = j_\ell(x) = j_\ell(k(\eta_0-\eta))$ is the spherical Bessel function, $j^*_\ell(x) = \mathrm{d}j_\ell(x)/\mathrm{d}x$, and $\tau(\eta)$ is the optical depth defined by

$$\tau(\eta) \equiv \int_\eta^{\eta_0} \mathrm{d}\eta a n_e \sigma_{\mathrm{T}}, \tag{152}$$

where η_0 is the comoving time today.

As discussed above, the conformal time η is the same in the Jordan and Einstein frames, as well as $an_e\sigma_{\mathrm{T}}$ and therefore τ. The spherical Bessel function is the radial part of the eigenfunction $Q^{(k)}$ of the comoving spatial d'Alembertian operator $a^2\hat{\Box}$, and thus is the same in the two frames. Perturbed variables σ_k, $\pi^{(\gamma)}_k$ and $v^{(b)}_k$ are frame-independent too, and so we just need to check that the remaining terms in Eq. (151) are frame independent—this can be done by integrating by part the term involving Δ_k in Eq. (151):

$$\begin{aligned} &\frac{1}{4}\int \mathrm{d}\eta e^{-\tau} a n_e\sigma_{\mathrm{T}}\Delta^{(\gamma)}_k j_\ell \\ &\quad = \frac{1}{4}\int \mathrm{d}\eta\frac{\mathrm{d}e^{-\tau}}{\mathrm{d}\eta}\Delta^{(\gamma)}_k j_\ell(k(\eta_0-\eta)) \\ &\quad = \frac{1}{4}\int \mathrm{d}\eta e^{-\tau}[k\Delta^{(\gamma)}_k j^*_\ell - (\Delta^{(\gamma)}_k)' j_\ell]. \end{aligned} \tag{153}$$

As combinations $\Delta_k - 4w_k$ and $\Delta_k + 4h_k$ are frame independent (which can be checked using relations derived from previous sections), we conclude that $\tilde{I}_{\ell,k} = I_{\ell,k}$ and so the CMB temperature anisotropies are the same in the Jordan and Einstein frames.

B. Other observables

Apart from the primary CMB power spectrum, which depends on the Weyl potential Φ_k through the Sachs-Wolfe effect, there are other observables which are directly determined by Φ_k. One is the integrated Sachs-Wolfe (ISW) effect, a secondary effect on the CMB temperature anisotropies caused by CMB photons gaining (or losing) energy by falling into and climbing out of time varying Weyl potentials, and which can be expressed as an integration over the time variation of Φ_k:

$$I^{\rm ISW}_{\ell,k} = 2\int_{\eta}^{\eta_0} {\rm d}\eta \Phi'_k j_\ell. \qquad (154)$$

As part of the CMB temperature anisotropies, it is frame independent as discussed in the previous subsection, and this can be seen directly as well, given that Φ_k, j_ℓ and η are the same in the two frames.

Another is gravitational lensing, which is the effect of the trajectories of photons from distant sources (such as galaxies or the last scattering surface) being deflected by the Weyl potential of foreground lenses (such as galaxy clusters, cosmic voids or more generally the intervening large-scale structure). This causes distortions of the images of the sources and amplifications of their magnitudes. The (unobserved) angular position of the source in the source plane, $\boldsymbol{\beta}$, is related to the observed angular position, $\boldsymbol{\theta}$, through

$$\beta^i = \theta^i - \frac{2}{c^2}\int_0^{\chi_s} {\rm d}\chi \frac{(\chi_s - \chi)\chi}{\chi_s} \nabla^{\beta^i}\Phi_{\rm Weyl}(\chi, \boldsymbol{\beta}(\chi)), \qquad (155)$$

where $\Phi_{\rm Weyl}$ is the Weyl potential (the real-space counterpart of Φ_k), χ is the comoving distance, χ_s is the comoving distance of the source and $i = (1, 2)$ represent the two axes in the plane perpendicular to the line of sight. The comoving distance is given by $\chi = c\Delta\eta$ where $\Delta\eta$ is the conformal time needed for light to travel between the objects, and is frame independent as c and η. The Weyl potential is also frame independent, so that it follows that the gravitational lensing calculated in the two frames are the same.

The pre-recombination interaction between baryons, electrons and photons not only lead to the CMB temperature anisotropies, but is also responsible for a baryonic acoustic oscillation (BAO) length scale which is imprinted in the late-time distribution of matter, and which can be used as a standard ruler to measure cosmological distances. The BAO scale is given by the maximum distance traveled by sound waves before recombination, and has a comoving size of

$$l_{\rm BAO} = \int_0^{\eta_{\rm rec}} c_{\rm s}{\rm d}\eta, \qquad (156)$$

where $\eta_{\rm rec}$ is the conformal time of recombination and $c_{\rm s}$ is the speed of sound waves, given by

$$c_{\rm s} = \frac{1}{\sqrt{3(1+\frac{\rho^{(b)}}{\rho^{(\gamma)}})}}. \qquad (157)$$

Because both $\rho^{(b)}$ and $\rho^{(\gamma)}$ transform in the same way in a conformal transformation, it follows that $c_{\rm s}$, and therefore $l_{\rm BAO}$, are frame independent. The comoving angular diameter distance for BAO features at a given time η, $d_{\rm A} = l_{\rm BAO}/\Theta$, where Θ is the angle subtended by the BAO pattern, is equal to $\chi(\eta)$ for a flat space, and is frame independent.

While the relation between the comoving distance and conformal time, $\chi(\eta)$, is frame independent, the same does not apply if other 'time' variables are used. For example, the scale factor, which is often used as a time variable in cosmology, is different in the two frames: $\tilde{a} = \sqrt{A}a$. From the equations

$$\tilde{a}_0 = \int_0^{\eta_0} {\rm d}\eta \frac{{\rm d}\tilde{a}}{{\rm d}\eta},$$
$$a_0 = \int_0^{\eta_0} {\rm d}\eta \frac{{\rm d}a}{{\rm d}\eta} = \int_0^{\eta_0} {\rm d}\eta \frac{\rm d}{{\rm d}\eta}(\tilde{a}A^{-\frac{1}{2}}), \qquad (158)$$

it can be seen that $\tilde{a}_0 = 1$ for today in the Jordan frame corresponds to $a_0 \neq 1$ in the Einstein frame, where today is characterized by $a_0 = A_0^{-\frac{1}{2}}$. This means that the comoving-distance-scale-factor relations are different in the frames. The physical times are also frame dependent as ${\rm d}\tilde{t} = \sqrt{A}{\rm d}t$. Because time measurements require the use of atomic transitions, which are not affected by the scalar field in the Jordan frame where matter is minimally coupled, we consider $\tilde{t}$ as the physical time, and it is convenient to define $\tilde{a}_0 = \tilde{a}(\tilde{t}_0) = 1$.

In the Jordan frame, cosmological redshift is given as usual: $\tilde{z} = 1/\tilde{a} - 1$. In the Einstein frame, it is a bit more complicated: due to the time evolution of particle masses, including the electron mass, in this frame, the frequency of an atomic transition as measured in the past (e.g., when the conformal time was η), $\nu(\eta)$, is not the same as the frequency of the same atomic transition measured in our labs today, ν_0, but the two are related by $\nu(\eta) = \sqrt{A(\eta_0)/A(\eta)}\nu_0$ since $\nu \propto m = \tilde{m}A^{-\frac{1}{2}}$ [19]. The total photon redshifting including this contribution is then $1 + z \equiv \nu(\eta)/\nu_0 = [A(\eta_0)^{\frac{1}{2}}a_0]/[A(\eta)^{\frac{1}{2}}a(\eta)] = \tilde{a}_0/\tilde{a} = 1/\tilde{a} = 1 + \tilde{z}$. Therefore, redshift is a frame-independent quantity. The luminosity distance of an object a photon emitted by which at time η is received by an observer today is given by $d_{\rm L} = (1 + z)\chi(\eta)$ where $\chi(\eta)$ is the comoving distance to η, and the $(1+z)$ factor comes from photon energy redshifting and time dilation, both of which are affected by the A-dependence of frequency—with the properly defined frame-independent redshift, the luminosity-distance-redshift relation $d_L(z)$ is the same in the two frames as well.

Finally, another commonly used cosmological observable is the two point correlation function of the matter overdensity field or its tracers, $\xi(|\mathbf{r}|) = \langle\Delta(\mathbf{x})\Delta(\mathbf{x}+\mathbf{r})\rangle$, or the matter power spectrum $P(k)$ given by $\langle\Delta_k(\mathbf{k})\Delta_k(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')P(|\mathbf{k}|)$. As we have seen in Eq. (124), Δ_k is frame dependent, which means that $P(k)$ depends on whether we are in the Einstein or the Jordan frames. Note that $P(k)$ is generally gauge dependent, but the effect of using different gauges is small on small scales.

C. A numerical example

From the discussion and comparisons above, it is apparent that the Jordan frame has the disadvantage of having complicated expressions. Taking Eq. (138) as example: the tilded dynamical quantities in the Jordan-frame version of these equations, given in Eqs. (129), (130) and (131), respectively, are lengthy and in the end they cancel each other in a combination. Therefore, the Einstein frame is computationally more convenient in practice.

It is often said that the calculation can be done in either of the frames, and physical observables should not depend on which frame is used. While this is true, there is a subtlety here—the two frames have the same redshift, but not the same values of the scale factor, i.e., $\tilde{a} \neq a$. To obtain observables today, such as the CMB power spectrum, the linear perturbation equations usually need to be integrated up to $z=0$ or $\tilde{a}=1$, and if the calculation is carried out in the Einstein frame the integration should be stopped at $a \neq 1$. Therefore, a recipe for linear perturbation calculation is to compute the background quantities using $\tilde{a}$ which has the advantage of being more directly related to redshifts—and the perturbation evolution using the conformal time η—which is the same in both frames. There is no need to derive the Jordan-frame background equations, even though they should be fairly simple, and in practice one can do everything in the Einstein frame: $\tilde{a}$ is given by $\sqrt{A}a$. One example to show why it is more convenient to use $\tilde{a}$ instead of a is the conformal time today, for which we need to integrate $\mathrm{d}\eta/\mathrm{d}\tilde{a}$ until $\tilde{a}=1$, while if we use a the integration should stop at $a=A^{-1/2}$ which is model dependent.[5]

As an numerical example, we consider the K-mouflage model [25,26] studied in [32]. In this model, the total action is given by Eq. (28) with

$$A(\phi) = \exp(2\beta M_{\mathrm{Pl}}^{-1}\phi), \tag{159}$$

$$\mathcal{L}_\phi = -M^4 K(\sigma), \tag{160}$$

where β is a dimensionless model parameter characterizing the coupling strength of the scalar field with matter in the Einstein frame, M is a model parameter of mass dimension that will be fixed given the fractional energy density of the scalar field today,

$$\sigma \equiv \frac{1}{2}M_{\mathrm{Pl}}^{-4}(\nabla\phi)^2, \tag{161}$$

$$K = -1 + \sigma + K_0\sigma^m, \tag{162}$$

where K_0, m are two other dimensionless parameters. This model has been described in detail in the above references, and as we only use it as an example to illustrate our numerical implementation, we shall keep things simple by only presenting the above equations. Note that σ here has no subscript $_k$, to be distinguished from the shear σ_k.

We have implemented this model in the publicly available linear Boltzmann code CAMB [33], using both the Einstein and the Jordan frames, and some numerical results are shown in Fig. 1. In the left panel, we have plotted the CMB temperature spectra for the K-mouflage model (see legends for model parameters) as the colored curves, and the corresponding ΛCDM model[6] as the black solid line. The blue dashed and green solid curves are obtained respectively by integrating Jordan- and Einstein-frame perturbation quantities to $\sqrt{A}a=1$ and $\tilde{a}=1$ (in this particular model we find that $a=1.0835$ at $\tilde{a}=1$), and they are identical as expected from the discussion above. The red solid line, in contrast, is obtained by integrating the Einstein-frame variables to $a=1$: here the background expansion history is incorrect (therefore the shift of CMB peaks due to wrong distances) and there is less time for the evolution than in the correct calculation (hence a weaker integrated Sachs-Wolfe effect).

In the right panel of Fig. 1, we show the linear matter power spectra for the same models and calculations. Again, because the red curve stops at $a=1$ rather than $a=1.0835$, there has been less time for the growth of matter density perturbations, which results in a smaller $P(k)$ than the correct prediction. The blue dashed and green solid lines are identical on small scales, while on very large scales they show mild difference. The density contrast, and therefore $P(k)$, is gauge-dependent, and this difference is expected unless one uses the gauge in which the scalar field is homogeneous ($\xi_k = 0$).

[5]In many places, the codes are default to integrate to $z = 1/\tilde{a} - 1 = 0$, e.g., the function `dtauda` in CAMB. If we integrate $\mathrm{d}\eta/\mathrm{d}a$, these places may all need to be changed.

[6]This is the ΛCDM model whose present-day density parameter Ω_Λ is equal to the current density parameter of the scalar field, $\Omega_{\phi 0}$, in the K-mouflage model, and all the non-Kmouflage model parameters are the same in the two models.

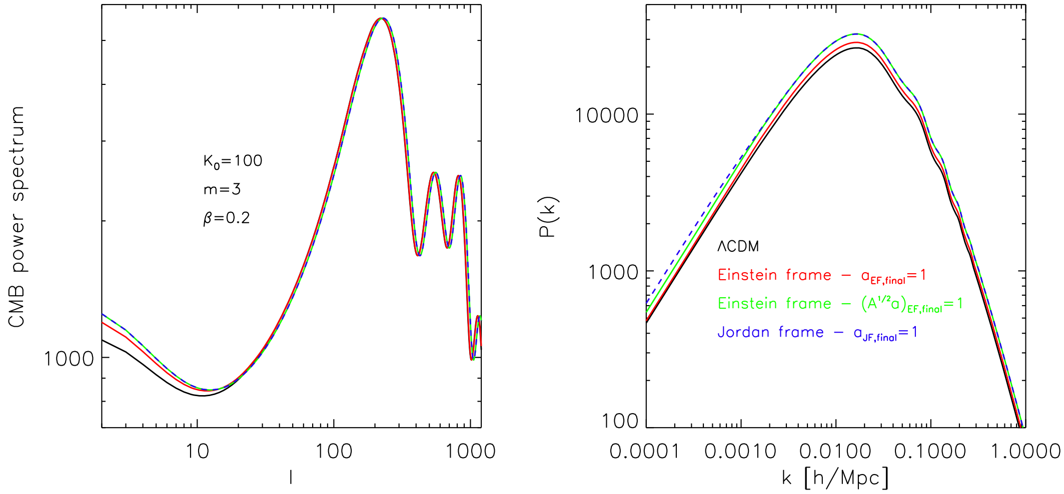

FIG. 1. *Left panel*: the CMB temperature power spectrum $C(l)$ in the different models/calculations. The black solid line is for the ΛCDM model for comparison, the blue dashed and green solid lines are the results computed from the Jordan-frame and Einstein-frame quantities, respectively, and the red solid line is computed in the Einstein frame but with the calculation stopping at $a = 1$ rather than $\sqrt{A}a = 1$. *Right panel*: the same as the left panel but for the matter power spectra $P(k)$.

VI. DISCUSSIONS AND CONCLUSIONS

In this paper, we have studied some aspects of the physics of the scalar-tensor theory in two conformally related frames, the so-called Einstein and Jordan frames. Many debates occurred in the community about their physical equivalence, and this work aims to confirm the idea that the two frames are physically equivalent, namely results of cosmological observations are the same in both frames, given that they are basically the same action expressed in different forms using field redefinition.

We have done this by a detailed comparison of the equations in the two frames at different levels. Starting from the original action written in the Einstein frame, we reexpress it in the Jordan frame and derive the Einstein and Klein-Gordon equations in the two frames, before checking that they are mathematically equivalent. This means that one can derive an equation of motion in one frame simply by starting from its counterpart in the other frame. In other words, working in parallel and independently in both frames, or working in a given frame and then moving to the other, are two equivalent approaches to study these theories.

We have then focused in Sec. IV on the links between physical quantities (that is, dynamical and kinematic quantities) in the two frames, and used these relations to show the equivalence of key linearized perturbed equations. Many physical quantities (such as the scale factor, the 4-acceleration or the energy density of a given matter specie for instance) have different expressions in the two frames, which could lead to the conclusion that the they are not physically equivalent. However, the crucial point to notice is that these quantities are not directly measurable by an observer. The cosmological observables arise from combinations of these quantities, and Sec. V shows that these combinations are generally frame independent. Hence, computations can be done in either frame without changing physical conclusions.

Physics in the Einstein frame is described by GR, to which a new specie of matter is added—a scalar field which will interact will usual species of matter. Hence, in the Einstein frame, Einstein equations have the same form as in GR, while conservation equations of different matter species have different forms from the standard model, since matter exchange energy and momentum with the scalar field. For cosmological perturbations, working in the Einstein frame has the advantage of substantially simplified field equations. In the Jordan frame, gravity is no longer described by GR, as the scalar-field is now non-minimally coupled to the gravitational part of the action so that the Einstein equations are more complicated. The corrections to the standard GR equations can be treated as an effective fluid which contributes terms to the total (effective) stress energy tensor. In the example of the constraint equation for the Weyl potential, such complicated additional terms cancel exactly, leaving the result unchanged from the much simpler Einstein-frame calculation.

Because the two frames are related by a conformal transformation, they share the same conformal time (η), which means that calculations using η as the time variable are not affected. However, the scale factor takes different values in the two frames, with the Jordan-frame scale factor $\tilde{a}$ considered as the physical one because matter particles follow their geodesics in this frame. The redshift, $\tilde{z} = z = \frac{1}{\tilde{a}} - 1$ is a physical observable that is agreed by both frames, even though $z \neq 1/a - 1$ in the Einstein frame. Instead, in the Einstein frame we have $z = 1/(\sqrt{A}a) - 1$; when working in this frame, the results of observables, such as the

CMB power spectrum, may not automatically be the same as from the Jordan-frame calculation, and care needs to be taken to ensure that integrations end at the correct time, as $a_0 = A_0^{-\frac{1}{2}}$ rather than $a_0 = 1$ represents the present day.

Before closing the paper, we would like to mention that the cosmological equivalence studied here is at the classical level. There have been interesting discussions at the quantum level, e.g., [34], and these are beyond the scope of this work.

Another interesting point which could motivate further works is the case when the two metrics are related by more complicated relations than the conformal one (7) studied all along this paper. For example, it is known that [35] the most general relation linking the metrics and a scalar field ϕ, compatible with causality and the weak equivalence principle, is a disformal transformation $\tilde{g}_{\mu\nu} = A(\phi)g_{\mu\nu} + B(\phi)\nabla_\mu\phi\nabla_\nu\phi$. As in the conformal case, $(\mathcal{M}, \tilde{g}_{\mu\nu})$ defines the Jordan frame, while $(\mathcal{M}, g_{\mu\nu})$ defines what we call the Einstein frame. While a purely conformal transformation is merely a rescaling of the metric, a disformal transformation contains both a conformal rescaling of the metric and a distortion of it. Such transformations have been considered in various circumstances, such as varying speed of light cosmologies [36], theories of massive gravity [37] or in the description of branes embedded in a higher-dimensional space [38], and the physical equivalence of the Einstein and Jordan frames in this general case could be particularly interesting to study, see, e.g., [39–43].

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APPENDIX A: USEFUL PERTURBATION RELATIONS

In this appendix, we present some useful relations that hold to first order in perturbations, which are useful for derivations and checks of the perturbed equations.

We start from the relation between the second-order covariant derivatives in the two frames $\tilde{\nabla}_\mu\tilde{\nabla}^\nu\psi$ and $\nabla_\mu\nabla^\nu\psi$, where ψ is a general scalar quantity:

$$\tilde{\nabla}_\mu\tilde{\nabla}^\nu\psi = \frac{1}{A}\nabla_\mu\nabla^\nu\psi - \frac{1}{2A}(\nabla_\mu A\nabla^\nu\psi + \nabla_\mu\psi\nabla^\nu A - \delta^\mu_\nu\nabla^\lambda A\nabla_\lambda\psi). \tag{A1}$$

Using the decomposition of these covariant derivatives in the two frames,

$$\nabla_\mu\nabla^\nu\psi = \hat{\nabla}_{\langle\mu}\hat{\nabla}^{\nu\rangle}\psi + \frac{1}{3}h^\nu_\mu\hat{\Box}\psi + u_\mu u^\nu\ddot{\psi} - 2u_{(\mu}\hat{\nabla}^{\nu)}\dot{\psi} + \frac{2}{3}\theta u_{(\mu}\hat{\nabla}^{\nu)}\psi - \sigma^\nu_\mu\dot{\psi} - \varpi^\nu_\mu\dot{\psi} - \frac{1}{3}\theta h^\nu_\mu\dot{\psi}, \tag{A2}$$

$$\tilde{\nabla}_\mu\tilde{\nabla}^\nu\psi = \hat{\tilde{\nabla}}_{\langle\mu}\hat{\tilde{\nabla}}^{\nu\rangle}\psi + \frac{1}{3}\tilde{h}_\mu{}^\nu\hat{\tilde{\Box}}\psi + \tilde{u}_\mu\tilde{u}^\nu\mathring{\mathring{\psi}} - 2\tilde{u}_{(\mu}\hat{\tilde{\nabla}}^{\nu)}\mathring{\psi} + \frac{2}{3}\tilde{\theta}\tilde{u}_{(\mu}\hat{\tilde{\nabla}}^{\nu)}\psi - \sigma_\mu{}^\nu\mathring{\psi} - \tilde{\varpi}_\mu{}^\nu\mathring{\psi} - \frac{1}{3}\tilde{\theta}\tilde{h}_\mu{}^\nu\mathring{\psi}, \tag{A3}$$

and that (to first order),

$$\nabla_\mu A\nabla^\nu\psi = \dot{A}\,\dot{\psi}\,u_\mu u^\nu - u_\mu\dot{A}\hat{\nabla}^\nu\psi - u^\nu\dot{\psi}\hat{\nabla}_\mu A,$$
$$\nabla_\mu\psi\nabla^\nu A = \dot{A}\,\dot{\psi}\,u_\mu u^\nu - u_\mu\dot{\psi}\hat{\nabla}^\nu A - u^\nu\dot{A}\hat{\nabla}_\mu\psi,$$
$$\nabla^\lambda A\nabla_\lambda\psi = -\dot{A}\,\dot{\psi},$$

it can be found that

$$\tilde{\theta} = \frac{1}{A^{\frac{1}{2}}}\left[\theta + \frac{3\dot{A}}{2A}\right], \tag{A4}$$

$$\tilde{\sigma}_\mu{}^\nu = \frac{1}{A^{\frac{1}{2}}}\sigma_\mu{}^\nu, \tag{A5}$$

$$\tilde{\varpi}_\mu{}^\nu = \frac{1}{A^{\frac{1}{2}}}\tilde{\varpi}_\mu{}^\nu, \tag{A6}$$

$$\hat{\tilde{\Box}}\psi = \frac{1}{A}\hat{\Box}\psi, \tag{A7}$$

$$\hat{\tilde{\nabla}}_{\langle\mu}\hat{\tilde{\nabla}}^{\nu\rangle}\psi = \frac{1}{A}\hat{\nabla}_{\langle\mu}\hat{\nabla}^{\nu\rangle}\psi, \tag{A8}$$

$$\hat{\tilde{\nabla}}_\mu\psi = \hat{\nabla}_\mu\psi, \tag{A9}$$

$$\hat{\tilde{\nabla}}^\mu\psi = \frac{1}{A}\hat{\nabla}^\mu\psi, \tag{A10}$$

$$\hat{\tilde{\nabla}}_\mu\mathring{\psi} = \frac{1}{A^{\frac{1}{2}}}\left[\hat{\nabla}_\mu\dot{\psi} - \frac{\dot{A}}{2A}\hat{\nabla}_\mu\psi\right], \tag{A11}$$

$$\hat{\tilde{\nabla}}^\mu\mathring{\psi} = \frac{1}{A^{\frac{3}{2}}}\left[\hat{\nabla}^\mu\dot{\psi} - \frac{\dot{A}}{2A}\hat{\nabla}^\mu\psi\right]. \tag{A12}$$

In the $3+1$ formalism, time and covariant spatial derivatives do not commute, but they satisfy the following relation which is useful in calculations:

$$\hat{\nabla}_\mu \dot{\psi} = (\hat{\nabla}_\mu \psi)^{\cdot} + \frac{1}{3}\theta \hat{\nabla}_\mu \psi - \dot{u}_\mu \dot{\psi}. \qquad \text{(A13)}$$

A similar relation exists for the Jordan frame quantities.

APPENDIX B: EQUIVALENCE OF THE KLEIN-GORDON EQUATIONS IN THE JORDAN AND EINSTEIN FRAMES

Let's start with the Klein-Gordon equation in the Jordan frame:

$$\tilde{\nabla}_\mu \left[\frac{\partial \tilde{\mathcal{L}}_{\tilde{\phi}}(\tilde{\phi}, (\tilde{\nabla}\tilde{\phi})^2)}{\partial(\tilde{\nabla}_\mu \tilde{\phi})} \right] = \frac{\partial \tilde{\mathcal{L}}_{\tilde{\phi}}}{\partial \tilde{\phi}} + \frac{1}{2}\frac{\mathrm{d}\ln A}{\mathrm{d}\tilde{\phi}}(\tilde{T}^{(m)} + \tilde{T}^{(\tilde{\phi})}). \qquad \text{(B1)}$$

To show that this equation is equivalent to its Einstein-frame counterpart, we first slightly rewrite Eq. (27) as

$$\tilde{\nabla}_\mu \left[\frac{\partial \tilde{\mathcal{L}}_{\tilde{\phi}}}{\partial(\tilde{\nabla}_\mu \tilde{\phi})} \right] = \frac{1}{A^{\frac{3}{2}}} \nabla_\mu \left[\frac{\partial \mathcal{L}_\phi}{\partial(\nabla_\mu \phi)} \right] + \frac{1}{2A^{\frac{3}{2}}} \frac{\mathrm{d}\ln A}{\mathrm{d}\phi} \frac{\partial \mathcal{L}_\phi}{\partial \sigma} (\nabla\phi)^2, \qquad \text{(B2)}$$

where we have used $\frac{\partial \mathcal{L}_\phi}{\partial(\nabla_\mu \phi)} \nabla_\mu \phi = \frac{\partial \mathcal{L}_\phi}{\partial \sigma}(\nabla\phi)^2$ where $\sigma \equiv \frac{1}{2}(\nabla\phi)^2$.

We will also use the relations $\partial\tilde{\phi}/\partial\phi = A^{-\frac{1}{2}}$, $\tilde{\mathcal{L}}_{\tilde{\phi}} = A^{-2}\mathcal{L}_\phi$, $\tilde{T}^{(i)} = A^{-2}T^{(i)}$ and $T^{(\phi)} = 4\mathcal{L}_\phi - \frac{\partial \mathcal{L}_\phi}{\partial \sigma}(\nabla\phi)^2$, where the last one comes from

$$T^{(\phi)}_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_\phi)}{\delta g^{\mu\nu}} = \mathcal{L}_\phi g_{\mu\nu} - \frac{\partial \mathcal{L}_\phi}{\partial \sigma} \nabla_\mu \phi \nabla_\nu \phi. \qquad \text{(B3)}$$

Finally, for a $\mathcal{L}_\phi$ that contains general functions of $\sigma = \frac{1}{2}(\nabla\phi)^2$, to ensure the correct dimension, the Lagrangian density can be written for example as

$$\mathcal{L}_\phi(\phi, \sigma) = M_*^4 K(M_*^{-4}\sigma) - V(\phi), \qquad \text{(B4)}$$

where M_* is a constant of mass dimension and $K(\cdots)$ is a dimensionless function. The Jordan-frame counterpart has the form

$$\tilde{\mathcal{L}}_{\tilde{\phi}}(\tilde{\phi}, \tilde{\sigma}) = (A^{-\frac{1}{2}}M_*)^4 K[(A^{-\frac{1}{2}}M_*)^{-4}\tilde{\sigma}] - \tilde{V}(\tilde{\phi}), \qquad \text{(B5)}$$

where $\tilde{\sigma} = \frac{1}{2}(\tilde{\nabla}\tilde{\phi})^2 = A^{-2}\sigma$. Therefore, we have

$$\frac{\partial \tilde{\mathcal{L}}_{\tilde{\phi}}}{\partial \tilde{\phi}} = -\frac{2}{A^{\frac{3}{2}}} \frac{\mathrm{d}\ln A}{\mathrm{d}\phi} \mathcal{L}_\phi + \frac{1}{A^{\frac{3}{2}}} \frac{\partial \mathcal{L}_\phi}{\partial \phi} + \frac{1}{A^{\frac{3}{2}}} \frac{\partial K}{\partial \sigma} \frac{\mathrm{d}\ln A}{\mathrm{d}\phi} (\nabla\phi)^2. \qquad \text{(B6)}$$

Note that $\partial K/\partial\sigma = \partial \mathcal{L}_\phi/\partial\sigma$.

Using the above relations, it is straightforward to check that Eq. (B1) can be rewritten as

$$\nabla_\mu \left[\frac{\partial \mathcal{L}_\phi(\phi, (\nabla\phi)^2)}{\partial(\nabla_\mu \phi)} \right] = \frac{\partial \mathcal{L}_\phi}{\partial \phi} + \frac{1}{2}\frac{\mathrm{d}\ln A}{\mathrm{d}\phi} T^{(m)}, \qquad \text{(B7)}$$

which is the Klein-Gordon equation in the Einstein frame.

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