Acceleration waves in a nonlinear Biot theory of porous media

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Abstract

We extend a theory of Biot to be applicable to nonlinear deformations of an elastic body which contains pores saturated by a fluid. A detailed acceleration wave analysis is presented for the full nonlinear theory.

Keywords: acceleration waves, porous media, nonlinear deformations

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1. Introduction

There is much interest in the propagation of waves in porous and acoustic media, see e.g. Biot [1], Brunnhuber and Jordan [2], Christov [3], Christov and Jordan [4], Christov et al. [5], Ciarletta and Straughan [6–8], Jordan [9–14], Jordan and Puri [15], Jordan and Saccomandi [16], Jordan et al. [17, 18], Paoletti [19], Rossmanith and Puri [20, 21], Wei and Jordan [22]. This interest is driven by the many real life applications this topic has.

Many of the early articles dealing with wave propagation in porous media were based on linear theories developed by Biot, see e.g. Biot [1].

To develop a fully nonlinear theory of acoustic wave propagation in a porous medium Jordan [10] used what may be termed an equivalent fluid theory and showed that we could analyze such propagation in a completely nonlinear framework by using an acceleration wave analysis, see also Ciarletta and Straughan [6]. These works assume the solid skeleton remains stationary.

In order to accommodate nonlinear wave motion in a porous medium with the skeleton allowed to deform or vibrate, two approaches have been employed. One is to employ a theory of a mixture of a fluid and of a solid, see e.g. De Boer and Liu [23]. The other is to employ a theory of nonlinear elasticity where the body includes voids, see e.g. Iesan [24], Ciarletta and Straughan [7, 8]. Biot [25] is critical of employing a mixture theory approach due to inherent difficulties with interacting continua based on a Eulerian description. He writes such a theory… "*lacks the required sophistication to account for all significant and essential properties of porous media.*" Chen [26] also raises doubts about the validity of acceleration wave analysis in mixture theories.

In this work we wish to address the issue of nonlinear wave motion in a porous body where we allow the body to undergo a finite deformation with an approach which is consistent with the original linear theory of Biot [1]. In order to achieve this we commence with work of Biot [25] where he develops a fully nonlinear theory for a porous medium by incorporating an equation for the pressure inside the pores in the material. Biot presents his theory in the quasi-static and isothermal context, and in particular he neglects the acceleration term in the momentum equation for the elastic body. However, he writes that his theory brings the mechanics of porous media… "*to the same level of development of the classical theory of finite deformations in elasticity.*" In this paper we generalize Biot [25] work and include the acceleration into the momentum equation.

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2. Generalized nonlinear Biot theory

To begin we employ standard indicial notation and the Einstein summation convention for repeated indices. Points in the reference configuration are denoted by X_A and these are mapped into the current configuration by the mapping

$$x_i = x_i(X_A, t). \tag{1}$$

The deformation gradient F_{iA} is defined by

$$F_{iA} = \frac{\partial x_i}{\partial X_A}. \tag{2}$$

The displacement vector u_i is given by

$$u_i = x_i - X_i. \tag{3}$$

We employ the ideas of Biot [25] but include acceleration in the momentum equation which may be written

$$\rho \ddot{x}_i = \frac{\partial \Pi_{Ai}}{\partial X_A} + \rho f_i. \tag{4}$$

In this equation ρ is the density, f_i is the body force and Π_{Ai} is the Piola-Kirchhoff stress tensor. In terms of the Helmholtz free energy function ψ the Piola-Kirchhoff stress tensor is defined as

$$\Pi_{Ai} = \rho \frac{\partial \psi}{\partial F_{iA}}. \tag{5}$$

Let the elastic body contain pores and denote the pressure in the pores by $p(\mathbf{X}, t)$. The theory of Biot [25] assumes the constitutive relation

$$\psi = \psi(F_{iA}, p, X_R), \tag{6}$$

where inclusion of X_R allows the body to be inhomogeneous.

We essentially follow Biot [25] and write a conservation law for the pressure as

$$\frac{\partial m}{\partial t} = \frac{\partial K_B}{\partial X_B}. \tag{7}$$

Here m is a function based on the pressure distribution in the pores and K_B is the associated flux. The constitutive theory for m and K_B is

$$\begin{aligned} K_B &= K_B(F_{iA}, p, p_{,C}, X_R), \\ m &= m(F_{iA}, p, X_R). \end{aligned} \tag{8}$$

The governing equations for the theory are thus (4), (7) and the balance of mass. By employing (5), (6) and (8), equation (4) and (7) may be written in expanded form as

$$\rho \ddot{x}_i = \frac{\partial \Pi_{Ai}}{\partial F_{jB}} F_{jB,A} + \frac{\partial \Pi_{Ai}}{\partial p} p_{,A}, \tag{9}$$

$$\frac{\partial m}{\partial F_{iA}} \dot{F}_{iA} + \frac{\partial m}{\partial p} \dot{p} = \frac{\partial K_B}{\partial F_{iA}} F_{iA,B} + \frac{\partial K_B}{\partial p} p_{,B} + \frac{\partial K_B}{\partial p_{,Q}} p_{,QB}. \tag{10}$$

In equation (9) we have taken the externally supplied body force f_i to be 0.

3. Nonlinear acceleration waves

An acceleration wave $\mathscr{S}$ for equation (9) and (10) is a two-dimensional surface in $\mathbb{R}^3$ such that x_i, p are C^1 in $\mathbf{X}$ and t everywhere whereas $\ddot{x}_i, \dot{x}_{i,A}, x_{i,AB}, \ddot{p}, \dot{p}_{,A}$ and $p_{,AB}$ suffer a finite discontinuity across $\mathscr{S}$, along with their higher derivatives.

The jump of a function $f(\mathbf{X}, t)$ across $\mathscr{S}$ is defined by

$$[f(t)] = f^-(\mathbf{X}, t) - f^+(\mathbf{X}, t), \qquad \mathbf{X} \in \mathscr{S},$$

where

$$f^-(\mathbf{X}, t) = \lim_{\mathbf{X} \to \mathscr{S}} f(\mathbf{X}, t)$$

approaching $\mathscr{S}$ from the left, whereas

$$f^+(\mathbf{X}, t) = \lim_{\mathbf{X} \to \mathscr{S}} f(\mathbf{X}, t)$$

approaching $\mathscr{S}$ from the right.

Given the definition of an acceleration wave for equations (9) and (10) we take the jumps of these equations to obtain

$$\rho[\ddot{x}_i] = \frac{\partial \Pi_{Ai}}{\partial F_{jB}} [x_{j,AB}] \tag{11}$$

and

$$\frac{\partial m}{\partial F_{iA}} [\dot{x}_{i,A}] = \frac{\partial K_B}{\partial F_{iA}} [x_{i,AB}] + \frac{\partial K_B}{\partial p_{,Q}} [p_{,QB}]. \tag{12}$$

To progress from this point we need compatibility conditions across $\mathscr{S}$, see Chen [27], section 4. In particular compatibility relations and the Hadamard condition lead to the equation

$$[x_{j,AB}] = \frac{N_A N_B}{U_N^2} [\ddot{x}_j] \tag{13}$$

where N_A is the unit normal at a point on $\mathscr{S}$ referred to the reference configuration and U_N is the speed of $\mathscr{S}$ at that point.

Define the amplitudes across $\mathscr{S}$, $A_i(t)$ and $P(t)$ by

$$A_i(t) = [\ddot{x}_i] \qquad \text{and} \qquad P(t) = [\ddot{p}]. \tag{14}$$

By employing (13) in (11) one may arrive at

$$(\rho U_N^2 \delta_{ij} - Q_{ij}) A_j = 0, \tag{15}$$

where Q_{ij} is the acoustic tensor defined by

$$Q_{ij} = \frac{\partial \Pi_{Ai}}{\partial F_{jB}} N_A N_B. \tag{16}$$

We see that for an acceleration wave to propagate, the elastic wave amplitude A_i has to be an eigenvector of the acoustic tensor Q_{ij}.

Let λ_i be the unit vector in the direction of A_i pointing to the right, so that $A_i = A(t)\lambda_i$. Truesdell [28] has established existence theorems for the propagation of acceleration waves in classical nonlinear elasticity. To relate these to (15) we note that the unit normals to $\mathscr{S}$ in the current and reference configurations are connected by the formula

$$\mathbf{N} = \frac{\mathbf{F}^T \mathbf{n}}{|\mathbf{F}^T \mathbf{n}|} \tag{17}$$

see Chen [27], p. 317. Thus we can write Q_{ij} in (16) as either a function of $\mathbf{N}$ or of $\mathbf{n}$. We follow Chen and write $\hat{\mathbf{Q}}(\mathbf{N})$ when referring to the reference configuration and $\bar{\mathbf{Q}}(\mathbf{n})$ when the current configuration is considered. Then one Truesdell result shows that if the body has positive longitudinal elasticity, i.e.

$$n_i \bar{Q}_{ij}(\mathbf{n}) n_j > 0$$

for all unit vectors n_i there is at least one direction in which a longitudinal wave may exist and propagate, see Chen [27], p. 317. In this case λ_i may be taken as n_i.

While a longitudinal wave has $\lambda_i = n_i$ there are also transverse waves for which $\lambda_i = t_i$ with $t_i n_i = 0$. If the material is strongly elliptic so that

$$m_i \bar{Q}_{ij}(\mathbf{n}) m_j > 0$$

for all unit vectors $\mathbf{m}$ and $\mathbf{n}$, then another theorem of Truesdell [28] shows that there is at least one direction in which a longitudinal wave and two transverse waves with orthogonal amplitude vectors will exist, see Chen [27], p. 322.

Thus, there are definite conditions guaranteeing the propagation of an acceleration wave from (15). From (15) the wavespeed U_N is given by

$$\rho U_N^2 = |\hat{\mathbf{Q}}(\mathbf{N})\mathbf{n}| \tag{18}$$

cf. Chen [27], p. 317, or if we know λ_i, then

$$\rho U_N^2 = Q_{ij}\lambda_i\lambda_j. \tag{19}$$

From equation (12) we may use the compatibility conditions and the definitions of A_i and P to derive the equation

$$\frac{\partial K_B}{\partial p_{,Q}} N_Q N_B P = -\left(N_A U_N \frac{\partial m}{\partial F_{iA}} + N_A N_B \frac{\partial K_B}{\partial F_{iA}} \right) A_i. \tag{20}$$

Thus, once we know A_i, the pressure amplitude P is likewise known.

We turn next to calculate the wave amplitude. To do this we restrict attention to a plane wave propagating in a single direction. This allows to work in one space dimension but we still retain key physics which is not obscured by differential geometry as in the full three-dimensional scenario, cf. Lindsay and Straughan [29].

4. Wave amplitudes

For a one-dimensional plane wave we may rewrite the governing equations (4) and (7) as

$$\rho \ddot{u} = \frac{\partial \Pi}{\partial X}, \qquad \frac{\partial m}{\partial t} = \frac{\partial K}{\partial X} \tag{21}$$

where $u = x - X$ and Π, m and K are the one-dimensional counterparts of Π_{Ai}, m and K_A. With $F = \partial x/\partial X$ the constitutive theory is

$$\Pi = \rho\psi_F, \quad \psi = \psi(F,p), \quad K = K(F,p,p_X), \quad m = m(F,p).$$

Then the wavespeed equation (15) is essentially replaced by

$$(\rho U_N^2 - \Pi_F)[u_{XX}] = 0,$$

and so

$$U_N^2 = \Pi_F/\rho. \tag{22}$$

The amplitudes are now given by

$$A(t) = [\ddot{u}], \qquad P(t) = [\ddot{p}]$$

and P is given in terms of A by

$$P = -\frac{m_F U_N + K_F}{K_{pX}}. \tag{23}$$

To calculate the amplitude A (and hence P) we differentiate $(21)_1$ with respect to t and we suppose the wave is moving into a region where F_X and p are constant. Then p^+ and $F^+ = u_X^+$ are constants. Upon taking the jump of the differentiated form of $(21)_1$ we then derive

$$\rho[\ddot{u}] = \Pi_F[\dot{u}_{XX}] + \Pi_p[\dot{p}_X] + \Pi_{FF}[u_{XX}\dot{u}_X]. \tag{24}$$

We next use the jump relation for a product

$$[fg] = f^+[g] + g^+[f] + [f][g]$$

in (24) together with the Hadamard relation and compatibility relations to derive from equation (24)

$$2\rho\frac{\delta A}{\delta t} = -\Pi_p\frac{P}{U_N} - \Pi_{FF}\frac{A^2}{U_N^3}, \tag{25}$$

where $\delta/\delta t$ is the intrinsic derivative at the wavefront. Finally we substitute for P from (23) to derive

$$\frac{\delta A}{\delta t} - \alpha A + \beta A^2 = 0, \tag{26}$$

where α and β are given by

$$\alpha = \frac{(m_F U_N + K_F)\Pi_p}{2\rho U_N K_{pX}}$$

and

$$\beta = \frac{\Pi_{FF}}{2\rho U_N^3}.$$

Given the conditions ahead of the wave the coefficients α and β are constants and so we solve (26) to obtain

$$A(t) = \frac{A(0)}{e^{-\alpha t} + \{\beta A(0)/\alpha\}(1 - e^{-\alpha t})}. \tag{27}$$

When $A(0) < 0$ equation (27) suggests that $A(t)$ blows up in a finite time $\mathcal{T}$ where

$$\mathcal{T} = \frac{1}{\alpha}\log\left(\frac{\beta A(0) - \alpha}{\beta A(0)}\right).$$

5. Conclusions

We have extended the development of Biot [25] for a fully nonlinear theory for a porous medium by including the acceleration term in the momentum equation. This then yields a fully nonlinear theory which is amenable to a complete analysis of wave motion by acceleration wave theory.

We analyzed wavespeed behaviour and determined the wavespeed exactly. The question of existence of a propagation direction is discussed, and we then obtain the wave amplitude exactly for a plane wave propagating in one dimension. It is noteworthy that our wave results are exact and no approximation nor linearization is performed.

For the amplitude of the wave we find that if $A(0) < 0$ then $A(t) \to \infty$ in a finite time. To interpret this we recall that

$$A(t) = [\ddot{u}] = \ddot{u}^- - \ddot{u}^+.$$

From the compatibility relations one has

$$[\ddot{u}] = -U_N[\dot{u}_X] = U_N^2[u_{XX}].$$

When the wave is moving into a region at rest one sees that

$$[\ddot{u}] = U_N^2 u_{XX}^-.$$

Thus $A(0) < 0$ is equivalent to $u_{XX}^-(0) < 0$ and under this condition we see that $u_{XX}^-(t) \to -\infty$ in a finite time. This suggests that blow up of the wave amplitude is leading to shock formation in a finite time.

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