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3d $\mathcal{N} = 4$ super-Yang-Mills on a lattice

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Abstract: In this paper we explore a new approach to studying three-dimensional $\mathcal{N} = 4$ super-Yang-Mills on a lattice. Our strategy is to complexify the Donaldson-Witten twist of four-dimensional $\mathcal{N} = 2$ super-Yang-Mills to make it amenable to a lattice formulation and we find that lattice gauge invariance forces the model to live in at most three dimensions. We analyze the renormalization of the lattice theory and show that uncomplexified three-dimensional $\mathcal{N} = 4$ super-Yang-Mills can be reached in the continuum limit by supplementing the lattice action with appropriate mass terms.

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Contents

1 Introduction

It has been said that 4d $\mathcal{N} = 4$ super-Yang-Mills (SYM) is the hydrogen atom of quantum field theories. This may be motivated by the fact that it is a finite, integrable theory (in the planar limit) about which so much is known thanks to the SU$(2, 2|4)$ superconformal symmetry. If there is any truth in this appelation, then it is of primary interest to develop the theories with less than the maximal amount of supersymmetry (SUSY), as these would correspond to, say, helium or lithium atoms, which are in many ways more interesting. In particular, $\mathcal{N} = 2$ SYM is rich with nonpertubative phenomena such as confinement, spontaneous chiral symmetry breaking and monopole condensation, as exhibited in the famous Seiberg-Witten solution [1].

For the study of such non-perturbative questions, supersymmetric lattice gauge theory is potentially a very powerful approach. There are two known lattice formulations of 4d $\mathcal{N} = 4$ SYM which preserve a subset of the supersymmetries: orbifolding matrix models [2], and applying geometric discretization to a topologically twisted theory [3]. Note that lattice models derived from twisting approach are free from the fermion doubling problem [4]. Of course the same will be true of the orbifold models due to their equivalence to twisted models [5]. For further details on these two approaches, see the review [6].

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In this paper, we will consider analogous constructions for supersymmetric gauge theories in three dimensions, notably 3d $\mathcal{N} = 4$ SYM. Since the gauge coupling has positive mass dimension in three dimensions, 3d gauge theories are asymptotically free and super-renormalizable. Hence, they can be thought of as toy models for 4d QCD. For example, many properties of high temperature QCD are captured by Euclidean 3d Yang-Mills theory coupled to scalar adjoint matter fields [7].

Gaiotto and Witten classified 3d $\mathcal{N} = 4$ SYM theories as good, bad, and ugly [8]. For U(N) gauge group and N_f flavours, a good theory corresponds to $N_f \geq 2N$ and flows to an IR theory whose superconformal R-symmetry is manifest in the UV. Ugly theories correspond to $N_f = 2N - 1$ and flow to an IR theory whose superconformal R-symmetry is manifest in the UV, plus a decoupled free sector. Finally, bad theories correspond to $N_f \leq 2N-2$ and do not flow to an IR theory whose superconformal R-symmetry is manifest in the UV, so the IR limit of bad theories is not as well understood. For recent progress in this direction, see [9].

Like 4d $\mathcal{N} = 4$ SYM, the lattice formulation of 3d $\mathcal{N} = 4$ SYM can be obtained by orbifolding [10] or topologically twisting followed by geometric discretization. As we will explain in the next section, there are two ways to topologically twist, one corresponding to the dimensional reduction of the Donaldson-Witten twist in 4d [11], and the other known as the Blau-Thompson twist [12]. A lattice formulation based on geometric discretization of the Blau-Thompson twist was previously proposed in [13], so in this paper we develop a lattice formulation based on dimensionally reducing the Donaldson-Witten twist. Alternative approaches to formulating lattice 3d $\mathcal{N} = 4$ SYM based on a lattice Leibnitz rule were considered in [14, 15], although these approaches have some unresolved aspects [16, 17].

Note that there are several advantages to the latter approach. First of all, whereas the Blau-Thompson twist utilizes an internal SU(2) symmetry that generically becomes spontaneously broken in the IR, the twist we consider involves SU(2) R-symmetry which is preserved.[1] Secondly, since our lattice model will arise from dimensional reduction, it will have a larger point-symmetry group than the lattice model arising from the Blau-Thompson twist, which is intrinsically three dimensional. Indeed, we proceed by applying geometric discretization of the Donaldson-Witten twist of 4d $\mathcal{N} = 2$ SYM, and then show that lattice gauge invariance only holds if the basis vectors of the lattice are linearly dependent, forcing the theory to live in lower dimensions.[2] One disadvantage of the twisting approach compared to the orbifolding approach in 3d however is that in the twisted approach we must complexify the fields in order to implement the geometric discretization, and subsequently must introduce mass terms in order to decouple unwanted fields in the continuum limit.

The structure of this paper is as follows. In section 2, we review the two approaches to twisting 3d $\mathcal{N} = 4$ SYM, focusing on the one we make use of this in this paperwhich is equivalent to dimensional reduction of the Donaldson-Witten twist of 4d $\mathcal{N} = 2$ SYM), and we describe a complexification of 3d $\mathcal{N} = 4$ SYM that will allow us to formulate the theory on a lattice. In section 3, we apply geometric descretization to the complexified theory

[1] We thank Stefano Cremonesi for pointing this out.

[2] Note that this is very similar to the lattice formulation of 4d $\mathcal{N} = 4$ SYM, which is initially formulated in 5d but is then forced to live in 4d by lattice gauge invariance.

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and show that lattice gauge invariance forces the resulting lattice gauge theory to live in at most three dimensions. In section 4, we discuss the renormalization of the lattice theory and propose mass terms to reach the desired continuum limit, and in section 5 we present our conclusions and future directions. We also have several appendices. In appendix A, we derive the discrete R-symmetries of the Donaldson-Witten twist, and in appendix B, we compare the two twists of 3d $\mathcal{N}=4$ SYM in greater detail.

2 Continuum theory

Consider Euclidean 3d $\mathcal{N}=4$ SYM. The global symmetries of this theory are $\mathrm{SU}(2)_E \times \mathrm{SU}(2)_N \times \mathrm{SU}(2)_I$, where the first SU(2) corresponds to the rotation group in three spatial dimensions, the second one corresponds to an internal rotation group which arises from the dimensional reduction from 6d, i.e. $\mathrm{SO}(6) \to \mathrm{SU}(2)_E \times \mathrm{SU}(2)_N$, and the third one is the R-symmetry group of the 6d theory [18]. The two known twists of this theory correspond to identifying $\mathrm{SU}(2)_E$ with $\mathrm{SU}(2)_N$ and $\mathrm{SU}(2)_R$, respectively. Whereas the first twist (constructed by Blau and Thompson [12]) is intrinsically three-dimensional, the second one can be obtained by dimensionally reducing the Donaldson-Witten twist of 4d $\mathcal{N}=2$ SYM [11]. A lattice theory based on the Blau-Thompson twist was previously constructed in [13]. In this paper, we will consider the approach based on dimensional reduction. We relate the two approaches in appendix B.

Since we will use dimensional reduction in this paper, let us describe the Donaldson-Witten twist in more detail. The global symmetries of 4d $\mathcal{N}=2$ SYM are $\mathrm{SU}(2)_l \times \mathrm{SU}(2)_r \times \mathrm{SU}(2)_R \times \mathrm{U}(1)$ where $\mathrm{SU}(2)_l \times \mathrm{SU}(2)_r$ is locally the 4d rotation group and $\mathrm{SU}(2)_R \times \mathrm{U}(1)$ is the R-symmetry group (the U(1) factor arises from dimensional reduction from 6d and is the analogue of $\mathrm{SU}(2)_N$ in 3d). The twist by $\mathrm{SU}(2)_R$ is accomplished by identifying the twisted rotation group, $\mathrm{SU}(2)' \equiv \mathrm{diag}[\mathrm{SU}(2)_r \times \mathrm{SU}(2)_R]$. Then the fields transform in the following representations of $\mathrm{SU}(2)_l \times \mathrm{SU}(2)' \times \mathrm{U}(1)$:

$$\begin{aligned} &\text{bosons}: \ (1/2,1/2)^0 + (0,0)^2 + (0,0)^{-2} \\ &\text{fermions}: \ (1/2,1/2)^1 + (0,1)^{-1} + (0,0)^{-1} \end{aligned} \tag{2.1}$$

Since $\mathrm{SU}(2)_l \times \mathrm{SU}(2)' \simeq \mathrm{SO}(4)'$, the $(1/2,1/2)$ correspond to four-dimensional vector representations of $\mathrm{SO}(4)'$. It will turn out that the $(0,1)$ corresponds to an antisymmetric self-dual tensor. Thus as usual in the twisted formulations, fermions no longer carry spinor indices, but appear as scalars, vectors and antisymmetric tensors. The twisted bosonic and fermionic fields enumerated in (2.1) shall subsequently be denoted as $\{A_\mu, \phi, \bar{\phi}\}$, and $\{\psi_\mu, \chi_{\mu\nu}, \eta\}$, respectively, where $\chi_{\mu\nu} = \star\chi_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\chi^{\rho\lambda}$, and $\bar{\phi} = \phi^\dagger$. Although twisted theories are usually considered in a curved background, we will be working in Euclidean flat space, so there will be no distinction between upper and lower Lorentz indices.

The Lagrangian for twisted 4d $\mathcal{N}=2$ SYM can be written as follows:

$$\begin{aligned} g^2\mathcal{L}_{4d}^{\mathcal{N}=2} = \mathrm{tr}\Big(&\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{1}{2}\mathcal{D}_\mu\bar{\phi}\mathcal{D}^\mu\phi - \alpha\left[\phi,\bar{\phi}\right]^2 \\ &-\frac{i}{2}\eta\mathcal{D}_\mu\psi^\mu + i\alpha\phi\{\eta,\eta\} - \frac{i}{2}\bar{\phi}\{\psi_\mu,\psi^\mu\} + \mathcal{L}_\chi\Big), \end{aligned} \tag{2.2}$$

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where $\mathcal{D}_\mu X = \partial_\mu X + \mathcal{A}_\mu X$, $\mathcal{F}_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu]$, and

$$\mathcal{L}_\chi = tr\left(\frac{i}{8}\phi\{\chi_{\mu\nu}, \chi^{\mu\nu}\} - i\chi^{\mu\nu}\mathcal{D}_\mu\psi_\nu\right). \tag{2.3}$$

Note that discrete R-symmetries fix $\alpha = \frac{1}{8}$, as we show in appendix A. (It is amusing that [19] also uses discrete R-symmetries to fix the form of the $\mathcal{N} = 2$ SYM Lagrangian, albeit in the usual untwisted form; see p. 161.) The equations of motion for χ are given by:

$$[\chi_{\mu\nu}, \phi] = 2\left(\mathcal{D}_{[\mu}\psi_{\nu]} + *\mathcal{D}_{[\mu}\psi_{\nu]}\right). \tag{2.4}$$

Plugging these equations of motion into (2.3) then gives

$$\begin{aligned} g^2\mathcal{L}_{4d}^{\mathcal{N}=2} = \mathrm{tr}\Big(&\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{1}{2}\mathcal{D}_\mu\bar{\phi}\mathcal{D}^\mu\phi - \alpha\left[\phi, \bar{\phi}\right]^2 - \frac{i}{2}\eta\mathcal{D}_\mu\psi^\mu \\ &-\frac{i}{2}\chi^{\mu\nu}\mathcal{D}_\mu\psi_\nu + i\alpha\phi\{\eta, \eta\} - \frac{i}{2}\bar{\phi}\{\psi_\mu, \psi^\mu\}\Big). \end{aligned} \tag{2.5}$$

This form of the Lagrangian is useful because it can be written in a way that makes a BRST symmetry manifest:

$$g^2\mathcal{L}_{4d}^{\mathcal{N}=2} = Q\,tr\left[\frac{1}{4}\chi_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{1}{2}\mathcal{D}_\mu\bar{\phi}\psi^\mu + \alpha\eta\left[\phi, \bar{\phi}\right]\right] - \frac{1}{4}tr\left[\star\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right], \tag{2.6}$$

where Q generates the following transformations:[3]

$$Q\,\phi = 0,\ Q\,\bar{\phi} = i\eta,\ Q\,\mathcal{A}_\mu = i\psi_\mu,\ Q\,\eta = \left[\bar{\phi}, \phi\right],\ Q\,\psi_\mu = \mathcal{D}_\mu\phi,\ Q\,\chi_{\mu\nu} = \mathcal{F}_{\mu\nu} + *\mathcal{F}_{\mu\nu}. \tag{2.7}$$

Let us verify the BRST symmetry. First note that the equations of motion in (2.4) are invariant under the transformations in (2.7). Furthermore, using equations (2.7) and (2.4) one finds that Q^2 generates a gauge transformation:

$$\begin{aligned} &Q^2\phi = 0, && Q^2\bar{\phi} = i\left[\bar{\phi}, \phi\right], && Q^2\mathcal{A}_\mu = i\mathcal{D}_\mu\phi, \\ &Q^2\eta = i\left[\eta, \phi\right], && Q^2\psi_\mu = i\left[\psi_\mu, \phi\right], && Q^2\chi_{\mu\nu} = i\left[\chi_{\mu\nu}, \phi\right]. \end{aligned}$$

Since Q^2 annihilates gauge-invariant operators, it follows that the first term in (2.6) is Q-exact. Moreover the second term is Q-closed since

$$Q\,tr\left[\star\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right] = 4\epsilon^{\mu\nu\rho\lambda}\mathcal{D}_\mu\psi_\nu\mathcal{F}_{\rho\lambda},$$

which vanishes after applying integration by parts and the Bianchi identity, so the twisted theory in (2.5) is indeed BRST invariant. A lattice formulation of this model was proposed in [20], although it was not based on geometric discretization. In this paper, we will employ geoemtric discretization and subsequently find that the lattice theory can live in at most three dimensions.

[3] At first sight the BRST transformations may not appear to be compatible with the constraint that $\bar{\phi}$ is the complex conjugate of ϕ, but as explained in [11] this is not a problem because conservation of the corresponding supercurrent is compatible with this constraint. In the next section, we will consider a complexification where ϕ and $\bar{\phi}$ are independent, so the BRST transformations will appear more natural.

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2.1 Complexification

In order to make the theory amenable to geometric discretization, we must complexify the fields. Furthermore, when we apply geometric discretization to formulate the model on a lattice, we will find that consistency of the self-duality constraint of χ with lattice gauge invariance implies that the model can live in at most three dimensions. Taking the lattice to be three-dimensional, the continuum limit will then correspond to complexified 3d $\mathcal{N}=4$ SYM. As we will show in section 4.1, the uncomplexified theory can be reached in the continuum limit by adding appropriate mass terms to the lattice action.

Consider the following complexification of twisted 4d $\mathcal{N}=2$ SYM:

$$g^2\mathcal{L}^* = \mathrm{tr}\Big(\frac{1}{4}\mathcal{F}_{\mu\nu}\bar{\mathcal{F}}^{\mu\nu} + \frac{1}{2}\bar{\mathcal{D}}_\mu\bar{\phi}\mathcal{D}_\mu\phi - \alpha\left[\phi,\bar{\phi}\right]^2 \\ -\frac{i}{2}\eta\bar{\mathcal{D}}_\mu\psi^\mu + i\alpha\phi\left\{\eta,\eta\right\} - \frac{i}{2}\bar{\phi}\left\{\psi_\mu,\bar{\psi}^\mu\right\} + \mathcal{L}_\chi\Big), \tag{2.8}$$

where

$$\mathcal{L}_\chi = tr\left[\frac{i}{8}\phi\left\{\chi_{\mu\nu},\bar{\chi}_{\mu\nu}\right\} - \frac{i}{2}\left(\chi^{\mu\nu}\mathcal{D}_\mu\psi_\nu + \bar{\chi}^{\mu\nu}\bar{\mathcal{D}}_\mu\bar{\psi}_\nu\right)\right]. \tag{2.9}$$

Note that all the fields are complex. In particular, $\bar{\mathcal{A}} = \mathcal{A}^\dagger$, $\bar{\eta} = \eta^\dagger$, $\bar{\psi} = \psi^\dagger$, and $\bar{\chi} = \chi^\dagger$, but $\bar{\phi} \neq \phi^\dagger$ so $\bar{\phi}$ and ϕ are independent. We define $\mathcal{D}_\mu X = \partial_\mu + [\mathcal{A}_\mu, X]$, $\bar{\mathcal{D}}_\mu X = \partial_\mu X + \left[\bar{\mathcal{A}}_\mu, X\right]$, $\mathcal{F}_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu]$, and $\bar{\mathcal{F}}_{\mu\nu} = \left[\bar{\mathcal{D}}_\mu, \bar{\mathcal{D}}_\nu\right]$. We also impose the Hodge-duality constraint $\chi_{\mu\nu} = *\bar{\chi}_{\mu\nu}$. A similar complexification of twisted 4d $\mathcal{N}=2$ SYM was considered in [21], where it was argued to be equivalent to twisted 4d $\mathcal{N}=4$ SYM. As mentioned above, consistency of this constraint with lattice gauge invariance implies that the lattice theory must be at most three-dimensional; details will be given the following section.

As before, we can integrate out χ to obtain the following equations of motion:

$$[\chi_{\mu\nu},\phi] = 2\left(\bar{\mathcal{D}}_{[\mu}\bar{\psi}_{\nu]} + \star\mathcal{D}_{[\mu}\psi_{\nu]}\right). \tag{2.10}$$

Plugging the equations of motion back into (2.9) then gives

$$g^2\mathcal{L}^* = \mathrm{tr}\Big(\frac{1}{4}\mathcal{F}_{\mu\nu}\bar{\mathcal{F}}^{\mu\nu} + \frac{1}{2}\bar{\mathcal{D}}_\mu\bar{\phi}\mathcal{D}_\mu\phi - \alpha\left[\phi,\bar{\phi}\right]^2 - \frac{i}{2}\eta\bar{\mathcal{D}}_\mu\psi^\mu \\ +i\alpha\phi\left\{\eta,\eta\right\} - \frac{i}{2}\bar{\phi}\left\{\psi_\mu,\bar{\psi}^\mu\right\} - \frac{i}{2}\chi^{\mu\nu}\mathcal{D}_\mu\psi_\nu\Big), \tag{2.11}$$

which can be written in a manifestly BRST-invariant form as follows:

$$g^2\mathcal{L}^* = Q\,tr\left[\frac{1}{4}\chi_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{1}{2}\bar{\mathcal{D}}_\mu\bar{\phi}\psi^\mu + \alpha\eta\left[\phi,\bar{\phi}\right]\right] - \frac{1}{4}tr\left(\star\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right), \tag{2.12}$$

where Q generates the following transformations:

$$\begin{aligned} &Q\,\phi = 0, && Q\,\bar{\phi} = i\eta, && Q\,\mathcal{A}_\mu = i\psi_\mu, && Q\,\bar{\mathcal{A}}_\mu = i\bar{\psi}_\mu, \\ &Q\,\eta = \left[\bar{\phi},\phi\right], && Q\,\psi_\mu = \mathcal{D}_\mu\phi, && Q\,\bar{\psi}_\mu = \bar{\mathcal{D}}_\mu\phi, && Q\,\chi_{\mu\nu} = \bar{\mathcal{F}}_{\mu\nu} + *\mathcal{F}_{\mu\nu}. \end{aligned} \tag{2.13}$$

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Note that the equations of motion in (2.10) are invariant under the transformations in (2.13). Using (2.13) and (2.10) it is not difficult to show that Q squares into a gauge transformation implying that the first term in (2.12) is BRST exact. Moreover the second term is BRST closed since acting with Q gives a term that vanishes by the Bianchi identity, as shown in the previous subsection. Hence the complexified theory in (2.8) is BRST invariant. Our next task will be to formulate this model on a lattice in such a way that the BRST symmetry is preserved and becomes enhanced to $\mathcal{N} = 2$ supersymmetry in the continuum limit.

3 Lattice theory

In this section, we will generalize the complexified Lagrangian in (2.8) to a lattice. The fields $\phi(n), \bar{\phi}(n), \eta(n)$ will reside on the lattice site $n = \sum_{\mu=1}^{d} n_\mu e_\mu$, $n_\mu \in \mathbb{Z}$, while the gauge field $\mathcal{A}_\mu(n)$ will be generalized to a Wilson line $\mathcal{U}_\mu(n)$ residing on the link $(n, n + e_\mu)$ as will $\psi_\mu(n)$, and the field $\chi_{\mu\nu}(n)$ will reside on the link $(n + e_\mu + e_\nu, n)$. In these expressions, e_μ are principle vectors of the lattice, which will turn out to be linearly dependent, and are proportional to the lattice spacing. Under a lattice gauge transformation, the fields transform as follows:

$$\begin{aligned}
\left\{\phi(n), \bar{\phi}(n), \eta(n)\right\} &\to G(n)\left\{\phi(n), \bar{\phi}(n), \eta(n)\right\} G^\dagger(n) \\
\left\{\mathcal{U}_\mu(n), \psi_\mu(n)\right\} &\to G(n)\left\{\mathcal{U}_\mu(n), \psi_\mu(n)\right\} G^\dagger\left(n + e_\mu\right) \\
\chi_{\mu\nu}(n) &\to G\left(n + e_\mu + e_\nu\right)\chi_{\mu\nu}(n) G^\dagger(n).
\end{aligned} \tag{3.1}$$

Here, $G(n)$ are elements of the gauge group of the target (continuum) theory. As is usual with the twisted (or orbifold) formulations, the gauge group must be U(N), in order to be consistent with the scalar supersymmetry algebra, (3.9) below. In particular, the relation $Q\,\mathcal{U}_\mu(n) = i\psi_\mu(n)$ must contain GL$(N, \mathbb{C})$ valued fields on both sides, because $\mathcal{U}_\mu(n) = 1 + a\mathcal{A}_\mu(n) + \cdots$ in the continuum limit, so $\psi_\mu(n)$ must also have an expansion that includes the unit matrix. The lattice covariant derivatives are defined to be

$$\begin{aligned}
\mathcal{D}^+_\mu f(n) &= \mathcal{U}_\mu(n) f\left(n + e_\mu\right) - f(n)\mathcal{U}_\mu(n) \\
\mathcal{D}^+_\mu f_\nu(n) &= \mathcal{U}_\mu(n) f_\nu\left(n + e_\mu\right) - f_\nu(n)\mathcal{U}_\mu\left(n + e_\nu\right) \\
\bar{\mathcal{D}}^-_\mu f_\mu(n) &= f_\mu(n)\bar{\mathcal{U}}_\mu(n) - \bar{\mathcal{U}}_\mu\left(n - e_\mu\right) f_\mu\left(n - e_\mu\right) \\
\bar{\mathcal{D}}^-_\mu f_{\nu\lambda}(n) &= f_{\nu\lambda}(n)\bar{\mathcal{U}}_\mu\left(n - e_\mu\right) - \bar{\mathcal{U}}_\mu\left(n + e_\nu + e_\lambda - e_\mu\right) f_{\nu\lambda}\left(n - e_\mu\right).
\end{aligned}$$

In terms of these lattice derivatives, the gauge field strength is then given by

$$\mathcal{F}_{\mu\nu}(n) = \mathcal{D}^+_\mu \mathcal{U}_\nu(n).$$

Similarly, we define the complex conjugate lattice derivatives as follows:

$$\begin{aligned}
\bar{\mathcal{D}}^+_\mu f(n) &= f\left(n + e_\mu\right)\bar{\mathcal{U}}_\mu(n) - \bar{\mathcal{U}}_\mu(n) f(n) \\
\bar{\mathcal{D}}^+_\mu f_\nu(n) &= f_\nu\left(n + e_\mu\right)\bar{\mathcal{U}}_\mu(n) - \bar{\mathcal{U}}_\mu\left(n + e_\nu\right) f_\nu(n) \\
\mathcal{D}^-_\mu f_\mu(n) &= \mathcal{U}_\mu(n) f_\mu(n) - f_\mu\left(n - e_\mu\right)\mathcal{U}_\mu\left(n - e_\mu\right) \\
\mathcal{D}^-_\mu f_{\nu\lambda}(n) &= \mathcal{U}_\mu\left(n - e_\mu\right) f_{\nu\lambda}(n) - f_{\nu\lambda}\left(n - e_\mu\right)\mathcal{U}_\mu\left(n + e_\nu + e_\lambda - e_\mu\right),
\end{aligned}$$

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in terms of which the complex conjugate field strength is given by

$$\bar{\mathcal{F}}_{\mu\nu}(n) = -\bar{\mathcal{D}}^{+}_{\mu}\bar{\mathcal{U}}_{\nu}(n).$$

For the complexified theory in (2.8), supersymmetry requires that $\chi_{\mu\nu} = *\bar{\chi}_{\mu\nu}$ in order to have an equal number of bosons and fermions. Let us therefore impose the following analogous constraint on the lattice fields χ and $\bar{\chi}$:

$$\chi_{\mu\nu}(n) = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\bar{\chi}_{\rho\lambda}\left(n + e_{\mu} + e_{\nu}\right). \tag{3.2}$$

If we apply the lattice gauge transformations in (3.1), we find that the above equation is left invariant if and only if the basis vectors of the lattice are linearly dependent:

$$\sum_{\mu=1}^{4} e_{\mu} = 0. \tag{3.3}$$

Note that the constraint in (3.2) is the unique choice which reduces to $\chi_{\mu\nu} = *\bar{\chi}_{\mu\nu}$ in the continuum limit and respects lattice gauge and Q-invariance. To see this, consider a more general ansatz:

$$\chi_{\mu\nu}(n) = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\bar{\chi}_{\rho\lambda}\left(n + \Delta\right).$$

Applying the gauge transformations in (3.1) then implies the constraints $\Delta = e_{\mu} + e_{\nu}$ and $\Delta + e_{\rho} + e_{\lambda} = 0$, from which (3.2) and (3.3) follow. One could consider generalizing this ansatz by dressing it with link variables $\left(\mathcal{U}_{\mu}, \bar{\mathcal{U}}_{\mu}\right)$, but this will spoil Q-inavariance of the lattice theory since the link variables must transform non-trivially under BRST transformations, as we will see shortly. One could also consider replacing $\bar{\chi}$ with χ on the right-hand-side of the above ansatz since this would imply the same reduction in the number of femrions, but in that case one finds that there is no choice of Δ consistent with the lattice gauge transformations in (3.1).

Hence, the lattice can be at most three-dimensional. In this case, the complexifed Lagrangian in (2.8) has the following lattice generalization:

$$\begin{aligned}\mathcal{L} = \mathrm{tr}\Bigg(&\frac{1}{4}\bar{\mathcal{F}}_{\mu\nu}(n)\mathcal{F}_{\mu\nu}(n) + \frac{1}{2}\bar{\mathcal{D}}^{+}_{\mu}\bar{\phi}(n)\mathcal{D}^{+}_{\mu}\phi(n) - \alpha\left[\phi(n), \bar{\phi}(n)\right]^{2} \\ &+ \frac{i}{2}\bar{\mathcal{D}}^{+}_{\mu}\eta(n)\psi_{\mu}(n) + i\alpha\phi(n)\left\{\eta(n), \eta(n)\right\} \\ &- \frac{i}{2}\bar{\phi}(n)\left(\psi_{\mu}(n)\bar{\psi}_{\mu}(n) + \bar{\psi}_{\mu}\left(n - e_{\mu}\right)\psi_{\mu}\left(n - e_{\mu}\right)\right)\Bigg) + \mathcal{L}_{\chi},\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}\mathcal{L}_{\chi} = \mathrm{tr}\Bigg[&\frac{i}{8}\left(\phi(n)\bar{\chi}_{\mu\nu}(n)\chi_{\mu\nu}(n) + \phi\left(n + e_{\mu} + e_{\nu}\right)\chi_{\mu\nu}(n)\bar{\chi}_{\mu\nu}(n)\right) \\ &- \frac{i}{2}\left(\bar{\chi}_{\mu\nu}(n)\bar{\mathcal{D}}^{+}_{\mu}\bar{\psi}_{\nu}(n) + \chi_{\mu\nu}(n)\mathcal{D}^{+}_{\mu}\psi_{\nu}(n)\right)\Bigg].\end{aligned} \tag{3.5}$$

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Using the constraints in (3.2) and (3.3), one obtains the following equations of motion for $\bar{\chi}$:

$$2\left(\bar{\mathcal{D}}^{+}_{[\mu}\bar{\psi}_{\nu]}(n)+\frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\mathcal{D}^{+}_{\rho}\psi_{\lambda}\left(n+e_{\mu}+e_{\nu}\right)\right)=\chi_{\mu\nu}(n)\phi(n)-\phi\left(n+e_{\mu}+e_{\nu}\right)\chi_{\mu\nu}(n). \quad (3.6)$$

Plugging the equations of motion back into (3.5) then gives

$$\mathcal{L}_{\chi}=-\frac{i}{4}\mathrm{tr}\left(\bar{\chi}_{\mu\nu}(n)\bar{\mathcal{D}}^{+}_{\mu}\bar{\psi}_{\nu}(n)+\chi_{\mu\nu}(n)\mathcal{D}^{+}_{\mu}\psi_{\nu}(n)\right). \quad (3.7)$$

The Lagrangian can subsequently be expressed in a manifestly BRST invariant form as follows:

$$\begin{aligned}\mathcal{L}=Q\,\mathrm{tr}\left(\frac{1}{4}\chi_{\mu\nu}(n)\mathcal{F}_{\mu\nu}(n)+\frac{1}{2}\bar{\mathcal{D}}^{+}_{\mu}\bar{\phi}(n)\psi_{\mu}(n)+\alpha\eta(n)\left[\phi(n),\bar{\phi}(n)\right]\right)\\ -\frac{1}{8}\epsilon_{\mu\nu\rho\lambda}\mathrm{tr}\left(\mathcal{F}_{\mu\nu}(n)\mathcal{F}_{\rho\lambda}\left(n+e_{\mu}+e_{\nu}\right)\right), \quad (3.8)\end{aligned}$$

where the BRST operator Q acts according to

$$\begin{aligned}Q\,\phi(n)&=0, & Q\,\bar{\phi}(n)&=i\eta(n),\\ Q\,\eta(n)&=\left[\bar{\phi}(n),\phi(n)\right], & &\\ Q\,\mathcal{U}_{\mu}(n)&=i\psi_{\mu}(n), & Q\,\bar{\mathcal{U}}_{\mu}(n)&=-i\bar{\psi}_{\mu}(n)\\ Q\,\psi_{\mu}(n)&=\mathcal{D}^{+}_{\mu}\phi(n), & Q\,\bar{\psi}_{\mu}(n)&=\bar{\mathcal{D}}^{+}_{\mu}\phi(n)\\ Q\,\chi_{\mu\nu}(n)&=\bar{\mathcal{F}}_{\mu\nu}(n)+\frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\mathcal{F}_{\rho\lambda}\left(n+e_{\mu}+e_{\nu}\right). & & \quad (3.9)\end{aligned}$$

To verify that the lattice theory is BRST-invariant, first note that the constraint in (3.2) and the equations of motion in (3.6) are invariant under the tranformations in (3.9), and the second term in (3.8) is Q-closed since

$$Q\sum_{n}\epsilon_{\mu\nu\rho\lambda}\mathrm{tr}\left[\mathcal{F}_{\mu\nu}(n)\mathcal{F}_{\rho\lambda}\left(n+e_{\mu}+e_{\nu}\right)\right]=-4i\sum_{n}\epsilon_{\mu\nu\rho\lambda}\psi_{\nu}\left(n-e_{\nu}\right)\mathcal{D}^{-}_{\mu}\mathcal{F}_{\rho\lambda}\left(n+e_{\mu}\right)=0,$$

where we used (3.3) and the lattice Bianchi identity $\epsilon_{\mu\nu\rho\lambda}\mathcal{D}^{-}_{\mu}\mathcal{F}_{\rho\lambda}\left(n+e_{\mu}\right)=0$. Furthermore, using (3.9) and (3.6), one finds that Q^2 generates a lattice gauge transformation:

$$\begin{aligned}Q^2\phi(n)&=0, & Q^2\bar{\phi}(n)&=i\left[\bar{\phi}(n),\phi(n)\right],\\ Q^2\eta(n)&=i\left[\eta(n),\phi(n)\right], & &\\ Q^2\mathcal{U}_{\mu}(n)&=i\mathcal{D}^{+}_{\mu}\phi(n), & Q^2\bar{\mathcal{U}}_{\mu}(n)&=-i\bar{\mathcal{D}}^{+}_{\mu}\phi(n).\\ Q^2\psi_{\mu}(n)&=i\left(\psi_{\mu}(n)\phi\left(n+e_{\mu}\right)-\phi(n)\psi_{\mu}(n)\right) & & \\ Q^2\bar{\psi}_{\mu}(n)&=i\left(\bar{\psi}_{\mu}(n)\phi\left(n\right)-\phi\left(n+e_{\mu}\right)\bar{\psi}_{\mu}(n)\right). & & \\ Q^2\chi_{\mu\nu}(n)&=2i\left(\bar{\mathcal{D}}^{+}_{[\mu}\bar{\psi}_{\nu]}(n)+\frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\mathcal{D}^{+}_{\rho}\psi_{\lambda}\left(n+e_{\mu}+e_{\nu}\right)\right). & & \end{aligned}$$

Hence the first term in (3.8) is Q-exact and the lattice theory in (3.4) is indeed BRST-invariant.

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3.1 Base space

To avoid unwanted renormalizations, it is advantageous to have maximal point group symmetry. Thus we would like to have principal vectors e_1, e_2, e_3 and e_4, arranged such that the symmetry is just S_4, the permutation group of four numbers, (1234) → (2134), etc. This is achieved with the A_3^* lattice [2, 22], which has the hyper-triangular constraints

$$e_\mu \cdot e_\nu = \delta_{\mu\nu} - \frac{1}{4}, \quad e_1 + e_2 + e_3 + e_4 = 0, \quad \sum_{\mu=1}^{4} e_\mu{}^i e_\nu{}^j = \delta^{ij}, \tag{3.10}$$

where $i, j = 1, 2, 3$ label the components of the four vectors. An explicit choice of the principal vectors is

$$\begin{aligned} e_1 &= \left(\sqrt{\frac{3}{4}}, 0, 0\right), & e_2 &= \left(-\frac{1}{\sqrt{12}}, \sqrt{\frac{2}{3}}, 0\right) \\ e_3 &= \left(-\frac{1}{\sqrt{12}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right), & e_4 &= \left(-\frac{1}{\sqrt{12}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{2}}\right) \end{aligned} \tag{3.11}$$

It can be checked that these satisfy (3.10).

The lattice Λ can be specified by

$$\Lambda = \{\, n_1 e_1 + n_2 e_2 + n_3 e_3 \mid n_i \in \mathbb{Z} \;\forall\; i = 1, 2, 3 \,\} \tag{3.12}$$

The vector $\mathbf{n} = (n_1, n_2, n_3)$ is then associated with a site on an *abstract* cubic lattice, which is how a computer code would "see" the lattice. In particular, periodic boundary conditions would be imposed *via* $n_i \simeq n_i + L_i$ with L_i the size of the abstract torus in the direction i. Note that the integer-valued vectors labelling the sites of abstract are related to the R-charges defined in the orbifold formulation [2, 10].

The four basic directions e_μ are to be associated with the link fields $\mathcal{A}_\mu$ and ψ_μ. Note that the S_4 point-symmetry group is a subgroup of the twisted rotation group SU(2)′ and the lattice fields transform in reducible representations of the point symmetry group. For example, the four-vector A_μ decomposes into a three-vector and a scalar, which corresponds to the S_4 symmetric linear combination $\sigma = \sum_{\mu=1}^{4} A_\mu$. The parts of A_μ that are orthogonal to this symmetric combination are the *bona fide* gauge fields of the 3d theory. Morover if we add mass terms for the imaginary parts of the fields (as we describe in section 4.1), in continuum limit we will be left with one real scalar and one gauge field along with two real scalars coming from $(\phi, \bar{\phi})$, which is precisely the bosonic field content of 3d $\mathcal{N} = 4$ SYM. For more details of the dimensional reduction, see appendix B.

4 Renormalization

To understand the renormalization of the lattice theory, first we have to outline the dimensions of fields and parameters. Note that, in terms of mass dimensions,

$$\left[\frac{1}{g^2}\right] = -1 \quad \Rightarrow \quad [g] = 1/2 \tag{4.1}$$

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This can be understood in terms of dimensional reduction, which yields

$$\frac{1}{g^2} = \frac{L_4}{g^2_{4\mathrm{d}}} \tag{4.2}$$

where L_4 is the size of the fourth dimension that is reduced out, and of course the 4d gauge coupling is dimensionless. It follows that, as written, all fields have the same dimensions as in 4d. However, this is not convenient for an analysis of relevant/marginal/irrelevant operator classification, so we scale out the coupling to get canonical kinetic terms before proceeding. For instance,

$$\mathcal{A}_\mu \to g\mathcal{A}_\mu, \quad \mathcal{F}_{\mu\nu} \to g\mathcal{F}_{\mu\nu}, \quad \phi \to g\phi, \quad \bar{\phi} \to g\bar{\phi}, \quad \psi_\mu \to g\psi_\mu, \quad \eta \to g\eta \ldots \tag{4.3}$$

Hence after the rescaling,

$$[\mathcal{A}_\mu] = 1/2, \quad [\mathcal{F}_{\mu\nu}] = 3/2, \quad [\phi] = [\bar{\phi}] = 1/2, \quad [\psi_\mu] = [\eta] = 1 \ldots . \tag{4.4}$$

Notice that the scalar potential term becomes $g^2[\phi, \bar{\phi}]^2$, so that the operator has mass dimension 2 and is *relevant.* This is very important for understanding the unwanted radiative corrections that occur when we flow to the long distance effective theory (given that the lattice regulator breaks SUSY explicitly and allows for non-SUSY renormalizations).

To further understand these matters with canonical normalization, notice the SUSY relation $Q\mathcal{A}_\mu = i\psi_\mu$ implies that the supercharge carries mass dimension, $[Q] = 1/2$. This will allow us to analyze the renormalizations from the point of view of Q-exact terms. Notice also that the SUSY variation of η becomes $Q\eta = g[\bar{\phi}, \phi]$. Thus we see also from this perspective that $Q(\eta[\phi, \bar{\phi}])$ is a relevant operator, with mass dimension 5/2, though the $[\bar{\phi}, \phi]^2$ part of it should really be counted as mass dimension 2, because of the appearance of g, for the purposes of RG analysis. In detail, taking into account the rescalings (4.3),

$$\frac{1}{g^2} Q\mathrm{tr}\eta[\phi, \bar{\phi}] \to gQ\mathrm{tr}\eta[\phi, \bar{\phi}] = g^2\mathrm{tr}[\phi, \bar{\phi}]^2 + ig\mathrm{tr}\eta[\phi, \eta] \tag{4.5}$$

so we see that we have both dimension 2 and 5/2 operators, with the additional dimensions (to get to 3) soaked up by powers of g, which has dimension 1/2.

Because in 3d the coupling carries mass dimension, renormalizations are highly constrained. If in the long distance effective theory a marginal operator is generated, it must enter at one loop as $\mathcal{O}(g^2 a)$, and with higher powers of $g^2 a$ at higher orders. Here, it is important that we keep track of the dimensionless quantity $g^2 a$, because the coefficient of a marginal operator must be dimensionless, yet loop corrections (with canonical normalization of kinetic terms) will be powers of g^2. Thus in the continuum limit $a \to 0$, all of these radiative corrections vanish. For this reason we do not have to worry about marginal operators in terms of recovering SUSY. Irrelevant operators simply come with additional powers of the lattice spacing a, so these are also not troublesome. Thus the operators that we must focus our attention on are the relevant operators, which in 3d with canonical kinetic terms are operators with $d_{\mathcal{O}} < 3$. In the remainder of this subsection we enumerate such operators allowed by the lattice symmetries, which will give us a count on the number

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of fine-tunings required to achieve the SUSY continuum limit. Thus we set aside a rather long list of marginal operators that are consistent with Q and S_4 invariance, such as:

$$Q\mathrm{tr}\mathcal{D}_\mu\bar{\phi}\psi_\mu, \quad Q\mathrm{tr}\left(\sum_\mu \mathcal{D}_\mu\phi\right)\eta, \quad Q\mathrm{tr}\left(\eta\sum_{\mu<\nu}\mathcal{F}_{\mu\nu}\right), \quad \ldots \tag{4.6}$$

Firstly, as noted above, $Q\mathrm{tr}(\eta[\phi,\bar{\phi}])$ is a relevant operator. In particular, the part of it that gives $[\phi,\bar{\phi}]^2$ could appear at one loop with g^2 and radiative effects will allow this to have a dimensionless coefficient that is different from the SUSY theory. So from this we get one fine-tuning, but it is not a new operator that we have to add to the lattice theory to achieve the SUSY long distance limit. In particular, for relatively coarse lattice spacings and weak $g\sqrt{a}$, we do not anticipate large deviations from SUSY occuring from this operator, based on experience in the 4d $\mathcal{N}=4$ lattice theory. With these sorts of considerations we finally find the following list of relevant operators that are allowed by lattice gauge invariance, Q symmetry, and the S_4 point group symmetry:

$$\begin{aligned} &Q\mathrm{tr}(\eta[\phi,\bar{\phi}]), \quad Q\mathrm{tr}(\eta\{\phi,\bar{\phi}\}), \quad Q\mathrm{tr}(\eta\phi), \quad Q\mathrm{tr}(\eta\bar{\phi}), \\ &Q(\mathrm{tr}\eta\mathrm{tr}\{\phi,\bar{\phi}\}), \quad Q(\mathrm{tr}\eta\mathrm{tr}\phi), \quad Q(\mathrm{tr}\eta\mathrm{tr}\bar{\phi}) \end{aligned} \tag{4.7}$$

The double trace operators, that involve the trace of a single field, are possible because the group is U(N). Note however that, in analogy to the 4d $\mathcal{N}=4$ theory, there is a fermionic shift symmetry in the bare theory,

$$\eta \to \eta + b\mathbb{I} \tag{4.8}$$

where b is a constant Grassmann number and $\mathbb{I}$ is an $N\times N$ unit matrix. This forbids all of the terms in (4.7) individually, except for $Q\mathrm{tr}(\eta[\phi,\bar{\phi}])$, which already appears in the tree action. On the other hand, again in analogy to the 4d $\mathcal{N}=4$ theory, there are three linear combinations of the other terms that are allowed by the shift symmetry:

$$\begin{aligned} &Q\mathrm{tr}(\eta\{\phi,\bar{\phi}\}) - \frac{1}{N}Q(\mathrm{tr}\eta\mathrm{tr}\{\phi,\bar{\phi}\}) \\ &Q\mathrm{tr}(\eta\phi) - \frac{1}{N}Q(\mathrm{tr}\eta\mathrm{tr}\phi) \\ &Q\mathrm{tr}(\eta\bar{\phi}) - \frac{1}{N}Q(\mathrm{tr}\eta\mathrm{tr}\bar{\phi}) \end{aligned} \tag{4.9}$$

These three *new* operators are allowed by all the lattice symmetries. They generate terms that are cubic and quartic in scalar fields, as well as a mass term for the SU(N) part of η. E.g.,

$$Q\mathrm{tr}(\eta\bar{\phi}) - \frac{1}{N}Q(\mathrm{tr}\eta\mathrm{tr}\bar{\phi}) = \frac{g}{N}\mathrm{tr}\left[\phi,\bar{\phi}\right]\mathrm{tr}\bar{\phi} \tag{4.10}$$

In each of the cases above, we end up with a contribution from dimension 2 operators after applying Q. Loop corrections beyond g^2 would necessarily come with positive powers of the lattice spacing a, and therefore do not appear in the continuum limit. The result of this is that all of the quantum corrections that have to be cancelled by counterterms appear at 1-loop. This renders the fine-tuning of the lattice theory quite manageable.

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4.1 Mass terms

A first, basic requirement of the lattice formulation is that it reproduce the desired continuum limit, notably $\mathcal{N} = 4$ SYM in 3d. As for lattice 4d $\mathcal{N} = 4$ SYM, we must add the following mass terms to ensure that the gauge link fields have the form $\mathcal{U}_\mu(n) = 1 + a\mathcal{A}_\mu(n) + \cdots$ in the continuum limit:

$$\mathcal{L}_U = m_U^2 \left(\frac{1}{N} \sum_\mu \mathrm{tr}\left(\mathcal{U}_\mu(n)\overline{\mathcal{U}}_\mu(n)\right) - 1 \right)^2 . \tag{4.11}$$

Let us denote the Hermitian part of $\mathcal{A}_\mu$ as B_μ. In appendix C, we show that (4.11) provides a mass term for the abelian part of B, which protects the dynamical lattice spacing a from uncontrolled fluctuations.

Because of the complexification that was introduced above, the continuum theory has double the desired spectrum, so we must add additional mass terms to decouple the unwanted fields. Note that abelian part of the B field is already decoupled by the mass terms in (4.11). To decouple the non-abelian part of B, we add the following additional mass terms:

$$\mathcal{L}_B = m_B^2 \sum_\mu \mathrm{tr}\left[\left(\mathcal{U}_\mu(n)\overline{\mathcal{U}}_\mu(n) - 1/N \mathrm{tr}\left(\mathcal{U}_\mu(n)\overline{\mathcal{U}}_\mu(n)\right)\right)^2 \right] . \tag{4.12}$$

For the scalar fields ϕ and $\bar{\phi}$, we take the mass terms to be

$$\mathcal{L}_\phi = m_\phi^2 \mathrm{tr} \left| \phi(n)^\dagger - \bar{\phi}(n) \right|^2 . \tag{4.13}$$

If we break up the scalar fields into real components

$$\phi = \phi_R + i\phi_I, \quad \bar{\phi} = \bar{\phi}_R + i\bar{\phi}_I, \tag{4.14}$$

the mass term then takes the form

$$(\phi_R, \phi_I, \bar{\phi}_R, \bar{\phi}_I) \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_R \\ \phi_I \\ \bar{\phi}_R \\ \bar{\phi}_I \end{pmatrix} . \tag{4.15}$$

This matrix has two eigenvectors with eigenvalue 2:

$$\phi_I^+ = \phi_I + \bar{\phi}_I, \quad \phi_R^- = \phi_R - \bar{\phi}_R \tag{4.16}$$

and two eigenvectors with eigenvalue 0 (i.e., states that will survive in the low energy spectrum):

$$\phi_I^- = \phi_I - \bar{\phi}_I, \quad \phi_R^+ = \phi_R + \bar{\phi}_R. \tag{4.17}$$

Unlike the scalars, the real and imaginary parts of the fermionic fields ψ and χ do not transform covariantly under lattice gauge transformations. On the other hand, the

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following fermionic fields do transform covariantly and reduce to their real and imaginary parts in the continuum limit:

$$\begin{aligned}\psi_\mu^\pm(n) &= \psi_\mu(n) \pm U_\mu(n)\bar{\psi}_\mu(n)U_\mu(n) \\ \chi_{\mu\nu}^\pm(n) &= \chi_{\mu\nu}(n) \pm \bar{U}_\mu(n+e_\nu)\bar{U}_\nu(n)\bar{\chi}_{\mu\nu}(n)\bar{U}_\mu(n+e_\nu)\bar{U}_\nu(n),\end{aligned} \tag{4.18}$$

where repeated indices aren't summed. We also define

$$\eta^\pm(n) = \eta(n) \pm \bar{\eta}(n).$$

Note that $\bar{\eta}^\pm(n) = \pm\eta(n)$. Also note that the Hodge-duality constraint on χ translates into the following constraint relating light fields to heavy fields:

$$\chi_{\mu\nu}^+(n) - \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\bar{\chi}_{\rho\lambda}^+(n+e_\mu+e_\nu) = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\bar{\chi}_{\rho\lambda}^-(n+e_\mu+e_\nu) - \chi_{\mu\nu}^-(n).$$

We then define the following fermionic mass terms:

$$\begin{aligned}\mathcal{L}_\Psi = m_\Psi \big[&\bar{\psi}_\mu^-(n)\psi_\mu^-(n) + \bar{\chi}_{\mu\nu}^-(n)\chi_{\mu\nu}^-(n) \\ &+\eta^-(n)\left(\psi_\mu^-(n)\bar{U}_\mu(n) + U_\mu(n)U_\nu(n+e_\nu)\chi_{\mu\nu}^-(n) + \text{c.c.}\right)\big],\end{aligned} \tag{4.19}$$

where repeated indices are now summed over.

After writing the Lagrnagian in terms of the light and heavy matter fields defined above, one can integrate out the heavy fields at nonzero lattice spacing to obtain a gauge-invariant Lagrangian in terms of light matter fields. Although it is not necessary to obtain the correct continuum limit, we may also take the real part of the Lagrangian in (3.4) as this will ensure that the action is real at non-zero lattice spacing, which is more convenient for numerical calculations. It may also be interesting to simulate the complex action using recently developed Lefschetz thimble and complex Langevin techniques (see for example [23]).

All of the mass parameters in (4.12), (4.13), and (4.19) are of the same order, large compared to the dynamical scale Λ_{dyn} of 3d $\mathcal{N} = 4$ SYM. Since the mass terms cannot be obtained from Q-exact terms, they will violate this symmetry. Note that taking the real part of the Lagrangian in (3.4) also breaks lattice susy.[4] On the other hand, since the model is super-renormalizable this will just introduce a finite number of additional counter-terms up to two loops so renormalizing the theory will still be a managable task. Since the unwanted fields are only coupled to the $\mathcal{N} = 4$ SYM sector through gauge interactions and scalar interactions, they decouple according to the Appelquist and Carazzone theorem [24] and we are therefore left with the 3d $\mathcal{N} = 4$ field content at low energies. Because we take $m_i \gg \Lambda_{dyn}$, where $\Lambda_{dyn} \sim g^2$ is the scale of the target 3d gauge theory, the effective gauge coupling g^2/m_i is weak at the scale where they decouple, and this perturbative analysis of decoupling is reliable. We leave a detailed analysis of perturbative renormalization for future work.

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5 Conclusion

In this paper, we explore a new approach to formulating 3d $\mathcal{N} = 4$ SYM on a lattice. Starting with a complexifcation of the Donaldson-Witten twist of 4d $\mathcal{N} = 2$ SYM, we apply geometric discretization and find that lattice gauge invariance is only consistent with a certain Hodge-duality constraint on the lattice fermions if the basis vectors of the lattice are linearly dependent, implying that the model can live in at most three dimensions. Choosing the basis vectors to form a tetrahedron (or equivalently to span an A_3^* lattice), the resulting lattice gauge theory has an S_4 point symmetry, in contrast to the lattice formulation based on the Blau-Thompson twist which has an S_3 point symmetry group. We then analyze the renormalization of the lattice theory, enumerating marginal operators consistent with the lattice symmetries. Thanks to the super-renormalizability of the theory, the counter-terms that need to be fine-tuned in order to restore full supersymmetry in the continuum limit can be fixed perturbatively at one loop. Since our lattice model was based on a complexification of 3d $\mathcal{N} = 4$ SYM, we also propose to add mass terms in order to decouple the unwanted fields in the continuum limit.

The study of 3d $\mathcal{N} = 4$ lattice gauge theories is still in its infancy, so there are many important directions to explore:

- Perhaps the most immediate task is to analyze perturbative renormalization of the lattice theory along the lines of [25]. With these fine-tunings in hand, we will then be in a position to simulate the model on a computer and check the predictions of [18].

- It would be interesting to generalize our lattice model to incorporate matter multiplets along the lines of [13, 26] in order to investigate 3d mirror symmetry, whereby two different 3d $\mathcal{N} = 4$ gauge theories flow to the same superconformal fixed point in the IR [27]. Note that under mirror symmetry, Wilson loops are exchanged with vortex loops [28]. Since lattice gauge theory is well-suited for the computation of loop operators, it should provide a powerful tool for testing such 3d dualities.

- Note that our construction can be generalized to $d = 2$ by taking only two basis vectors of the lattice to be independent. It would therefore be interesting to study relation to the lattice formulations proposed in [10, 20] and invetigate 2d mirror symmetry [29].

- Using the gauge-invariant Hamiltonian formulation of Yang-Mills-Chern-Simons theories with $0 \leq \mathcal{N} \leq 4$ supersymmetry, it has been argued that a mass gap is present for $\mathcal{N} \leq 1$ and absent for extended supersymmetry [30, 31]. It would therefore be interesting to explore how to formulate non-abelian Chern-Simons theory on a lattice and couple it to our model in order to test these arguments and explore the existence of a mass gap. Note that a lattice formulation of abelian Chern-Simons theory was proposed in [32].

- Holographic duals of 3d $\mathcal{N} = 4$ superconformal field theories were proposed in [33]. Given that such theories arise as the IR fixed points of 3d $\mathcal{N} = 4$ gauge theory, it

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would be very interseting to use lattice techniques to test these proposals. Morover, it would be interesting to use lattice techniques to simulate certain 3d Euclidean Yang-Mills theories coupled to scalars and fermions which provide a holographic description of inflationary cosmology [34].

Ultimately, we hope that this work will provide a useful starting point for studying the non-pertubative dynamics of 3d $\mathcal{N} = 4$ gauge theories using lattice techniques, as well as many other important questions in quantum field theory and holography.

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A R-symmetry

In this appendix, we analyze the discrete R-symmetries of the Donaldson-Witten twist of 4d $\mathcal{N} = 2$ SYM. There are seven discrete R-charges. Four of them can be combined into a 1-form R_μ and the remaining three into a self-dual 2-form $R_{\mu\nu} = -R_{\nu\mu} = \star R_{\mu\nu}$. We will determine the discrete R-symmetries and deduce the parameter α in the twisted Lagrangian in (2.2) following the approach in [35]. The other seven supersymmetries can then be obtained by conjugating the nilpotent supersymmetry by the seven discrete R-symmetries. For convenience, we reproduce the Lagrangian in (2.2) below:

$$\begin{aligned} g^2 \mathcal{L}_{4d}^{\mathcal{N}=2} = & \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{1}{2}\mathcal{D}_\mu\bar{\phi}\mathcal{D}^\mu\phi - \alpha\left[\phi,\bar{\phi}\right]^2 \\ & - i\chi^{\mu\nu}\mathcal{D}_\mu\psi_\nu - \frac{i}{2}\eta\mathcal{D}_\mu\psi^\mu - \frac{i}{2}\bar{\phi}\left\{\psi_\mu,\psi^\mu\right\} + i\alpha\phi\left\{\eta,\eta\right\} + \frac{i}{8}\phi\left\{\chi_{\mu\nu},\chi^{\mu\nu}\right\}. \end{aligned} \tag{A.1}$$

A.1 R_μ

Let us make the following ansatz for the transformations generated by R_μ:

$$\begin{aligned} \eta &\to \beta_1\psi_\mu \\ \psi_\mu &\to \beta_1^{-1}\eta \\ \psi_\nu &\to \beta_2\chi_{\mu\nu} \\ \chi_{\mu\nu} &\to \beta_2^{-1}\psi_\nu \\ \chi_{\nu\rho} &\to \beta_2^{-1}\epsilon_{\mu\nu\rho\lambda}\psi_\lambda \end{aligned}$$

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$$\phi \to \bar{\phi}$$
$$\bar{\phi} \to \phi$$
$$\mathcal{A} \to \mathcal{A}$$

where $\nu, \rho \neq \mu$. Demanding that the Lagrangian in (A.1) is invariant under these transformations fixes the coefficients to be

$$\alpha = \frac{1}{8}, \quad \beta_1 = \pm 2i, \quad \beta_2 = \pm i.$$

A.2 $R_{\mu\nu}$

Let us make the following ansatz for the transformations generated by $R_{\mu\nu}$:

$$\begin{aligned}
\eta &\to \gamma_1 \chi_{\mu\nu} \\
\chi_{\mu\nu} &\to -\gamma_1^{-1} \eta \\
\psi_\mu &\to \gamma_2 \psi_\nu \\
\psi_\nu &\to -\gamma_2^{-1} \psi_\mu \\
\psi_\rho &\to \epsilon_{\mu\nu\rho\lambda} \psi^\lambda \\
\chi_{\mu\rho} &\to \gamma_3 \chi_{\nu\rho} \\
\chi_{\nu\rho} &\to -\gamma_3^{-1} \chi_{\mu\rho} \\
\chi_{\rho\lambda} &\to -\gamma_1^{-1} \epsilon_{\rho\lambda\mu\nu} \eta \\
\{\mathcal{A}, \phi, \bar{\phi}\} &\to \{\mathcal{A}, \phi, \bar{\phi}\}
\end{aligned}$$

where $\rho, \lambda \neq \mu, \nu$. Demanding that (A.1) is invariant under the above transformations fixes the coefficients to be

$$\alpha = \frac{1}{8}, \quad \gamma_1 = 2, \quad \gamma_2 = \gamma_3 = 1.$$

B Comparison of twists

In this appendix, we will compare the two twists of 3d $\mathcal{N} = 4$ SYM. Recall that prior to twisting, the fermions and scalars transform in the $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $(0, 1, 0)$ representation of the global symmetry group $\mathrm{SU}(2)_E \times \mathrm{SU}(2)_N \times \mathrm{SU}(2)_R$, respectively, where $\mathrm{SU}(2)_E$ is the group of Euclidean rotations, $\mathrm{SU}(2)_N$ is an internal symmetry group that has a 6d origin, and $\mathrm{SU}(2)_R$ is the R-symmetry group. Twisting by the R-symmetry group amounts to breaking $\mathrm{SU}(2)_E \times \mathrm{SU}(2)_R \to \mathrm{SU}(2)' = \mathrm{diag}\,(\mathrm{SU}(2)_E \times \mathrm{SU}(2)_R)$, after which the fields transform in the following representations of $\mathrm{SU}(2)' \times \mathrm{SU}(2)_N$:

$$\text{fermions}: \left(0, \frac{1}{2}\right) \oplus \left(1, \frac{1}{2}\right), \qquad \text{bosons}: (1, 0) \oplus (0, 1).$$

Note that this can be obtained from dimensionally reducing the Donaldson-Witten twist of 4d $\mathcal{N} = 2$ SYM, which is our starting point for defining the lattice theory. On the other hand, it is also possible to twist by $\mathrm{SU}(2)_N$ as shown by Blau-Thompson in [12]. Moreover, a lattice theory based on this twist was developed in [13]. Twisting by $\mathrm{SU}(2)_N$ breaks

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$\mathrm{SU}(2)_E \times \mathrm{SU}(2)_N \to \mathrm{SU}(2)' = \mathrm{diag}\,(\mathrm{SU}(2)_E \times \mathrm{SU}(2)_N)$ after which the fields transform in the following representations of $\mathrm{SU}(2)' \times \mathrm{SU}(2)_R$:

$$\text{fermions}: \left(0, \frac{1}{2}\right) \oplus \left(1, \frac{1}{2}\right), \quad \text{bosons}: (1,0) \oplus (1,0).$$

Note that in the Blau-Thompson twist, there are no bosonic scalars.

In the remainder of this appendix, we will derive the Lagrangians for the two twists of 3d $\mathcal{N} = 4$ SYM by dimensionally reducing 6d $\mathcal{N} = 1$ SYM, demonstrating that they indeed describe the same underlying theory even though their Lagrangians take different forms and realize BRST symmetry in different ways (whereas the BRST charge is nilpotent in the Blau-Thompson twist, it squares to a gauge transformation in the Donaldson-Witten twist). We will also show that the 3d Donaldson-Witten twist can be obtained by dimensionally reducing the Lagrangian in equation (2.2).

The Lagrangian for 6d $\mathcal{N} = 1$ SYM is given by

$$\mathcal{L}_{6d} = \mathrm{tr}\left(\frac{1}{4} F_{MN} F^{MN} - i \Psi_R^\dagger \Gamma^M D_M \Psi_L\right) \tag{B.1}$$

where $M = 1, \ldots, 6$, $D_M X = \partial_M X + [A_M, X]$, and $F_{MN} = [D_M, D_N]$. Since we are working in Euclidean signature, the Dirac matrices obey

$$\{\Gamma_M, \Gamma_N\} = \delta_{MN}.$$

Morover$\Psi_{L/R}$ are chiral spinors satisfying

$$\Gamma_7 \Psi_{L/R} = \pm \Psi_{L/R}$$

where $\Gamma_7 = \Gamma_1 \ldots \Gamma_6$. In the following we will present explicit formulas for the Dirac matrices as tensor products of the Pauli matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Blau-Thompson twist. Our derivation of the Blau-Thompson twist of 3d $\mathcal{N} = 4$ SYM will closely follow the original derivation in [12]. Consider the following representation of the Dirac matrices which naturally split 6d into $3 + 3$, making the $\mathrm{SU}(2)_E \times \mathrm{SU}(2)_N$ symmetry manifest:

$$\Gamma_{k=1,2,3} = \sigma_1 \otimes \sigma_0 \otimes \sigma_k, \quad \Gamma_{a=4,5,6} = \sigma_2 \otimes \sigma_{a-3} \otimes \sigma_0, \quad \Gamma_7 = \sigma_3 \otimes \sigma_0 \otimes \sigma_0.$$

In particular, $\mathrm{SU}(2)_E$ is generated by $\frac{1}{2}[\Gamma_k, \Gamma_l]$ with $k, l \in \{1, 2, 3\}$, and $\mathrm{SU}(2)_N$ is generated by $\frac{1}{2}[\Gamma_a, \Gamma_b]$ with $a, b \in \{3, 4, 5\}$. With this choice of Dirac matrices, the fermions can be written as follows:

$$\Psi_L = \begin{pmatrix} \psi^{\alpha\dot{\alpha}} \\ 0 \end{pmatrix}, \quad \Psi_R^\dagger = \begin{pmatrix} 0 & \chi^{\alpha\dot{\alpha}} \end{pmatrix},$$

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where $\alpha, \dot{\alpha}$ are spinor indices for $\mathrm{SU}(2)_E \times \mathrm{SU}(2)_N$. To dimensionally reduce to three dimensions, we take the fields to be independent of x^4, x^5, x^6, after which the covariant derivatives along the internal directions reduce to

$$D_a X \to [A_a, X]\,.$$

If we then twist by $\mathrm{SU}(2)_N$, this breaks $\mathrm{SU}(2)_E \times \mathrm{SU}(2)_N \to \mathrm{diag}\,(\mathrm{SU}(2)_E \times \mathrm{SU}(2)_N)$ which amounts to identifying the α and $\dot{\alpha}$ indices. It is then convenient to decompose the fermions into spin-1 and spin-0 parts as follows:

$$\psi^{\alpha\beta} = \psi^k\,(\sigma_k)^{\alpha\beta} + \epsilon^{\alpha\beta}\eta, \quad \chi^{\alpha\beta} = \chi^\mu\,(\sigma_k)^{\alpha\beta} + \epsilon^{\alpha\beta}\bar{\eta}.$$

After doing so, the fermionic terms in (B.1) reduce to

$$\mathcal{L}^f_{3d} = \mathrm{tr}\left(-2\chi^{kl}\mathcal{D}_k\psi_l - 2i\chi^k\bar{\mathcal{D}}_k\eta + 2i\bar{\eta}\bar{\mathcal{D}}_k\psi^k\right) \tag{B.2}$$

where we have defined $\chi_{kl} = \epsilon_{klm}\chi^m$. Morover, after dimenional reduction and twisting by $\mathrm{SU}(2)_N$, the internal components of A_M become a vector in three dimensions:

$$A_M = (A_k, A_a) \to (A_k, V_k)\,.$$

The bosonic terms in (B.1) then reduce to

$$\begin{aligned}\mathcal{L}^b_{3d} &= \mathrm{tr}\left(\frac{1}{4}F_{kl}^2 + \frac{1}{4}\,[V_k, V_l]^2 + \frac{1}{2}\,(D_k V_l)^2\right)\\ &= \mathrm{tr}\left(\frac{1}{4}\,(F_{kl} - [V_k, V_l])^2 + \frac{1}{4}\left(D_{[k}V_{l]}\right)^2 + \frac{1}{2}\left(D_k V^k\right)^2\right),\end{aligned}$$

where we used integration by parts to obtain the second equality. It it then convenient to define the following covariant derivatives:

$$\mathcal{D}_k X = \partial_k X + [(A + iV)_k\,, X]\,, \qquad \bar{\mathcal{D}}_k X = \partial_k X + [(A - iV)_k\,, X]\,,$$

in terms of which the bosonic terms in the Lagrangian can be written compactly as follows:

$$\mathcal{L}^b_{3d} = \mathrm{tr}\left(\frac{1}{4}\bar{\mathcal{F}}_{kl}\mathcal{F}^{kl} + \frac{1}{2}\left(D_k V^k\right)^2\right), \tag{B.3}$$

where $\mathcal{F}_{kl} = [\mathcal{D}_k, \mathcal{D}_l]$ and $\bar{\mathcal{F}}_{kl} = \left[\bar{\mathcal{D}}_k, \bar{\mathcal{D}}_l\right]$.

Donaldson-Witten twist. Our strategy for deriving the Donaldson-Witten twist of 3d $\mathcal{N} = 4$ SYM will be to first dimensionally reduce the 6d Lagrangian in (B.1) to 4d to make $\mathrm{SU}(2)_R$ manifest, and then dimensionally reduce to three dimensions and twist by $\mathrm{SU}(2)_R$. We therefore choose the following representation for the Dirac matrices which make the $4 + 2$ split manifest:

$$\Gamma^\mu = \sigma_0 \otimes \sigma_1 \otimes \sigma^{\mu-1}, \quad \Gamma^5 = \sigma_1 \otimes \sigma_3 \otimes \sigma_0, \quad \Gamma^6 = \sigma_2 \otimes \sigma_3 \otimes \sigma_0, \quad \Gamma^7 = -\sigma_3 \otimes \sigma_3 \otimes \sigma_0$$

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where $\mu = 1, 2, 3, 4$. Note that $\frac{1}{2}[\Gamma_\mu, \Gamma_\nu]$ generate the 4d Euclidean rotation group which is locally $\mathrm{SU}(2)_l \times \mathrm{SU}(2)_r$. For this choice of Dirac matrices, the fermions can be written as

$$\Psi_L = \begin{pmatrix} 0 \\ \lambda^\alpha \\ \bar{\omega}^{\dot{\alpha}} \\ 0 \end{pmatrix}, \qquad \Psi_R^\dagger = \begin{pmatrix} \bar{\lambda}^{\dot{\alpha}} & 0 & 0 & \omega^\alpha \end{pmatrix},$$

where $\alpha, \dot{\alpha}$ are spinor indices for $\mathrm{SU}(2)_l \times \mathrm{SU}(2)_r$. If we take the fields to be independent of the x^5, x^6 directions, (B.1) reduces to the following 4d Lagrangian:

$$\mathcal{L}_{4d} = \mathrm{tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi D^\mu\bar{\phi} - \frac{1}{8}\left[\phi,\bar{\phi}\right]^2 - i\bar{\psi}^{I\dot{\alpha}}D_{\alpha\dot{\alpha}}\psi_I^\alpha - \frac{i}{2}\left\{\psi^{I\alpha},\psi_{I\alpha}\right\}\bar{\phi} + \frac{i}{2}\left\{\bar{\psi}^{I\dot{\alpha}},\bar{\psi}_{I\dot{\alpha}}\right\}\phi\right), \tag{B.4}$$

where $\phi = A_5 - iA_6$, $\bar{\phi} = \phi^\dagger$, $D_{\alpha\dot{\alpha}} = \frac{1}{2}\sigma^\mu_{\alpha\dot{\alpha}}D_\mu$, and I is an SU(2) R-symmetry index; in particular the fermionic fields are given by

$$\psi^{I\alpha} = \sqrt{2}\left(\lambda^\alpha, \omega^\alpha\right), \quad \bar{\psi}^{I\dot{\alpha}} = \sqrt{2}\left(\bar{\lambda}^{\dot{\alpha}}, \bar{\omega}^{\dot{\alpha}}\right).$$

Next, we dimensionally reduce to three dimensions by taking the fields to be independent of x^4. Relabelling the fields as $\left(\phi, \bar{\phi}\right) = (B_1 + iB_2, B_1 - iB_2)$ and $A^4 = B_3$, the bosonic terms in the Lagrangian reduce to

$$\mathcal{L}_{3d}^b = \mathrm{tr}\left(\frac{1}{4}F_{kl}^2 + \frac{1}{2}\sum_{i=1}^{3}\left(D_k B_i\right)^2 + \frac{1}{2}\sum_{i<j}\left[B_i, B_j\right]^2\right). \tag{B.5}$$

Note that dimensional reduction to 3d breaks $\mathrm{SU}(2)_l \times \mathrm{SU}(2)_r \to \mathrm{SU}(2)_E = \mathrm{diag}\left(\mathrm{SU}(2)_l \times \mathrm{SU}(2)_r\right)$, which amounts to identifying the α and $\dot{\alpha}$ indices. If we then twist the 3d rotation group by $\mathrm{SU}(2)_R$, this breaks $\mathrm{SU}(2)_E \times \mathrm{SU}(2)_R \to \mathrm{diag}\left(\mathrm{SU}(2)_E \times \mathrm{SU}(2)_R\right)$ so the R-symmetry index I can be identified with the $\mathrm{SU}(2)_E$ index α. It is then convenient to decompose the fermions into spin-1 and spin-0 parts as follows:

$$\psi^{\alpha\beta} = \psi^k\left(\sigma_k\right)^{\alpha\beta} + \epsilon^{\alpha\beta}\psi^4, \quad \bar{\psi}^{\alpha\beta} = \chi^k\left(\sigma_k\right)^{\alpha\beta} + \frac{1}{2}\eta\epsilon^{\alpha\beta}.$$

After doing so, the fermionic terms in (B.4) become

$$\begin{aligned}\mathcal{L}_{3d}^f = \mathrm{tr}\Big(&-i\chi^{kl}D_k\psi_l - i\chi^k D_k\psi^4 + i\chi^k\left[B_3,\psi_k\right] - \frac{i}{2}\psi^k D_k\eta - \frac{i}{2}\psi^4\left[B_3,\eta\right] \\ &+\frac{i}{8}\phi\left\{\eta,\eta\right\} - \frac{i}{2}\bar{\phi}\left\{\psi^k,\psi_k\right\} - \frac{i}{2}\bar{\phi}\left\{\psi^4,\psi_4\right\} + \frac{i}{2}\phi\left\{\chi^k,\chi_k\right\}\Big),\end{aligned} \tag{B.6}$$

where we defined $\chi_{kl} = \epsilon_{klm}\chi^m$. The Lagrangian in equations (B.5) and (B.6) is precisely what we obtain after dimensionally reducing the twisted 4d Lagrangian in (2.2).

C Link field potential

Here we delve into details of the link field potential (4.11), which is of the same form as is used in 4d $\mathcal{N} = 4$ SYM, and thus well understood from those previous studies. To further

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understand this functional, we use the lattice spacing a to rescale the link fields to have canonical dimension 1, so that the potential then takes the form

$$\mathcal{V} = m_U^2 \sum_{\mu,x} \left(\frac{1}{N} \mathrm{tr} \overline{\mathcal{U}}_\mu \mathcal{U}_\mu - \frac{1}{a^2} \right)^2 \tag{C.1}$$

For this we explore the continuum limit in the linear formulation

$$\mathcal{U}_\mu(x) = \frac{1}{a} + A_\mu(x) + iB_\mu(x), \quad \overline{\mathcal{U}}_\mu(x) = \frac{1}{a} - A_\mu(x) + iB_\mu(x) \tag{C.2}$$

Here, B_μ are scalar fields that must be lifted to very large masses in order to obtain the target continuum theory, whereas A_μ contain the physical gauge fields. In addition, one linear combination of these also contains a third scalar field (in addition to $\phi, \bar{\phi}$) that should be retained in the low energy spectrum,

$$\phi_3 = \frac{1}{2} \sum_{\mu=1}^{4} A_\mu = \sum_{\mu=1}^{4} P_{4\mu} A_\mu \tag{C.3}$$

The latter notation indicates a projection. The actual gauge fields V_i are obtained from orthogonal projections:

$$V_i = \sum_{\mu=1}^{4} P_{i\mu} A_\mu, \quad i = 1, 2, 3 \tag{C.4}$$

where $\sum_\mu P_{\alpha\mu} P_{\beta\mu} = \delta_{\alpha\beta}$; $\alpha, \beta = 1, \ldots, 4$.

Multiplying everything out and tracing, throwing out the gauge fields V_μ, we obtain the quantity

$$\mathrm{tr} \mathcal{U}_\mu \overline{\mathcal{U}}_\mu = \frac{N}{a^2} - \frac{2\sqrt{N}}{a} B_a^0 + \frac{1}{4} \phi_3^A \phi_3^A + B_\mu^A B_\mu^A \tag{C.5}$$

where $A = 0, 1, \ldots, N^2 - 1$ correspond to the U(N) generators t^A which satisfy $\mathrm{tr} t^A t^B = -\delta^{AB}$. Note in particular that the U(1) generator is

$$t^0 = \frac{i}{\sqrt{N}} \tag{C.6}$$

Thus the scalar potential is

$$\mathcal{V} = m_U^2 \sum_{\mu,x} \left[\frac{4}{a^2 N} (B_\mu^0)^2 - \frac{4 B_\mu^0}{aN\sqrt{N}} \left(\frac{1}{4} \phi_3^A \phi_3^A + B_\mu^A B_\mu^A \right) + \frac{1}{N^2} \left(\frac{1}{4} \phi_3^A \phi_3^A + B_\mu^A B_\mu^A \right)^2 \right] \tag{C.7}$$

What we see is the following: the 4 U(1) scalars in B_μ^0 get a positive mass-squared term, driving them to zero at small field values. This is good because they would shift the lattice spacing if they had a nonzero vacuum expectation value. We see that at large field values where the quartic term dominates, there are no flat directions. This is highly desireable, because we do not want scalars wandering along flat directions during a simulation, but rather stuck near a point in moduli space. All six U(N) scalars are lifted and there is no runaway. At intermediate fields the cubic term plays an important role and there will be

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saddle points. At first the potential decreases as $\sum_\mu B_\mu^0$ is increased, but then finally the quartic term takes over to prevent runaway.

Next notice what happens in the limit of very small lattice spacing. The B_μ^0 mass term is dominant and lifts these modes from the low-energy spectrum. Also, these fields are driven to zero in the semi-classical approximation. The subdominant (in lattice spacing) cubic term is therefore driven to zero, and is negligible. If we scale m_U^2 (dimensionless) to smaller values as we decrease a, such that m_U^2/a^2 is nevertheless very large, then the quartic term is also negligible. In this way the non-mass potential terms do not disturb the scalar potential of the desired low energy theory.

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