Bidispersive thermal convection

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Abstract

We obtain the linear instability and nonlinear stability thresholds for a problem of thermal convection in a bidispersive porous medium with a single temperature. It is important to note that we show that the linear instability threshold is the same as the nonlinear stability one. This means that the linear theory is capturing completely the physics of the onset of thermal convection. This result contrasts with the general theory of thermal convection in a bidispersive porous material where the temperatures in the macropores and micropores are allowed be different. In that case the coincidence of the stability boundaries has not been proved.

Keywords: Bidispersive porous media; natural convection; two-velocity; stability.

1 Introduction

A bidispersive porous medium is one where the solid skeleton contains two types of pores. One type consists of the usual pores one finds regularly. However, there are in addition micropores which may be cracks in the skeleton or may be deliberately created in a man made bidispersive product. For example, very small glass beads may be joined together to create an almost overall spherical shape, and these larger spheres then assembled together to form the bidispersive porous medium, see the picture given on page 3069 of Nield and Kuznetsov [1].

The porosity associated with the macropores is denoted by ϕ, i.e. ϕ is the ratio of the volume of the macropores to the total volume of the saturated porous material. In addition the micropores generate a porosity ε which is the ratio of the volume occupied by the micropores to the volume of the porous body which remains once the macropores are removed. This leads to the fraction of volume occupied by the micropores being $\varepsilon(1-\phi)$ while the fraction of the volume occupied by the solid skeleton is $(1-\varepsilon)(1-\phi)$.

Theoretical work on fluid flow in bidispersive porous media commenced with work of Nield and Kuznetsov [2]. Further work followed see e.g. Nield [3], Nield and Bejan [4], Nield and Kuznetsov [2, 5, 1, 6, 7, 8], Straughan [9, 10], chapter 13, with fundamental work on the thermal convection problem arising in Nield and Kuznetsov [1]. These works all employ different velocities U_i^f and U_i^p in the macro and micropores, and likewise different temperatures T^f and T^p.

One of the major reasons for analysing the behaviour of bidispersive (alternatively double porosity) porous media is the multitude of applications which are arising. For example, landslides are being modelled with this theory, see e.g. Montrasio et al [11], Sanavia and Schrefler [12], with other forms of land movement also relevant, see e.g. Hammond and Barr [13]. Bidispersive porous media are believed to be important to understanding hydraulic fracturing ('fracking') where underground rocks or soil are deliberately disturbed to release trapped gas for human consumption, see e.g. Huang et al [14], Kim and Moridis [15]. Finally, we mention an application to the vital production of clean drinking water, see e.g. Ghasemizadeh et al [16], Zuber and Matyka [17], although many further applications are discussed in Straughan [10], chapter 13.

In this paper we shall investigate thermal convection in a bidispersive porous medium one when temperature is employed and the horizontal layer containing the porous material is heated from below. To achieve our results we use the one temperature model of Falsaperla et al [18] which is based on the full model of Nield and Kuznetsov [1].

2 Equations for thermal convection

We suppose the porous medium is contained in the horizontal layer $0 < z < d$ with the temperature at $z = d$ is kept fixed at $T_L{}^\circ C$ while the temperature at $z = d$ is kept fixed at $T_U{}^\circ C$ with $T_L > T_U$. Denote by U_i^f and U_i^p the velocities of the fluid in the macropores and in the micropores, respectively. The temperature in the porous medium is denoted by $T(\mathbf{x}, t)$. A Boussinesq approximation is used whereby the density is constant except in the buoyancy forces which are linear in temperature. The relevant equations may then be derived, cf. Falsaperla et al [18],

$$
\begin{aligned}
&-\frac{\mu}{K_f}U_i^f - \zeta(U_i^f - U_i^p) - p^f_{,i} + \rho_F \alpha T k_i = 0, \quad U^f_{i,i} = 0,\\
&-\frac{\mu}{K_p}U_i^p - \zeta(U_i^p - U_i^f) - p^p_{,i} + \rho_F \alpha T k_i = 0, \quad U^p_{i,i} = 0, \qquad (1)\\
&(\rho c)_m T_{,t} + (\rho c)_f (U_i^f + U_i^p) T_{,i} = \kappa_m \Delta T\,.
\end{aligned}
$$

In these equations $\mathbf{k} = (0, 0, 1)$, μ, K_f, K_p and ζ are, respectively, dynamic viscosity, permeability in the macropores, permeability in the micropores, and an interaction coefficient. In addition ρ_F is a reference density, α is the coefficient of thermal expansion of the fluid, p^f, p^p are the pressures in the macro and micropores, ρ denotes density, c specific heat at constant pressure, κ_m is a thermal conductivity and Δ is the three dimensional Laplace operator. Standard indicial notation is employed throughout.

In terms of the components in the solid, macro and micro phases, $(\rho c)_m$ and

κ_m are given by, see [18],

$$
\begin{aligned}
&(\rho c)_m = (1-\phi)(1-\varepsilon)(\rho c)_s + \phi(\rho c)_f + \varepsilon(1-\phi)(\rho c)_p\,,\\
&\kappa_m = (1-\phi)(1-\varepsilon)\kappa_s + \phi\kappa_f + \varepsilon(1-\phi)\kappa_p.
\end{aligned}
\tag{2}
$$

To investigate thermal convection we study stability of the steady solution

$$
\bar{U}_i^f \equiv 0\,, \quad \bar{U}_i^p \equiv 0\,, \quad \bar{T} = T_L - \beta z\,, \tag{3}
$$

where β is the temperature gradient

$$
\beta = \frac{T_L - T_U}{d}\,. \tag{4}
$$

Now let u_i^f, u_i^p, θ, π^f, π^p be perturbations to the steady solutions and then non-dimensionalize the resulting perturbation equations with the substitutions

$$
x_i = x_i^* d\,, \quad t = t^*\mathcal{I}\,, \quad \frac{\mu}{\zeta K_f} = \gamma_1\,, \quad \frac{\mu}{\zeta K_p} = \gamma_2\,, \tag{5}
$$

where

$$
\mathcal{I} = \frac{d^2(\rho c)_m}{\kappa_m} \tag{6}
$$

and where the velocity scale U and Rayleigh number Ra are given by

$$
U = \frac{\kappa_m}{d(\rho c)_f} \tag{7}
$$

$$
Ra = R^2 = \frac{(\rho c)_f \beta d^2 \rho_F g \alpha}{\kappa_m \zeta}. \tag{8}
$$

The resulting perturbation equations have form

$$
\begin{aligned}
&\gamma_1 u_i^f + (u_i^f - u_i^p) = -\pi^f_{,i} + R\theta k_i, \quad u^f_{i,i} = 0\,,\\
&\gamma_2 u_i^p - (u_i^f - u_i^p) = -\pi^p_{,i} + R\theta k_i, \quad u^p_{i,i} = 0\,,\\
&\theta_{,t} + (u_i^f + u_i^p)\theta_{,i} = R(w^f + w^p) + \Delta\theta\,,
\end{aligned}
\tag{9}
$$

where $\mathbf{u}^f = (u^f, v^f, w^f)$ and $\mathbf{u}^p = (u^p, v^p, w^p)$. These equations hold in the domain $(x,y) \in \mathbb{R}^2$, $\{z \in (0,1)\}$, $t > 0$. The boundary conditions are

$$
u_i^f n_i = 0, \quad u_i^p n_i = 0, \quad \theta = 0, \quad \text{on } z = 0, 1\,, \tag{10}
$$

where n_i is the unit outward normal to $z = 0$ or $z = 1$, and u_i^f, u_i^p, θ, π^f, π^p satisfy a plane tiling shape in the (x,y) plane. The periodic convection cell which arises will be denoted by V.

3 Linear instability

To find the linear instability boundary we discard the nonlinear terms in $(9)_3$ and seek a solution in which u_i^f, u_i^p, θ, π^f, π^f have time dependence like $e^{\sigma t}$. This results in a system of equations of form

$$\begin{aligned}
&\gamma_1 u_i^f + (u_i^f - u_i^p) = -\pi_{,i}^f + R\theta k_i, \quad u_{i,i}^f = 0\,, \\
&\gamma_2 u_i^p - (u_i^f - u_i^p) = -\pi_{,i}^p + R\theta k_i, \quad u_{i,i}^p = 0\,, \qquad (11) \\
&\sigma\theta = R(w^f + w^p) + \Delta\theta.
\end{aligned}$$

We shall show that the strong form of the principle of exchange of stabilities holds. Multiply equation $(11)_1$ by u_i^{f*}, the complex conjugate of u_i^f, and integrate over the period cell V. Multiply $(11)_2$ by u_i^{p*} and multiply $(11)_3$ by θ^* and integrate each over the period cell V. Denote by $(\cdot,\cdot)$ and $\parallel \cdot \parallel$ the inner product and norm on the complex Hilbert space $L^2(V)$ and then one may show after some integration by parts and use of the boundary conditions,

$$\begin{aligned}
&\gamma_1 \parallel \mathbf{u}^f \parallel^2 + \parallel \mathbf{u}^f \parallel^2 - (u_i^p, u_i^{f*}) = R(\theta, w^{f*}), \\
&\gamma_2 \parallel \mathbf{u}^p \parallel^2 + \parallel \mathbf{u}^p \parallel^2 - (u_i^f, u_i^{p*}) = R(\theta, w^{p*}), \qquad (12) \\
&\sigma \parallel \theta \parallel^2 = - \parallel \nabla\theta \parallel^2 + R(w^f, \theta^*) + R(w^p, \theta^*).
\end{aligned}$$

Next, add the three equations in (12) to obtain

$$\begin{aligned}
\sigma \parallel \theta \parallel^2 = & - \parallel \nabla\theta \parallel^2 \\
& -(\gamma_1 + 1) \parallel \mathbf{u}^f \parallel^2 - (\gamma_2 + 1) \parallel \mathbf{u}^p \parallel^2 \\
& + (u_i^p, u_i^{f*}) + (u_i^f, u_i^{p*}) \qquad (13) \\
& + R\left[(\theta, w^{f*}) + (w^f, \theta^*)\right] \\
& + R\left[(\theta, w^{p*}) + (w^p, \theta^*)\right].
\end{aligned}$$

Write $\mathbf{u}^f$, $\mathbf{u}^p$ and θ in their real and imaginary parts and put $\sigma = \sigma_r + i\sigma_1$. Take the imaginary part of equation (13) to find

$$\sigma_1 \parallel \theta \parallel^2 = 0. \qquad (14)$$

We require $\parallel \theta \parallel \neq 0$ and so $\sigma_1 = 0$. Thus, $\sigma \in R$ and oscillatory convection does not hold. Therefore, to find the linear instability boundary we analyze system (11) with $\sigma = 0$.

To proceed we remove u^f, u^p, v^f, v^p from equation (11) by taking the double curl of each of $(11)_1$ and $(11)_2$ and focus on the third components of the resulting equations. We thus have to solve

$$
\begin{aligned}
&\gamma_1 \Delta w^f + \Delta w^f - \Delta w^p = R\Delta^* \theta\,, \\
&\gamma_2 \Delta w^p - (\Delta w^f - \Delta w^p) = R\Delta^* \theta\,, \\
&\Delta\theta + R(w^f + w^p) = 0\,,
\end{aligned}
\tag{15}
$$

where $\Delta^* = \partial^2/\partial x^2 + \partial^2/\partial y^2$. By employing normal modes in (15) one may reduce the calculation for the critical Rayleigh number to

$$
R^2 = \frac{\Lambda_n^2}{a^2}\left(\frac{\gamma_1\gamma_2 + \gamma_1 + \gamma_2}{4 + \gamma_1 + \gamma_2}\right), \tag{16}
$$

where $\Lambda_n = n^2\pi^2 + a^2$. One finds the minimum in (16) is for $n = 1$ and for $a = \pi$. Thus, the critical Rayleigh number is

$$
R^2 = 4\pi^2\left(\frac{\gamma_1\gamma_2 + \gamma_1 + \gamma_2}{4 + \gamma_1 + \gamma_2}\right). \tag{17}
$$

4 Nonlinear stability

Equation (17) yields the linear instability threshold. It gives no information on stability. In order to achieve this we return to the full nonlinear equations (9). Multiply $(9)_1$ by u_i^f and integrate over the period cell V. Then multiply $(9)_2$ by u_i^p and $(9)_3$ by θ and integrate each over V. After some integration by parts we obtain the energy identities

$$
\begin{aligned}
&\gamma_1 \parallel \mathbf{u}^f \parallel^2 + \Big(u_i^f, \{u_i^f - u_i^p\}\Big) = R(\theta, w^f), \\
&\gamma_2 \parallel \mathbf{u}^p \parallel^2 - \Big(u_i^p, \{u_i^f - u_i^p\}\Big) = R(\theta, w^p), \\
&\frac{d}{dt}\frac{1}{2} \parallel \theta \parallel^2 = R(w^f + w^p, \theta) - \parallel \nabla\theta \parallel^2,
\end{aligned}
\tag{18}
$$

where now $(\cdot,\cdot)$ and $\parallel \cdot \parallel$ denote the inner product and norm on the real Hilbert space $L^2(V)$. Add the three equations in (18) to find

$$
\frac{d}{dt}\frac{1}{2} \parallel \theta \parallel^2 = RI - D\,, \tag{19}
$$

where

$$
I = 2(w^f + w^p, \theta) \tag{20}
$$

and

$$
D = \parallel \nabla\theta^2 \parallel + \gamma_1 \parallel \mathbf{u}^f \parallel^2 + \gamma_2 \parallel \mathbf{u}^p \parallel^2 + \parallel \mathbf{u}^f - \mathbf{u}^p \parallel^2 . \tag{21}
$$

Define

$$
\frac{1}{R_E} = \max_H \frac{I}{D} \tag{22}
$$

where H consists of L^2 functions for u_i^f and u_i^p and H^1 functions for θ. Then from (19) we may obtain

$$\frac{d}{dt}\frac{1}{2} \parallel \theta \parallel^2 \leq -D\left(1-\frac{R}{R_E}\right). \tag{23}$$

If $R < R_E$, say $1 - R/R_E = a > 0$, then by using Poincaré's inequality in (23) we find

$$\frac{d}{dt}\frac{1}{2} \parallel \theta \parallel^2 \leq -a\pi^2 \parallel \theta \parallel^2 . \tag{24}$$

Thus integrate to see that

$$\parallel \theta(t) \parallel^2 \leq \parallel \theta(0) \parallel^2 \exp(-2a\pi^2 t). \tag{25}$$

Thus, inequality (25) shows that $\parallel \theta(t) \parallel$ decays exponentially provided $R < R_E$.

Next, add $(18)_1$ and $(18)_2$ to find

$$\gamma_1 \parallel \mathbf{u}^f \parallel^2 + \gamma_2 \parallel \mathbf{u}^p \parallel^2 + \parallel \mathbf{u}^f - \mathbf{u}^p \parallel^2 = R(\theta, w^f + w^p). \tag{26}$$

Employ the arithmetic-geometric mean inequality on the right hand side of (26) to see that

$$\gamma_1 \parallel \mathbf{u}^f \parallel^2 + \gamma_2 \parallel \mathbf{u}^p \parallel^2 \leq R^2(\gamma_1^{-1} + \gamma_2^{-1}) \parallel \theta \parallel^2 . \tag{27}$$

Now from (25) and (27) we deduce that $R < R_E$ also guarantees decay of $\mathbf{u}^f$ and $\mathbf{u}^p$. Thus $R < R_E$ represents a global (for all initial data) nonlinear stability threshold. To proceed we need to solve the maximum problem (22). The Euler-Lagrange equations which arise from (22) are, for Lagrange multipliers λ^f and λ^p,

$$\begin{aligned} &R_E\theta k_i - \gamma_1 u_i^f - (u_i^f - u_i^p) = \lambda^f_{,i}\,, \\ &R_E\theta k_i - \gamma_2 u_i^f + (u_i^f - u_i^p) = \lambda^p_{,i}\,, \\ &(w^f + w^p)R_E + \Delta\theta = 0. \end{aligned} \tag{28}$$

Equations (28) are the same as equations (11) with $\sigma = 0$. Thus the nonlinear energy stability threshold is identical to the linear instability one given by (17).

5 Conclusions

We have found the linear instability threshold and the nonlinear one for the problem of thermal convection in a bidispersive porous medium with a single temperature. A strong result is proved which demonstrates that the linear instability threshold coincides with the nonlinear stability one. This result is optimal and shows that in this case the linear theory captures the physics of the onset of convection correctly. It is worth stressing that this not true in the equivalent bidispersive convection problem when there are different temperatures T^f and T^p for the macro and micropores phases, see Nield and Kuznetsov [1], Straughan [9, 10], chapter 13.

For the local thermal non-equilibrium convection problem with one porosity but different solid and fluid temperature one also has strong exchange of stabilities and coincidence of linear/nonlinear thresholds, see Banu and Rees [19], Postelnicu and Rees [20], Straughan [21]. This then demonstrates that the general bidispersive convection problem with T^f and T^p is somewhat different from the single temperature or single porosity cases and may be worthy of investigation for new thermal effects.

The global stability/instability threshold is given by (17), but to bring it into line with other analyses we might consider another Rayleigh number which in a sense does not involve the interaction coefficient ζ. Define the Rayleigh number Ra_{cl} (classical) by

$$Ra_{cl} = \frac{\beta d^2 \rho_F g \alpha K_f}{k_m \mu} \,, \tag{29}$$

where $k_m = \kappa_m/(\rho c)_m$. Then the Rayleigh number in (17) is

$$Ra = \gamma_1 Ra_{cl} \tag{30}$$

This then allows us to rewrite (17) as

$$Ra_{cl} = 4\pi^2 \left(\frac{1 + \gamma_2 + K_r^{-1}}{4 + \gamma_1 + \gamma_2} \right) , \tag{31}$$

where $K_r = \gamma_1/\gamma_2 = K_p/K_f$. For calculation with specific values of μ, K_f and K_p the Rayleigh number Ra_{cl} may be more useful when dealing with experiments.

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