Horizontally isotropic bidispersive thermal convection

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Abstract

A bidispersive porous material is one which has usual pores but additionally contains a system of micro pores due to cracks or fissures in the solid skeleton. We present general equations for thermal convection in a bidispersive porous medium when the permeabilities, interaction coefficient, and thermal conductivity are anisotropic but symmetric tensors. In this case we show exchange of stabilities holds and fluid movement will commence via stationary convection, and additionally we show the global nonlinear stability threshold is the same as the linear instability one. Attention is then focussed on the case where the interaction coefficient and thermal conductivity are isotropic, and the permeability is isotropic in the horizontal directions, although the permeability in the vertical direction is different. The nonlinear stability threshold is calculated in this case and numerical results are presented and discussed in detail.

1 Introduction

Double porosity, or bidispersive, materials are occupying much research attention. A double porosity material is one where there are the usual pores, known as macro pores, but there are also cracks or fissures in the solid skeleton and these give rise to a micro porosity. Theories for non-isothermal fluid flow in a bidispersive porous material were developed by Nield & Kuznetsov [1, 2, 3], see also Nield [4], and these theories which allow for independent velocity, pressure and temperature fields in the macro and micro phases are described in detail in the books by Straughan [5, 6]. A simpler theory which retains independent velocity and pressure fields but restricts attention to a single temperature field, $T(\mathbf{x}, t)$, where $\mathbf{x}$ is a spatial variable and t denotes time, is developed from the Nield & Kuznetsov [1] model in Falsaperla *et al.* [7] and further studies in this theory are analysed in Gentile & Straughan [8, 9].

Fluid flow, and in particular, thermal convection in a single porosity fluid saturated porous medium has been the subject of much recent research. Particular interest has been in cases where the permeability is anisotropic, cf. Capone *et al.* [10, 11, 12], Harfash [13], Harfash & Hill [14], Hill & Morad [15], Karmakar & Raja Sekhar [16], Rees *et al.* [17, 18], Straughan & Walker [19]. A case of special practical interest is where the permeability is isotropic in the horizontal, x, y, directions but has a different value in the vertical, z, direction. This is known as horizontally isotropic permeability and thermal convection in this theory is analysed in e.g. Capone *et al.* [10, 11, 12], and in Hill & Morad [15]. In general when the permeability in one direction is different from that in the orthogonal directions the material is known as transversely isotropic. We adopt the notation horizontally isotropic permeability to distinguish with the general case of transverse isotropy where the axis is not necessarily in the vertical direction, cf. Tyvand & Storesletten [20], Straughan & Walker [19].

The major reason for the interest in horizontally isotropic permeability is that many real life porous materials possess this property, such as soils or rocks, and as Karmakar & Raja Sekhar [16] point out, analysis of such porous media are important in hydrocarbon recovery.

Let us denote by K_V the value of the vertical permeability and by K_H the value of the horizontal permeability in a horizontally isotropic porous medium. We define the ratio ℓ^2 by $\ell^2 = K_H/K_V$. Fazelalani [21] analyses various real rocks and while K_V/K_H is often less than one there are cases where $K_V/K_H = 2.4484$ and $K_V/K_H = 82.624$. Ayan *et al.* [22] study many routine core samples and find K_V/K_H often has small values, for example, for limestone they report values of 0.0362, 0.03679 and 0.270. They also write that, *...the importance of permeability anisotropy to sound reservoir management is not in dispute ... so that reservoir models may be refined, leading to better field development strategies, such as enhanced oil recovery programs and infill well placement.* Widarsono *et al.* [23] analyse rock samples from Indonesia and they report a range of K_V/K_H values for sandstone and for carbonate rocks, the values they find being typically in the range $0 < K_V/K_H < 1.2$ for sandstone, whereas $0 < K_V/K_H < 4.2$ for carbonate rocks.

Application areas for double porosity media, and especially those with anisotropic permeability, are many and include carbon sequestration, Hill & Morad [15], Carneiro [24]; landslides, Borja *et al.* [25], Montrasio *et al.* [26], Scotto di Santolo & Evangelista [27]; hydraulic fracturing for natural gas ("fracking"), Kim & Moridis [28]; drinking water recovery from an aquifer, Zuber & Motyka [29], Ghasemizadeh *et al.* [30]; oil recovery from an underground reservoir, Olusola *et al.* [31], Karmakar & Raja Sekhar [16]; and nuclear waste treatment, Said *et al.* [32]. These and many other applications are discussed in the books by Straughan [5, 6].

The purpose of the present work is to present what we believe is the first analysis of thermal convection in a bidispersive porous medium allowing for an anisotropic permeability. We begin with a general case where the permeability tensors for the macro and micro phases are symmetric along with the interaction coefficients between the macro and micro velocities. We show in the general case that the onset of convection is by stationary convection and oscillatory convection will not occur. We then show that an analysis of the theory of linear instability will also guarantee nonlinear stability. This is a powerful result which shows the linear theory is correctly capturing the physics of the onset of thermal convection. To establish the coincidence of the linear instability and nonlinear global stability threshold we employ an energy method. Such techniques are very much in vogue in the literature, see e.g. Hill & Morad [15], Hill & Malashetty [33], Hill & Carr [34, 35], Matta *et al.* [36], Amendola & Fabrizio [37], Amendola *et al.* [38], Deepika & Narayana [39], Nandal & Mahajan [40]; see also the books by Straughan [5, 41]. We conclude with a detailed numerical analysis of thermal convection in a horizontally isotropic bidispersive porous medium where we study the effects of the term $\ell^2 = K_H/K_V$, the constant interaction parameter which arises due to fluid interactions between the macro and micro pores, and the ratio of permeabilities in the macro and micro phases.

2 Equations of motion

We let the porous medium be contained in the horizontal layer $0 < z < d$ and the temperatures on the boundaries are kept at fixed constant values, $T = T_L$ at $z = 0$, $T = T_U$ at $z = d$, where $T_L > T_U$, where all unscaled temperatures are in degrees Celsius.

Let the porosity of the macropores be ϕ, i.e. ϕ is the ratio of the volume of the macropores to the total volume of the fluid saturated porous material. The microporosity is denoted by ϵ, i.e. ϵ is the volume occupied by the micropores to the volume of the porous body which remains once the macropores are removed. Hence, the fraction of volume occupied by the micropores is $\epsilon(1-\phi)$ and the fraction of volume occupied by the solid skeleton is $(1-\epsilon)(1-\phi)$.

We let U_i^f and U_i^p be fluid velocities in the macropores and micropores, respectively. In general, a sub or superscript f denotes fluid in the macropores whereas a sub or superscript p denotes fluid in the micropores. The temperature in the porous medium is T. If we allow for general anisotropic permeabilities and interaction coefficients then the governing equations for non-isothermal flow in an anisotropic bidispersive porous medium, employing a Boussinesq approximation, may be derived from Gentile & Straughan [8] as

$$\begin{aligned}
&- M_{ij}^f U_j^f - \zeta_{ij}(U_j^f - U_j^p) - p_{,i}^f + \rho_F \alpha g T k_i = 0, \qquad U_{i,i}^f = 0,\\
&- M_{ij}^p U_j^p - \zeta_{ij}(U_j^p - U_j^f) - p_{,i}^p + \rho_F \alpha g T k_i = 0, \qquad U_{i,i}^p = 0,\\
&(\rho c)_m T_{,t} + (\rho c)_f (U_i^f + U_i^p) T_{,i} = \kappa_{ij}^m T_{,ij}\,.
\end{aligned} \tag{1}$$

In equations (1), p^f and p^p are the pressures in the macro and micro pores, ζ_{ij} are interaction coefficients, ρ_F, α, g are a reference density, coefficient of thermal expansion, and gravity, and $\mathbf{k} = (0, 0, 1)$. Throughout we employ standard indicial notation in conjunction with the Einstein summation convention, and subscript $,i$ denotes $\partial/\partial x_i$. The quantities $(\rho c)_m$ and κ_{ij}^m are given by, cf.

Falsaperla *et al.* [7],

$$\begin{aligned}(\rho c)_m &= (1-\phi)(1-\epsilon)(\rho c)_s + \phi(\rho c)_f + \epsilon(1-\phi)(\rho c)_p\,,\\ \kappa^m_{ij} &= (1-\phi)(1-\epsilon)\kappa^s_{ij} + \phi\kappa^f_{ij} + \epsilon(1-\phi)\kappa^p_{ij}\,,\end{aligned} \tag{2}$$

where c denotes the specific heat at constant pressure, and s denotes the solid skeleton. The tensors M^f_{ij} and M^p_{ij} are given by $M^f_{ij} = \mu(K^f_{ij})^{-1}$ and $M^p_{ij} = \mu(K^p_{ij})^{-1}$, where K^f_{ij} and K^p_{ij} are the permeability tensors for the macropores and the micropores, and μ is the dynamic viscosity of the saturating fluid. We shall require that K^f_{ij}, K^p_{ij} and κ^m_{ij} are symmetric and positive-definite tensors while ζ_{ij} is a symmetric positive semi-definite tensor. The coefficients in $K^f_{ij}, K^p_{ij}, \kappa^m_{ij}$ and ζ_{ij} are here assumed constant.

Equations (1) posses the steady conduction solution

$$\bar{\mathbf{U}}^f \equiv 0, \quad \bar{\mathbf{U}}^p \equiv 0, \qquad \bar{T} = -\beta z + T_L\,, \tag{3}$$

where $\beta = (T_L - T_U)/d$ is the temperature gradient.

We derive the equations for a perturbation to (3), $(u^f_i, u^p_i, \pi^f, \pi^p, \theta)$ where

$$\begin{aligned}&U^f_i = \bar{U}^f_i + u^f_i, \qquad U^p_i = \bar{U}^p_i + u^p_i, \qquad p^f = \bar{p}^f + \pi^f,\\ &p^p = \bar{p}^p + \pi^p, \qquad T = \bar{T} + \theta,\end{aligned}$$

and we non-dimensionalize the perturbation equations with length scale d, time scale $(\rho c)_m d^2/\kappa^m_{11}$, velocity scale $U = \kappa^m_{11}/(\rho c)_f d$, temperature scale

$$T^\sharp = U\sqrt{\frac{\beta d^2 m_{11}}{k_m \rho_F \alpha g}}$$

where $k_m = \kappa^m_{11}/(\rho c)_f$, and where $m_{11} = M^f_{11}$. We rescale M^f_{ij} by taking out a factor m_{11} and likewise use the same factor on M^p_{11} to write $M^p_{11} = \omega m_{11}$. Define the Rayleigh number $Ra = R^2$ by

$$Ra = \frac{\beta d^2 \rho_F \alpha g}{k_m m_{11}}\,.$$

Put $\lambda_{ij} = \zeta_{ij}/m_{11}$ and denote $\mathbf{u}^f = (u^f, v^f, w^f)$, $\mathbf{u}^p = (u^p, v^p, w^p)$. Then the non-dimensional perturbation equations may be written in the form

$$\begin{aligned}&-M^f_{ij}u^f_j - \lambda_{ij}(u^f_j - u^p_j) - \pi^f_{,i} + R\theta k_i = 0, \qquad u^f_{i,i} = 0,\\ &-\omega M^p_{ij}u^p_j - \lambda_{ij}(u^p_j - u^f_j) - \pi^p_{,i} + R\theta k_i = 0, \qquad u^p_{i,i} = 0,\\ &\theta_{,t} + (u^f_i + u^p_i)\theta_{,i} = R(w^f + w^p) + \kappa_{ij}\theta_{,ij}\,,\end{aligned} \tag{4}$$

where κ_{ij} is the non-dimensional form of κ^m_{ij}. The domain of equations (4) is $\{(x,y) \in \mathbb{R}^2\} \times \{0 < z < 1\} \times \{t > 0\}$. We suppose the solution satisifes a plane tiling periodicity in the (x,y) directions and we denote the periodic cell by V. The boundary conditions are

$$w^f = w^p = \theta = 0, \qquad \text{on } z = 0, 1. \tag{5}$$

3 Exchange of stabilities

We linearize equations (4) and write the solutions as

$$
\begin{aligned}
&u_j^f = e^{\sigma t} u_j^f(\mathbf{x}), \quad u_j^p = e^{\sigma t} u_j^p(\mathbf{x}), \quad \theta = e^{\sigma t}\theta(\mathbf{x}),\\
&\pi^f = e^{\sigma t}\pi^f(\mathbf{x}), \quad \pi^p = e^{\sigma t}\pi^p(\mathbf{x}),
\end{aligned}
$$

cf. Chandrasekhar [42]. Then we derive the linear system of equations

$$
\begin{aligned}
&-M_{ij}^f u_j^f - \lambda_{ij}(u_j^f - u_j^p) - \pi_{,i}^f + R\theta k_i = 0, \qquad u_{i,i}^f = 0,\\
&-\omega M_{ij}^p u_j^p - \lambda_{ij}(u_j^p - u_j^f) - \pi_{,i}^p + R\theta k_i = 0, \qquad u_{i,i}^p = 0,\\
&\sigma\theta = R(w^f + w^p) + \kappa_{ij}\theta_{,ij}\,.
\end{aligned} \tag{6}
$$

According to Chandrasekhar [42] one says the principle of exchange of stabilities holds if σ is real and the marginal states are characterized by $\sigma = 0$, i.e. $Re(\sigma) = 0, Im(\sigma) = 0$. We say that if $\sigma \in \mathbb{R}$ then one says the strong form of the principle of exchange of stabilities holds, cf. Chandrasekhar [42], p. 24. The importance of this is that if we can show exchange of stabilities then we know thermal convection commences by stationary convection. We now show the strong form of the principle of exchange of stabilities holds for equations (6). In the interests of clarity it is worth observing that there is a weaker statement of the principle of exchange of stabilities which requires $Im(\sigma) \neq 0$ implies $Re(\sigma) < 0$. This was used by E.A. Speigel in penetrative convection, see Veronis [43], and further use is described in Davis [44], and in Straughan [41], pp. 84,85.

Let $*$ denote the complex conjugate of a quantity and let $< \cdot >$ denote integration over a period cell V. Multiply $(6)_1$ by u_i^{f*}, $(6)_2$ by u_i^{p*}, and $(6)_3$ by θ^*, and integrate each over V. After use of the boundary conditions we may show

$$
\begin{aligned}
&-< M_{ij}^f u_j^f u_i^{f*} > - < \lambda_{ij}(u_j^f - u_j^p)u_i^{f*} > + R < \theta w^{f*} > = 0,\\
&-\omega < M_{ij}^p u_j^p u_i^{p*} > - < \lambda_{ij}(u_j^p - u_j^f)u_i^{p*} > + R < \theta w^{p*} > = 0,\\
&\sigma < \theta\theta^* > = R\big[< w^f\theta^* > + < w^p\theta^* >\big] - < \kappa_{ij}\theta_{,j}\theta_{,i}^* > \,.
\end{aligned}
$$

Upon addition these three equations lead to

$$
\begin{aligned}
\sigma < \theta\theta^* > = &- < M_{ij}^f u_j^f u_i^{f*} > - \omega < M_{ij}^p u_j^p u_i^{p*} > - < \kappa_{ij}\theta_{,j}\theta_{,i}^* >\\
&- < \lambda_{ij}(u_j^f - u_j^p)(u_i^{f*} - u_i^{p*}) >\\
&+ R\big[< \theta w^{f*} > + < w^f\theta^* > + < \theta w^{p*} > + < w^p\theta^* >\big].
\end{aligned} \tag{7}
$$

Now, put $\sigma = \sigma_r + i\sigma_1$ and take the imaginary part of (7) to obtain

$$
\sigma_1 < \theta\theta^* > = 0.
$$

We know $< \theta\theta^* > \neq 0$ and so $\sigma_1 = 0$ and $\sigma \in \mathbb{R}$.

4 Nonlinear stability

Let $\|\cdot\|$ denote the norm on $L^2(V)$. To investigate nonlinear stability we multiply equation $(4)_1$ by u_i^f, equation $(4)_2$ by u_i^p, and $(4)_3$ by θ and integrate each in turn over the domain V. By integrating by parts, using the boundary conditions, and using the fact that u_i^f and u_i^p are solenoidal we arrive at three equations. Add the first two of these and then one may derive the following energy identities,

$$-< M_{ij}^f u_i^f u_j^f > -\omega < M_{ij}^p u_i^p u_j^p > - < \lambda_{ij}(u_j^f - u_j^p)(u_i^f - u_i^p) > +R < \theta(w^f + w^p) > = 0, \tag{8}$$

and

$$\frac{d}{dt}\frac{1}{2}\|\theta\|^2 = R < \theta(w^f + w^p) > - < \kappa_{ij}\theta_{,i}\theta_{,j} > . \tag{9}$$

One now adds (8) and (9) to arrive at the energy identity

$$\frac{dE}{dt} = RI - D, \tag{10}$$

where

$$\begin{aligned} E =& \frac{1}{2}\|\theta\|^2, \qquad I = 2 < \theta(w^f + w^p) >, \\ D =& < M_{ij}^f u_i^f u_j^f > +\omega < M_{ij}^p u_i^p u_j^p > \\ &+ < \lambda_{ij}(u_j^f - u_j^p)(u_i^f - u_i^p) > + < \kappa_{ij}\theta_{,i}\theta_{,j} > . \end{aligned} \tag{11}$$

Define R_E by

$$\frac{1}{R_E} = \max_H \frac{I}{D}, \tag{12}$$

where H is the space of admissible solutions, i.e. u_i^f, u_i^p are in $L^2(V)$, are divergence free, θ is in $H^1(V)$, all satisfy the boundary conditions (5) together with periodicity in (x, y).

From (10) we derive

$$\frac{dE}{dt} \le \frac{RD}{R_E} - D = -D\Big(1 - \frac{R}{R_E}\Big). \tag{13}$$

When $R < R_E$ put $a = 1 - R/R_E$ and use Poincaré's inequality to deduce from (13)

$$\frac{d}{dt}\frac{1}{2}\|\theta\|^2 \le -\pi^2\kappa_0 a\|\theta\|^2 , \tag{14}$$

where $\kappa_0 > 0$ is the constant in the positive-definiteness of κ_{ij}. Since $a > 0$ an integration of (14) shows

$$\|\theta(t)\|^2 \le \|\theta(0)\|^2 \exp(-2\pi^2\kappa_0 at), \tag{15}$$

and so $a > 0$ guarantees rapid decay of $\|\theta(t)\|$. Let M_1 and M_2 be constants such that

$$M_{ij}^f\xi_i\xi_j \ge M_1\xi_i\xi_i, \qquad M_{ij}^p\xi_i\xi_j \ge M_2\xi_i\xi_i, \qquad \text{for all } \xi_i,$$

then from (8) recollecting λ_{ij} is positive semi-definite, we may use the arithmetic-geometric mean inequality to find

$$M_1\|\mathbf{u}^f\|^2 + \omega M_2\|\mathbf{u}^p\|^2 \le R^2\Big(\frac{1}{\omega M_2} + \frac{1}{M_1}\Big)\,\|\theta\|^2. \tag{16}$$

If we now employ (15) in (16) we see that $a > 0$ also guarantees decay of $\|\mathbf{u}^f(t)\|$ and $\|\mathbf{u}^p(t)\|$. Thus, the global nonlinear stabilitiy threshold is R_E.

To determine R_E one calculates the Euler-Lagrange equations from (12), and for Lagrange multipliers λ^f and λ^p one finds these are

$$\begin{aligned}
&-M^f_{ij}u^f_j - \lambda_{ij}(u^f_j - u^p_j) + R_E\theta k_i = \lambda^f_{,i}, \qquad u^f_{i,i} = 0,\\
&-\omega M^p_{ij}u^p_j - \lambda_{ij}(u^p_j - u^f_j) + R_E\theta k_i = \lambda^p_{,i}, \qquad u^p_{i,i} = 0,\\
&R_E(w^f + w^p) + \kappa_{ij}\theta_{,ij} = 0\,.
\end{aligned} \tag{17}$$

One may observe that (17) are identical to equations (6) when $\sigma = 0$ and since we have shown exchange of stabilities holds we conclude that the nonlinear stability boundary $Ra_E = R_E^2$ is the same as the linear instability boundary $Ra_L = R^2$.

5 Horizontally isotropic equations

We have shown the linear instability boundary for the conduction solution (3) is the same as the global nonlinear stability one in the case of general symmetry of $M^f_{ij}, M^p_{ij}, \lambda_{ij}$ and κ_{ij}. We now specialize to the case where $\lambda_{ij} = \lambda\delta_{ij}$ and $\kappa^m_{ij} = \kappa_m\delta_{ij}$, δ_{ij} being the Kronecker delta, with $\lambda > 0$, $\kappa_m > 0$ constants, and we analyse the case where M^f_{ij} and M^p_{ij} represent horizontally isotropic permeabilities. We explicitly calculate the critical Rayleigh number for thermal convection in this important physical case.

We thus suppose

$$M^f_{ij} = \mu(K^f_{ij})^{-1} = diag(\mu/K^f_H, \mu/K^f_H, \mu/K^f_V),$$

where K^f_H and K^f_V are the horizontal and vertical permeabilities associated with the macro porosity. We shall suppose M^p_{ij} has the same structure and assume $M^p_{ij} = \omega M^f_{ij}$. We could allow M^f_{ij} and M^p_{ij} to be both of horizontally isotropic type, i.e. $\mathbf{M}^f = diag(a_1, a_1, a_3)$, $\mathbf{M}^p = diag(b_1, b_1, b_3)$, and not impose the restriction involving ω. However, this introduces a further parameter into the analysis and as this is the first time we have seen such work we believe it is acceptable to present the simpler theory. Let $a_1 = \mu/K^f_H$ and $a_3 = \mu/K^f_V$ and recall $\ell^2 = a_3/a_1 = K^f_H/K^f_V$. Thus, ℓ^2 is a measure of the horizontal to vertical permeability in both the macro porosity and micro porosity systems. As we suppose M^p_{ij} has the same geometric structure as M^f_{ij}, it follows that

$$M^p_{ij} = \mu(K^f_{ij})^{-1} = diag(b_1, b_1, b_3) = diag\left(\frac{\mu}{K^p_H}, \frac{\mu}{K^p_H}, \frac{\mu}{K^p_V}\right),$$

where K_H^p and K_V^p are the horizontal and vertical permeabilities associated with the micro porosity. We thus see that $\omega = K_H^f/K_H^p = K_V^f/K_V^p$ is a measure of the ratio of horizontal or vertical permeabilities in the macro to micro porosity systems. Note also that $\ell^2 = K_H^f/K_V^f = K_H^p/K_V^p$.

The conduction solution is again (3) and the governing equations are still (1), *mutatis mutandis*. One may write out the perturbation equations and in this case we use the non-dimensional variables of d for length, $\mathcal{T}, \mathcal{P}, U$ and $T^\sharp$ for time, pressure, velocity and temperature, where

$$\begin{aligned} \mathcal{T} &= \frac{(\rho c)_m d^2}{\kappa_m}\,, \qquad P = dUa_1, \\ U &= \frac{\kappa_m}{d(\rho c)_f}\,, \qquad T^\sharp = U\sqrt{\frac{a_1(\rho c)_f \beta d^2}{\rho_F \alpha g \kappa_m}}\,. \end{aligned}$$

Define $\lambda = \zeta/a_1$ where the interaction coefficient in equations (1) is $\zeta_{ij} = \zeta\delta_{ij}$, and define the Rayleigh number by

$$Ra = R^2 = \frac{\rho_F \alpha g \beta d^2}{a_1 \kappa_m}\,. \tag{18}$$

The relevant non-dimensional perturbation equations for the horizontally isotropic case are

$$\begin{aligned} &- D_{ij}u_j^f - \lambda(u_i^f - u_i^p) - \pi_{,i}^f + R\theta k_i = 0, \qquad u_{i,i}^f = 0, \\ &- \omega D_{ij}u_j^p - \lambda(u_i^p - u_i^f) - \pi_{,i}^p + R\theta k_i = 0, \qquad u_{i,i}^p = 0, \\ &\theta_{,t} + (u_i^f + u_i^p)\theta_{,i} = R(w^f + w^p) + \Delta\theta, \end{aligned} \tag{19}$$

where $D_{ij} = diag(1, 1, \ell^2)$. These equations hold on the domain $\{(x, y) \in \mathbb{R}^2\} \times \{z \in (0, 1)\} \times \{t > 0\}$ together with the boundary conditions (5) and periodicity in (x, y).

6 Stability threshold

To find the linear instability (and global nonlinear stability) threshold value critical Rayleigh number from (19) we discard the nonlinear terms and seek a time dependence like $e^{\sigma t}$ for $u_i^f, u_i^p, \theta, \pi^f$ and π^p. We then discard the resulting $\sigma\theta$ term since exchange of stabilities holds. We then take the double curl of $(19)_1$ and $(19)_2$ and there remains the following system of equations

$$\begin{aligned} &w_{zz}^f + \ell^2\Delta^* w^f + \lambda(\Delta w^f - \Delta w^p) - R\Delta^*\theta = 0, \\ &\omega(w_{zz}^p + \ell^2\Delta^* w^p) - \lambda(\Delta w^f - \Delta w^p) - R\Delta^*\theta = 0, \\ &R(w^f + w^p) + \Delta\theta = 0, \end{aligned} \tag{20}$$

where Δ is the three-dimensional Laplacian and $\Delta^* = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplacian.

Let f be a planform for the solution, cf. Chandrasekhar [42], pp. 43-52, so that $\Delta^* f = -a^2 f$, where a is the wavenumber. Then we write $w^f = W^f \sin n\pi z\, f(x,y)$ for W^f an amplitude, with a similar representation for w^p and θ, cf. Chandrasekhar [42]. After some manipulation we find

$$R^2 = \frac{\lambda(1+\omega)\Lambda_{\ell n}\Lambda_n^2 + \omega\Lambda_n\Lambda_{\ell n}^2}{a^2[4\lambda\Lambda_n + (1+\omega)\Lambda_{\ell n}]}\,,$$

where $\Lambda_n = n^2\pi^2 + a^2$ and $\Lambda_{\ell n} = n^2\pi^2 + \ell^2 a^2$. One may show $\partial R^2/\partial n^2 \geq 0$ and so since we must minimize R^2 one selects $n = 1$. Then

$$R^2 = \frac{\lambda(1+\omega)\Lambda_\ell\Lambda^2 + \omega\Lambda\Lambda_\ell^2}{a^2[4\lambda\Lambda + (1+\omega)\Lambda_\ell]}\,, \tag{21}$$

where $\Lambda = \pi^2 + a^2$ and $\Lambda_\ell = \pi^2 + \ell^2 a^2$.

The critical value of $Ra = R^2$ is found by fixing λ, ω and ℓ^2 and minimizing R^2 as given by (21) in a^2. This we do numerically and results are reported in the next section.

7 Numerical results and conclusions

We now report numerical results for the horizontally isotropic bidispersive convection problem described in sections 5 and 6. We have computed many results and those presented represent a selection chosen to describe the type of behaviour found.

The numerical results presented are displayed in a series of tables. Tables 1 - 3 show critical Rayleigh and wavenumbers, Ra, a^2 for various values of ω, λ and ℓ^2. In table 4 we again show Ra and a^2 but now we keep λ fixed and vary ω for a selection of ℓ^2 values. In table 5 we show detail of a^2 for λ fixed with ω varying for some ℓ^2 values. Table 6 shows the variation in Ra and a^2 as the interaction parameter λ is varied for fixed ω and for two values of ℓ^2. Table 7 displays Ra and a^2 for fixed ω and λ when ℓ^2 is very small or relatively large. Finally in tables 8 and 9 we fix λ and show the variation of Ra and a^2 when ℓ^2 is varied for a selection of ω values. Recall from section 5 that $\omega = K_H^f/K_H^p = K_V^f/K_V^p$ and so ω is a measure between the permeability in the macro and micro states. Also $\ell^2 = K_H^f/K_V^f = K_H^p/K_V^p$ and so ℓ^2 measures the permeability between the horizontal and vertical directions in the porous layer.

For many of the rocks discussed in the Introduction ℓ^2 is relatively large and certainly $\ell^2 > 1$. In tables 1 - 3 ℓ^2 takes values 2,3 and 10, and in all cases we see that increasing ω from 0.5 to 1.5 results in a relatively strong increase in the critical Rayleigh number, Ra. This means that as ω increases Ra increases and it becomes more difficult for convection to occur. Thus increasing the permeability ratio from the macro to micro phases results in convection occurring less easily. The wavenumber shows little variation as ω increases although there is a minimum or maximum achieved and this is discussed further below.

In table 4 we see specifically how Ra increases as ω increases for fixed ℓ^2 values of 0.6, 5 and 10. The Rayleigh number increases relatively strongly in all cases as ω increases. For each fixed value of ℓ^2 we see there is little variation in a^2 as ω increases. However, when $\ell^2 < 1$, a^2 decreases from $\omega = 0.5$ to a minimum when $\omega = 1$ and thereafter increases. When $\ell^2 = 1$, a^2 stays the same. When $\ell^2 > 1$, a^2 increases from $\omega = 0.5$ to a maximum when $\omega = 1$ and thereafter decreases. The aspect ratio of a convection cell, L, is the width/depth ratio and $L \propto 1/a$. Thus, when $\ell^2 < 1$ the cells increase in width as ω increases toward 1 and then decrease in width afterward. For $\ell^2 > 1$ the effect is exactly the opposite. Table 5 presents details of the wavenumber variation as ω increases with $\lambda = 0.1$ and ℓ^2 taking the values 0.9, 1.0 and 1.1. Again, we observe a maximum in a^2 at $\omega = 1.0$ when $\ell^2 = 0.9$ and a minimum in a^2 at $\omega = 1.0$ when $\ell^2 = 1.1$. When $\ell^2 = 1.0$ the wavenumber always stays the same.

Table 6 shows that as λ increases from 0.1 to 10, Ra increases, but relatively slowly, although the Ra values depend strongly on the value of ℓ^2. The wavenumber also displays little variation over the same range of λ, and so one may conclude that the interaction effect displayed via the λ term is having less of an effect on convection and convection cell shape than variation in ω or ℓ^2. We have computed the λ variation for several other values of ℓ^2 and the effect observed in table 6 persists. Namely, for $\ell^2 < 1$, a^2 displays a minimum value as λ increases, with $\lambda \neq 1$ at the minimum, and when $\ell^2 > 1$, a^2 displays a maximum value, again with $\lambda \neq 1$. When $\ell^2 = 1$, a^2 is always found to have value 9.86960 (to 5 decimal places), regardless of the value of λ.

In table 7 we fix λ and ω and show critical Ra and a^2 values for $\ell^2 = 0.02, 0.03, 0.04$ and $\ell^2 = 200, 300, 400$. The same trend as already reported is again found with Ra being relatively small whereas a^2 is relatively large for small ℓ^2, but Ra is relatively large and a^2 relatively small when ℓ^2 is large. This shows that the horizontal to vertical permeability variation plays a major role in quantitative assessment of bidispersive thermal convection. Thus, in interpreting any experimental results it is very important to have accurate values for the horizontal and vertical permeabilities.

Tables 8 and 9 show that as ℓ^2 increases Ra increases relatively strongly and a^2 decreases relatively strongly. These values again show how important the horizontal to vertical permeability ratio is upon thermal convection.

Table 1: Critical values of Rayleigh number and wavenumber, for varying ω and λ, $\ell^2 = 2.0$

Ra	a^2	ω	λ	ℓ^2
19.554850	7.01932	0.5	0.1	2
25.609738	6.98248	0.8	0.1	2
30.139889	6.97945	1.1	0.1	2
34.660713	6.98822	1.5	0.1	2
19.830782	7.03838	0.5	0.2	2
25.642756	6.98446	0.8	0.2	2
30.146293	6.97980	1.1	0.2	2
34.779883	6.99423	1.5	0.2	2
20.337366	7.05188	0.5	0.5	2
25.706946	6.98643	0.8	0.5	2
30.159319	6.98021	1.1	0.5	2
35.034058	7.00227	1.5	0.5	2

Table 2: Critical values of Rayleigh number and wavenumber, for varying ω and λ, $\ell^2 = 3.0$

Ra	a^2	ω	λ	ℓ^2
24.965048	5.74510	0.5	0.1	3
32.787747	5.70236	0.8	0.1	3
38.596547	5.69889	1.1	0.1	3
44.356344	5.70880	1.5	0.1	3
25.277619	5.77045	0.5	0.2	3
32.824776	5.70489	0.8	0.2	3
38.603670	5.69932	1.1	0.2	3
44.487732	5.71616	1.5	0.2	3
25.889329	5.79484	0.5	0.5	3
32.901132	5.70797	0.8	0.5	3
38.618972	5.69993	1.1	0.5	3
44.782247	5.72752	1.5	0.5	3

Table 3: Critical values of Rayleigh number and wavenumber, for varying ω and λ, $\ell^2 = 10.0$

Ra	a^2	ω	λ	ℓ^2
57.547897	3.16051	0.5	0.1	10
76.056358	3.12445	0.8	0.1	10
89.575925	3.12158	1.1	0.1	10
102.79763	3.12951	1.5	0.1	10
58.018776	3.18858	0.5	0.2	10
76.111008	3.12703	0.8	0.2	10
89.586245	3.12201	1.1	0.2	10
102.98426	3.13643	1.5	0.2	10
59.094795	3.23390	0.5	0.5	10
76.240829	3.13170	0.8	0.5	10
89.611471	3.12284	1.1	0.5	10
103.45329	3.15067	1.5	0.5	10

Table 4: Critical values of Rayleigh number and wavenumber, for varying ω. Here $\lambda = 0.1$, $\ell^2 = 0.6, \ell^2 = 5.0, \ell^2 = 10.0$, in the columns for Ra, a^2 moving left to right

ω	Ra	a^2	Ra	a^2	Ra	a^2
0.5	10.692	12.666	34.910	4.460	57.548	3.161
0.6	11.845	12.702	39.018	4.437	64.433	3.141
0.7	12.894	12.723	42.688	4.425	70.563	3.130
0.8	13.852	12.735	45.988	4.418	76.056	3.124
0.9	14.731	12.740	48.970	4.415	81.008	3.122
1.0	15.541	12.742	51.678	4.414	85.493	3.121
1.1	16.288	12.740	54.148	4.414	89.576	3.122
1.2	16.980	12.737	56.411	4.416	93.308	3.123
1.3	17.623	12.733	58.491	4.418	96.732	3.125
1.4	18.222	12.728	60.410	4.421	99.885	3.127
1.5	18.782	12.722	62.186	4.424	102.798	3.130
2.0	21.101	12.691	69.388	4.440	114.567	3.142

Table 5: Critical values of wavenumber, for varying ω. Here $\lambda = 0.1$, $\ell^2 = 0.9, \ell^2 = 1.0, \ell^2 = 1.1$, in the columns for a^2 moving left to right

ω	a^2 ($\ell^2 = 0.9$)	a^2 ($\ell^2 = 1.0$)	a^2 ($\ell^2 = 1.1$)
0.5	10.39206	9.86960	9.41914
0.6	10.39753	9.86960	9.41489
0.7	10.40069	9.86960	9.41244
0.8	10.40243	9.86960	9.41111
0.9	10.40325	9.86960	9.41047
1.0	10.40348	9.86960	9.41030
1.1	10.40330	9.86960	9.41043
1.2	10.40286	9.86960	9.41077
1.3	10.40224	9.86960	9.41124
1.4	10.40149	9.86960	9.41181
1.5	10.40067	9.86960	9.41243

Table 6: Critical values of Rayleigh number and wavenumber, for varying λ. Here $\omega = 1.5$, $\ell^2 = 0.6, \ell^2 = 10.0$, in the columns for Ra, a^2 moving left to right

λ	Ra	a^2	Ra	a^2
0.1	18.781790	12.72231	102.79763	3.12951
0.2	18.875915	12.71362	102.98426	3.13643
0.5	19.043527	12.70810	103.45329	3.15067
1.0	19.172314	12.71233	104.02485	3.16193
2.0	19.274362	12.72081	104.73788	3.16669
5.0	19.357210	12.73098	105.65291	3.15784
10.0	19.389926	12.73580	106.15992	3.14585

Table 7: Critical values of Rayleigh number and wavenumber, for large and small values of ℓ^2. Here $\omega = 1.5$, $\lambda = 0.5$.

Ra	a^2	ℓ^2
7.9880209	68.55	0.02
8.4296560	56.04	0.03
8.8112876	48.59	0.04
1360.7783	0.71	200.0
1991.2202	0.58	300.0
2615.6553	0.50	400.0

Table 8: Critical values of Rayleigh number and wavenumber, for varying ℓ^2. Here $\lambda = 0.1$, $\omega = 0.5, \omega = 0.7, \omega = 0.9$, in the columns for Ra, a^2 moving left to right

ℓ^2	Ra	a^2	Ra	a^2	Ra	a^2
0.6	10.692000	12.66594	12.894017	12.72286	14.731457	12.74007
0.7	11.433015	11.74950	13.805739	11.78488	15.779334	11.79549
0.8	12.145798	11.00806	14.682581	11.02806	16.787046	11.03402
0.9	12.835472	10.39206	15.530905	10.40069	17.761933	10.40325
1.0	13.505774	9.86960	16.355344	9.86960	18.709337	9.86960
1.1	14.159532	9.41914	17.159401	9.41244	19.633294	9.41047
1.2	14.798945	9.02549	17.945794	9.01350	20.536937	9.00998
1.3	15.425762	8.67760	18.716688	8.66138	21.422758	8.65663
1.4	16.041405	8.36723	19.473838	8.34758	22.292780	8.34184
1.5	16.647045	8.08808	20.218692	8.06561	23.148668	8.05907

Table 9: Critical values of Rayleigh number and wavenumber, for varying ℓ^2. Here $\lambda = 0.1$, $\omega = 1.1, \omega = 1.3, \omega = 1.5$, in the columns for Ra, a^2 moving left to right

ℓ^2	Ra	a^2	Ra	a^2	Ra	a^2
0.6	16.287980	12.74042	17.623435	12.73311	18.781790	12.72231
0.7	17.446716	11.79571	18.874021	11.79124	20.109615	11.78464
0.8	18.561037	11.03415	20.076731	11.03165	21.386711	11.02798
0.9	19.639059	10.40330	21.240303	10.40224	22.622321	10.40067
1.0	20.686691	9.86960	22.371103	9.86960	23.823183	9.86960
1.1	21.708394	9.41043	23.473938	9.41124	24.994385	9.41243
1.2	22.707634	9.00991	24.552542	9.01135	26.139883	9.01347
1.3	23.687168	8.65653	25.609886	8.65847	27.262826	8.66131
1.4	24.649231	8.34172	26.648380	8.34406	28.365765	8.34747
1.5	25.595663	8.05893	27.670010	8.06159	29.450806	8.06547

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