Hyperbolic diffusion with Christov-Morro theory

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Abstract

We employ recent ideas of C.I. Christov and of A. Morro to develop a theory for diffusion of a solute in a Darcy porous medium taking convection effects into account. The key point is that the solute evolution is not governed by a parabolic system of equations. Indeed, the theory developed is basically hyperbolic. This still leads to a model wich allows for convective (gravitational) overturning in a porous layer, but in addition to the classical mode of stationary convection instability there is the possibility of oscillating convection being dominant for a lower salt Rayleigh number, if the relaxation time is sufficiently large.

Keywords: Convection in porous media; Hyperbolic diffusion; Christov concentration flux equation; Linear Instability; Oscillatory convection.

1 Introduction

The problem of movement of a solute in a porous medium is one with many applications in pollution, transport of radio-active waste, and many other things. There are many models for movement of a solute but typically these are based on Fick's law and lead to a parabolic equation, or a system of such equations, cf. Graffi [16], Prouse & Zaretti [26], Franchi & Straughan [12], Straughan [28]. Also, there are many detailed numerical analyses and simulations of solute transport in such systems, see e.g. Ewing et al. [7, 8, 9].

Recently, there has been a lot of interest in developing and analysing models for solute transport which have hyperbolic characteristics rather than parabolic. For example, the early work of Galenko & Danilov [13], Sobolev [27], and a very useful review may be found in Galenko & Jou [14]. A parallel mathematical analysis development has also taken place, see e.g. Gatti et al. [15], Bonetti et al. [1], Grasselli et al. [17], Wu et al. [32], and a good review of the mathematical literature is contained in Jiang [20], see also section 9 of Straughan [31]. The development of hyperbolic transport equations follows the analogous hyperbolic heat propagation theory which is analysed in the artcles of Christov & Jordan [3], Christov & Jordan [6], Jordan [22], Jordan et al [21] and the many references therein.

There are many novel applications of a hyperbolic theory of solute transport. For example, we mention material transfer in stars, Herrera & Falcón [19], Falcón [10], Straughan [30], drug delivery in the human body, Ferreira & de Oliveira [11], and other biological and medical applications are considered in chapter 9 of the book by Straughan [31]. Thus, we believe there is strong motivation to develop and analyse a hyperbolic theory of convective overturning of a solute in a layer of a porous material. To do this we employ recent work of Christov [4, 5] who advocate the use of a Lie derivative and Morro [23, 24] who develops a thermodynamically consistent theory which is compatible with Christov's derivative.

2 The model

Let $\hat{\phi}$ be the porosity in a porous medium, cf. Straughan [28], p.1, and let V_i be the actual velocity of fluid in the pores of a saturated porous medium. Denote by $v_i = \hat{\phi} V_i$ the pore averaged velocity. Then, for a solute dissolved in the fluid, with concentration $C(\mathbf{x}, t)$ Darcy's law governing the velocity field is, cf. Straughan [28], pp.10-12,

$$0 = -\frac{\partial p}{\partial x_i} - \frac{\mu}{k} v_i - \rho_0 g k_i \alpha C \,, \tag{1}$$

where $p, \mu, k, \rho_0, g, \alpha$ are the pore averaged pressure, dynamic viscosity of the fluid, the permeability, the constant density (employing a Boussinesq approximation), gravity, and the coefficient of salt dependence in the density law. Here $\mathbf{k} = (0, 0, 1)$ and standard indicial notation will be employed throughout. In addition the fluid is incompressible, so

$$v_{i,i} = 0 \,. \tag{2}$$

To describe solute movement in the pores we suppose the solute concentration satisfies the equations

$$\frac{\partial C}{\partial t} + V_i \frac{\partial C}{\partial x_i} = -\frac{\partial J_i}{\partial x_i} \,, \tag{3}$$

and

$$\begin{gathered} \tau \left(\frac{\partial J_i}{\partial t} + V_j \frac{\partial J_i}{\partial x_j} - J_j \frac{\partial V_i}{\partial x_j} \right) = \\ -k_c \frac{\partial C}{\partial x_i} - J_i + \xi_1 \Delta J_i + \xi_2 J_{k,ki} \,. \end{gathered} \tag{4}$$

In these equations $\mathbf{J}$ is the solute flux, equation (3) expresses concentration balance, and (4) is a generalization of Fick's law. The parameter $\tau(> 0)$ is a relaxation time, k_c is a coefficient of solute diffusion, the derivative on the left in (4) is proposed by Christov [4], $\xi_1, \xi_2 > 0$ are coefficients corresponding to terms introduced by Morro [23]. (We remark that when $\tau = 0$ and $\xi_1 = \xi_2 = 0$, equations (3) and (4) reduce to the classical equation of Fick's law and parabolic transport of solute.)

In keeping with equation (1), it is convenient to rewrite (3) and (4) in terms of the pore averaged velocity v_i so that we have

$$\hat{\phi}\frac{\partial C}{\partial t} + v_i C_{,i} = -\hat{\phi} J_{i,i}\,, \tag{5}$$

$$\begin{gathered}\tau\left(\hat{\phi}\frac{\partial J_i}{\partial t} + v_j\frac{\partial J_i}{\partial x_j} - J_j\frac{\partial v_i}{\partial x_j}\right) = \\ -\hat{\phi}J_i - \hat{\phi}k_c C_{,i} + \hat{\phi}\xi_1\Delta J_i + \hat{\phi}\xi_2 J_{k,ki}\,.\end{gathered} \tag{6}$$

The complete model for solute transport in a porous material consists of equations (1),(2),(5) and (6), and is a system in the variables v_i, p, C and J_i.

3 Overturning instability

Suppose a porous medium governed by equations (1),(2),(5) and (6) occupies the region $\mathbb{R}^2 \times \{z \in (0,d)\}$, with gravity acting in the downward direction. We are primarily interested in whether a top heavy layer will be unstable.Thus, we consider the situation with the boundary conditions

$$v_i n_i = 0\,, \quad z = 0, d, \tag{7}$$

and

$$C = C_L,\, z = 0\,, \quad C = C_U,\, z = d\,, \tag{8}$$

where $C = C_L$ and $C = C_U$ are constants and $\mathbf{n}$ is the unit outward normal. For conditions on the flux $\mathbf{J}$ we assume zero tangential component of flux so that, in general, $\epsilon_{ijk}J_j n_k = 0$. In the case of a plane layer this results in

$$J_1 = J_2 = 0\,, \quad z = 0, d\,. \tag{9}$$

We seek a steady solution $\{\bar{v}_i, \bar{C}, \bar{J}_i, \bar{p}\}$ to (1),(2), (5)-(9), such that $\bar{v}_i \equiv 0$ and $\bar{C} = \bar{C}(z)$. Hence, $\bar{v}_i \equiv 0$

$$\bar{C} = C_L + \beta z\,, \tag{10}$$

where $\beta = (C_U - C_L)/d$, and $\bar{J}_3 = -\beta k_c$, $\bar{J}_1 = \bar{J}_2 = 0$. For the most part we consider $\beta > 0$, although the case $\beta < 0$ (which should be stable) is briefly investigated.

Next, introduce perturbations $\{u_i, \phi, \delta_i, \pi\}$ to $\{\bar{v}_i, \bar{C}, \bar{J}_i, \bar{p}\}$ such that

$$v_i = \bar{v}_i + u_i\,, \quad C = \bar{C} + \phi\,,$$

$$J_i = \bar{J}_i + \delta_i\,, \quad p = \bar{p} + \pi\,.$$

The perturbation equations are found to be

$$
\begin{aligned}
&0 = -\pi_{,i} - \frac{\mu}{k} u_i - \rho_0 g \alpha k_i \phi \,, \\
&u_{i,i} = 0 \,, \\
&\hat{\phi}\phi_{,t} + u_i \phi_{,i} + \beta w = -\hat{\phi}\delta_{i,i} \,, \\
&\tau \left(\hat{\phi}\delta_{i,t} + u_j \delta_{i,j} + \beta k_c u_{i,z} - \delta_j u_{i,j} \right) = \\
&-\hat{\phi}\delta_i - \hat{\phi} k_c \phi_{,i} + \hat{\phi}\xi_1 \, \Delta\delta_i + \hat{\phi}\xi_2 \, \delta_{k,ki} \ .
\end{aligned}
\tag{11}
$$

where $w = u_3$.
These equations are non-dimensionalized by writing

$$
\begin{aligned}
&u_i = u_i^* V, \ \phi = \phi^* C^\sharp, \ \delta_i = \delta_i^* \Xi, \ \pi = \pi^* P \,, \\
&\Xi = \frac{k_c C^\sharp}{d}, \ x_i = x_i^* d, \ t = t^* \mathcal{T}, \ Da = \frac{k}{d^2} \,, \\
&Sg = \frac{\tau\mu}{\rho_0 d^2}, \ Ps = \frac{\mu}{\rho_0 k_c}, V = \frac{\hat{\phi} k_c}{d} \,, \\
&R = \sqrt{\frac{d^2 \beta \rho_0 g \alpha k}{\hat{\phi}\mu k_c}}, \ C^\sharp = V \sqrt{\frac{d^2 \beta \mu}{\hat{\phi}\rho_0 g \alpha k k_c}} \,, \\
&\mathcal{T} = \frac{\rho_0 k}{\mu}, \ P = \frac{\mu V d}{k}, \ \lambda_1 = \frac{\xi_1}{d^2}, \ \lambda_2 = \frac{\xi_2}{d^2} \,,
\end{aligned}
$$

the asterisks denoting non-dimensional quantities. Here, Da is the Darcy number, Ps is the salt Prandtl number, $Ra = R^2$ is the salt Rayleigh number, and Sg is a parameter introduced in Papanicolaou et al. [25].

For convenience we now drop all asterisks and the non-dimensional perturbation equations become,

$$
\begin{aligned}
&\frac{Sg}{Da}\delta_{i,t} + \frac{Sg}{Ps}\left(u_j \delta_{i,j} - \delta_j u_{i,j} \right) \\
&+ \frac{Sg\, R}{Ps} u_{i,z} = -\delta_i - \phi_{,i} + \lambda_1 \Delta\delta_i + \lambda_2 \delta_{k,ki} \,, \\
&u_{i,i} = 0 \,, \\
&\frac{Ps}{Da}\phi_{,t} + u_i \phi_{,i} = -Rw - \delta_{i,i} \,, \\
&0 = -\pi_{,i} - u_i - k_i R \phi \,.
\end{aligned}
\tag{12}
$$

(When $\beta < 0$ the Rw term in $(12)_3$ has a positive sign).

The appropriate boundary conditions become:

$$w = 0,\ \theta = 0,\ \delta_1 = \delta_2 = 0,\ z = 0, 1\,, \tag{13}$$

and $u_i, \theta, \delta_i, \pi$ satisfy a plane tiling periodicity in x and y.

We next take $\partial/\partial x_i$ of $(12)_1$, set $\zeta = \delta_{i,i}$ and then remove π by taking curlcurl of $(12)_4$ and retaining the third component. Next linearize and assume forms $w = e^{\sigma t}w(\mathbf{x})$, $\phi = e^{\sigma t}\phi(\mathbf{x})$, $\zeta = e^{\sigma t}\zeta(\mathbf{x})$, and this results in the system

$$\begin{aligned} 0 &= \Delta w + R\Delta^* \phi\,, \\ \sigma\frac{Ps}{Da}\phi &= -Rw - \zeta\,, \\ \sigma\frac{Sg}{Da}\zeta &= -\zeta - \Delta\phi + (\lambda_1 + \lambda_2)\Delta\zeta\,, \end{aligned} \tag{14}$$

where $\Delta^* = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

Now eliminate ϕ and ζ and then write $w = W(z)f(x, y)$ where f is a plane tiling planform such that

$$\Delta^* f = -a^2 f.$$

Then write $W(z) = \sin n\pi z$. We may do this since we only seek the most unstable mode, and from (14) by repeated differentiation and evaluation on the boundaries $z = 0, 1$, a sin series representation suffices. In this manner we reduce (14) to

$$\begin{aligned} &\sigma^2 \frac{Ps\,Sg}{Da^2}\Lambda + \sigma\Big[\frac{Ps}{Da}(\lambda_1+\lambda_2)\Lambda^2 + \frac{Ps}{Da}\Lambda \\ &- \frac{Sg}{Da}R^2a^2\Big] + \Lambda^2 = R^2a^2\big[(\lambda_1+\lambda_2)\Lambda + 1\big]\,, \end{aligned} \tag{15}$$

where $\Lambda = n^2\pi^2 + a^2$.

For stationary convection one puts $\sigma = 0$. Then (15) yields

$$R^2 = \frac{\Lambda^2}{a^2[(\lambda_1+\lambda_2)\Lambda + 1]}\,. \tag{16}$$

By taking $\partial R^2/\partial n^2$ we see $n = 1$ is the most unstable and put $n = 1$. Then dR^2/da^2 leads to the critical value a_c of a as

$$a_c^2 = \frac{\pi^2[(\lambda_1+\lambda_2)\pi^2 + 1]}{[1 - (\lambda_1+\lambda_2)\pi^2]} \tag{17}$$

and so we require $\lambda_1 + \lambda_2 < \pi^{-2}$. Then substitution of (17) in (16) yields

$$R_{sc}^2 = \frac{4\pi^2}{[1 + (\lambda_1+\lambda_2)\pi^2]^2} \tag{18}$$

Remarks

1. When $\lambda_1 = \lambda_2 = 0$ we recover $R^2_{sc} = 4\pi^2$, cf Straughan [29].
2. When we consider $C_L > C_U$, (16) leads to $R^2 < 0$ and the system is, as expected, stable.

To study oscillatory convection we put $\sigma = i\sigma_1$, $\sigma_1 \in \mathbb{R}$ cf. Chandrasekhar [2]. Then taking the imaginary part of (15) we find

$$R^2 = \frac{Ps}{Sg}(\lambda_1 + \lambda_2)\frac{(\pi^2 + a^2)^2}{a^2} + \frac{Ps}{Sg}\,\frac{\pi^2 + a^2}{a^2}\,. \tag{19}$$

By calculating dR^2/da^2 we then find

$$a_c^2 = \pi\sqrt{\frac{(\lambda_1 + \lambda_2)\pi^2 + 1}{\lambda_1 + \lambda_2}}\,, \tag{20}$$

and thus

$$\begin{aligned} R^2_{osc} \quad &= \frac{Ps}{Sg}\left[\frac{X\left(\pi^2 + \sqrt{\pi^4 + \pi^2/X}\right)^2}{\sqrt{\pi^4 + \pi^2/X}} + \frac{\pi^2 + \sqrt{\pi^4 + \pi^2/X}}{\sqrt{\pi^4 + \pi^2/X}}\right] \\ &= \frac{Ps}{Sg}\left(\frac{\pi^3 X^{3/2}}{\sqrt{\pi^2 X + 1}} + 2\pi^2 X + \pi\sqrt{X}\sqrt{\pi^2 X + 1} + \frac{\pi\sqrt{X}}{\sqrt{\pi^2 X + 1}} + 1\right)\,, \end{aligned} \tag{21}$$

where $X = \lambda_1 + \lambda_2$.

Remarks

1. When $C_U < C_L$, we are again led to $R^2 \leq 0$ in (19) and so no instability.
2. When $X = 0\,,\ R^2_{osc} = Ps/Sg$ cf. Straughan [29].

4 Conclusions

We have developed a model for convective overturning of a top heavy fluid saturated porous medium allowing hyperbolic diffusion of the salt field according to the prescription of Christov [4] and of Morro [23].

Equations (18) and (21) show that for λ_1, λ_2 and Sg sufficiently small the critical salt Rayleigh number is given by (18). However from (21) when X is small there is a critical value of $Sg = Sg_c$ (and hence τ) such that once $Sg > Sg_c$ instability is by oscillatory convection rather than stationary convection. For $Sg > Sg_c$ the critical value of R^2 is given by (21) and R^2_{crit} decreases as Sg increases (other parameters fixed). This means that as τ increases the system ceases to be governed by the parabolic character of the problem and for τ sufficiently large the eigenvalues generating instability swap places and the complex eigenvalue dominates. Thus, the hyperbolic nature of the system is then dictating instability. However, we observe that the effect of the λ_1, λ_2 terms is to increase R^2_{osc} and so make the likelihood of oscillatory convection less probable. Thus, there is an important interplay between Sg, λ_1 and λ_2 in determining whether stationary or oscillatory convection occurs.

At present accurate values of τ, ξ_1, ξ_2 are not easy to find. However, for a binary alloy system one has $10^{-11}s < \tau < 10^{-7}s$, cf. Straughan [31], p.253, whereas $\xi_1, \xi_2 = O(\ell^2)$ where ℓ is the mean free path of the heat carriers (phonons). For carbon nanotubes $\ell = O(10^{-9}m)$, cf. Hepplestone & Srivastava [18], and for a layer of depth 1cm we might then expect values of $\lambda_1, \lambda_2 = O(10^{-8})$ while $Sg = O(10^{-5})$. This suggests stationary convection is still likely to be the dominant mechanism in laboratory experiments. However, the situation may be very different in stars, Herrera & Falcón [19].

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