On the efficiency of the first price auction[*]

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Abstract

We provide a natural setting in privatizations in which the equilibrium of the first price auction gives greater expected surplus than any equilibrium of the open ascending auction.

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1   Introduction

The auction theorist's toolkit proposes an open ascending auction, see for instance Maskin (1992), or any of its variations, like proxy auctions or clock auctions, when the objective is efficiency and more sophisticated mechanisms like a Vickrey-Clark-Groves auction are impracticable. Indeed, Krishna (2003) shows for the case of an indivisible unit that under additive separability of the value function (which can be interpreted as a first order approach) and a minor monotonicity condition, the open ascending auction implements the ex post efficient allocation whenever the ex post efficient allocation is implementable. In the same case, the alternative of using a first price auction is considered to be less efficient for its inability to deal with asymmetries across bidders, see for instance Maskin and Riley (2000).

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In contrast to those conclusions, our main result shows that the first price auction may induce greater ex ante expected surplus than the open ascending auction when the ex post efficient allocation is not implementable.[1] Interestingly, this finding is derived for a realistic model of privatizations in which asymmetries across bidders arise because an incumbent has better information and lower setup costs than the other bidders, the entrants.

We are not aware of such a result in the literature. The only related observation is that the first price auction may Pareto dominate the open ascending auction when both the buyer and the seller are risk averse, see Holt (1980), Riley and Samuelson (1981), Matthews (1983) and Maskin and Riley (1984). The connection, however, is weak since we assume risk neutrality.

2   The Model

Our model captures a common situation in privatizations: an incumbent has private information about a common component of the value, and entrants have higher setup costs but possibly lower variable costs than the incumbent.[2]

More specifically, we assume that a set of firms indexed by $i \in \{1, 2, \ldots, n+1\}$, $n > 1$,[3] compete for a privatized service that gives profits $\pi(D, C_i) > 0$ after the firm incurs in a setup cost k_i, and where D denotes a demand shifter and C_i an individual variable cost shifter, e.g. Firm i's marginal cost. $\pi(D, C_i)$ is continuous, increasing in D, decreasing in C_i, and strictly submodular in (D, C_i): $\frac{\partial^2 \pi(d,c)}{\partial d \partial c} < 0$. The interpretation of submodularity is that the higher the demand, the more beneficial is for the firm to have lower variable costs. This assumption is satisfied by most of the economic models that give rise to the function π.[4]

Firm 1 is an incumbent already operating the service. It privately knows the demand for the service D, which is assumed to be an independent random variable that follows a distribution F with density and support $[\underline{d}, \overline{d}]$. In general, we use capital letters to denote the random variable and lower case letters to denote a realization of the random variable.

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[1]Actually, the first price auction maximizes the ex ante expected surplus subject to implementability in our setting, as one can show adapting the results of Hernando-Veciana and Michelucci (2017).

[2]Our model is inspired by an example proposed by Maskin (1992), pages 127-8, to illustrate that the ex post efficient allocation may not be implementable. This is also the case here. The explanation is the same as in Maskin's example.

[3]The case of only two bidders is special as in this case the open ascending auction maximizes the expected surplus subject to implementability under fairly general conditions, see Hernando-Veciana and Michelucci (2011).

[4]For instance, suppose the sale of a license to operate an unregulated monopoly with linear demand $Q(p) = D - p$ and constant marginal cost C_i. In this case, $\pi(D, C_i) = \frac{(D - C_i)^2}{2}$, which satisfies our assumptions.

Each of the other firms, the entrants, have the same setup cost $k_i = k$ greater than the incumbent's, i.e. $k > k_1$. To simplify the notation we let $k_1 = 0$.[5] Setup costs are commonly known but variable costs are the firm's private information. Each C_i is assumed to be a random variable that follows an independent distribution G_i with positive density over the support $[\underline{c}_i, \overline{c}_i]$. To simplify notation, we assume all the entrant's G_i's have the same support $[\underline{c}, \overline{c}]$.

Our results are derived under three main assumptions:[6]

A1. $E[\pi(D, \overline{c})] - k > \pi(\overline{d}, \underline{c}_1)$.

A2. $\pi(\underline{d}, \underline{c}) - k < \pi(\underline{d}, \overline{c}_1)$.

A3. All the entrants variable costs follow the same distribution, $G_i = G$ for all $i \neq 1$.

A1 says that the average value of an entrant with maximum variable cost is greater than the maximum incumbent's value. Since π is decreasing in its second argument, A1 is satisfied if k is not too large and $\underline{c}_1$ is sufficiently larger than $\overline{c}$, i.e. if the incumbent's setup cost advantage is small and entrants have sufficiently lower variable costs than the incumbent. A2 says that in the case of the minimum demand, the entrants have lower value than the incumbent with probability one. Since π is strictly submodular, A2 only requires a sufficiently small $\underline{d}$ if we also assume that $\underline{c} < \overline{c}_1$. A3 is a requirement that entrants are ex ante symmetric.

A1 is the most demanding of our assumptions. It guarantees that there exists an equilibrium of the first price auction in which the range of the entrants' bid function and the incumbent's do not overlap. This equilibrium has a straightforward characterisation. We expect our ranking to be robust to a weakening of A1 that allows the incumbent to win with positive probability in equilibrium. However, the derivation of the equilibrium becomes substantially more complex and requires of an adaptation of the complex analysis of asymmetric first price auctions, see, for instance, Lebrun (2006).

A2 is also stronger than it is needed. Lemma 2 (b) only requires that a sufficiently small level for the demand exists such that a set of types of the entrants with positive probability have a lower value than the incumbent. By assuming this happens with probability one, we can shorten substantially the proof of Lemma 2. A3 guarantees the existence of an equilibrium of the first price auction in which entrants use the same strictly monotone bid function and, consequently, that the entrant with lowest

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[5]This is without loss of generality as the setup cost of the incumbent can always be incorporated in the function π. In this case, the interpretation of k is the difference between the entrant's setup cost and the incumbent's.

[6]$E[\ldots]$ denotes the expected value of the expression between the brackets.

cost outbids the other entrants. More generally, our efficiency ranking should hold true provided that asymmetries are not too large.

3   The Analysis

We start our comparison with the study of the first price auction,[7] see Krishna (2010) for a description of the rules. Our first result shows that this auction format has an equilibrium in strictly monotone strategies in which all the entrants use the same bid function. The key feature is that this equilibrium displays that the incumbent is outbid with probability one.[8]

Lemma 1. *There exists an equilibrium of the first price auction in which the entrant with the highest value wins with probability one.*

Proof. We propose some strategies that satisfy the conditions of the proposition, and argue that they constitute an equilibrium. We propose that the incumbent bids his value $\pi(D, C_1)$ and the entrants play the unique symmetric equilibrium strategy of a first price auction, see for instance Krishna (2010), with n bidders with value function $V(c) \equiv E[\pi(D, c)] - k$. Since the entrants' strategy is strictly decreasing and the entrant with largest cost bids $V(\bar{c}) \equiv E[\pi(D, \bar{c})] - k$, A1 means that the entrant with lowest cost wins with probability one as required. A1 also means that there is no price at which the incumbent can win with positive profits. Hence, the incumbent has no incentive to deviate. Entrants do not have incentives to deviate within the range of their bid function by construction, whereas deviations outside the range of their bid function are not strictly profitable: lower bids always lose and higher bids do not win with higher probability than with the maximum bid in the range but mean paying a higher price.   ∎

In what follows, we assume that this is the equilibrium played in the first price auction.

Next, we turn to the open ascending auction, see Krishna (2010) for a description of the general rules. We assume the usual uniformly-random tie-breaking rule, see Hernando-Veciana and Michelucci (2017) for the details.[9] Note that the incumbent knows his value and thus its unique weakly dominant

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[7]The proof of Lemma 1 can easily be adapted to show that the second price auction is also more efficient than the open ascending auction in our setting. However, we find the comparison with the first price auction more striking.

[8]Although, we do not claim uniqueness, we expect that A1 is sufficient to rule out that the incumbent wins in equilibrium. More generally, we expect that the equilibrium outcome is unique although its proof may be cumbersome.

[9]Hernando-Veciana and Michelucci (2017) also discusses other tie-breaking rules.

strategy is to remain in the auction until his value is reached. In what follows, we restrict to equilibria with this feature, that is we restrict to equilibria in weakly undominated strategies.

Lemma 2. *There is no equilibrium of the open ascending auction in which either: (a) the incumbent wins with positive probability, or (b) the entrant with highest value wins with probability one.*

Proof. To prove (a), note that the incumbent can win with positive probability if and only if the last entrant remaining active quits with positive probability before the incumbent does. We argue that this cannot happen in equilibrium: if the incumbent and only one entrant are active, the entrant has a strict incentive to remain active until the incumbent quits. This is because the entrant's expected utility when she stays active until the incumbent quits is equal to:

$$
\begin{aligned}
E\left[\pi(D,c) - k - \pi(D,C_1) \middle| \pi(D,C_1) \geq p\right] &\geq E\left[\pi(D,c) - k - \pi(\overline{d},\underline{c}_1) \middle| \pi(D,C_1) \geq p\right] \\
&\geq E\left[\pi(D,c) - k - \pi(\overline{d},\underline{c}_1)\right] \\
&> 0,
\end{aligned}
$$

where c denotes the entrant's cost and p the current price, and recall that $\pi(D,C_1)$ is the incumbent's value and hence his bid. The first inequality is a direct consequence of the monotonicity of π, and the last one follows from Assumption A1. To prove the second inequality, we show that the distribution of D conditional on the event $\{\pi(D,C_1) \geq p\}$ first order stochastically dominates its unconditional distribution. Consider the difference between both distributions evaluated at a point $d \in [\underline{d},\overline{d}]$:

$$
\Gamma(d) \equiv \int_{\underline{d}}^{d} \frac{f(x)G_1(\psi(x))}{\int_{\underline{d}}^{\overline{d}} f(\tilde{x})G_1(\psi(\tilde{x}))d\tilde{x}}dx - \int_{\underline{d}}^{d} f(x)dx = \int_{\underline{d}}^{d} f(x)\left(\frac{G_1(\psi(x))}{\int_{\underline{d}}^{\overline{d}} f(\tilde{x})G_1(\psi(\tilde{x}))d\tilde{x}} - 1\right)dx, \quad (1)
$$

where $\psi(x)$ is equal to the value of c_1 that solves $\pi(x,c_1) = p$ if any, and otherwise either $\underline{c}_1$ or $\overline{c}_1$ depending on whether $\pi(x,c_1) < p$ or $\pi(x,c_1) > p$ for all c_1. To complete the proof, just note that $\Gamma(d) \leq 0$ because $\Gamma(\underline{d}) = \Gamma(\overline{d}) = 0$ and Γ is quasi-convex since its derivative crosses zero only once and from below because ψ is an increasing function.

To prove that there is no equilibrium that satisfies (b), we argue by contradiction. First, we show that any equilibrium that satisfies (b) also satisfies another property to which we refer to as (P). Next, we argue that (b) and (P) imply that there is a profitable deviation. The property (P) is that only one entrant remains active along the equilibrium path as the price goes above the minimum value of the incumbent $\pi(\underline{d},\overline{c})$. To prove that (b) implies (P) recall that the incumbent quits at price $\pi(D,C_1)$. Thus, the expected profit of winning for an entrant with cost c at a price $p \geq \pi(D,C_1)$ after the

incumbent has quits at price $\tilde{p}$ is equal to:

$$E[\pi(D, c) - k - p | \pi(D, C_1) = \tilde{p}] \leq E[\pi(D, c) - k - \tilde{p} | \pi(D, C_1) = \tilde{p}],$$

which is strictly negative when $\tilde{p}$ is close to $\pi(\underline{d}, \overline{c}_1)$ by A2. This means that any entrant still active quits immediately after the incumbent when the incumbent quits at a price close to $\pi(\underline{d}, \overline{c}_1)$. Consequently, the tie-breaking rule means that only one entrant can remain active if (b) is to be satisfied.

Finally, we provide a contradiction by arguing that there is a profitable deviation for an entrant with type $\overline{c}$ from any equilibrium that satisfies (b) and (P). This entrant must lose with probability one when (b) is satisfied. However, she has a profitable deviation consisting on remaining in the auction until all the other bidders quit. In this case, (P) means that the deviating entrant wins with probability one and pays the incumbent's bid $\pi(D, C_1)$. This gives a expected payoff of:

$$E[\pi(D, \overline{c}) - k - \pi(D, C_1)],$$

which is strictly positive by Assumption A1. ∎

The proof of the lemma exploits the fact, already pointed out in Hernando-Veciana and Michelucci (2017), that open ascending auctions are prompt to inefficient rushes when the ex post efficient allocation is not implementable.

Lemmas 1 and 2 imply the following corollary as our main result.

Corollary 1. *The equilibrium of the first price auction gives greater expected surplus than any equilibrium of the open ascending auction.*

4   Conclusions

This note adds to the current literature a somewhat surprising ranking between the open ascending auction and first price auction by providing a realistic environment for which the latter auction dominates the former in terms of expected surplus.

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