

Time-varying Consumption Tax, Productive Government Spending, and Aggregate Instability*

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Abstract

In this paper we investigate if government balanced-budget rules together with endogenous taxation may lead to aggregate instability in an endogenous growth framework. After highlighting the differences with the exogenous growth framework, we prove that under counter-cyclical consumption taxes, while there exists a unique balanced growth path, sunspot equilibria based on self-fulfilling expectations occur through a form of global indeterminacy. In addition, we argue that this result is empirically plausible for a large set of OECD countries and that it may also emerge with endogenous income taxes.

JEL Classification C62, E32, H20, O41

Keywords Endogenous growth, time-varying consumption tax, global indeterminacy, self-fulfilling expectations, sunspot equilibria.

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1 Introduction

Balanced-budget rules recommendations to governments has been a recurrent debate after the starting of the last financial crisis, concerning mainly their consequences in terms of government debt sustainability. As shown by Schaechter *et al.* [26], in 2012 approximately 60 countries, mostly advanced, have adopted a type of balanced-budget rule either at the national or supra-national level. Balanced-budget rules can be implemented as overall balance, structural or cyclically adjusted balance, and balance “over the cycle”. Since the beginning of the Great Recession in 2008, the first type of rule has been debated throughout Europe and adopted by some countries like Germany or Switzerland as a “Golden Rule”.

But this type of balanced-budget rule has been criticized concerning its economic stabilization features. Since the paper of Schmitt-Grohé and Uribe [28], it is a well established fact that balanced-budget rules may lead to belief-driven aggregate instability and endogenous sunspot fluctuations. However, depending on the fiscal policy, aggregate instability occurs under different types of preferences. While it requires a large enough income effect when labor income taxes are considered (see Abad *et al.* [1]),¹ low enough income effects are necessary under consumption taxes (see Nourry *et al.* [25]). Such a conclusion has strong policy implications as for a given specification of preferences, one type of fiscal policy must be preferred to the other if the government is willing to avoid endogenous fluctuations. For instance, under a standard additively-separable utility function, Giannitsarou [13] suggests that consumption taxes must be favoured with respect to income or capital taxes as they reduce the possible occurrence of aggregate instability.

These results can be criticized in two dimensions. First, as clearly mentioned by Schmitt-Grohé and Uribe [28], they are partially based on the assumption that tax rates are not predetermined,² while taxes are in practice typically set in advance.³ Second, they are established within stationary models without long-run growth. The aim of this paper is to revisit the issue of aggregate instability coming from balanced-budget rules focusing on consumption taxes compatible with endogenous

¹Actually, local indeterminacy requires that consumption and labor are Edgeworth substitutes or weak Edgeworth complements (see also Linnemann [19]). These properties are associated to a Jaimovich-Rebelo [16] utility function characterized by a large enough income effect.

²The initial value of the tax rate is indeed a function of a forward variable (*i.e.*, consumption or labor).

³It is however claimed in Schmitt-Grohé and Uribe [28] that their main conclusions are robust to the consideration of a discrete-time reformulation of their model with tax rates set $k \geq 1$ periods in advance to that in each period $t \geq 0$, the tax rates for periods $t, \dots, t+k-1$ are pre-determined.

growth. In practice, this requirement implies that the tax rate depends on de-trended consumption to have a constant tax on a balanced growth path. As a consequence, we consider a time-varying consumption tax which is a predetermined variable and we are thus able to solve the two main weaknesses of the standard literature.

We consider a standard neoclassical growth model augmented with a government that provides a constant stream of expenditures financed through consumption taxes and a balanced budget rule. Endogenous growth is obtained from assuming a Barro-type [3] production function in which government spending acts as an external productive input. In order to have a constant tax on a balanced growth path, the tax rate needs to depend on de-trended consumption and thus becomes a state variable with a given initial condition. Finally, we consider a representative household characterized by a CRRA utility function and inelastic labor. Such a formulation is known to rule out the existence of endogenous fluctuations in exogenous growth models (see Giannitsarou [13]).

We first prove that there exists a unique Balanced Growth Path (BGP) along which the common growth rate of consumption, capital, GDP and government spending is constant. The particularity of such a BGP is that the equilibrium tax rate is just equal to its initial value. A consequence of this property is that, as in the Barro [3] model, there is no transitional dynamics with respect to this unique unstable BGP and, therefore, there exists a unique initial choice of consumption such that the economy evolves along its BGP. This conclusion is thus similar to the one reached by Giannitsarou [13]: there is a priori no room for endogenous fluctuations.

However, we can prove that the BGP is not the unique long run solution of our model. Indeed, if the tax rule is counter-cyclical with respect to consumption, for any arbitrary initial value of the tax rate, close enough to its initial condition, there exists a corresponding value for the tax rate, consumption, capital and the constant growth rate that can be an asymptotic equilibrium of our economy, namely an Asymptotic Balanced Growth Path (ABGP). An ABGP is not itself an equilibrium as it does not respect the initial conditions. However we prove that some transitional dynamics exist with a unique equilibrium path converging toward this ABGP. Moreover, we show that there exist a continuum of such ABGP and of equilibria each of them converging over time to a different ABGP.⁴

The existence of an equilibrium path converging to an ABGP is associated to the

⁴On the other hand, we can prove that if the consumption tax is procyclical then the BGP is the unique equilibrium path.

existence of consumers' beliefs that are different from those associated to the BGP. Indeed, they may believe that the consumption tax profile will not remain constant but rather change over time and eventually converge to a positive value different from the initial condition. A specific form of global indeterminacy emerges since from a given initial tax rate, the representative agent can choose an initial consumption to be immediately on the unique BGP or alternatively an initial consumption consistent with any other equilibrium converging to an ABGP. Again different choices reveal different consumers' beliefs of the long run outcome of the economy.

Because this specific form of global indeterminacy is fundamentally related to expectations, one may wonder about the possible existence of sunspot equilibria and endogenous fluctuations based on self-fulfilling beliefs. To this purpose, we adapt existing results (e.g. Shigoka [29], Benhabib *et al.* [7], Cazzavillan [9]) and we show that sunspot equilibria can be obtained by randomizing over the deterministic equilibria converging to the ABGPs. From an analytical viewpoint, we assume that the sunspot variable is a continuous time homogenous Markov chain and we use the generator of the chain as proposed by Grimmett and Stirzaker [14] to prove the existence of sunspot equilibria. We then conclude that in an endogenous growth framework, contrary to the conclusions of Giannitsarou [13], endogenous sunspot fluctuations may arise under a balanced-budget rule and consumption taxes although there exists a unique underlying BGP equilibrium. It is also worth noting that contrary to Drugeon and Wigniolle [11] or Nishimura and Shigoka [24], our methodology allows to prove the existence of sunspots in a non-stationary economic environment while the steady state (BGP) is unstable.

Our results can be compared to some recent conclusions provided by Angeletos and La'O [2] and Benhabib *et al.* [5] within infinite horizon models with sentiments. They show that endogenous fluctuations, based on a certain type of extrinsic shocks called "sentiments", can be accommodated in unique-equilibrium, rational-expectations, macroeconomic models like those in the RBC/DSGE paradigm provided there is some mechanism that prevents the agents from having identical equilibrium expectations. Of course, our framework is still based on the existence of externalities as we need to generate a form of Ak technology to get endogenous growth. But, contrary to the standard literature which is based on the existence of local indeterminacy (see Benhabib and Farmer [4]), we find sunspot fluctuations while there exists a unique deterministic BGP without transitional dynamics. The existence of the continuum of ABGPs and of equilibrium paths converging to these, is also fundamentally based on the expectations of agents.

The rest of the paper is organized as follows. Section 2 presents the firms and

households' behaviors. Section 3 discusses fiscal policies under balanced-budget rules comparing exogenous and endogenous growth models and presents the consumption tax rule that we consider in the paper. Section 4 defines the intertemporal equilibrium. Section 5 proves the existence and uniqueness of a BGP. In Section 6 its stability is investigated and it is also shown that depending on agents' expectations, there may exist a continuum of other equilibria that converge toward some ABGPs. Based on this conclusion, we prove in Section 7 that sunspot equilibria and endogenous fluctuations based on self-fulfilling beliefs occur and we show through a numerical exercise that the existence of aggregate instability and sunspot fluctuations driven by consumption tax rates is empirically plausible for a large set of OECD countries. Section 8 proves the robustness of our results showing that the same conclusions can be obtained under income taxes. Section 9 contains a conclusion and all the proofs are provided in a final Appendix.

2 Description of the private sector

In this section the endogenous growth model originally developed by Barro [3] is modified by assuming that the government levies a time-varying consumption tax to finance its spending. In this economy the government spending is productive since it is a public good provided by the government to the firms which use it as an essential input of production. For this reason, our paper is different from Giannitsarou [13] and Nourry *et al.* [25] where the government spending is just a pure waste of resources. As in Barro [3], productive government spending is the source of endogenous growth in our model.

2.1 Firms

A representative firm produces the final good y using a Cobb-Douglas technology with constant returns at the private level but which is also affected by a public good externality, $y = Ak^\alpha(L\mathcal{G})^{1-\alpha}$, where $\alpha \in (0, 1)$ is the share of capital income in GDP, \mathcal{G} is the per capita quantity of government purchases of goods and services and A is the constant TFP. We assume that population is normalized to one, $L = 1$, so that we get a standard Barro-type [3] formulation such that $y = Ak^\alpha\mathcal{G}^{1-\alpha}$. Profit maximization then respectively gives the rental rate of capital and the wage rate:

$$r = A\alpha \left(\frac{k}{\mathcal{G}}\right)^{\alpha-1}, \quad w = A(1 - \alpha)k \left(\frac{k}{\mathcal{G}}\right)^{\alpha-1} \quad (1)$$

2.2 Households

We consider a representative household endowed with a fixed amount of labor and an initial stock of private physical capital which depreciates at rate $\delta > 0$. His instantaneous utility function is consistent with endogenous growth and given by

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma} \quad (2)$$

with $\sigma > 0$ the inverse of the elasticity of intertemporal substitution in consumption.

The representative household derives income from wage and capital. Denoting $\tau > 0$ the tax rate on consumption, his budget constraint is given by:

$$(1 + \tau)c + \dot{k} = rk + w - \delta k \quad (3)$$

with r and w as given by (1).

The representative household then solves the following problem taking as given the prices r and w , and the time-varying path of τ :

$$\begin{aligned} \max \quad & \int_0^\infty \frac{c^{1-\sigma}}{1-\sigma} e^{-\rho t} dt \\ \text{s.t.} \quad & \dot{k} = rk + w - \delta k - (1 + \tau)c \\ & k \geq 0, \quad c \geq 0 \\ & k(0) = k_0 > 0 \text{ given} \end{aligned}$$

where the set of admissible parameters is so defined

$$\Theta \equiv \{(\alpha, \rho, \delta, \sigma, A) : \alpha \in (0, 1), \rho > 0, \delta > 0, \sigma > 0 \text{ and } A > 0\}.$$

The current value Hamiltonian associated to this problem is

$$\mathcal{H} = \frac{c^{1-\sigma}}{1-\sigma} + \lambda[rk + w - \delta k - (1 + \tau)c]$$

where λ is the utility price of the final good. Considering (1), the first order conditions with respect to the control, c , and the state, k , write respectively

$$c^{-\sigma} = \lambda(1 + \tau) \quad (4)$$

$$-\frac{\dot{\lambda}}{\lambda} = \alpha A \left(\frac{k}{\mathcal{G}}\right)^{\alpha-1} - \delta - \rho \quad (5)$$

Differentiation of equation (4) gives

$$\frac{\dot{c}}{c} = -\frac{1}{\sigma} \left[\frac{\dot{\lambda}}{\lambda} + \frac{\dot{\tau}}{1 + \tau} \right] \quad (6)$$

Let us then substitute equation (5) into (6). It follows that, given an initial capital stock k_0 , the tax and government spending path $(\tau(t), \mathcal{G}(t))_{t \geq 0}$, the representative household maximizes his/her utility by choosing any path $(c(t), k(t))_{t \geq 0}$ which solves the system of ODEs

$$\dot{k} = Ak^\alpha \mathcal{G}^{1-\alpha} - \delta k - (1 + \tau)c \quad (7)$$

$$\frac{\dot{c}}{c} = \frac{1}{\sigma} \left[\alpha A \left(\frac{k}{\mathcal{G}} \right)^{\alpha-1} - \delta - \rho - \frac{\dot{\tau}}{1 + \tau} \right] \quad (8)$$

respects the positivity constraints $k \geq 0$, $c \geq 0$, and the transversality condition

$$\lim_{t \rightarrow +\infty} \frac{k}{c^\sigma (1 + \tau)} e^{-\rho t} = 0 \quad (9)$$

3 Balanced-budget rule and fiscal policy

This section is organized as it follows. First, we emphasize the deep difference between exogenous and endogenous growth models. In particular, we show that global indeterminacy emerges naturally in an endogenous growth model if the government balances its budget in each period and the tax rate is endogenous and time-varying as usually assumed in exogenous growth models (e.g. Giannitsarou [13]). This result does not depend on the type of taxation and, for example, it still holds with an income tax.

Secondly, we argue that a natural attempt to rule out this global indeterminacy consists in introducing a fiscal policy rule. Therefore, we design a fiscal policy rule such that the tax rate is still endogenous, time-varying and also predetermined. Later, and specifically in Section 7, we prove under which conditions the fiscal policy rule is indeed effective in ruling out global indeterminacy and therefore aggregate instability. To do so, we will have to investigate first all the possible equilibria which may exist.

3.1 Exogenous vs endogenous growth models

In exogenous growth models where labor supply is endogenous and government spending is unproductive, local indeterminacy may emerge if the government balances its budget in each period and the tax on labor income is time-varying and endogenous (e.g. Schmitt-Grohe and Uribe [28]). Formally, local indeterminacy may emerge under the following balanced-budget rule $\mathcal{G} = \tau wL$ where \mathcal{G} is usually

exogenously given and possibly time-varying.⁵ Observe that the tax rate is endogenous and time-varying because it must adjust in each period to balance the budget. In the same framework, changing the labor income tax with a consumption tax rules out local indeterminacy when the utility function is CES (see Giannitsarou [13]).

Let us now investigate what happens to the Barro's model if the tax rate is allowed to be time-varying and endogenous and the government balances its budget in each period. Similarly to Giannitsarou [13], the balanced-budget rule is

$$\mathcal{G} = \tau c. \quad (10)$$

Then the dynamics of the economy can be described by combining the Euler equation, the capital accumulation equation and the balanced-budget rule. In particular the following equation in the variable τ and the new variable $z \equiv \frac{k}{\bar{g}}$ describes the dynamics of the economy:

$$\frac{\dot{z}}{z} + \frac{\dot{\tau}}{\tau} = (1 - \alpha)Az^{\alpha-1} + \rho - \frac{\dot{\tau}}{\tau(1 + \tau)} - \frac{1 + \tau}{\tau} \frac{1}{z} \quad (11)$$

Therefore, the model is globally indeterminate because equation (11) is underdetermined (i.e. there are more variables than equations). Intuitively different households' beliefs about the evolution of the tax rate can be self-fulfilled by choosing opportunely different paths of the capital-government ratio.

A natural attempt to avoid this form of pervasive indeterminacy consists in specifying a fiscal policy rule. In fact, with a fiscal policy rule we may circumvent the underdetermined issue previously explained. The next section is aimed at designing a fiscal policy rule.

3.2 Designing a fiscal policy rule

The objective of this section is to design a fiscal policy rule such that the tax rate is:

- i) endogenous,
- ii) time-varying but constant in the long run and
- iii) predetermined.

⁵An exception is considered in the quantitative analysis performed by Schmitt-Grohe and Uribe [28] where they assume that \mathcal{G} is endogenous and depends on the level of income.

Features *i)* and *ii)* are necessary to be consistent with the existing literature on aggregate instability. The last feature is also desirable because tax rates are typically set in advance (see for example Schmitt-Grohe and Uribe [28] - page 993).

Consistently with the households problem we need to specify a fiscal policy rule for the consumption tax. In Section 8 we will also provide an insight on what happens if there is an income tax instead of a consumption tax. We assume that the fiscal policy rule requires that the tax rate is a function of detrended consumption, i.e. $\tau = \tau(\tilde{c})$, and therefore the government balanced-budget rule becomes:

$$\mathcal{G} = \tau(\tilde{c})c \quad (12)$$

where $\tilde{c} \equiv ce^{-\gamma t}$ indicates de-trended consumption with γ the (endogenous) asymptotic and constant growth rate of the economy.⁶ The fiscal instrument τ is clearly time-varying and endogenously determined; in fact, it is similar to the fiscal policy suggested by Nourry *et al.* [25] among others. In particular, Nourry *et al.* [25] study, in an exogenous growth model, the case $\mathcal{G}(c) = \tau(c)c$ while we need that the tax rate depends on de-trended consumption to have a constant tax on a balanced growth path.

In addition, we will also assume, from now on, the following:

Assumption 1. *The elasticity of the tax rate with respect to de-trended consumption is constant and given by*

$$\phi \equiv \frac{d\tau}{d\tilde{c}} \frac{\tilde{c}}{\tau} \quad (13)$$

It is worth noting that such a restriction is common in the literature. In their seminal contribution, Schmitt-Grohé and Uribe [28] consider a tax on labor income with constant government spending such that $\tau(wl) = \mathcal{G}/wl$ which has a constant elasticity with respect to its tax base equal to -1 . The same property is assumed by Giannitsarou [13] with a consumption tax satisfying $\tau(c) = \mathcal{G}/c$. In Nourry *et al.* [25] however, the government spending is assumed to vary with consumption and the elasticity of the tax rate $\tau(c) = \mathcal{G}(c)/c$ is equal to $\eta - 1$ with η the constant elasticity of government spending with respect to consumption.

⁶More precisely, γ is the (constant) growth rate if the economy is on a BGP or is the asymptotic (constant) growth rate if the economy is not on a BGP but converges over time to an asymptotic BGP (see Definition 3). In fact, at this stage of the analysis, we cannot exclude *a priori* the existence of a subset of initial conditions such that the economy is not on a BGP at $t = 0$ but rather converges to it over time as it happens, for example, in an endogenous growth model with a Jones and Manuelli [17] production function.

As a consequence of Assumption 1 we have that

$$\dot{\tau} \equiv \frac{d\tau}{dt} = \frac{d\tau}{d\tilde{c}} \dot{\tilde{c}} = \frac{d\tau}{d\tilde{c}} \tilde{c} \left(\frac{\dot{\tilde{c}}}{\tilde{c}} - \gamma \right) \quad (14)$$

and therefore

$$\frac{\dot{\tau}}{\tau} \equiv \phi \left(\frac{\dot{\tilde{c}}}{\tilde{c}} - \gamma \right) \quad (15)$$

Integrating (15) leads to

$$\tau(t) \equiv B \left(c(t) e^{-\gamma t} \right)^\phi \quad (16)$$

with B a generic (and endogenously determined) constant. Therefore, the last expression (16) is rather a menu of fiscal policies. To select just one of them (and avoiding in this way to introduce a trivial form of indeterminacy in the model) we assume that $\tau(0) = \tau_0 > 0$ is exogenously given. By doing so, we may find the value of B and observe that the last expression, and therefore the fiscal rule (15), is equivalent to

$$\tau(t) \equiv \tau_0 \left(\frac{\tilde{c}(t)}{c_0} \right)^\phi = \tau_0 \left(\frac{c(t)}{c_0 e^{\gamma t}} \right)^\phi \quad (17)$$

where $B = \tau_0/c_0^\phi$. Clearly the tax rate, τ , is a predetermined variable, in the sense that the initial value τ_0 is given, which is consistent with the fact that tax rates are typically set in advance and it seems even more compelling in our model where the tax base is not predetermined since it depends on consumption. It is also worth to underline that identities (15) and (17) are a direct consequence of Assumption 1. Note that this formulation is consistent and indeed includes, under the restriction $\phi = 0$, the case of a constant and exogenously given tax rate $\tau = \tau_0$ briefly mentioned by Barro [3]. Moreover, the fiscal rule (15) is pro(counter)-cyclical if $\phi > 0$ ($\phi < 0$) since it increases (decreases) when consumption grows faster (slower) than γ .⁷

It is also worth noting that while our formulation is very similar to the one of Nourry *et al.* [25], there is a strong qualitative difference: here we postulate a specific form of the tax function as given by (17) and government spending adjusts accordingly along the balanced-budget rule (12), while in Nourry *et al.* [25] the government spending rule is postulated as in (10) and the tax rate adjusts accordingly since $\tau(c(t)) = \mathcal{G}/c(t)$. In this case, the tax rate is not predetermined as $\tau(0) = \mathcal{G}/c(0)$.

Before concluding this section, we notice that one could be tempted to assume an exogenous target value for γ . This would lead to two undesirable consequences:

⁷The definition of pro(counter)-cyclical is based on a comparison of the growth rates. This is consistent with the real business cycle literature where an economy is said to be in recession if it grows more slowly than at its trend.

first, the model becomes an exogenous growth model since it emerges immediately from the fiscal rule that the growth rate of consumption (and therefore of capital) will not be anymore determined, as usual in an endogenous growth model, by a combination of parameters but rather by the target value itself otherwise the tax rate will be growing in the long run.⁸ Secondly it can be easily proved that an equilibrium path will exist only for a zero-measure set of parameters.

4 Intertemporal equilibrium

Given an initial condition of capital $k_0 > 0$ and of the consumption tax $\tau_0 > 0$, an intertemporal equilibrium is any path $(c(t), k(t), \tau(t), \mathcal{G}(t))_{t \geq 0}$ which satisfies the system of equations (7), (8), (12) and (15), respects the inequality constraints $k \geq 0$, $c \geq 0$, and the transversality condition (9). Put differently, equations (7)-(9) with (17) then provide a set of necessary and sufficient conditions for an intertemporal equilibrium starting from a given pair (k_0, τ_0) .

Therefore we may define the control-like variable $x \equiv \frac{c}{k}$ and observe that the intertemporal equilibrium can be derived studying the following system of nonlinear ODEs in the variables (x, τ) :

$$\begin{aligned} \frac{\dot{x}}{x} &= \frac{[(1+\tau)(1-\sigma) - \phi\tau] [\alpha A(x\tau)^{1-\alpha} - \delta - \rho - \sigma\gamma]}{\sigma(1+\tau) + \phi\tau} \\ &+ \gamma(1-\sigma) + (1+\tau)x - (1-\alpha)A(x\tau)^{1-\alpha} - \rho \end{aligned} \quad (18)$$

$$\frac{\dot{\tau}}{\tau} = \frac{\phi(1+\tau)}{\sigma(1+\tau) + \phi\tau} [\alpha A(x\tau)^{1-\alpha} - \delta - \rho - \sigma\gamma] \quad (19)$$

The interested reader may find in Appendix A the detailed procedure to obtain this system starting from equations (7), (8), (12) and (15). It is also worth noting that x_0 is not predetermined since it depends on c_0 while τ_0 is exogenously given and therefore predetermined.

5 Balanced growth paths

A balanced growth path (BGP) is an intertemporal equilibrium where consumption, and capital are purely exponential functions of time t , namely:

$$k(t) = k_0 e^{\gamma t} \quad \text{and} \quad c(t) = c_0 e^{\gamma t} \quad \forall t \geq 0. \quad (20)$$

⁸To see this point even more explicitly, observe that the fiscal rule could be rewritten as $\tau(t) = \tau_0 (c(t)/z(t))^\phi$ with $z(t) = c_0 e^{\bar{\gamma} t}$. Then all the aggregate variables will grow at the rate of the variable $z(t)$ whose growth rate $\bar{\gamma}$ has been given exogenously.

From equation (15) it follows immediately that along a BGP the consumption tax is constant and equal to

$$\tau(t) = \hat{\tau} = \tau_0 \quad \forall t \geq 0$$

with the hat symbol indicating, from now on, the value of a variable on a BGP. Along the BGP, the tax rate is therefore constant and equal to its initial value. Also, government spending will be purely exponential with a growth rate equal to γ consistently with the balanced-budget rule (12). Therefore, equations (18) and (19) rewrite

$$0 = (\alpha - \sigma)A(x\hat{\tau})^{1-\alpha} + \sigma(1 + \hat{\tau})x - \rho - \delta(1 - \sigma) \equiv g(x) \quad (21)$$

$$\gamma = \frac{1}{\sigma} [\alpha A(x\hat{\tau})^{1-\alpha} - \delta - \rho]. \quad (22)$$

Studying the zeros of equation (21) is the necessary step to prove existence and uniqueness of a balanced growth path. Moreover, along a BGP, the transversality condition (9) becomes

$$\lim_{t \rightarrow +\infty} \frac{k_0}{c_0^\sigma (1 + \tau_0)} e^{-[\rho - \gamma(1 - \sigma)]t} = 0 \quad (23)$$

It follows that along a BGP, condition (23) holds if and only if $\rho - \gamma(1 - \sigma) > 0$. As we restrict our analysis to endogenous positive long-run growth,⁹ any value of γ solution of equation (22) needs to satisfy $\gamma \geq 0$ when $\sigma \geq 1$ and $\gamma \in [0, \rho/(1 - \sigma))$ when $\sigma < 1$.

Proposition 1 (Existence and Uniqueness of a BGP). *Given any initial condition of capital $k_0 > 0$ and the tax rate $\tau_0 > 0$, there exist $\underline{A} > 0$, $\underline{\tau} > 0$ and $\bar{\tau}(\sigma) \in (0, +\infty]$ with $\bar{\tau}(\sigma) > \underline{\tau}$ such that when $A > \underline{A}$ and one of the following conditions holds:*

- i) $\sigma \geq 1$ and $\tau_0 > \underline{\tau}$,
- ii) $\sigma \in (0, 1)$ and $\tau_0 \in (\underline{\tau}, \bar{\tau}(\sigma))$,

there is a unique balanced growth path where the ratio of consumption over capital is constant and equal to \hat{x} – the unique positive root of equation (21) – and the growth rate of the economy is

$$\hat{\gamma} = \frac{1}{\sigma} [\alpha A(\hat{x}\hat{\tau})^{1-\alpha} - \delta - \rho] > 0, \quad (24)$$

with $\hat{\tau} = \tau_0$.

⁹We could indeed also focus on negative growth rates $\gamma < 0$ implying to consider a long-run values for capital and consumption equal to zero. As these solutions are ruled out by the Inada conditions satisfied by the utility and production functions specifications, we do not consider such possibility.

Proof. See Appendix A.2. ■

Discussion of these conditions is in order. The requirements of a level of technology greater than \underline{A} and of a tax rate larger than $\underline{\tau}$ guarantee positive economic growth. The first one is indeed a condition similar to the one in the AK model while the second one allows to provide a large enough government spending to sustain growth through the technology $y = Ak^\alpha \mathcal{G}^{1-\alpha}$. Also the condition of a tax rate lower than $\bar{\tau}(\sigma)$ when $\sigma < 1$ guarantees that the transversality condition is respected and therefore the utility is bounded. Therefore, Proposition 1 shows that if $A > \underline{A}$ and one of the conditions $i) - ii)$ holds, a unique BGP exists. Of course, uniqueness depends on the existence of a unique value \hat{x} which implies a unique specification of initial consumption for any exogenously given initial condition of the capital stock and consumption tax. For any given k_0 and τ_0 , we have indeed $c_0 = k_0 \hat{x}$ and $\tau(\tilde{c}) = \tau_0$ so that the stationary value of de-trended consumption \tilde{c} corresponding to the BGP is derived from (20) and such that $\tilde{c} = c_0 = k_0 \hat{x}$.

Clearly the value of the positive real zero \hat{x} and of $\hat{\gamma}$ depend on the exogenously given parameters but also on the initial condition of the consumption tax τ_0 .¹⁰ For this reason we may explicitly write \hat{x} and $\hat{\gamma}$ as continuous and differentiable functions of these values, i.e. $\hat{x} = \hat{x}(\alpha, \tau_0, \rho, \delta, \sigma)$ and $\hat{\gamma}(\alpha, \tau_0, \rho, \delta, \sigma)$. Given a generic capital stock k_0 , the balanced growth path is

$$\hat{k} = k_0 e^{\hat{\gamma}(\alpha, \tau_0, \rho, \delta, \sigma)t} \quad \text{and} \quad \hat{c} = \hat{x}(\alpha, \tau_0, \rho, \delta, \sigma) k_0 e^{\hat{\gamma}(\alpha, \tau_0, \rho, \delta, \sigma)t}$$

where the growth rate is positive if and only if the conditions of Proposition 1 hold. For example, it is easy to check numerically that given $\hat{\tau} = \tau_0 = 0.2$, the following parameter's values, $\alpha = 1/3$, $\rho = 0.01$, $\delta = 0.025$ and $A = 2.3$, imply $\underline{\tau} = 0.139$ and $(\hat{x}, \hat{\gamma}) = (0.143, 0.0184)$ when $\sigma = 2$, $(\hat{x}, \hat{\gamma}) = (0.106, 0.0239)$ when $\sigma = 1$, and $(\hat{x}, \hat{\gamma}) = (0.083, 0.025)$ with $\bar{\tau}(\sigma) = 0.2$ when $\sigma = 0.6$.

It is also worth noting that at the BGP the consumption tax is constant and therefore not distorting (i.e. lump-sum). Therefore the BGP analysis just done is identical to an economy where the consumption tax is set to be constant over time.

We conclude this section with some comparative statics results that provide sufficient conditions for the growth rate $\hat{\gamma}$ and welfare to be increasing functions of the tax rate $\tau_0 = \hat{\tau}$. Indeed, we can easily compute welfare along the BGP characterized by the stationary values of the growth rate $\hat{\gamma}$ and the ratio of consumption over

¹⁰The fact that the taxation enters in the equation of the consumption-capital ratio and therefore affects also the growth rate is not surprising and consistent with previous contributions (e.g. Barro [3]).

capital \hat{x} , namely

$$W(\hat{\gamma}, \hat{x}) = \frac{(\hat{x}k_0)^{1-\sigma}}{(1-\sigma)[\rho - \hat{\gamma}(1-\sigma)]} \quad (25)$$

We then get the following result:

Corollary 1. *Let the conditions of Proposition 1 hold. There exist $A_{min} > \underline{A}$ and $\underline{\sigma} > 0$ such that when $\tau_0 < (1-\alpha)/\alpha$, $\sigma > \underline{\sigma}$ and $A > A_{min}$, then*

$$\frac{d\hat{x}}{d\tau_0} > 0, \quad \frac{d\hat{\gamma}}{d\tau_0} > 0 \quad \text{and} \quad \frac{dW(\hat{\gamma}, \hat{x})}{d\tau_0} > 0$$

Proof. See Appendix A.3. ■

Obviously, as shown by expression (25), along the BGP welfare is an increasing function of both the growth rate $\hat{\gamma}$ and the ratio of consumption over capital \hat{x} . Because of the public good externality in the production function, a large growth factor allows to generate an increasing amount of public good which improves the aggregate production level and thus consumption. Corollary 1 then provides conditions for a positive impact of the tax rate τ_0 on the growth rate, consumption over capital and welfare. In particular, such a conclusion requires a low enough elasticity of intertemporal substitution in consumption $1/\sigma$ which prevents a too large consumption smoothing over time in order to ensure a larger consumption in the long run, i.e. along the BGP.

6 Transitional dynamics

6.1 Local determinacy of the steady state $(\hat{x}, \hat{\tau})$

In this section we start by investigating the local stability properties of the steady state $(\hat{x}, \hat{\tau})$ (with $\hat{\tau} = \tau_0$) which characterizes the unique BGP of our economy. Let us recall that in the formulation considered by Barro [3] where the tax rate τ is constant, there is no transitional dynamics and the economy directly jumps on the BGP from the initial date $t = 0$. In our framework we get similar conclusions:

Proposition 2. *Consider the steady state $(\hat{x}, \hat{\tau})$ (with $\hat{\tau} = \tau_0$) which characterizes the unique BGP of our economy. For any given initial conditions (k_0, τ_0) , there is no transitional dynamics, i.e. there exists a unique $c_0 = k_0\hat{x}$ such that the economy directly jumps on the BGP from the initial date $t = 0$.*

Proof. See Appendix A.4. ■

As shown in the proof of Proposition 2, the steady state $(\hat{x}, \hat{\tau})$ is locally saddle-path stable if and only if $\phi \in (-\sigma(1 + \hat{\tau})/\hat{\tau}, 0)$ and locally unstable otherwise.¹¹ However, in both cases, we find the same conclusion as in Barro [3]: there is no transitional dynamics with respect to the BGP as any initial choice of $c(0)$ different from $c_0 = k_0\hat{x}$ leads to trajectories diverging from $(\hat{\tau}, \hat{x})$. This is not really surprising since the tax rate on the BGP is exactly $\hat{\tau} = \tau_0$ which is the initial condition of the state variable of our problem.

Note however that the reasons for the absence of transitional dynamics in our framework is different than the ones in Barro [3]. In his model, τ is constant by definition and so is the consumption growth rate \dot{c}/c . In our case, the tax rate can a priori exhibit counter or procyclicality and there is no transitional dynamics because of the local stability properties of the BGP.

To make this argument more explicit, consider Figure 1 which illustrates Proposition 2 and shows the phase diagrams when the parameters are set as in the previous section. The initial conditions are $k_0 = 1$ and $\tau_0 = 0.2$, $\sigma = 1$ and ϕ is equal to 0.5 (left diagram) and -0.01 (right diagram). According to the directions of the arrows it is clear that in both phase diagrams, any choice of $x_0 \neq \hat{x}_0$ along the vertical line $\hat{\tau} = \tau_0$ leads to paths which cannot converge to $(\hat{x}, \hat{\tau})$. In this case we have local determinacy of the steady state $(\hat{x}, \hat{\tau})$ since given any k_0 and τ_0 satisfying the conditions of Proposition 1, there exists a unique choice of $x_0 = \hat{x}_0$ and therefore of $c_0 = k_0\hat{x}$ which pins down an equilibrium path corresponding to the BGP described in the previous section.

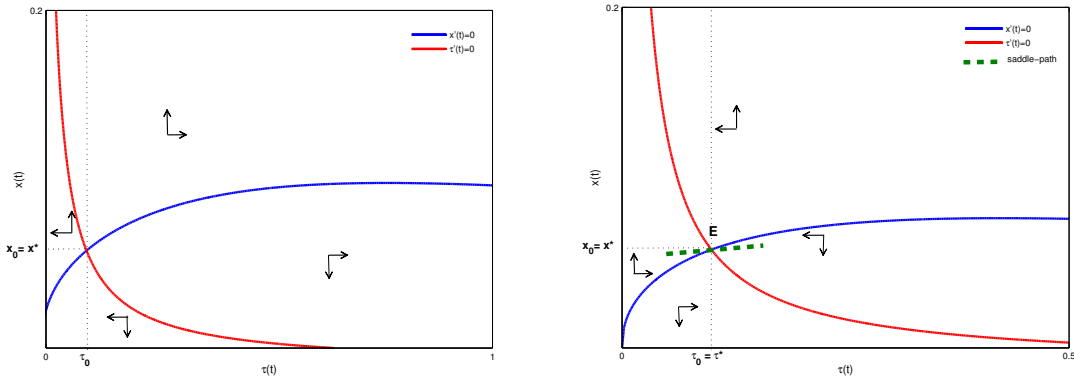


Figure 1: Phase Diagrams when $\phi = 0.5$ (left) and $\phi = -0.01$ (right) and $\gamma = \hat{\gamma}^0$

¹¹Note that the change in stability at $\phi = -\sigma(1 + \hat{\tau})/\hat{\tau}$ occurs through a discontinuity in a similar way as in the model of Benhabib and Farmer [4] (see for example figure 2, page 34) since one of the eigenvalues of the Jacobian matrix changes its sign from $+\infty$ to $-\infty$.

6.2 Existence of other equilibria

As we have shown in the previous subsection, the unique steady state $(\hat{x}, \hat{\tau})$ may be a saddle-point or totally unstable depending on whether $\phi \in (-\sigma(1 + \hat{\tau})/\hat{\tau}, 0)$ or not. While these two possible configurations do not alter the fact that when $\tau_0 = \hat{\tau}$, the economy directly jumps on the BGP from the initial date $t = 0$, we can prove that contrary to Barro [3], some particular transitional dynamics may occur in our model. Indeed, the BGP as defined by $(\hat{x}, \hat{\tau})$ and $\hat{\gamma}$ is not the unique possible equilibrium of our economy. In this section, depending on the value of ϕ , we look for the existence of equilibrium paths $(x_t, \tau_t)_{t \geq 0}$ which may eventually converge to an Asymptotic BGP, denoted from now on ABGP, defined as follows:

Definition 1 (ABGP). *An ABGP is any path $(x(t), \tau(t))_{t \geq 0} = (x^*, \tau^*)$ such that:*

- a) τ^* is a positive arbitrary constant sufficiently close to (but different from) τ_0 ;*
- b) (x^*, τ^*) is a steady state of (18)-(19) with $x^* > 0$ and $\gamma^* > 0$ solution of*

$$0 = (\alpha - \sigma)A(x^*\tau^*)^{1-\alpha} + \sigma(1 + \tau^*)x^* - \rho - \delta(1 - \sigma) \quad (26)$$

$$\gamma^* = \frac{1}{\sigma} [\alpha A(x^*\tau^*)^{1-\alpha} - \delta - \rho]. \quad (27)$$

- c) (x^*, τ^*) satisfies the transversality condition.*

Crucially an ABGP is not an equilibrium since it does not satisfy the initial condition $\tau(0) = \tau_0$. An ABGP in terms of the original variables is a path

$$k^* = k_0 e^{\gamma^*(\alpha, \tau^*, \rho, \delta, \sigma)t} \quad \text{and} \quad c^* = x^*(\alpha, \tau^*, \rho, \delta, \sigma) k_0 e^{\gamma^*(\alpha, \tau^*, \rho, \delta, \sigma)t} \quad (28)$$

which is defined as a steady state (x^*, τ^*) of the system (18)-(19) but is *not an equilibrium* because τ^* is generically different from the exogenously given initial condition of the consumption tax, τ_0 . If such an ABGP exists, the asymptotic value (x^*, τ^*) as well as the asymptotic growth rate of the economy γ^* will not be pinned down by τ_0 through equations (21) and (22) as before, but rather from the asymptotic value of the consumption tax τ^* and then by equations (26)-(27).

The existence of an equilibrium path converging to an ABGP is associated to the existence of consumers' beliefs that are different from those associated to the BGP. Indeed, they may believe that the consumption tax profile will not remain constant but rather change over time and eventually converge to a positive value $\tau^* \neq \tau_0$. Therefore, the consumption over capital ratio and the growth rate will converge to x^* and γ^* respectively. Based on that we will prove in the next Proposition that

under some conditions on ϕ the consumers may indeed decide a consumption path which makes this belief self-fulfilling.

Building on Propositions 1 and 2 we can prove the following result:

Proposition 3. *Given any initial condition $k_0 > 0$ and $\tau_0 > 0$, consider $\underline{\tau}$ and $\bar{\tau}(\sigma)$ as defined by Proposition 1. There exist $\underline{A} > 0$ such that when $A > \underline{A}$, there is a unique equilibrium path $(x_t, \tau_t)_{t \geq 0}$ converging over time to the ABGP (x^*, τ^*) if and only if $\phi \in (-\sigma(1 + \tau^*)/\tau^*, 0)$ and one of the following conditions holds:*

- i) $\sigma \geq 1$ and $\tau^* > \underline{\tau}$,
- ii) $\sigma \in (0, 1)$ and $\tau^* \in (\underline{\tau}, \bar{\tau}(\sigma))$.

Proof. See Appendix A.5. ■

Discussion of these conditions is again in order. First of all, if the consumers believe that the asymptotic tax rate will be τ^* then a unique ABGP exists if one of the conditions i)-ii) holds. As already discussed previously, the requirement of a level of technology greater than \underline{A} is a standard condition for AK models and a tax rate larger than $\underline{\tau}$ provides a large enough government spending to sustain growth through the technology $y = Ak^\alpha \mathcal{G}^{1-\alpha}$. Also the condition $\tau < \bar{\tau}(\sigma)$ when $\sigma < 1$ guarantees that the transversality condition is respected and thus the utility is bounded.

Furthermore, given (k_0, τ_0) there exists a unique equilibrium path converging to the ABGP (x^*, τ^*) if and only if $\phi \in (-\sigma(1 + \tau^*)/\tau^*, 0)$. To provide an intuition for this result let us assume for simplicity that $\sigma = 1$. Considering the expression of the tax rate as given by (17), let us denote $g(c) \equiv (1 + \tau(\tilde{c}))c$ with $\tilde{c} = ce^{-\gamma t}$. It follows that the elasticity of $g(c)$ is given by

$$\varepsilon_{gc} \equiv \frac{g'(c)c}{g(c)} = 1 + \frac{\phi\tau}{1+\tau}$$

Since $\phi > -(1 + \tau^*)/\tau^*$, we get $\varepsilon_{gc} > 0$. Consider then the system of ODEs (7)-(8) with $\sigma = 1$ which can be written as

$$\dot{k} = Ak^\alpha \mathcal{G}^{1-\alpha} - \delta k - g(c) \quad (29)$$

$$\frac{\dot{c}}{c} = r - \delta - \rho - \frac{\dot{\tau}}{1 + \tau} \quad (30)$$

If households expect that in the future the consumption tax rate will be above average, then they expect to consume less in the future and thus, considering that $\varepsilon_{gc} > 0$, we derive from (29) that $g(c)$ is decreasing and thus investment is increasing. This implies that the rental rate of capital r is decreasing and thus through equation

(30) that consumption is also decreasing. We conclude from the balanced-budget rule that the tax rate is decreasing and that the initial expectation is self-fulfilling. Of course this mechanism requires a low enough elasticity of intertemporal substitution in consumption, i.e. a large enough value of σ , to avoid intertemporal consumption's compensations associated to the initial expected decrease of c .

Figure 2 shows the phase diagrams which implicitly account for the consumers' beliefs of a steady state where the same parameters' values as in Figure 1 except that here $(x^*, \tau^*) = (0.123, 0.25)$ and therefore a growth rate $\gamma = 0.037$.

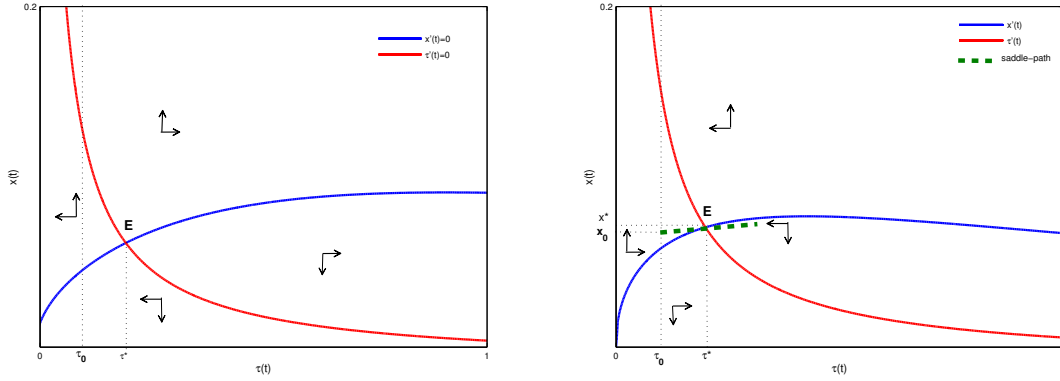


Figure 2: Phase Diagrams when $\phi = 0.5$ (left) and $\phi = -0.01$ (right) and $\gamma = \hat{\gamma}^1$

Observe that both the locus $\dot{x} = 0$ and $\dot{\tau} = 0$ are shifted with respect to the previous case to reflect the different beliefs.¹² According to the directions of the arrows it is clear that in the case of a pro-cyclical consumption tax (i.e. $\phi = 0.5$), there does not exist an equilibrium path which makes this belief self-fulfilling. On the other hand, in the case of a counter-cyclical consumption tax (i.e. $\phi = -0.01$) an equilibrium path converging to the steady state may exist as shown by the golden path in the Figure. In this case the consumers' belief is indeed self-fulfilling.

6.3 Overall (deterministic) dynamics

Proposition 3 actually proves that there exists a continuum of equilibria each of them converging to a different ABGP. In fact, any value of τ^* in a neighborhood of the given initial value τ_0 can be a self-fulfilling belief for the consumers if the conditions of the Proposition are met. Of course this implies a form of global indeterminacy since from a given τ_0 , one can select either the unique BGP by jumping on it from the

¹²This is indeed obvious from equations (18) and (19) since the growth rate, γ , enters explicitly in both of them.

initial date or select any other equilibrium converging to an ABGP. Again different choices reveal different consumers' beliefs of the long run outcome of the economy.

Combining the results found in section 3 and subsections 4.1 and 4.2 allows to state the following Theorem which fully characterizes the dynamics of the economy.

Theorem 1. *Given the initial conditions k_0 and τ_0 , let $\tau_{inf} = \tau_0 - \epsilon > 0$ and $\tau_{sup} = \tau_0 + \epsilon$ with $\epsilon, \epsilon > 0$ small enough. Consider $\underline{\tau}$ and $\bar{\tau}(\sigma)$ as defined by Proposition 1. There exists $\underline{A} > 0$ such that if $A > \underline{A}$, $\phi \in \left(-\frac{\sigma(1+\tau_{sup})}{\tau_{sup}}, 0\right)$ and one of the following conditions holds:*

i) $\sigma \geq 1$ and $\tau_{inf} > \underline{\tau}$,

ii) $\sigma \in (0, 1)$, $\tau_{inf} > \underline{\tau}$ and $\tau_{sup} < \bar{\tau}(\sigma)$,

*then there is a continuum of equilibrium paths, indexed by the letter j , departing from (τ_0, x_0^j) , each of them converging to a different ABGP (τ^{*j}, x^{*j}) with $\tau^{*j} \in (\tau_{inf}, \tau_{sup})$, i.e. the dynamics of the economy is globally, but not locally, indeterminate.*

Proof. See Appendix A.6. ■

To fully understand the dynamic behavior of the economy we can write explicitly the solution of the linearized system:

$$\tilde{\tau} = b_1 v_{11} e^{\lambda_1 t} + b_2 v_{21} e^{\lambda_2 t} \quad (31)$$

$$\tilde{x} = b_1 v_{12} e^{\lambda_1 t} + b_2 v_{22} e^{\lambda_2 t} \quad (32)$$

where $\mathbf{v}_i \equiv (v_{i1}, v_{i2})^T$ is the eigenvector associated to the eigenvalue λ_i , with $i = 1, 2$ while b_i are arbitrary constants. If $\phi \in (-\sigma(1 + \tau^*)/\tau^*, 0)$ and assuming without loss of generality that $\lambda_2 > 0$, the saddle-path solution can be easily found imposing $b_2 = 0$. Combining (31) and (32) and imposing $b_2 = 0$ we get that

$$\tilde{\tau}_0 = \frac{v_{11}}{v_{12}} \tilde{x}_0 \quad (33)$$

with $\tilde{\tau}_0 = \tau_0 - \tau^*$ and $\tilde{x}_0 = x_0 - x^*$. Therefore, given any initial condition k_0 and τ_0 we have the following solution of x converging over time to (x^*, τ^*) :

$$x = x^* + \frac{v_{12}}{v_{11}} \tilde{\tau}_0 e^{\lambda_1 t} \quad (34)$$

Of course as $t \rightarrow \infty$ we have that $\tilde{\tau} \rightarrow \tau^*$, meaning that c converges to the corresponding ABGP since $x \rightarrow x^*$ also converges to the corresponding ABGP. Observe also that the initial level of consumption for this equilibrium path can be obtained from (34) evaluated at $t = 0$, taking into account (28), and it is equal to

$$c_0 = c^* + \frac{v_{12}}{v_{11}} (\tau_0 - \tau^*) k_0 \quad (35)$$

Remark 1. Note that we have a constraint on the initial choice of c_0 (and therefore on x_0) because initial consumption cannot be higher than the initial wealth, $c_0 \leq y_0 - \delta k_0 - \tau_0 c_0$ which at the equilibrium implies that

$$x_0 \leq A \frac{A \tau_0^{\frac{1-\alpha}{\alpha}}}{(1+\tau_0)^{\frac{1}{\alpha}}} - \delta$$

Figure 3 shows the presence of global indeterminacy in the phase diagram $(x(t), \tau(t))$.

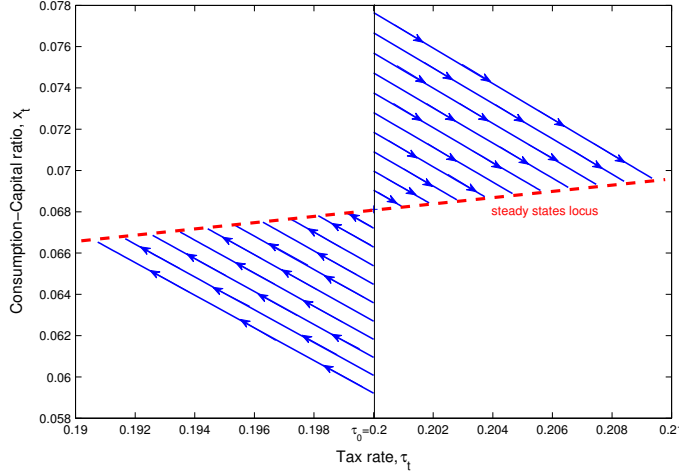


Figure 3: Converging equilibria for different initial values of c_0 .

The initial tax rate is assumed to be equal to $\tau_0 = 0.2$, and the parameters are chosen as in the balanced growth path section with ϕ set to -0.5 . Different initial choices of c_0 pins down different equilibrium paths of the tax rate and of the consumption-capital ratio, each of them converging to a different steady state, characterized by a different growth rate of consumption and capital. Similarly Figure 4 illustrates the emergence of global indeterminacy in the spaces $(t, x(t))$ and $(t, \tau(t))$.

Global indeterminacy arises when the government uses counter-cyclical consumption tax. Under these circumstances, the long run growth rate as well as the consumption over capital ratio cannot be univocally determined within the model. Therefore an economy characterized by these features and an initial value of the tax rate $\tau(0)$ can remain on a balanced growth path but can also follow alternative paths towards different ABGPs each of them characterized by a different (asymptotic) growth rate.

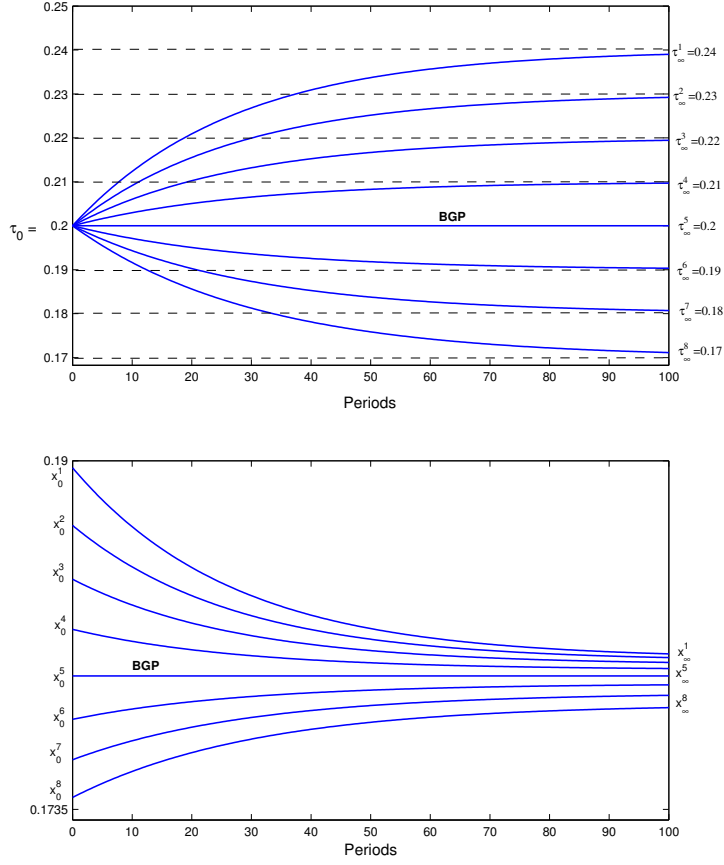


Figure 4: Dynamics of the Tax Rate and of the Consumption-capital Ratio for Fiscal Policy (15)

7 Aggregate instability and stabilization policy

7.1 Sunspot equilibria

Suppose that the households choose an initial value of consumption such that they are at $t = 0$ in (τ_0, x_0^j) . From Theorem 1, we know that if some conditions on parameters are respected then the deterministic dynamical system (18)-(19) has a unique solution around the steady state (τ^{*j}, x^{*j}) converging to it over time. Let us call this path $(x, \tau) = \phi_j(t) \equiv (\phi_{j,x}(t), \phi_{j,\tau}(t))$ as shown on the following Figure 7:

As observed before, this is indeed the unique equilibrium consistent with an asymptotic growth rate γ^j . Clearly, aggregate instability cannot emerge unless

- a) countercyclical taxation is implemented;
- b) extrinsic uncertainty is introduced through a sunspot variable.

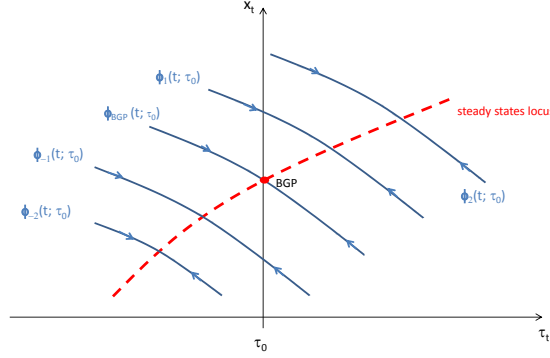


Figure 5: Examples of Saddle-Paths

Condition a) is obtained by setting appropriately the elasticity of the tax rate with respect to de-trended consumption. This is indeed described in details in Theorem 1.

On the other hand, extrinsic uncertainty can be introduced in the model through a sunspot variable. Formally, a sunspot variable can be represented by a continuous-time homogeneous Markov chain $\{\varepsilon_t\}_{t \geq 0}$ with $p_{ij}(t-s)$ indicating the transition probability to move from state i at time s to state j at time t with $s \leq t$ while the initial probability distribution of ε_0 is denoted by $\pi = (\pi_1, \dots, \pi_N)$ with $\pi_j = \mathbb{P}(\varepsilon_0 = z_j)$. More precisely, we need to consider a probability space $(\Omega, B_\Omega, \mathbb{P})$ where Ω is the sample space, B_Ω is a σ -field associated to Ω , and \mathbb{P} is a probability measure. We assume also that the state space is a countable subset of \mathbb{R} :

$$Z \equiv \{z_1, \dots, z_{\bar{t}}, \dots, z_N\} \subset \mathbb{R}$$

with $-\bar{\varepsilon} \leq z_1 < \dots < z_{\bar{t}} = 0 < \dots < z_N \leq \bar{\varepsilon}$. Then each random variable ε_t is a function from $\Omega \rightarrow Z$ which we assume to be B_Ω -measurable.

As explained in Shigoka [29] and Benhabib and Wen [6], the extrinsic uncertainty modifies the deterministic dynamics described by system (18)-(19), from now on $f(x, \tau)$, as it follows:

$$\begin{pmatrix} dx \\ \dot{\tau} dt \end{pmatrix} = f(x, \tau) dt + m \begin{pmatrix} d\varepsilon_t \\ 0 \end{pmatrix}, \quad (36)$$

where m is a constant measuring the weight of the sunspot variable. Two considerations are appropriate: i) the deterministic dynamics can be obtained by setting $m = 0$ ii) the sunspot variable does not affect the fundamental of the economy. In fact, the sunspot variable is a device to randomize among the deterministic equilibria. See the example in the Supplementary Material.

Before proving the existence of sunspot equilibria, several intermediary steps need to be done. In the following we explain carefully these steps and at the end we provide a Theorem which proves our main result.

Differently from the discrete-time case, the evolution of a continuous-time Markov chain cannot be described by the initial distribution π and the $n - step$ transition probability matrix, \mathbf{P}^n , since there is no implicit unit length of time. However, it is possible to define a matrix \mathbf{G} (generator of the chain) which takes over the role of \mathbf{P} . This procedure can be found in Grimmett and Stirzaker [14] among others and, as far as we know, our paper represents the first economic application of this procedure.

Let \mathbf{P}_t be the $N \times N$ matrix with entries $p_{ij}(t)$. The family $\{\mathbf{P}_t\}_{t \geq 0}$ is the *transition stochastic semigroup* of the Markov chain (see Supplementary Material) and the evolution of $\{\varepsilon_t\}_{t \geq 0}$ depends on $\{\mathbf{P}_t\}_{t \geq 0}$ and the initial distribution π of ε_0 . Let us also assume from now on that the transition stochastic semigroup $\{\mathbf{P}_t\}_{t \geq 0}$ is standard, i.e. $\lim_{t \rightarrow 0} \mathbf{P}_t - \mathbf{I} = \mathbf{0}$ or

$$\lim_{t \rightarrow 0} p_{ii}(t) = 1 \quad \text{and} \quad \lim_{t \rightarrow 0} p_{ij}(t) = 0 \text{ for } i \neq j$$

Under these assumptions on the semigroup the following result can be proved:

Proposition 4. *Consider the interval $(t, t + h)$ with h small. Then*

$$\lim_{h \rightarrow 0} \frac{1}{h} (\mathbf{P}_h - \mathbf{I}) = \mathbf{G}$$

i.e. there exists constants $\{g_{ij}\}$ such that

$$p_{ii}(h) \simeq 1 + g_{ii}h \quad \text{and} \quad p_{ij}(h) \simeq g_{ij}h \text{ if } i \neq j \quad (37)$$

with $g_{ii} \leq 0$ and $g_{ij} > 0$ for $i \neq j$. The matrix $\mathbf{G} = (g_{ij})$ is called the generator of the Markov chain $\{\varepsilon_t\}_{t \geq 0}$.

Proof. See Grimmett and Stirzaker [14], Chapter VI, page 256-258. ■

Therefore, the continuous-time Markov chain $\{\varepsilon_t\}_{t \geq 0}$ has a generator \mathbf{G} which can be used together with the initial probability distribution π to describe the evolution of the chain. For this purpose, the following definition will turn out to be useful:

Definition 2. *Let $\varepsilon_s = z_i$, we define the “holding time” as*

$$\mathcal{T}_i \equiv \inf\{t \geq 0 : \varepsilon_{s+t} \neq z_i\}$$

Therefore the “holding time” is a random variable describing the further time until the Markov chain changes its state. The following Proposition is crucial to understand the evolution of the chain from a generic initial state $\varepsilon_s = z_i$.

Proposition 5. *Under the assumptions on the Markov chain introduced so far, the following results hold:*

- 1) *The random variable \mathcal{T}_i is exponentially distributed with parameter g_{ii} . Therefore,*

$$p_{ii}(t) = \mathbb{P}(\varepsilon_{s+t} = z_i | \varepsilon_s = z_i) = e^{g_{ii}t}.$$

- 2) *If there is a jumps, the probability that the Markov chain jumps from z_i to $z_j \neq z_i$ is $-\frac{g_{ij}}{g_{ii}}$.*

Proof. See Grimmett and Stirzaker [14], Chapter VI, page 259-260. ■

Through the last Proposition we can fully describe the evolution of the Markov chain and therefore we have all the ingredients to build sunspot equilibria. Before doing that we define a sunspot equilibrium as it follows:

Definition 3 (Sunspot Equilibrium). *A sunspot equilibrium is a stochastic process $\{(\tau_t, x_t, \varepsilon_t)\}_{t \geq 0}$ which solves the system of stochastic differential equations (36), respect the inequality constraints $\tau_t, x_t > 0$ and the transversality condition.*

We are now ready to prove the following theorem on the existence of sunspot equilibria while Figure 6 provides an example of one of these equilibria.

Theorem 2 (Existence of Sunspot Equilibria). *Assume that all the conditions for indeterminacy in Theorem 1 hold and a sunspot variable is introduced in the model through the continuous-time Markov chain $\{\varepsilon_t\}_{t \geq 0}$. Then sunspot equilibria exist.*

Proof. See Appendix A.7. ■

Theorem 1 shows how to build sunspot equilibria starting from our deterministic model characterized by a unique deterministic and locally determinate BGP and a continuum of other equilibria each of them converging over time to a different ABGP.

Contrary to the standard literature where sunspot equilibria are based on the existence of local indeterminacy (see Benhabib and Farmer [4]), we find sunspot fluctuations while there exists a unique unstable BGP as well as a continuum of

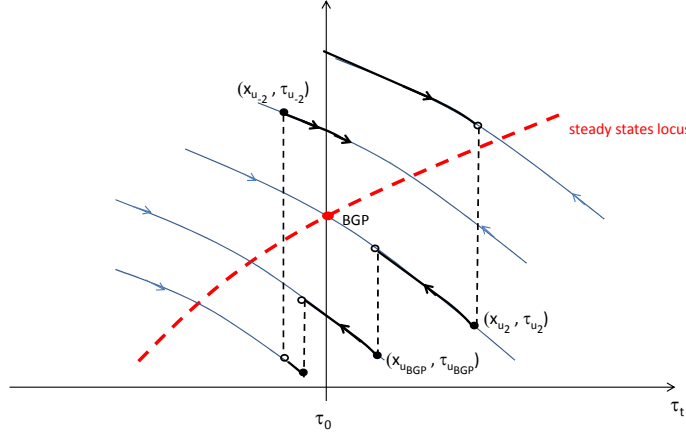


Figure 6: Example of a Sunspot Equilibrium

other equilibria converging to the ABGPs. From this point of view, our conclusions share some similarities with Farmer [12] where expectations-driven fluctuations, in an economy with a continuum of steady states, are generated from the existence of a continuum of equilibrium unemployment rates. However the presence of a continuum of steady states is deeply different in the two frameworks because in our case it crucially depends on the presence of endogenous growth and of endogenous countercyclical taxation when the government balances its budget. In Farmer [12], the existence of a continuum of steady states comes from the fact that there is one less equation than unknown. This under-determinacy arises from the absence of markets to allocate search intensity between the time of searching workers and the recruiting activities of firms. To close the model, Farmer then needs to introduce beliefs about the future value of asset prices, measured relative to the wage, and this justifies the existence of sunspots.

The existence of sunspot fluctuations with a unique deterministic equilibrium is also obtained by Dos Santos Ferreira and Lloyd-Braga [10] but again under a quite different mechanism. Here, the authors consider free entry oligopolistic equilibria where firms, producing under increasing returns to scale, compete in prices in contestable markets. Multiple free entry equilibria may exist, each one characterized by a number of producing firms that varies according to the (correct) conjectures of all the competitors. This multiplicity generates a static indeterminacy on which sunspot equilibria may be constructed while the intertemporal equilibrium is unique.

A direct consequence of Theorem 1 and of the description of sunspot equilibria done so far, lead to the following result.

Corollary 2 (Stabilizing fiscal policy) *Assume that the fiscal policy is procyclical, i.e. $\phi \in (0, +\infty)$, then aggregate instability cannot emerge.*

Of course the reason behind this result is that the dynamics of the economy with procyclical taxation is globally and locally determinate and therefore it is not possible to use a sunspot variable to randomize among the deterministic equilibria because there is only one of them. In fact, in this case we have two strictly positive eigenvalues and therefore two explosive paths to be ruled out by setting $b_1 = b_2 = 0$ in system (31)-(32). In this case the economy has no transitional dynamics and the only solution is the balanced growth path solution described in the previous section.

7.2 Policy implications

In order to check the empirical plausibility of our results, we provide now a simple numerical exercise. On the basis of quarterly data, we consider the benchmark parameterization $(s, \delta, \rho) = (1/3, 0.025, 0.01)$. Concerning the elasticity of intertemporal substitution in consumption, there is no agreement in the empirical literature about its precise value. While early studies such that Campbell [8], Kocherlakota [27] and Vissing-Jorgensen [30] suggest quite low values, more recent contributions, e.g. Mulligan [23], Vissing-Jorgensen and Attanasio [31] and Gruber [15], provide robust estimates of this elasticity between 1 and 2. In light of all these studies, we may assume that a plausible range for ϵ_{cc} is $(0.5, 2)$. Consumption tax rates have been estimated by Mendoza *et al.* [21, 22],¹³ and more recently by Volkerink and De Haan [32]. They provide ranges of tax rates for each OECD countries as given in Table 1:

$\tau \in (0.05, 0.1)$	Japan, US, Switzerland
$\tau \in (0.1, 0.15)$	Australia, Canada, Italy, Spain
$\tau \in (0.15, 0.2)$	Belgium, Germany, Greece, Netherlands, New Zealand, Portugal, UK
$\tau \in (0.2, 0.25)$	Austria, France, Iceland, Luxembourg, Sweden
$\tau \in (0.25, 0.3)$	Finland, Ireland
$\tau \in (0.3, 0.35)$	Denmark, Norway

Table 1: Consumption tax rates of OECD countries

As shown in Section 5, choosing $A = 2.3$ implies a minimal level of tax rate for indeterminacy such that $\underline{\tau} = 0.139$, with $\hat{\gamma} = 1.84\%$ when $\sigma = 2$, $\hat{\gamma} = 2.39\%$ when

¹³Updated estimates up to 1996 are available online from the authors.

$\sigma = 1$ and $\hat{\gamma} = 2.5\%$ with $\bar{\tau}(\sigma) = 0.2$ when $\sigma = 0.6$. Except for Japan, US and Switzerland, we then conclude that, depending on the value of σ , our conditions on the tax rate for the existence of aggregate instability are empirically plausible for most OECD countries.

Let us finally consider empirically plausible values of the elasticity ϕ of the consumption tax rate. Up to our knowledge there is no direct estimates of this parameter available in the literature. However, Lane [18] provides some empirical estimates of the elasticity of government expenditures with respect to output growth. Since consumption is almost perfectly correlated with output, we use Lane's results as a proxy for the elasticity of government expenditures with respect to consumption, as given by

$$\eta \equiv \mathcal{G}'(c)c/\mathcal{G}(c) \quad (38)$$

In the following we will say that public expenditures are counter(pro)-cyclical when $\eta < 0$ ($\eta > 0$). Consider then the balanced-budget rule (12) together with the fiscal rule (17). We get

$$\mathcal{G}(c) = \tau_0 (c_0 e^{\gamma t})^{-\phi} c^{1+\phi}$$

and thus

$$\eta = 1 + \phi \text{ or equivalently } \phi = \eta - 1$$

From Lane's approximation of η we can then evaluate empirically plausible values of ϕ . The tax rule (17) is therefore counter-cyclical for $\eta < 1$, pro-cyclical for $\eta > 1$ and constant for $\eta = 1$.

Using annual data over 1960 – 1998 for 22 OECD countries, Lane [18] shows that in most OECD countries government spending is counter-cyclical, i.e. $\eta \leq 0$.

$\eta \in (-0.1, 0]$	Australia, Greece, Japan, New Zealand, Spain
$\eta \in (-0.2, -0.1)$	Austria, US
$\eta \in (-0.4, -0.2)$	Belgium, Finland, Germany, Netherlands, Sweden
$\eta \in (-0.7, -0.4)$	Canada, France, Italy, UK

Table 2: Elasticity of government expenditure of OECD countries

We conclude that the elasticity of the consumption tax rate ϕ is negative implying thus counter-cyclicity, for most OECD countries. As a whole we have shown that the existence of aggregate instability and sunspot fluctuations driven by consumption tax rates is empirically plausible for a large set of OECD countries. According to the predictions of our model (see Corollary 2), the output volatility of these countries would reduce in response of a switch from a countercyclical to a procyclical taxation.

8 Income tax vs consumption tax

One could think that our main results strongly rely on the consideration of a consumption tax. This is actually not the case. If we assume that government expenditures are financed through a tax on income, i.e. $\mathcal{G} = \tau(\tilde{y})y$, with $\tilde{y} = ye^{\gamma t}$, and that the elasticity of the tax rate with respect to detrended output is constant and equal to ϕ , then we find similar results. Indeed, the model is basically the same except that the capital accumulation equation becomes now

$$\dot{k} = (1 - \tau)Ak^\alpha \mathcal{G}^{1-\alpha} - \delta k - c$$

Moreover, solving $\mathcal{G} = \tau y = \tau Ak^\alpha \mathcal{G}^{1-\alpha}$ with respect to \mathcal{G} gives $\mathcal{G} = (\tau A)^{1/\alpha} k$. From the corresponding first order conditions, straightforward computations then lead to the following dynamical system

$$\frac{\dot{x}}{x} = \frac{1}{\sigma} \left[(1 - \tau)A(\tau A)^{\frac{1-\alpha}{\alpha}} (\alpha - \sigma) - \delta(1 - \sigma) - \rho + \sigma x \right] \quad (39)$$

$$\frac{\dot{\tau}}{\tau} = \frac{\phi\alpha}{\alpha - \phi(1 - \alpha)} \left[(1 - \tau)A(\tau A)^{\frac{1-\alpha}{\alpha}} - \delta - \gamma - x \right] \quad (40)$$

with $x \equiv \frac{c}{k}$. Along a BGP as defined by (20), we find again that τ is constant and equal to its initial value τ_0 . It follows that there exists a unique steady-state of the dynamical system characterized by stationary values of x and γ , i.e. \hat{x} and $\hat{\gamma}$, that depend on τ_0 . But there also exist a continuum of ABGP characterized by an asymptotic value of τ , namely $\tau^* \neq \tau_0$ with τ^* sufficiently close to τ_0 , and of equilibrium path each of them converging to a different ABGP provided the saddle-point property holds.

9 Conclusion

We have considered a Barro-type [3] endogenous growth model in which a government provides as an external productive input a constant stream of expenditures financed through consumption taxes and a balanced-budget rule. In order to have a constant tax on a balanced growth path, the tax rate needs to depend on de-trended consumption and thus becomes a state variable with a given initial condition. We also consider a representative household characterized by a CRRA utility function and inelastic labor. Such a formulation is known to rule out the existence of endogenous fluctuations in a standard stationary framework (see Giannitsarou [13]).

We have proved that there exists a unique Balanced Growth Path (BGP) along which the common growth rate of consumption, capital, GDP and government

spending is constant. Moreover, as in the Barro [3] model, there is no transitional dynamics with respect to this unique BGP. However, we have shown that the BGP is not the unique long run solution of our model. Indeed, if the tax rule is counter-cyclical with respect to consumption, for any arbitrary initial value of the tax rate, close enough to its initial condition, there exists a corresponding value for the tax rate, consumption, capital and the constant growth rate that can be an asymptotic equilibrium of our economy, namely an Asymptotic Balanced Growth Path (ABGP). An ABGP is not itself an equilibrium as it does not respect the initial conditions. However, some transitional dynamics exist with a unique equilibrium path converging toward this ABGP, and we prove that there exist a continuum of such ABGP and of equilibria each of them converging over time to a different ABGP.

The existence of an equilibrium path converging to an ABGP is associated to the existence of consumers' beliefs that are different from those associated to the BGP. Indeed, they may believe that the consumption tax profile will not remain constant but rather change over time and eventually converge to a positive value different from the initial condition. Based on this property, we prove the existence sunspot equilibria and thus that endogenous sunspot fluctuations may arise under a balanced-budget rule and consumption taxes although there exists a unique underlying BGP equilibrium. Moreover, a simple numerical exercise shows that our conclusions are compatible with empirically realistic values of the main structural parameters and tax rates for many OECD countries.

A Appendix

A.1 Derivation of equations (18) and (19)

Dividing equation (7) by k and using the balanced-budget rule we may rewrite the system of equations (7), (8) as :

$$\frac{\dot{k}}{k} = A \left(\frac{k}{c\tau} \right)^{\alpha-1} - \delta - (1 + \tau) \frac{c}{k} \quad (41)$$

$$\frac{\dot{c}}{c} = \frac{1}{\sigma} \left[\alpha A \left(\frac{k}{c\tau} \right)^{\alpha-1} - \delta - \rho - \frac{\dot{\tau}}{1 + \tau} \right] \quad (42)$$

Substituting equation (42) into the fiscal policy rule (15) and solving for $\dot{\tau}/\tau$, we derive

$$\frac{\dot{\tau}}{\tau} = \frac{\phi(1 + \tau)}{\sigma(1 + \tau) + \phi\tau} \left[\alpha A \left(\frac{k}{\tau c} \right)^{\alpha-1} - \delta - \rho - \sigma\gamma \right] \quad (43)$$

Now let us define the control-like variable $x \equiv \frac{c}{k}$ which implies that $\frac{\dot{x}}{x} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k}$. Subtracting equation (41) from equation (42) and using the definition of the new variable x together with equation (43) we find

$$\begin{aligned} \frac{\dot{x}}{x} &= \frac{(1+\tau)(1-\sigma) - \phi\tau}{\sigma(1+\tau) + \phi\tau} [\alpha A(\tau x)^{1-\alpha} - \delta - \rho - \sigma\gamma] \\ &+ \gamma(1-\sigma) + (1+\tau)x - (1-\alpha)A(\tau x)^{1-\alpha} - \rho \end{aligned} \quad (44)$$

$$\frac{\dot{\tau}}{\tau} = \frac{\phi(1+\tau)}{\sigma(1+\tau) + \phi\tau} [\alpha A(\tau x)^{1-\alpha} - \delta - \rho - \sigma\gamma] \quad (45)$$

■

A.2 Proof of Proposition 1

The proof is articulated in four steps.

The first step of the proof consists in showing that there exists a positive solution of the equation $g(x) = 0$. Note first that $g(0) = -\rho - \delta(1-\sigma) < 0$ if and only if $\sigma < \sigma_0 \equiv (\delta + \rho)/\delta$, $\lim_{x \rightarrow +\infty} g(x) = +\infty$, and $g'(x) = (\alpha - \sigma)(1 - \alpha)A\hat{\tau}^{1-\alpha}x^{-\alpha} + \sigma(1 + \hat{\tau})$. If $\alpha \geq \sigma$, $g'(x) > 0$ for any x and the uniqueness of the solution is ensured. On the contrary, if $\sigma > \alpha$ we get $g'(x) = 0$ if and only if

$$x = x_{min} = \left(\frac{(1-\alpha)(\sigma - \alpha)A\hat{\tau}^{1-\alpha}}{\sigma(1 + \hat{\tau})} \right)^{\frac{1}{\alpha}} > 0$$

Since $g(x)$ is a continuous function, we conclude that $g'(x) < 0$ when $x \in (0, x_{min})$ and $g'(x) > 0$ when $x > x_{min}$. Moreover, we get

$$g(x_{min}) = -\sigma \left[\frac{\alpha(1+\hat{\tau})}{1-\alpha} \left[A\hat{\tau}^{1-\alpha}(1-\alpha)^{\frac{1-\frac{1}{\sigma}}{1+\hat{\tau}}} \right]^{\frac{1}{\alpha}} + \delta \left(\frac{1}{\sigma} - 1 \right) + \frac{\rho}{\sigma} \right]$$

with $\partial g(x_{min})/\partial \sigma < 0$, $g(x_{min})|_{\sigma=1} < 0$ and

$$\lim_{\sigma \rightarrow +\infty} \left[\frac{\alpha(1+\hat{\tau})}{1-\alpha} \left[A\hat{\tau}^{1-\alpha}(1-\alpha)^{\frac{1-\frac{1}{\sigma}}{1+\hat{\tau}}} \right]^{\frac{1}{\alpha}} + \delta \left(\frac{1}{\sigma} - 1 \right) + \frac{\rho}{\sigma} \right] = \alpha A^{\frac{1}{\alpha}} \left(\frac{\hat{\tau}(1-\alpha)}{1+\hat{\tau}} \right)^{\frac{1-\alpha}{\alpha}} - \delta \quad (46)$$

It follows that when $A > A_1$ with

$$A_1 \equiv \left(\frac{\delta}{\alpha} \right)^{\alpha} \left(\frac{1+\hat{\tau}}{\hat{\tau}(1-\alpha)} \right)^{1-\alpha},$$

the expression (46) is positive and $\lim_{\sigma \rightarrow +\infty} g(x_{min}) = -\infty$ so that $g(x_{min}) < 0$ for any $\sigma > \alpha$. Therefore, from all these results we conclude the following:

- if $\sigma \in (\alpha, \sigma_0)$ then $g(0) < 0$ and there also exists a unique \hat{x} solution of $g(x) = 0$;
- if $\sigma > \sigma_0$, then $g(0) > 0$, $g(x_{min}) < 0$ and there exists two solutions of $g(x) = 0$, namely \tilde{x} and \hat{x} with $\tilde{x} < \hat{x}$.

The second step of the proof is to verify that the steady state value of x , in particular in the case of multiplicity, leads to a constant growth rate γ which is positive and satisfies the transversality condition (23). We need to check that $\gamma > 0$ when $\sigma \leq 1$ and $\gamma \in (0, \rho/(1 - \sigma))$ when $\sigma < 1$. Since

$$\gamma = \frac{1}{\sigma} [\alpha A (x\hat{\tau})^{1-\alpha} - \delta - \rho]$$

the inequality $\gamma > 0$ is equivalent to

$$x > \frac{1}{\hat{\tau}} \left(\frac{\delta + \rho}{\alpha A} \right)^{\frac{1}{1-\alpha}} \equiv \underline{x}$$

A sufficient condition for the existence of a steady state value for x , \hat{x} , such that $\gamma(\hat{x}) > 0$ is $\hat{x} > \underline{x}$. Since $\hat{\tau} = \tau_0$, this inequality is obtained if $g(\underline{x}) < 0$, i.e.

$$\left(\frac{\delta + \rho}{\alpha A} \right)^{\frac{1}{1-\alpha}} < \tau_0 \left[\frac{\delta(1-\alpha) + \rho}{\alpha} - \left(\frac{\delta + \rho}{\alpha A} \right)^{\frac{1}{1-\alpha}} \right] \quad (47)$$

It follows that when $A > A_2$ with

$$A_2 \equiv \left(\frac{\alpha}{\delta(1-\alpha) + \rho} \right)^{1-\alpha} \frac{\delta + \rho}{\alpha}$$

the right-hand-side of (47) is positive. Then, in this case, $g(\underline{x}) < 0$ if and only if

$$\tau_0 > \frac{\left(\frac{\delta + \rho}{\alpha A} \right)^{\frac{1}{1-\alpha}}}{\frac{\delta(1-\alpha) + \rho}{\alpha} - \left(\frac{\delta + \rho}{\alpha A} \right)^{\frac{1}{1-\alpha}}} \equiv \underline{\tau} \quad (48)$$

It is worth noting that when $g(\underline{x}) < 0$, uniqueness of \hat{x} is also ensured. Indeed, in the case where $\sigma > \sigma_0$, we have shown previously that a second solution \tilde{x} of $g(x) = 0$ occurs with $\tilde{x} < \hat{x}$. It is obvious to derive that if $g(\underline{x}) < 0$ then $\underline{x} > \tilde{x}$ and \tilde{x} is characterized by a negative growth rate.

Let us consider finally the restriction to satisfy the transversality condition when $\sigma < 1$, namely

$$\gamma = \frac{1}{\sigma} [\alpha A (x\hat{\tau})^{1-\alpha} - \delta - \rho] < \rho/(1 - \sigma)$$

This inequality is equivalent to

$$x < \frac{1}{\hat{\tau}} \left(\frac{\delta(1-\sigma) + \rho}{\alpha A(1-\sigma)} \right)^{\frac{1}{1-\alpha}} \equiv \bar{x}$$

We then need to check that $\hat{x} < \bar{x}$. This inequality is obtained if $g(\bar{x}) > 0$, i.e.

$$1 > \tau_0 \left[(1 - \alpha) A^{\frac{1}{1-\alpha}} \left(\frac{\delta(1-\sigma) + \rho}{\alpha(1-\sigma)} \right)^{\frac{-\alpha}{1-\alpha}} - 1 \right] \quad (49)$$

The right-hand-side of this inequality is decreasing with respect to σ and negative when $\sigma = 1$. Moreover we get

$$\lim_{\sigma \rightarrow 0} \left[(1 - \alpha) A^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{\delta + \rho} \right)^{\frac{1}{1-\alpha}} - 1 \right]$$

which can be positive or negative. When this expression is negative, then (49) holds for any τ_0 . When this expression is positive, there exists $\sigma_1 \in (0, 1)$ such that if $\sigma > \sigma_1$, (49) again holds for any τ_0 . On the contrary, when $\sigma \in (0, \sigma_1)$, (49) holds if $\tau_0 < \bar{\tau}(\sigma)$ with

$$\bar{\tau}(\sigma) \equiv \frac{1}{(1-\alpha) A^{\frac{1}{1-\alpha}} \left(\frac{\delta(1-\sigma)+\rho}{\alpha(1-\sigma)} \right)^{\frac{1}{1-\alpha}} - 1}$$

To simplify the formulation, we have then proved that when $\sigma < 1$, there exists $\bar{\tau}(\sigma) \in (0, +\infty]$ such that $\hat{x} < \bar{x}$ and the corresponding growth rate $\hat{\gamma}$ satisfies the transversality condition if and only if $\tau_0 < \bar{\tau}(\sigma)$. But to complete the proof, we need to show in this case that $\underline{\tau} < \bar{\tau}(\sigma)$. We know that $\bar{\tau}(\sigma)$ is an increasing function of σ over $(0, \sigma_1)$ and that $\bar{\tau}(\sigma_1) = +\infty$. Moreover straightforward computations show that $\underline{\tau} < \bar{\tau}(0)$. Therefore, $\underline{\tau} < \bar{\tau}(\sigma)$ for any $\sigma \in (0, 1)$. The conclusions of the Proposition follow denoting $\underline{A} = \max\{A_1, A_2\}$. ■

A.3 Proof of Corollary 1

Under the conditions of Proposition 1, consider $\hat{x} = \hat{x}(\alpha, \tau_0, \rho, \delta, \sigma)$ the solution of equation (21) and recall that $\hat{\tau} = \tau_0$. Let us also denote equation (21) as follows

$$h(x, \tau_0) \equiv (\alpha - \sigma) A (x \tau_0)^{1-\alpha} + \sigma(1 + \tau_0)x - \rho - \delta(1 - \sigma) = 0 \quad (50)$$

We first get

$$\frac{\partial h}{\partial x} = (1 - \alpha) A \frac{(x \tau_0)^{1-\alpha}}{x} (\alpha - \sigma) + \sigma(1 + \tau_0)$$

From equation (18) evaluated along the steady state $(\hat{x}, \hat{\gamma})$ we derive

$$(1 + \tau_0)\hat{x} = (1 - \alpha) A (\hat{x} \tau_0)^{1-\alpha} + \rho - \hat{\gamma}(1 - \sigma) = \frac{(1-\alpha) A (\delta + \rho + \sigma \hat{\gamma})}{\alpha} + \rho - \hat{\gamma}(1 - \sigma) \quad (51)$$

and thus

$$\frac{\partial h}{\partial x} \Big|_{x=\hat{x}} = \frac{(1 - \alpha) A (\delta + \rho + \sigma \hat{\gamma}) + \alpha \sigma (1 + \tau_0) [\rho - \hat{\gamma}(1 - \sigma)]}{\alpha \hat{x}} > 0$$

as the transversality condition (23) holds.

Second we also compute from (50)

$$\frac{\partial h}{\partial \tau_0} = (1 - \alpha) A \frac{(x \tau_0)^{1-\alpha}}{\hat{\tau}} (\alpha - \sigma) + \sigma x$$

Using again (50) evaluated along the steady state $(\hat{x}, \hat{\gamma})$ we get

$$(\alpha - \sigma)A(x\tau_0)^{1-\alpha} = \rho + \delta(1 - \sigma) - \sigma(1 + \tau_0)\hat{x}$$

and thus

$$\left. \frac{\partial h}{\partial \tau_0} \right|_{x=\hat{x}} = \frac{(1 - \alpha)(\delta + \rho) - \sigma\{\hat{x}[(1 - \alpha) - \tau_0\alpha] + \delta(1 - \alpha)\}}{\tau_0} \quad (52)$$

If $\tau_0 < (1 - \alpha)/\alpha$ and $\sigma > \underline{\sigma}$ with

$$\underline{\sigma} \equiv \frac{(1 - \alpha)(\delta + \rho)}{\hat{x}[(1 - \alpha) - \tau_0\alpha] + \delta(1 - \alpha)}$$

then the expression (52) is negative. To be consistent with Proposition 1, we need now to check that $\underline{\tau} < (1 - \alpha)/\alpha$. Using (48), we conclude that this inequality holds if and only if $A > A_{min}$ with

$$A_{min} \equiv \left(\frac{\alpha}{(1 - \alpha)[\delta(1 - \alpha) + \rho]} \right)^{1 - \alpha} \frac{\delta + \rho}{\alpha} > \underline{A}$$

Finally, under all these conditions, we conclude from the implicit function theorem that

$$\frac{d\hat{x}}{d\tau_0} = - \frac{\partial h / \partial \tau_0|_{x=\hat{x}}}{\partial h / \partial x|_{x=\hat{x}}} > 0$$

The result on the growth rate can be found immediately by differentiating equation (24) with respect to $\hat{\tau}$:

$$\frac{d\hat{\gamma}}{d\tau_0} = \alpha(1 - \alpha)(\hat{x}\tau_0)^{-\alpha} \left(\left. \frac{d\hat{x}}{d\tau_0} \right|_{x=\hat{x}} + \hat{x} \right)$$

Let us finally consider the expression of welfare along the BGP as given by (25). We derive

$$\frac{dW(\hat{\gamma}, \hat{x})}{d\tau_0} = (\hat{x}k_0)^{1-\sigma} \left[\frac{d\hat{x}}{d\tau_0} \frac{\rho - \hat{\gamma}(1 - \sigma)}{\hat{x}} + \frac{d\hat{\gamma}}{d\tau_0} \right] > 0$$

and the result follows. ■

A.4 Proof of Proposition 2

Let us consider the system (18)-(19)

$$\begin{aligned} \dot{x} = & \left\{ \frac{[(1 + \tau)(1 - \sigma) - \phi\tau] [\alpha A(x\tau)^{1-\alpha} - \delta - \rho - \sigma\gamma]}{\sigma(1 + \tau) + \phi\tau} \right. \\ & \left. + \gamma(1 - \sigma) + (1 + \tau)x - (1 - \alpha)A(x\tau)^{1-\alpha} - \rho \right\} x \equiv \varphi(\tau, x) \end{aligned} \quad (53)$$

$$\dot{\tau} = \frac{\phi\tau(1 + \tau)}{\sigma(1 + \tau) + \phi\tau} [\alpha A(x\tau)^{1-\alpha} - \delta - \rho - \sigma\gamma] \equiv \phi(\tau, x) \quad (54)$$

Linearizing this system around the steady state $(\hat{x}, \hat{\tau})$ gives

$$\begin{pmatrix} \dot{x} \\ \dot{\tau} \end{pmatrix} = \begin{pmatrix} \frac{[(1+\hat{\tau})(\alpha-\sigma)-\phi\hat{\tau}](1-\alpha)(\delta+\rho+\sigma\hat{\gamma})}{\alpha[\sigma(1+\hat{\tau})+\phi\hat{\tau}]} + (1+\hat{\tau})\hat{x} & \frac{[(1+\hat{\tau})(\alpha-\sigma)-\phi\hat{\tau}](1-\alpha)(\delta+\rho+\sigma\hat{\gamma})}{\alpha[\sigma(1+\hat{\tau})+\phi\hat{\tau}]\hat{\tau}}\hat{x} + \hat{x}^2 \\ \frac{\phi\hat{\tau}(1+\hat{\tau})(1-\alpha)(\delta+\rho+\sigma\hat{\gamma})}{[\sigma(1+\hat{\tau})+\phi\hat{\tau}]\hat{x}} & \frac{\phi(1+\hat{\tau})(1-\alpha)(\delta+\rho+\sigma\hat{\gamma})}{\sigma(1+\hat{\tau})+\phi\hat{\tau}} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{\tau} \end{pmatrix}$$

where $\tilde{\tau} \equiv \tau - \hat{\tau}$ and $\tilde{x} \equiv x - \hat{x}$.

The proof consists in studying the determinant and trace of the Jacobian matrix, \mathbf{J} , which is the coefficient's matrix of the linearized system. After some tedious but straightforward computations, the trace and determinant of the Jacobian can be found to be equal to

$$\begin{aligned} \det(\mathbf{J}) &= \frac{\phi(1+\hat{\tau})(1-\alpha)(\delta+\rho+\sigma\hat{\gamma})}{[\sigma(1+\hat{\tau})+\phi\hat{\tau}]} \hat{x} \\ \text{tr}(\mathbf{J}) &= \frac{\phi(1+\hat{\tau})(1-\alpha)(\delta+\rho+\sigma\hat{\gamma})}{\sigma(1+\hat{\tau})+\phi\hat{\tau}} + \frac{[(1+\hat{\tau})(\alpha-\sigma)-\phi\hat{\tau}](1-\alpha)(\delta+\rho+\sigma\hat{\gamma})}{\alpha[\sigma(1+\hat{\tau})+\phi\hat{\tau}]} + (1+\hat{\tau})\hat{x} \end{aligned}$$

It follows immediately that

$$\det(\mathbf{J}) < 0 \quad \Leftrightarrow \quad \frac{\phi}{\sigma(1+\hat{\tau})+\phi\hat{\tau}} < 0 \quad \Leftrightarrow \quad \phi \in \left(-\frac{\sigma(1+\hat{\tau})}{\hat{\tau}}, 0 \right).$$

In this case the steady state $(\hat{x}, \hat{\tau})$ is a saddle-point. If on the contrary, $\phi \in \left(-\infty, -\frac{\sigma(1+\hat{\tau})}{\hat{\tau}} \right) \cup (0, \infty)$ we need to study the sign of the trace of the Jacobian. Assume first that $\phi < -\sigma(1+\hat{\tau})/\hat{\tau}$. We then get

$$(1+\hat{\tau})(\alpha-\sigma) - \phi\hat{\tau} > \alpha(1+\hat{\tau})$$

which implies that $\text{tr}(\mathbf{J}) > 0$ and the steady state $(\hat{x}, \hat{\tau})$ is totally unstable. Assume finally that $\phi > 0$. From equations (53)-(54) evaluated at the steady state $(\hat{x}, \hat{\tau})$ we get

$$(1+\tau)\hat{x} = (1-\alpha)A(\hat{x}\hat{\tau})^{1-\alpha} + \rho - \gamma(1-\sigma) = \frac{(1-\alpha)(\delta+\rho+\sigma\hat{\gamma})}{\alpha} + \rho - \gamma(1-\sigma)$$

From this expression we derive that

$$\frac{[(1+\hat{\tau})(\alpha-\sigma)-\phi\hat{\tau}](1-\alpha)(\delta+\rho+\sigma\hat{\gamma})}{\alpha[\sigma(1+\hat{\tau})+\phi\hat{\tau}]} + (1+\hat{\tau})\hat{x} = \frac{(1+\hat{\tau})(1-\alpha)(\delta+\rho+\sigma\hat{\gamma}) + [\rho - \hat{\gamma}(1-\sigma)][\sigma(1+\hat{\tau}) + \phi\hat{\tau}]}{\sigma(1+\hat{\tau}) + \phi\hat{\tau}} > 0$$

since the transversality condition implies $\rho - \hat{\gamma}(1-\sigma) > 0$. It follows again that $\text{tr}(\mathbf{J}) > 0$ and the steady state $(\hat{x}, \hat{\tau})$ is totally unstable.

We have then proved that for any value of σ , the steady state $(\hat{x}, \hat{\tau})$ is either a saddle-point or totally unstable. Since the stationary value of the tax rate $\hat{\tau}$ is given by its initial value τ_0 , in both of these configurations, the only initial value of $x(t)$ compatible with the transversality condition is $x(0) = \hat{x}$ and the economy immediately jumps on the BGP from the initial date $t = 0$. ■

A.5 Proof of Proposition 3

Given a τ^* the proof and existence and uniqueness of an ABGP is basically the same as the proof of Proposition 1 once we have substituted τ^* to $\hat{\tau}$. In particular the value of A_1 is now substituted by

$$A_1^* \equiv \left(\frac{\delta}{\alpha}\right)^\alpha \left(\frac{1+\tau^*}{\tau^*(1-\alpha)}\right)^{1-\alpha} \quad (55)$$

The critical values $\underline{\tau}$ and $\bar{\tau}(\sigma)$ have the same expressions as in the proof of Proposition 1 while $\underline{A} = \max\{A_1^*, A_2\}$.

Concerning the asymptotic stability of (x^*, τ^*) , the computations given in the proof of Proposition 2 applies so that (x^*, τ^*) is saddle-path stable if and only if $\phi \in (-\sigma(1+\tau^*)/\tau^*, 0)$. In this case, for a given τ_0 close enough to τ^* , there exists a unique value of $x(0)$ such that the equilibrium path $(x(t), \tau(t))$ converges towards (x^*, τ^*) . Note that if on the contrary, $\phi \in (-\infty, -\sigma(1+\tau^*)/\tau^*) \cup (0, +\infty)$, then (x^*, τ^*) is totally unstable and, therefore, an equilibrium $(x_t, \tau_t)_{t \geq 0}$ converging to the ABGP does not exist. Indeed, in this case, as $\tau(0) = \tau_0 \neq \tau^*$, the only equilibrium path is to jump on the unique BGP as given by $(\hat{x}, \hat{\tau})$. ■

A.6 Proof of Theorem 1

The result follows from Propositions 1, 2 and 3. Note that the value of A_1 or A_1^* is now substituted by

$$A_1^{max} \equiv \min_{\tau \in (\tau_{inf}, \tau_{sup})} \left(\frac{\delta}{\alpha}\right)^\alpha \left(\frac{1+\tau}{\tau(1-\alpha)}\right)^{1-\alpha} = \left(\frac{\delta}{\alpha}\right)^\alpha \left(\frac{1+\tau_{inf}}{\tau_{inf}(1-\alpha)}\right)^{1-\alpha} \quad (56)$$

The critical values $\underline{\tau}$ and $\bar{\tau}(\sigma)$ have the same expressions as in the proof of Proposition 1 while $\underline{A} = \max\{A_1^{max}, A_2\}$. Moreover, the condition on ϕ becomes $\phi \in (-\underline{\phi}, 0)$ with

$$\underline{\phi} = \min_{\tau \in (\tau_{inf}, \tau_{sup})} \frac{\sigma(1+\tau)}{\tau} = \frac{\sigma(1+\tau_{sup})}{\tau_{sup}}$$

Assuming also $\tau_{inf} > \underline{\tau}$ and $\tau_{sup} < \bar{\tau}(\sigma)$, we get that for any $\tau^{*j} \in (\tau_{inf}, \tau_{sup})$, the conditions for the existence of a solution of system (26)-(27) as given in Propositions 1 and 3, and the restriction on ϕ as given in Proposition 2 hold.

When $\phi \in \left(-\frac{\sigma(1+\tau_{sup})}{\tau_{sup}}, 0\right)$, local determinacy means that for any given τ_0 there is a unique equilibrium path either jumping on the unique BGP $(\hat{x}, \hat{\tau})$ or converging toward some ABGP (τ^{*j}, x^{*j}) with $\tau^{*j} \in (\tau_{inf}, \tau_{sup})$. Global indeterminacy means that while the initial tax rate τ_0 is given, the economy may converge to different asymptotic equilibria depending on the beliefs of the agents. Also we have not only multiple equilibria but a continuum of them. In fact, under the condition on ϕ we

have that variations of τ^* make x , $\det(\mathbf{J})$, $\text{tr}(\mathbf{J})$ change continuously. The last two changes imply a continuous changes of the eigenvalues, λ_1 , λ_2 , and of the associated eigenvectors. Therefore, the solution of x will also change continuously.

Finally, when $\phi \in (0, +\infty)$, the unique equilibrium path consists in jumping on the unique BGP from the initial date. There is thus local and global determinacy. ■

A.7 Proof of Theorem 2

Assuming that all the conditions in Theorem 1 are respected, a sunspot equilibrium starting at $t = 0$ in state j with probability π_j can be summed up as follows:¹⁴

$$(x_t, \tau_t) = \phi_j(t), \quad \text{for } t \in [0, t_j) \quad (57)$$

$$(x_{t_j}, \tau_{t_j}) = (\phi_{i,x}(t_j), \phi_{j,\tau}(t_j)), \quad \text{with probability } -\frac{g_{ji}}{g_{jj}} \quad (58)$$

$$(x_t, \tau_t) = \phi_i(t), \quad \text{for } t \in [t_j, t_i) \quad (59)$$

with t_m a value taken by the random variable $\mathcal{T}_m \sim e^{g_{mm}t}$ with $m = i, j$ for any $i, j = 1, \dots, N$ and $i \neq j$. The jump in the control-like variables at date t_i have size $s(z_i - z_j)$, and the resulting path respects the inequalities constraints as long as $\underline{\varepsilon}$, $\bar{\varepsilon}$ and s are sufficiently small. An example of a sunspot equilibrium is drawn in Figure 8. ■

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¹⁴See also the example in the Supplementary Material.

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Supplementary Material

An illustrative example

Before studying the existence of sunspot equilibria in our continuous time environment, it is useful to provide an example in discrete time assuming a period length equal to a positive integer, h .¹ Due to the presence of extrinsic uncertainty, the discrete time counterpart of the dynamical system (36)-(37) takes the form:

$$\begin{pmatrix} \mathbb{E}_t(\Delta x_{t+h}) \\ \Delta \tau_{t+h} \end{pmatrix} = F(x_t, \tau_t)h \quad \text{with } (x_0, \tau_0) \text{ given}$$

where $\Delta \chi_{t+h} \equiv \chi_{t+h} - \chi_t$ with $\chi = x, \tau$, \mathbb{E}_t is a conditional expectation operator, and $F(x_t, \tau_t)$ is the left hand side of the continuous time dynamical system, after (36) has been multiplied by x and (37) by τ . Since there is no intrinsic uncertainty in our model, then such system can be rewritten as

$$\begin{pmatrix} \Delta x_{t+h} \\ \Delta \tau_{t+h} \end{pmatrix} = F(x_t, \tau_t)h + s \begin{pmatrix} \Delta \varepsilon_{t+h} \\ 0 \end{pmatrix} \quad \text{with } (x_0, \tau_0) \text{ given} \quad (60)$$

where $\mathbb{E}_t(\Delta \varepsilon_{t+h}) = 0$ with ε_t the sunspot variable (see Shigoka [29] or Benhabib and Wen [6], among others).

Suppose now that the sunspot variable, ε_t takes the values $(0, z_1, z_2)$ at dates $(0, t_1 + h, t_2 + h)$ respectively.² Therefore, the dynamics of the system in the interval of time $t \in [0, t_1 - h]$ will be described by the initial value problem (from now on IVP)

$$\begin{pmatrix} \Delta x_{t+h} \\ \Delta \tau_{t+h} \end{pmatrix} = F(x_t, \tau_t)h \quad \text{with } (x_0, \tau_0) \text{ given}$$

¹In the following we assume that the dynamics is indeed invariant to the choice of time.

²Two remarks are in order. First, a sunspot is, as usual, an unanticipated random shock from the household's perspective. Second, the values taken by the sunspot variable as well as the time of the arrival of a sunspot, are exogenously given in the example. This will be relaxed in the next section.

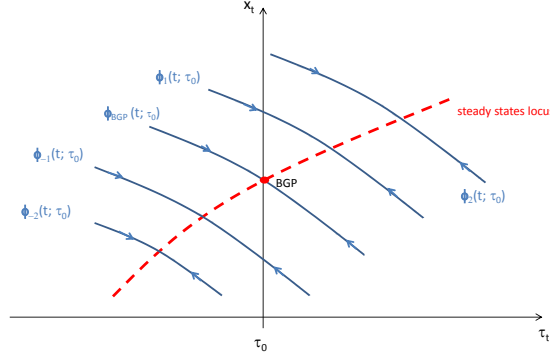


Figure 7: Examples of Saddle-Paths

which we know from Theorem 1 to have a unique solution in a neighbourhood of the steady state. In fact, such theorem tells us that *given an initial condition* the economy is on its BGP or there is a unique equilibrium path converging to an ABGP (see Figure 7). Let us indicate this equilibrium path with

$$\{x_t, \tau_t\}_{t=0}^{t_1} = \{\phi_{1x}(t), \phi_{1\tau}(t)\}_{t=0}^{t_1}.$$

At date $t_1 + h$, the second sunspot arrives and, therefore, we have that

$$\begin{pmatrix} \Delta x_{t_1+h} \\ \Delta \tau_{t_1+h} \end{pmatrix} = F(x_{t_1}, \tau_{t_1})h + s \begin{pmatrix} z_1 - 0 \\ 0 \end{pmatrix}$$

which clearly implies that

$$\begin{pmatrix} x_{t_1+h} \\ \tau_{t_1+h} \end{pmatrix} = \begin{pmatrix} \phi_{1x}(t_1) \\ \phi_{1\tau}(t_1) \end{pmatrix} + F(\phi_{1x}(t_1), \phi_{1\tau}(t_1))h + s \begin{pmatrix} z_1 \\ 0 \end{pmatrix} \quad (61)$$

where the first two terms on the RHS are obtained considering the equilibrium path found previously. Based on this observation, it follows that the dynamics of the system in the interval of time $t \in [t_1 + h, t_2 - h]$ will be given by the IVP

$$\begin{pmatrix} \Delta x_{t+h} \\ \Delta \tau_{t+h} \end{pmatrix} = F(x_t, \tau_t)h \quad \text{with } (x_{t_1+h}, \tau_{t_1+h}) \text{ given by (61)}. \quad (62)$$

Again from Theorem 1, we know that there exists a unique solution of this IVP in a neighborhood of the steady state. In fact, we proved that the dynamical system has a continuum of solutions, each of them associated with a different initial condition and, crucially, converging to a different ABGP with a different asymptotic growth rate. The presence of the sunspot variable has just modified the deterministic framework by allowing a “jump” at date $t_1 + h$ of size sz_1 in the no-predetermined variable,

as it emerges from (61),³ while the dynamics of the economy is still described by $F(\cdot)$ since the uncertainty is extrinsic and does not affect the fundamentals. The equilibrium path in the interval $t \in [t_1 + h, t_2]$ will be the solution of (62):

$$\{x_t, \tau_t\}_{t=t_1+h}^{t_2} = \{\phi_{2x}(t), \phi_{2\tau}(t)\}_{t=t_1+h}^{t_2}. \quad (63)$$

Of course, the realizations of the sunspot variable and of the parameter, s , have to be chosen, as usual, sufficiently small to guarantee that the resulting equilibrium respect all the inequalities constraints.

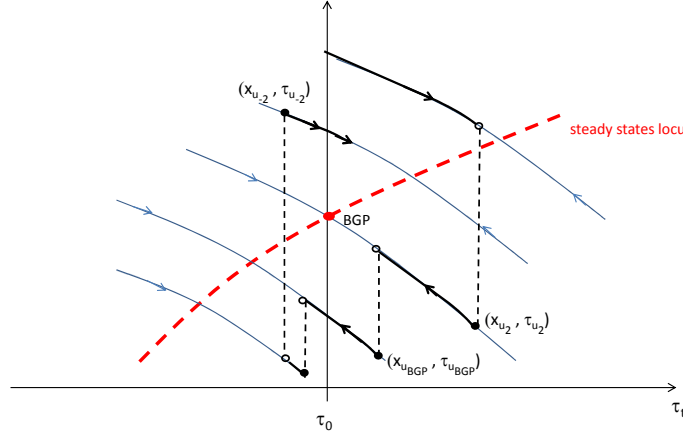


Figure 8: Example of a Sunspot Equilibrium

Similarly the equilibrium path in the interval $t \in [t_2 + h, \infty]$, will be derived as the solution of the same dynamical system $F(x_t, \tau_t)h$, where this time the given initial condition, $(x_{t_2+h}, \tau_{t_2+h})$, was obtained as it follows

$$\begin{pmatrix} x_{t_2+h} \\ \tau_{t_2+h} \end{pmatrix} = \begin{pmatrix} \phi_{2x}(t_2) \\ \phi_{2\tau}(t_2) \end{pmatrix} + F(\phi_{2x}(t_2), \phi_{2\tau}(t_2))h + s \begin{pmatrix} z_2 - z_1 \\ 0 \end{pmatrix} \quad (64)$$

Again, the presence of the sunspot variable and specifically of the change in its realization at date $t_2 + h$ implies a “jump” of size $s(z_2 - z_1)$ in the no-predetermined variable but no change in the fundamentals and hence no change in $F(\cdot)$. The equilibrium path in the interval $t \in [t_2 + h, \infty]$ will be:

$$\{x_t, \tau_t\}_{t=t_2+h}^{\infty} = \{\phi_{3x}(t), \phi_{3\tau}(t)\}_{t=t_2+h}^{\infty}. \quad (65)$$

³In continuous time, the jump of the no-predetermined (control) variable at time t_1 can be easily derived from equation (61) considering the limit $h \rightarrow 0^+$.

Under the conditions in Theorem 1, this equilibrium path will converge over time to an ABGP with an asymptotic growth rate γ_3 . An example of the resulting path is shown in Figure 8. From this example, two considerations: first, a sunspot equilibrium will be indeed a randomization over the deterministic equilibrium paths found in the previous sections; secondly, the continuous time case can be naturally derived by considering the limit $h \rightarrow 0$.

Further details on sunspot equilibria

Definition 4 (Markov property). *A family of random variables $\{\varepsilon_t\}_{t \geq 0}$ satisfies the Markov property if*

$$\mathbb{P}(\varepsilon_{t_n} = z_j \mid \varepsilon_{t_1} = z_1, \dots, \varepsilon_{t_{n-1}} = z_{n-1}) = \mathbb{P}(\varepsilon_{t_n} = z_j \mid \varepsilon_{t_{n-1}} = z_{j_{n-1}})$$

for all $z_1, \dots, z_{n-1} \in Z$ and any sequence $t_1 < t_2 < \dots < t_n$ of times.

Definition 5 (Homogeneity). *A Markov chain is homogenous if given the transition probability*

$$p_{ij}(s, t) \equiv \mathbb{P}(\varepsilon_t = z_j \mid \varepsilon_s = z_i) \quad \text{for } s \leq t$$

we have that

$$p_{ij}(s, t) = p_{ij}(0, t - s) \quad \forall i, j, s, t,$$

and we write $p_{ij}(t - s)$ for $p_{ij}(s, t)$.

Definition 6 (Stochastic semigroup). *A transition semigroup $\{\mathbf{P}_t\}_{t \geq 0}$ is stochastic if it satisfies the following:*

- i) $\mathbf{P}_0 = \mathbf{I}$;*
- ii) \mathbf{P}_t is stochastic (i.e. $p_{ij}(t) \geq 0$ and $\sum_{j=1}^N p_{ij}(t) = 1$)*
- iii) $\mathbf{P}_{s+t} = \mathbf{P}_s \mathbf{P}_t$ if $s, t \geq 0$*