A new study on reliability importance analysis of phased mission systems

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Abstract—Reliability importance which serves to quantify the influence of each component (or each type of components) in each phase on the reliability of a phased mission system (PMS) plays an important role in security assessment and risk management. In this paper, we present a new and efficient method for reliability importance analysis of PMSs using the theory of survival signature. A new kind of survival signature is applied to assess the reliability of PMS with multiple types of components. A closed-form formula is derived to predict reliability importance of the PMS with respect to each type of components in each phase. The Birnbaum importance model is further extended to calculate the reliability importance of the PMS with respect to each component in each phase. Finally, two numerical examples are used to demonstrate the validity and effectiveness of the proposed approaches.

Index Terms—Phased mission system; Reliability importance; Survival signature; System reliability; Structure function

NOMENCLATURE

PMS Phased mission system.

BDD Binary decision diagram.

RBD Reliability block diagram.

CDF Cumulative distribution function.

i, u Subscript: index of phases.

j Subscript: index of components.

k, v Subscript: index of types of components.

N Number of phases.

n_i Number of types of components in phase i.

$X_{i,j}$ State of component j in phase i.

$\boldsymbol{X}_i$ States of all components in phase i

$\boldsymbol{X}$ States of all components during the mission.

$\varphi_i(\cdot)$ State of the system in phase i.

$\varphi_s(\cdot)$ State of the phased mission system.

$\Phi_s(\cdot)$ Survival signature of the phased mission system.

$\Phi_1(\cdot)$ Survival signature of the first phase.

$\Phi_{1,2}(\cdot)$ Survival signature of the first two phase.

$S_{l_{1,1},\ldots,l_{N,n_N}}$ The set of all state vectors for the whole system.

$l_{i,k}$ Number of components of type k that function in phase i.

$P(\cdot)$ Probability function.

$m_{i,k}$ Number of components of type k that function at the beginning of phase i.

$m_{i,k}^{(\mathrm{a})}$ Number of components of type k that appeared in a phase before phase i that function at the beginning of phase i.

$m_{i,k}^{(\mathrm{n})}$ Number of components of type k that have not appeared in any phase before phase i that function at the beginning of phase i.

$i^{(\mathrm{a})}$ The nearest phase that the component appeared before phase i.

τ_i The end time of phase i.

$N(t)$ The phase that the system is in at time t.

$C_{i,k}(\cdot)$ Number of components of type k that function in phase i.

$F_{i,k}(\cdot)$ Conditional CDF of the life time of the components of type k in phase i.

$R_{i,k}(\cdot)$ Conditional reliability of the lifetime of the components of type k in phase i.

$R_s(\cdot)$ Reliability of the PMS.

$RI_{u,v}^{T}(t)$ Reliability importance of the components of type v in phase u.

$RI_{i,j}(\cdot)$, $RI_{i,j}^{(*)}$ Reliability importance of component j in phase i.

1. Introduction

In many practical applications, systems are designed to accomplish a mission by performing a sequence of tasks. For example, the mission of an aircraft flight can roughly be divided into seven tasks as follows: taxi to runway, take-off, ascent, cruise, descent, land, and taxi to terminal. The periods in which each of these successive tasks take place are known as phases and these systems are often called phased-mission systems (PMSs). Since a mission is successfully completed if and only if all its phases are successfully completed without failure, it is technically more difficult to ensure high reliability of a PMS. Consequently, reliability modelling and management of PMSs are of critical importance for safe operation and risk prevention.

Generally, reliability modelling of PMSs is more challenging than single-phased systems due to the dynamic behavior and dependence of the system in different phases [1, 2]. Over the past few decades, many efforts have been made to develop reliability modelling of PMSs. Existing modelling methods can be broadly classified into two types: state-space oriented models based on Markov chains or Petri nets and combinatorial methods based on binary decision diagram (BDD) or other decision diagrams. Moreover, for some special cases, space-oriented approaches and combinatorial methods can be combined effectively to gain the advantages of both.

In state space-oriented approaches, complex dependencies among system components can be explicitly modelled by using Markov chains or Petri nets. However, they suffer space explosion problem when modelling large-scale systems due to the fact that the cardinality of the state space becomes exponentially large as the number of components increases. Compared to space-oriented approaches, the combinatorial methods are more effective by exploiting Boolean algebra and various forms of decision diagrams to reduce the computational complexity.

Zang et al. [3] first proposed a method for reliability assessment of PMSs using BDD. Tang et al. [4] extended the BDD based method to analyze the reliability of PMSs with multiple failure mode components. Mo [5], Reed et al. [6] and Li et al. [7] further improved the efficiency of BDD analysis of PMSs with multiple failure mode components. Wang and Trivedi [8] and Lu et al. [9] proposed BDD based methods for the reliability analysis of PMSs with repairable components. Levitin et al. [10], Xing et al. [11] and Wang et al. [12] proposed methods for the reliability evaluation of PMSs with common cause failures using BDD. Zhai et al. [13] proposed an aggregated BDD method for reliability analysis of PMSs subject to dynamic demand requirements. Peng et al. [14] proposed a universal generating function-based method for the reliability analysis of the capacitated series-parallel PMSs with the consideration of imperfect fault coverage. Peng et al. [15] proposed a combinatorial method based on multi-valued decision diagrams for the reliability analysis of the capacitated series-parallel PMSs subjected to fault level coverage. Remenyte-Prescott et al. [16] performed reliability analysis for autonomous systems using the BDD based methodologies developed for phased mission analysis.

Li et al. [17] developed multi-state multivalued decision diagram algorithms for the reliability modelling of multi-state PMSs and partially repairable PMSs.

Recent studies of PMS are mostly based on BDD and the contributions mainly focus on improving computational efficiency and modelling the PMS with special features (e.g. common cause failures, repairable component or multiple failure mode). Until now, not much work has focused on reliability importance analysis of PMS. However, reliability importance which serves to quantify the degree of the influence of each component (or each type of components) in each phase on the reliability of the PMS plays an important role in security assessment and risk management [18-20]. For example, the results of a reliability importance analysis may be useful to the system designer, by informing to what extent the reliability of the PMS changes with perturbations to the reliability of the components in each phase. In addition, maintenance technicians can use reliability importance results to allocate resources for inspection, maintenance, and repair activities in an optimal manner over the life-time of a system.

In this paper, we propose a new survival signature methodology for reliability importance analysis of PMSs. The reminder of the paper is organized as follows: section 2 gives a brief background on PMSs; section 3 first shows how the survival signature can be used to evaluate reliability of PMSs, before providing a novel methodology which facilitates reliability importance analysis of PMSs with respect each type of components in each phase. Section 4 further extends the Birnbaum importance model to study reliability importance analysis of PMSs with respect to each component in each phase. Section 5 presents illustrative examples to show the application of the proposed method. Finally, section 6 closes the paper with conclusions.

2. Phased mission system

A PMS performs a sequence of functions or tasks during consecutive time periods to accomplish a specific mission, where each period is regarded as a phase. As a simple example, Figure 1 shows the reliability block diagram (RBD) of a five-component system with three phases. In Figure 1, it is assumed that the components of the system are divided into two types. Components 1 and 2 are classified as type 1 and the rest as type 2. From Figure 1, we can see that each phase of the PMS is corresponding to one configuration and the configuration changes from phase to phase. Moreover, the states of the same component in different phases are dependent of each other.

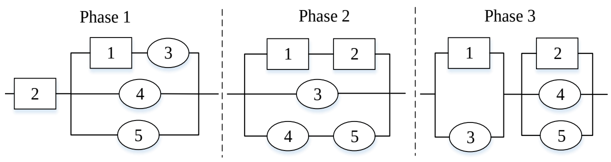

Figure 1 A PMS with multiple types of components

Let us consider a system which performs a phased mission with $N \geq 2$ phases, and there are n_i types of components in phase i, $i \in \{1, 2, . . . , N\}$. The state of component j, $j \in \{1, 2, . . . , n_i\}$ in phase i can be represented as a binary variable $X_{i,j}$

$$X_{i,j} = \begin{cases} 1 & \text{if component } j \text{ is functioning in phase } i \\ 0 & \text{if component } j \text{ is failed in phase } i \end{cases} \quad (1)$$

The state of the system in phase i can then be described as a binary function

$$\varphi_i(\boldsymbol{X}_i) = \varphi_i(X_{i,1}, \ldots, X_{i,n_i}) \quad (2)$$

where vector $\boldsymbol{X}_i=[X_{i,1}, \ldots, X_{i,n_i}]$ represents the states of all components in phase i, $\varphi_i(\boldsymbol{X}_i)=1$ represents success for functioning of the system during phase i and $\varphi_i(\boldsymbol{X}_i)=0$ represents the failure of the system in phase i.

Similarly, the state of the PMS is also a binary function which is completely determined by the states of all the components during the mission

$$\varphi_s\left(\boldsymbol{X}\right)=\varphi_s\left(X_{1,1},\dots,X_{1,n_1},\dots,X_{N,1},\dots,X_{N,n_N}\right) \qquad (3)$$

where vector $\boldsymbol{X}$=[$\boldsymbol{X}_1$,…, $\boldsymbol{X}_N$]=[$X_{1,1}$, …, X_{1,n_1} ,…, $X_{N,1}$, …, X_{N,n_N}] represents the states of the components during the mission. Because a PMS is functioning if and only if all its phases are completed without failure, the structure function of the PMS can be written as

$$\varphi_s(\boldsymbol{X})=\prod_{i=1}^{N}\varphi_i\left(X_{i,1},\dots,X_{i,n_i}\right) \qquad (4)$$

Since the states of the same component in different phases are dependent, the structure function (4) cannot be directly used to calculate the reliability of the PMS.

For the calculation of the reliability of the PMS, we could use a truth table to tabulate all possible combinations of the states of the components in each phase to realize the expression of the PMS and then calculate the reliability of the PMS by adding all functioning state probabilities. What is important and needs to be emphasized is that, in this paper, both the system and its components are assumed to be non-repairable during the mission, so if a component is failed in a certain phase, it cannot work again in subsequent phases. Therefore, when constructing the truth table, for component j in phases 1, 2,…, N, the following conditions should be satisfied: $X_{1,j}\geq X_{2,j}\geq\ldots\geq X_{N,j}$. The state combinations of the components that violate these conditions cannot be included in the truth table.

3. Reliability importance of each type of components

For a larger system, working with the full truth table is complicated, and one may particularly only need a summary of the truth table in case that the system has exchangeable components of one or more types. Recently, the theory of the survival signature has attracted increasing attention for performing reliability analysis of larger systems due to its high efficiency and low complexity [21-25]. In this section, a new survival signature is applied for reliability and reliability importance analysis of PMSs.

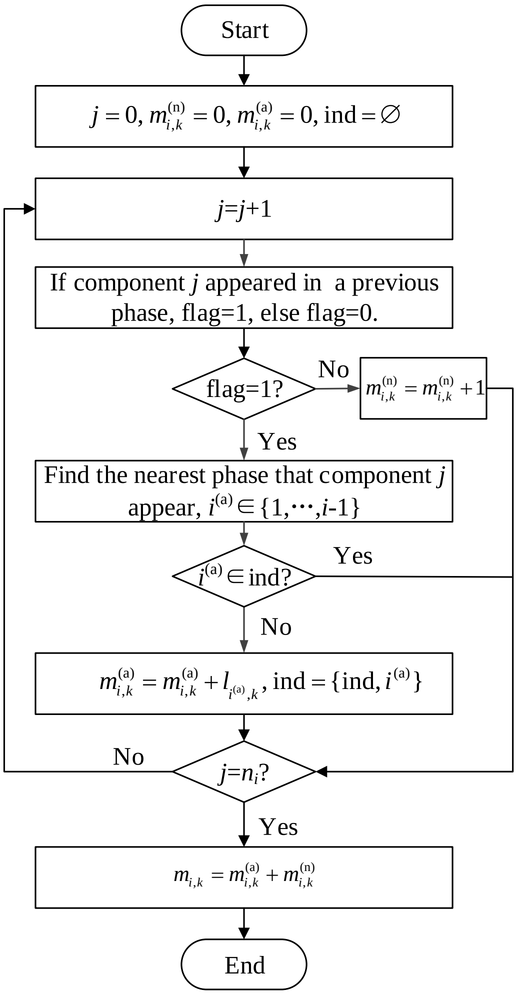

Figure 2 Procedure for the calculation of $m_{i,k}$

Let $\Phi_s(l_{1,1},\dots,l_{1,n_1},\dots,l_{N,1},\dots,l_{N,n_N})$ denote the probability that the PMS functions given that precisely $l_{i,k}$, $k\in\{1, 2, \dots, n_i\}$, components of type k function in phase i. Since the failure of the components in each phase is assumed to be independent and exchangeable, the survival signature of the PMS can be derived as follows [26]:

$$\begin{aligned}&\Phi_s(l_{1,1},\dots,l_{1,n_1},\dots,l_{N,1},\dots,l_{N,n_N})\\&=\prod_{i=1}^{N}\prod_{k=1}^{n_i}\binom{m_{i,k}}{l_{i,k}}^{-1}\times\sum_{\boldsymbol{X}\in S_{l_{1,1},\dots,l_{N,n_N}}}\varphi_s\left(\boldsymbol{X}\right)\end{aligned} \qquad (5)$$

where $\varphi_s\left(\boldsymbol{X}\right)$ is the structure function of the PMS defined in Section 2, $S_{l_{1,1},\ldots,l_{N,n_N}}$ denotes the set of all state vectors for the whole system. $m_{i,k}$ is the number of components of type k that function at the beginning of phase i. For phase i=1, $m_{1,k}$ is the number of the components of type k in phase 1. Generally, the number of components of type k that function at the beginning of phase k can be obtained as follows

$$m_{i,k} = m_{i,k}^{(a)} + m_{i,k}^{(n)} \tag{6}$$

where $m_{i,k}^{(a)}$ is the number of the components of type k that appeared in a phase before phase i that function at the beginning of phase i. While $m_{i,k}^{(n)}$ is the number of components of type k that have not appeared in any phase before phase i. As shown in Figure 2, the basic steps for calculating $m_{i,k}^{(a)}$ and $m_{i,k}^{(n)}$ can be explained as follows: (1) Check whether component j in phase i has appeared in a phase before phase i. If component j hasn't appeared, $m_{i,k}^{(n)} = m_{i,k}^{(n)} + 1$. (2) If component j has appeared in the phases before phase i, find the nearest phase that it appeared, $i^{(a)}$, $m_{i,k}^{(a)} = m_{i,k}^{(a)} + l_{i^{(a)},k}$, where $l_{i^{(a)},k}$ is the number of the components of type k function in phase $i^{(a)}$. Only one time of calculation is needed when more than one components have appeared in the same nearest phase. Readers are referred to the work of Huang et al. [26] for more details.

Let phase i run from fixed time τ_{i-1} to fixed time τ_i with $\tau_0 \equiv 0$ and $\tau_{i-1} < \tau_i \ \forall\ i$, the reliability of the system at time t can be expressed as:

$$R_s(t) = \sum_{l_{1,1}=0}^{m_{1,1}} \cdots \sum_{l_{N(t),n_{N(t)}}=0}^{m_{N(t),n_{N(t)}}} \left[\Phi(l_{1,1},\ldots,l_{1,n_t},\ldots,l_{N(t),1},\ldots,l_{N(t),n_{N(t)}}) P\left(\bigcap_{i=1}^{N(t)} \bigcap_{k=1}^{n_i} \left\{ C_{i,k}(t) = l_{i,k} \right\} \right) \right] \tag{7}$$

where $N(t) \le N$ is the phase that the system is in at time t,

$$N(t) = \begin{cases} 1, & t \in [\tau_0, \tau_1) \\ 2, & t \in [\tau_1, \tau_2) \\ \cdots \\ N, & t \in [\tau_{N-1}, \tau_N) \end{cases} \tag{8}$$

$\Phi(l_{1,1},\ldots,l_{1,n_t},\ldots,l_{N(t),1},\ldots,l_{N(t),n_{N(t)}})$ is the survival signature of the first $N(t)$ phases, $C_{i,k}(t)$ is the number of components of type k that function in phase i at time $t \in [\tau_{i-1}, \tau_i)$, $n_{N(t)}$ is the number of types of components in phase $N(t)$.

If the life times of the components of type k have a known conditional cumulative distribution function (CDF), Equation (7) can be rewritten as follows:

$$R_s(t) = \sum_{l_{1,1}=0}^{m_{1,1}} \cdots \sum_{l_{N(t),n_{N(t)}}=0}^{m_{N(t),n_{N(t)}}} \left[\Phi(l_{1,1},\ldots,l_{1,n_1},\ldots,l_{N(t),1},\ldots,l_{N(t),n_{N(t)}}) \prod_{i=1}^{N(t)} \prod_{k=1}^{n_i} \left[\binom{m_{i,k}}{l_{i,k}} \left(R_{i,k}(t)\right)^{l_{i,k}} \left(1 - R_{i,k}(t)\right)^{m_{i,k}-l_{i,k}} \right] \right] \tag{9}$$

where $R_{i,k}(t)=1-F_{i,k}(t)$, $F_{i,k}(t)$ is the CDF of the life time of the components of type k in phase i conditioned on that these components work at the beginning of phase i.

As an example, now let us describe the procedure to calculate the reliability the PMS in Figure 1. Firstly, Equation (5) is used to calculate the survival signature of the system. The survival signatures of phase 1, the first two phases and all phases are expressed as $\Phi_1(l_{1,1}, l_{1,2})$, $\Phi_{1,2}(l_{1,1},\ldots,l_{2,2})$ and $\Phi_s(l_{1,1},\ldots,l_{3,2})$, respectively. Then, the reliability of the system is derived as follows:

$$R_s(t) = \begin{cases} \sum_{l_{1,1}=0}^{m_{1,1}} \sum_{l_{1,2}=0}^{m_{1,2}} \Phi_1(l_{1,1}, l_{1,2}) \prod_{k=1}^{2} \left[\binom{m_{1,k}}{l_{1,k}} \left(R_{1,k}(t)\right)^{l_{1,k}} \left(1 - R_{1,k}(t)\right)^{m_{1,k}-l_{1,k}} \right], & t \in [\tau_0, \tau_1) \\ \sum_{l_{1,1}=0}^{m_{1,1}} \cdots \sum_{l_{2,2}=0}^{m_{2,2}} \left[\Phi_{1,2}(l_{1,1},\ldots,l_{2,2}) \prod_{i=1}^{2} \prod_{k=1}^{2} \left[\binom{m_{i,k}}{l_{i,k}} \left(R_{i,k}(t)\right)^{l_{i,k}} \left(1 - R_{i,k}(t)\right)^{m_{i,k}-l_{i,k}} \right] \right], & t \in [\tau_1, \tau_2) \\ \sum_{l_{1,1}=0}^{m_{1,1}} \cdots \sum_{l_{3,2}=0}^{m_{3,2}} \left[\Phi_s(l_{1,1},\ldots,l_{3,2}) \prod_{i=1}^{3} \prod_{k=1}^{2} \left[\binom{m_{i,k}}{l_{i,k}} \left(R_{i,k}(t)\right)^{l_{i,k}} \left(1 - R_{i,k}(t)\right)^{m_{i,k}-l_{i,k}} \right] \right], & t \in [\tau_2, \tau_3) \end{cases} \tag{10}$$

In reliability engineering, reliability importance measures can be used to prioritize components in a system during reliability improvement and maintenance planning of the system. For these kinds of purposes, a number of measures have been suggested in the literature [27]. Well known reliability importance measures mainly include the Birnbaum importance [28-29], Barlow-Proschan importance [30, 31], Fussell-Vesely importance [32], differential importance [33, 34], cost-based importance [35] and joint importance [36]. The Birnbaum importance measure for system reliability is undoubtedly the fundamental because many importance measures refer to it. In this paper, the theory of Birnbaum importance measure is applied to propose a practical and efficient method for reliability importance analysis of PMSs.

Mathematically, the Birnbaum importance is the partial derivative of the system reliability with respect to the reliability of an individual component [28, 29]. Therefore, for the PMS the reliability importance of the components of type $v\in\{1, 2, . . . , n_u\}$ of phase u, $u\in\{1, 2, . . . , N\}$ can be derived from Equation (9) as follows:

$$RI_{u,v}^{T}(t)=\frac{\partial R_s(t)}{\partial R_{u,v}(t)}=\sum_{l_{1,1}=0}^{m_{1,1}}\cdots\sum_{l_{N(t),n_{N(t)}}=0}^{m_{N(t),n_{N(t)}}}\left\{\frac{l_{u,v}-m_{u,v}R_{u,v}(t)}{R_{u,v}(t)\left(1-R_{u,v}(t)\right)}\Phi(l_{1,1},...,l_{1,n_1},...,l_{N(t),1},...,l_{N(t),n_{N(t)}})\times \prod_{i=1}^{N(t)}\prod_{k=1}^{n_i}\left[\binom{m_{i,k}}{l_{i,k}}\left(R_{i,k}(t)\right)^{l_{i,k}}\left(1-R_{i,k}(t)\right)^{m_{i,k}-l_{i,k}}\right]\right\},\quad 0<R_{u,v}(t)<1 \tag{11}$$

where $R_{u,v}(t)$ is the conditional reliability of the components of type v in phase u, $l_{u,v}$ is the number of components of type v function in phase u, $m_{u,v}$ is the number of components of type v that function at the beginning of phase u.

4. Reliability importance of each component

Most practical PMSs for which the reliability importance is investigated consist of multiple components in each type. Therefore, a more interesting challenge is to develop the theory of reliability importance analysis of PMSs with respect to each component in each phase. According to the definition of Birnbaum importance [28, 29], the reliability importance index of a PMS with respect to component j, $j\in\{1, 2, . . . , n_i\}$ in phase i, $i\in\{1, 2, . . . , N\}$, denoted by $RI_{i,j}(t)$, can be defined as follows:

$$RI_{i,j}(t)=P\left(T_s>t\mid X_{i,j}=1\right)-P\left(T_s>t\mid X_{i,j}=0\right) \tag{12}$$

where $P(T_s>t|X_{i,j}=1)$ represents the probability that the system functions on condition that the jth component works in phase i; $P(T_s>t|X_{i,j}=0)$ represents the probability that the system functions knowing that the jth component is failed in phase i.

As described in section 2, both the system and its components are assumed to be non-repairable during the mission, so if a component is failed in a certain phase, it cannot work again in subsequent phases. Moreover, if a component works in a certain phase, it should work in previous phases. For example, the RBDs for calculating the reliability importance of component 4 in phase i, $RI_{i,4}(t)$, for i=1, 2 and 3, can be depicted as Figures 3-5. In Figure 3, (*a*) is used for interpreting the RBD of the PMS that component 4 works in phase 1 and (*b*) is used for showing the RBD of the PMS that component 4 is failed in phase 1. Because component 4 works in phase 1, it is represented by a pathway. Because component 4 is failed in phase 1, it cannot work again in phases 2 and 3. So, component 4 is represented by a break in phases 1, 2 and 3. Similarly, in Figures 4 and 5, (*a*) is used for interpreting the case that component 4 works in phase 2 or 3 and (*b*) is used for showing the case that component 4 is failed in phase 2 or 3.

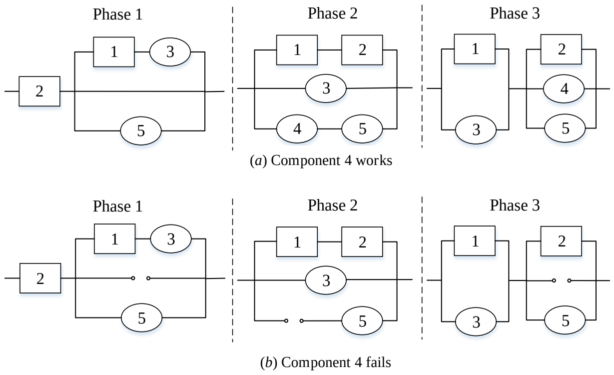

Figure 3 RBD of the PMS for calculating $RI_{1,4}(t)$ and $RI^*_{1,4}(t)$

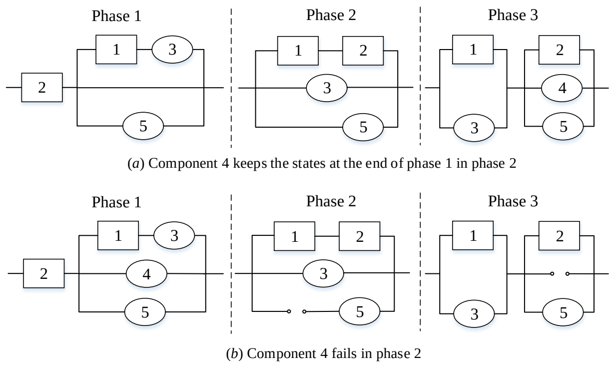

Figure 4 RBD of the PMS for calculating $RI_{2,4}(t)$

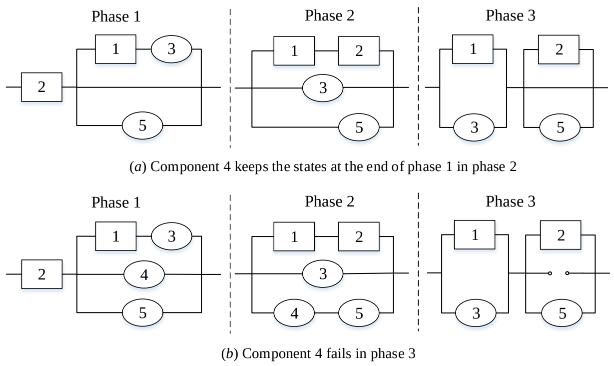

Figure 5 RBD of the PMS for calculating $RI_{3,4}(t)$

In order to calculate the reliability importance of component 4 of phase i, for i=1, 2, 3, the survival signatures of the PMSs shown in Figures 3-5 are calculated using Equation (5). Then Equation (9) is applied to calculate $P(T_s>t|X_{i,j}=1)$ and $P(T_s>t|X_{i,j}=0)$ to obtain the reliability importance of component 4 in each phase. The reliability of a component in a certain phase can only affect the reliability of the system in current and subsequent phases and has no effect on the reliability of the system in previous phases. Therefore, the reliability importance of component 4 of phase i can be obtained as follows

$$\begin{cases} RI_{1,4}(t)=P\left(T_s>t\,|\,X_{1,4}=1\right)-P\left(T_s>t\,|\,X_{1,4}=0\right),\ t\in[\tau_0,\tau_3) \\ RI_{2,4}(t)=P\left(T_s>t\,|\,X_{2,4}=1\right)-P\left(T_s>t\,|\,X_{2,4}=0\right), t\in[\tau_1,\tau_3) \\ RI_{3,4}(t)=P\left(T_s>t\,|\,X_{3,4}=1\right)-P\left(T_s>t\,|\,X_{3,4}=0\right), t\in[\tau_2,\tau_3) \end{cases} \tag{13}$$

As can be seen from Equation (12) and Figures 3 and 4, the calculation of $P(T_s>t|X_{i,j}=1)$ is based on the condition that component j works in all the phases from 1 to i. This model will clearly overestimate the reliability importance of component j of phase i, for $2\le i\le N$. In order to solve the problem, we propose a new reliability importance measure for PMSs as follows:

$$RI^*_{i,j}(t)=P\left(T_s>t\,|\,X_{i,j}=X_{i-1,j}\right)-P\left(T_s>t\,|\,X_{i,j}=0\right) \tag{14}$$

where $P(T_s>t|X_{i,j}=X_{i-1,j})$ represents the probability that the system functions if component j in phase i keeps the same state as in phase i-1, for phase i=1, $P(T_s>t|X_{i,j}=X_{i-1,j})=P(T_s>t|X_{i,j}=1)$. $P(T_s>t|X_{i,j}=0)$ represents the probability that the system functions knowing that component j is failed in phase i.

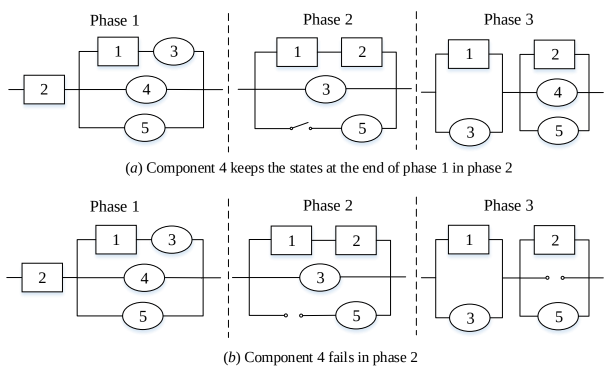

Figure 6 RBD of the PMS for calculating $RI^*_{2,4}(t)$

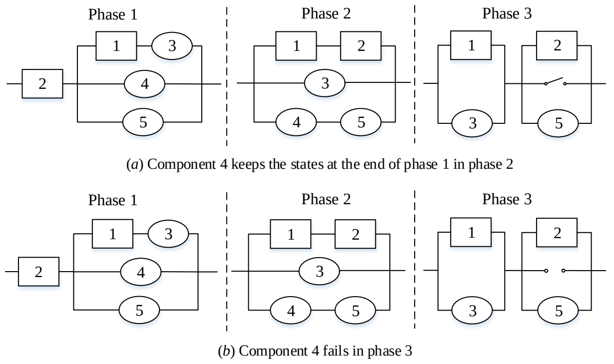

Figure 7 RBD of the PMS for calculating $RI^*_{3,4}(t)$

In this case, the RBD for calculating $RI^*_{1,4}(t)$ is the same as shown in Figure 3. The RBDs for calculating $RI^*_{2,4}(t)$ and $RI^*_{3,4}(t)$ are shown in Figures 6 and 7. In Figures 6 and 7, the switch symbol is used to represent that component 4 in a phase keeps the same state as in the previous phase. Figure (*a*) is used for interpreting the case that component 4 in phase *i* keeps the same state as in phase *i*-1 and Figure (*b*) is used for showing the case that component 4 is failed in phase *i*. The survival signatures of the PMSs shown in Figures 3, 6 and 7 can be calculated using Equation (5). Then the reliability importance of component 4 of phase *i* is obtained by calculating $P(T_s>t|X_{i,j}=X_{i-1,j})$ and $P(T_s>t|X_{i,j}=0)$ as follows:

$$\begin{cases} RI_{1,4}(t)=P\left(T_s>t|X_{1,4}=1\right)-P\left(T_s>t|X_{1,4}=0\right),\ t\in[\tau_0,\tau_3) \\ RI_{2,4}(t)=P\left(T_s>t|X_{2,4}=X_{1,4}\right)-P\left(T_s>t|X_{2,4}=0\right), t\in[\tau_1,\tau_3) \\ RI_{3,4}(t)=P\left(T_s>t|X_{3,4}=X_{2,4}\right)-P\left(T_s>t|X_{3,4}=0\right), t\in[\tau_2,\tau_3) \end{cases} \tag{15}$$

From Equations (11) and (14), we can learn that the reliability importance of each component in each phase is a function of time and it measures the degree of the influence of the reliability of the component in that phase, i.e., the bigger the value is, the bigger the influence of the component on the reliability of the PMS at a specific time *t* is, and vice versa. At each point in time the largest reliability importance over all components shows the most “critical” component. This helps to allocate resources for inspection, maintenance and repair in an optimal manner over the life-time of a system.

5. Numerical examples

Example 1 For the PMS shown in Figure 1, it is known that phases 1, 2 and 3 last for 10, 270 and 20 hours respectively. The conditional distributions of the life-time of the components in each phase can be divided into two types. The lifetime of the first type of components follows the same distribution in all phases. Table 1 summarizes the distribution information. For Weibull distribution, α and β are the scale parameter and shape parameter, respectively; for exponential distribution, λ is the failure rate.

Table 1 Conditional distribution information of the components in each phase

Type	Component	Distribution	Phase 1	Phase 2	Phase 3
1	1, 2	Weibull		α=400, β=3.2	
2	3, 4, 5	Exponential	$\lambda=5\times10^{-2}$	$\lambda=1\times10^{-4}$	$\lambda=2\times10^{-4}$

The survival signatures of the PMS can be obtained using Equation (7). All the results are shown in Table A1. In the table, {*c d*} represents the integers between *c* and *d*. For example, {0 3} represents the integers 0, 1, 2 and 3. This symbol is used to simplify the list of survival signature. For example, in the first phase, $l_{1,1}$ =1 and $l_{1,2}$ ={2 3} represent the combinations $[l_{1,1}, l_{1,2}]$=[1, 2] and $[l_{1,1}, l_{1,2}]$=[1, 3]. Φ_1, $\Phi_{1,2}$ and Φ_s are the survival signatures of the first phase, the first two phases and all phases, respectively. Rows with values $\Phi_1(l_{1,1},l_{1,2})=0$, $\Phi_{1,2}(l_{1,1},l_{1,2},l_{2,1},l_{2,2})=0$ and $\Phi_s(l_{1,1},l_{1,2},l_{2,1}, l_{2,2}, l_{3,1}, l_{3,2})=0$ are omitted.

Table 2 Reliability of the PMS at the time of phase switches

t	0	10^-	10^+	20^-	20^+	30
R_S	1	0.939	0.939	0.842	0.776	0.753

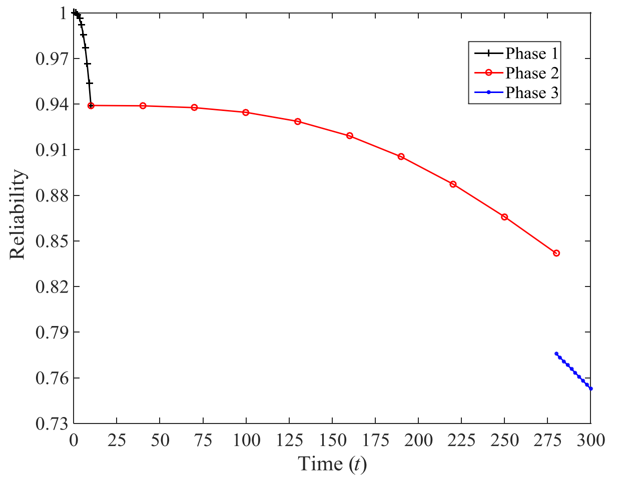

Figure 8 Reliability of the PMS

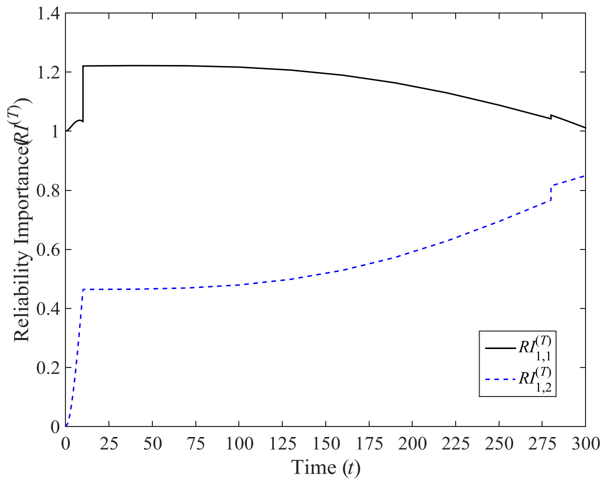

Figure 9 Reliability importance of each type of components of phase 1

We can obtain the conditional CDF of the life-time of each type of components in each phase by using the parameters shown in Table 1. Then the reliability of the PMS can be obtained by substituting the survival signatures and the conditional life-time CDFs of the components into Equation (9). The results are shown in Table 2 and Figure 8. From the results, we can learn that there is a reliability jump at t=280. The reason is that if components 1 and 3 or components 2, 4 and 5 are failed simultaneously in phase 2, the PMS may still function in phase 2, however, the PMS will be failed immediately when it steps into phase 3. Therefore, there is a sharp reliability jump between phases 2 and 3.

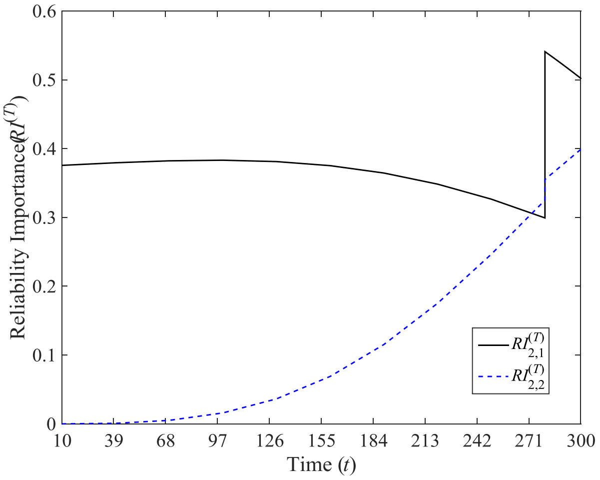

Figure 10 Reliability importance of each type of components of phase 2

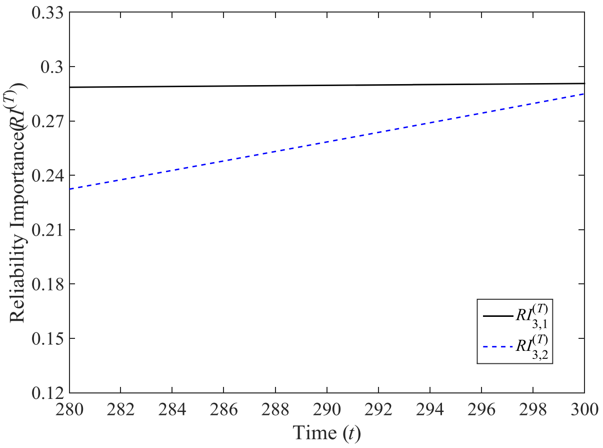

Figure 11 Reliability importance of each type of components of phase 3

Reliability importance analysis of the PMS is conducted by using Equation (11). Figures 9-11 show the reliability importance of each type of components in each phase, respectively. Since all the values of the reliability importance are positive, an increase in the reliability of each type of components in each phase increases the overall reliability of the PMS. In general, the components of type 1 are more important than those of type 2 in each phase. At the end of the phased mission (t=300), these results provide an ordering $RI_{1,1}^{(T)} > RI_{1,2}^{(T)} > RI_{2,1}^{(T)} > RI_{2,2}^{(T)} > RI_{3,1}^{(T)} > RI_{3,2}^{(T)}$. This indicates that for this PMS, the reliability of components in the earlier phases are more important than that in the later phases for completing the phased mission successfully. Therefore, if it is possible, we may be wise to focus on improving the reliability of the first type of components, especially in the first phase, to improve the reliability of this PMS. Moreover, we can find that the reliability of the components in a certain phase can only affect the reliability of the system in the current and subsequent phases and has no influence on the reliability of the system in previous phases.

To rank the importance of each component, we use Equation (12) to calculate the reliability importance of each component in each phase, the results are shown as the solid lines in Figures 12-14. The results show that the reliability of the components change with time, especially at the time of switches of phases. For example, component 1 in phase 2 is more important than component 3 in phase 2 at the beginning of phase 2, however, the order is reversed at the end of the phased mission. In general, component 2 in phase 1 is most important. This result is the same as our perception. If component 1, 3, 4 or 5 is failed in phase 1, there are still some chances that the phased mission can be completed successfully, however, if component 2 is failed in phase 1, the PMS will be failed immediately. Therefore, in engineering practice, it would be wise to give priorities to component 1 in phase 1 to effectively increase the chance to complete the phased mission successfully. In addition, we can also find that the reliability importance of components 4 and 5 is always equal. The reason is that the life-time distribution of components 4 and 5 is the same and their positions are equal in each phase. At the end of the phased mission, these results provide an ordering $RI_{1,2}>RI_{1,3}>RI_{2,3}>RI_{3,3}>RI_{3,1}> RI_{2,1}>RI_{1,1}>RI_{2,2}>RI_{1,4}=RI_{1,5}>RI_{3,2}>RI_{2,4}=RI_{2,5}>RI_{3,4}=RI_{3,5}$.

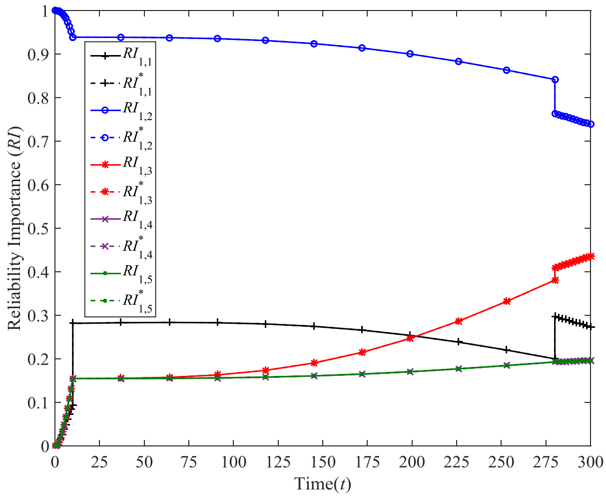

Figure 12 Reliability importance of each component of phase 1

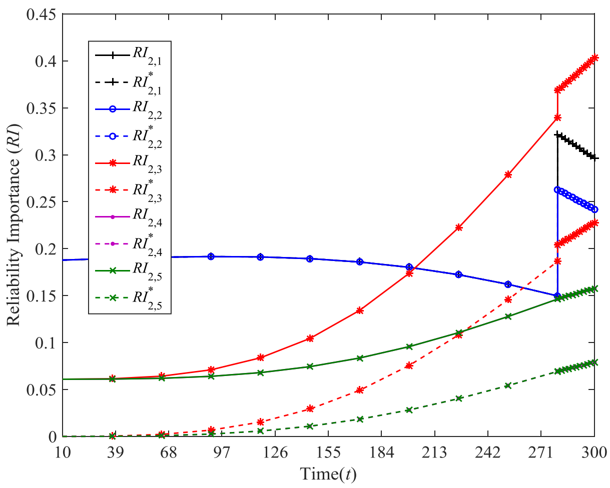

Figure 13 Reliability importance of each component of phase 2

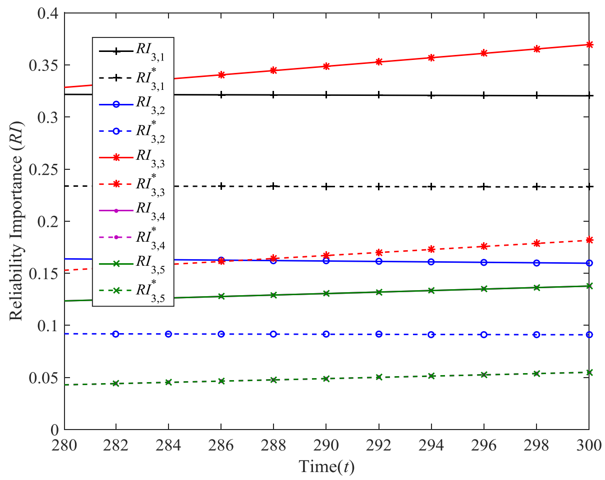

Figure 14 Reliability importance of each component of phase 3

Since Equation (12) tends to overestimate the importance of the components in all phases except phase 1, Equation (13) is used to calculate the reliability importance of each component in each phase, the results are shown as the dashed lines in Figures 12-14. The results show that the overestimation is eliminated successfully. At the end of PMS, the importance of each component in each phase is ordered as follows: $RI^*_{1,2} > RI^*_{1,3} > RI^*_{2,1} > RI^*_{1,1} > RI^*_{2,2} > RI^*_{3,1} > RI^*_{2,3} > RI^*_{1,4} = RI^*_{1,5} > RI^*_{3,3} > RI^*_{3,2} > RI^*_{2,4} = RI^*_{2,5} > RI^*_{3,4} = RI^*_{3,5}$. This order is quite different from that from Equation (12). In engineering practice, it would be wise to give priorities to these components in different phases to effectively increase the chance to complete the phased mission successfully. With the help of the information of reliability importance of the PMS, engineers could design different maintenance strategies in distinct phases to reduce the risk to the lower extent.

Further study shows that since the life-time distribution of components 4 and 5 is the same and their positions are equal in each phase, the reliability importance of components 4 and 5 is equal at any time. Moreover, an interesting and important conclusion can be drawn from Figures 9-14. If we use the second definition of the component reliability importance, the reliability importance of type k components, for k =1 and 2, is equal to the sum the reliability importance of each component of type k. For example, $RI^{(T)}_{i,1}(t) = RI^*_{i,1}(t) + RI^*_{i,2}(t)$, $RI^{(T)}_{i,2}(t) = RI^*_{i,3}(t) + RI^*_{i,4}(t) + RI^*_{i,5}(t)$, for i=1, 2 and 3. Similar conclusions can be drawn in (joint) reliability importance analysis of coherent systems [36]. Since the failure of the components in each phase is assumed to be independent and exchangeable, this result is obviously correct. This further verifies the correctness of the proposed reliability importance analysis method.

Example 2 In this example, the space application mission discussed by Zang et al. [3] is used to demonstrate the application of the proposed method for complex PMS with different types of components. The space application can be divided into five phases. Launch is the first phase, followed by Hibern.1, Asteroid, Hibern.2, Comet. The RBD of space application is shown as Figure 15. It is known that the five phases last for 48, 17520, 672, 26952 and 672 hours, respectively. The lifetimes of all the components follow exponential distributions, and the failure rates of the components in each phase are given in Table 3.

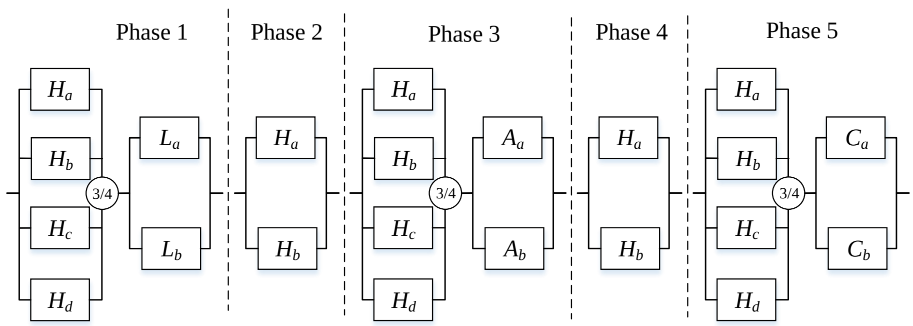

Figure 15 RBD of space application

Table 3 Failure rates of the components

	Phase1	Phase 2	Phase 3	Phase 4	Phase 5
H_a, H_b, H_c, H_d	1×10^{-5}	1×10^{-6}	1×10^{-5}	1×10^{-6}	1×10^{-5}
L_a, L_b	5×10^{-5}	0	0	0	0
A_a, A_b	0	0	1×10^{-5}	0	0
C_a, C_b	0	0	0	0	1×10^{-4}

Table 4 Types of components in Exp.3

Type 1	Type 2	Type 3	Type 4	Type 5
H_a, H_b	H_c, H_d	L_a, L_b	A_a, A_b	C_a, C_b

Table 5 Reliability importance of the PMS in Exp.3

$RI_{1,1}^{(T)}$	$RI_{1,2}^{(T)}$	$RI_{1,3}^{(T)}$	$RI_{2,1}^{(T)}$	$RI_{3,1}^{(T)}$	$RI_{3,2}^{(T)}$	$RI_{3,4}^{(T)}$	$RI_{4,1}^{(T)}$	$RI_{5,1}^{(T)}$	$RI_{5,2}^{(T)}$	$RI_{5,5}^{(T)}$
0.1520	0.2315	0.0047	0.1546	0.1530	0.2330	0.0133	0.1561	0.1530	0.2330	0.1292

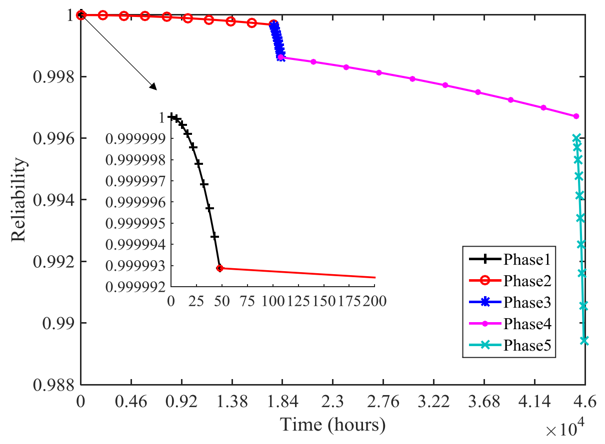

Figure 16 Reliability of the PMS

As shown in Table 4, in order to calculate the reliability of the PMS, we divided the components into 5 types. The reliability of the PMS is shown in Figure 15. The results are the same as these from BDD based method [3]. Table 5 shows the results of reliability importance analysis. From the results we can learn that H_c and H_d have the most significant influence on the reliability of the PMS. For example, if we have a chance to reduce the failure rates of H_c and H_d by half in phases 1, 3 and 5 (that is to say the failure rates of them in these phases are 0.5×10^{-5}), the reliability of PMS becomes R_s=0.99101. And if we manage to reduce the failure rates of H_a and H_b by half in phases 1, 3 and 5, the reliability of PMS will be R_s=0.990447. This further verifies the correctness of the proposed reliability importance analysis method.

6 Conclusion

The reliability importance of a PMS quantifies the influence of the reliability of each component (or each type of components) on the reliability of the PMS. The higher the value of the reliability importance of a component is, the greater is the influence of the reliability of this component on the reliability of the PMS, and vice versa. Therefore, reliability importance analysis is often critical towards understanding the PMS underlying failure and provides useful information for reliability improvement and risk reduction. In this paper, a new and efficient method for reliability importance analysis of PMS is proposed based on the theory of survival signature.

A new kind of survival signature proposed by the authors [26] is applied to calculate the reliability of the PMS. The proposed approach could separate the system structure from the component probabilistic failure distribution, thereby reducing the overall computational complexity. The reliability importance of different types of components in each phase is derived analytically in this paper to evaluate the relative importance of the components with respect to the reliability of the PMS. In comparison with reliability analysis, reliability importance analysis doesn't need more computation. This is another advantage of the proposed method. Moreover, the Birnbaum importance model is further extended to calculate the reliability importance of the PMS with respect to each component in each phase. An interesting and important conclusion can be drawn: the reliability importance of a type of components is equal to the sum the reliability importance of each component of this type. Since the failure of the components in each phase is assumed to be independent and exchangeable, this result is obviously correct. This further verifies the correctness of the proposed reliability importance analysis method.

However, it should be noted that the reliability importance discussed in this paper is based on Birnbaum importance. Therefore, reliability importance obtained in this study is the local result which is valid when the reliability is changed by a small amount. If one has an opportunity to improve reliability of components for a large amount, then of course the reliability importance should be calculated multiple times. Moreover, the reliability and reliability importance analysis of PMSs with multiple failure mode components are not studied in this paper. In practice the components may perhaps have more than one failure modes. And global reliability importance analysis of PMS with multiple failure mode components is the subject of current research by the authors. In general, however, this paper presents a practical method for reliability importance analysis of PMSs using the theory of survival signature.

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Appendix

Table A1 Survival signature of the PMS shown in Figure 1

The first phase			The first two phases					All phases						
$l_{1,1}$	$l_{1,2}$	Φ_1	$l_{1,1}$	$l_{1,2}$	$l_{2,1}$	$l_{2,2}$	$\Phi_{1,2}$	$l_{1,1}$	$l_{1,2}$	$l_{2,1}$	$l_{2,2}$	$l_{3,1}$	$l_{3,2}$	Φ_s
1	1	1/3	1	{2 3}	{0 1}	1	1/6	1	2	1	{1 2}	1	1	1/6
1	{2 3}	1/2	2	{1 3}	{0 1}	1	1/3	1	3	1	{1 3}	1	1	1/6
2	{1 3}	1	1	2	{0 1}	2	1/2	2	{1 3}	1	1	1	1	1/6
			1	3	{0 1}	{2 3}	1/2	1	2	{0 1}	2	0	2	1/3
			2	1	2	{0 1}	1	1	2	1	2	1	2	1/3
			2	2	{0 1}	2	1	1	3	0	{2 3}	0	2	1/3
			2	2	2	{0 2}	1	1	3	1	{2 3}	{0 1}	2	1/3
			2	3	{0 1}	{2 3}	1	1	3	0	3	0	3	1/2
			2	3	2	{0 3}	1	1	3	1	3	{0 1}	3	1/2
								2	1	2	1	1	1	1/2
								2	3	2	{1 3}	1	1	1/2
								2	2	{1 2}	2	1	1	1/2
								2	3	1	{2 3}	1	1	1/2
								2	2	{0 2}	2	0	2	2/3
								2	3	{0 2}	{2 3}	0	2	2/3
								2	2	{1 2}	2	1	2	5/6
								2	3	{1 2}	{2 3}	1	2	5/6
								2	{1 3}	2	0	2	0	1
								2	{1 3}	2	1	2	{0 1}	1
								2	{2 3}	2	2	2	{0 2}	1
								2	3	0	3	0	3	1
								2	3	{1 2}	3	{0 1}	3	1
								2	3	2	3	2	{0 3}	1