

# Long term behaviour of two interacting birth-and-death processes

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## Abstract

In this paper we study the long term evolution of a continuous time Markov chain formed by two interacting birth-and-death processes. The interaction between the processes is modelled by transition rates which are functions with suitable monotonicity properties. This is in line with the approach proposed by Gauss G.F. and Kolmogorov A.N. for modelling interaction between species in ecology. We obtain conditions for transience/recurrence of the Markov chain and describe in detail its asymptotic behaviour in special transient cases. In particular, we find that in some of these cases the Markov chain escapes to infinity in an unusual way, and the corresponding trajectories can be rather precisely described.

## 1 Introduction

A birth-and-death process on  $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$  is a continuous time Markov chain (CTMC) that evolves as follows. Given a current state  $k$  it jumps either to  $k + 1$ , or to  $k - 1$  (if  $k > 0$ ) at certain state dependent rates. The long term behaviour of a birth-and-death process is well known. Namely, given a set of transition rates one can, in principle, determine whether the corresponding birth-and-death process is recurrent/positive recurrent, or transient/explosive, and compute various characteristics of the process. These results can be found in many books (e.g., see [6], [8] and [10]). The long term behaviour of multivariate Markov processes with similar dynamics is less known.

In this paper we study the long term behaviour of CTMC  $\xi(t) = (\xi_1(t), \xi_2(t)) \in \mathbb{Z}_+^2$ , evolving as follows. Given  $\xi(t) = (x, y) \in \mathbb{Z}_+^2$  the Markov chain jumps to  $(x + 1, y)$  and to  $(x, y + 1)$  at rates  $F(x)G(y)$  and  $F(y)G(x)$  respectively, where  $F$  and  $G$  are positive functions on  $\mathbb{R}_+ = [0, \infty)$ . Also, the Markov chain jumps from  $(x, y)$  to  $(x - 1, y)$  at the constant rate of 1, provided  $x > 0$ , and it jumps to  $(x, y - 1)$  at the same constant rate of 1, provided that  $y > 0$ .

The Markov chain is a two-dimensional analogue of integer valued birth-and-death processes, and can be interpreted in terms of two interacting birth-and-death processes. The

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construction of the birth rates allows to model various types of both individual dynamics and interaction between the Markov chain components. Function  $F$  determines, in terms of statistical physics, the free dynamics of a component. Interaction between components is modelled by choosing an appropriate function  $G$ . If, say,  $G \equiv 1$ , then  $\xi_1(t)$  and  $\xi_2(t)$  are independent identically distributed birth-and-death processes. Given  $F$ , one can choose a decreasing  $G$  in order to model a competitive interaction. If  $G$  is increasing, then a component's growth is accelerated by its neighbour.

Recall that a birth-and-death process on  $\mathbb{Z}_+$  is a classic probabilistic model for the size of a population. Therefore CTMC  $\xi(t)$  can be regarded as a stochastic model for two interacting populations. The model is related to stochastic population models formulated in terms of two interacting birth-and-death processes (e.g., see [1], [2], [3], [13], [14] and references therein). In these models, which are stochastic versions of the famous Lotke-Volterra model, a pair of birth-and-death processes typically evolves as follows. Given a current state  $(x, y)$  of the components, the individual transition rates are linear in  $x$  and  $y$ , while interaction terms, included usually in death rates (i.e. competitive interaction), are proportional to  $xy$ . Our model is in the spirit of the more general approach proposed by Gauss G.F. ([7]) and Kolmogorov A.N. ([9]) for modelling interactions in ecology. Although they considered deterministic population models, the idea is rather general. According to this approach, the interaction between species should be modelled by transition rates specified by general functions with suitable (suggested by a motivating application) monotonicity properties. A brief, but informative presentation of these ideas is given in [15], where further references can be found. In our model the interaction is built into the birth rates, though the model can be generalised by allowing for non-constant death rates. We do not explore further the relationship of the Markov chain with stochastic population models and focus on its long term behaviour which is of interest from a mathematical point of view.

It should be also noted that our Markov chain is a particular example of non-homogeneous random walks. The long term behaviour of non-homogeneous random walks is much less studied (e.g., see [11] and references therein) in contrast to homogeneous random walks in domains with boundaries (e.g., see [5] and references therein).

We systematically apply the Lyapunov function approach in our proofs. This approach is well known and widely used for determining whether a Markov process is recurrent or transient (e.g., see [5], [11] and references therein). In Theorem 1 we establish whether the Markov chain is transient or recurrent under fairly general assumptions on functions  $F$  and  $G$ . Though the asymptotic behaviour of the Markov chain in this theorem can be guessed from approximate sketches of the vector field of its mean infinitesimal jumps (see Figures 1 and 2), the Lyapunov function approach helps to formalize these intuitive ideas. In Theorems 2 and 3 we obtain a more detailed description of the long term behaviour of the Markov chain in some transient cases. It should be noted that the Lyapunov function method is also a powerful tool for detecting phenomena that might not be immediately visible and are more refined than just recurrence/transience. Theorem 3 below provides an example of such a phenomenon. In particular we show that in a transient case specified by polynomial functions  $F$  and  $G$  the Markov chain with probability one escapes to infinity in the following way. Namely, the Markov chain is eventually absorbed to either a horizontal strip  $\{(x, y) : y \leq k\}$ , or a vertical strip  $\{(x, y) : x \leq k\}$ , where  $k$  is explicitly computable. Moreover, being eventually adsorbed by the horizontal (vertical) strip, the Markov chain

visits every line  $y = i$ ,  $i = 0, \dots, k$  ( $x = i$ ,  $i = 0, \dots, k$ ) infinitely often.

## 2 Results

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space on which the Markov chain is defined. Denote by  $\mathbb{E}$  the expectation with respect to probability measure  $\mathbb{P}$ . Recall that the embedded Markov chain, corresponding to a CTMC, is a discrete time Markov chain (DTMC) with the same state space, and that makes the same jumps as the CTMC with probabilities proportional to the corresponding jump rates. Let  $\zeta(t) = (\zeta_1(t), \zeta_2(t)) \in \mathbb{Z}_+^2$  be the DTMC corresponding to the CTMC  $\xi(t)$ . Note that we use the same symbol  $t$  for discrete time. Given a real valued function  $f$  on  $\mathbb{Z}_+^2$  denote

$$m_f(x, y, t) = \mathbb{E}(f(\zeta_1(t), \zeta_2(t)) | \zeta(0) = (x, y)) - f(x, y), \quad (x, y) \in \mathbb{Z}_+^2, t \in \mathbb{Z}_+. \quad (1)$$

It is easy to see that

$$m_f(x, y, 1) = \frac{\mathbb{L}f(x, y)}{\gamma(x, y)}, \quad (2)$$

where

$$\begin{aligned} \mathbb{L}f(x, y) = & (f(x+1, y) - f(x, y))F(x)G(y) + (f(x-1, y) - f(x, y))1_{\{x>0\}} \\ & + (f(x, y+1) - f(x, y))F(y)G(x) + (f(x, y-1) - f(x, y))1_{\{y>0\}} \end{aligned} \quad (3)$$

is the generator of CTMC  $\xi(t)$ , and

$$\gamma(x, y) = F(x)G(y) + F(y)G(x) + 1_{\{x>0\}} + 1_{\{y>0\}}, \quad (4)$$

is the total intensity of jumps of CTMC  $\xi(t)$ . In the last two equations and in what follows,  $1_A$  denotes the indicator function of a set  $A$ . Note that  $\gamma(x, y) = \gamma(y, x)$ .

Recall that a real valued function  $g$  is called non-decreasing (non-increasing) on a set  $A \subseteq \mathbb{R}$ , if  $g(x) \leq g(y)$  ( $g(x) \geq g(y)$ ) for all  $x, y \in A$ , such that  $x \leq y$ . Finally, throughout the text we denote by  $C_i$ ,  $i = 1, 2, \dots$ , or, just  $C$ , various constants, whose exact values are immaterial.

We are ready now to formulate the findings of our paper. We start with the classification of the long term behaviour of the Markov chain under fairly general assumptions on functions  $F$  and  $G$ .

**Theorem 1** *Let functions  $F$  and  $G$  be positive.*

1) *Let function  $F$  be non-increasing and  $\lim_{x \rightarrow \infty} F(x) = 0$ .*

a) *If one of the following two assumptions holds*

– *function  $G$  is non-increasing and  $\lim_{x \rightarrow \infty} G(x) = 0$ ,*

– *function  $G$  is non-decreasing,  $\lim_{x \rightarrow \infty} G(x) = \infty$  and  $\lim_{x \rightarrow \infty} F(x-1)G(x) = 0$ ,*

*then CTMC  $\xi(t)$  is positive recurrent.*

b) *If function  $G$  is non-decreasing and  $\lim_{x \rightarrow \infty} F(x)G(x) = \infty$ , then CTMC  $\xi(t)$  is transient.*

2) If function  $F$  is non-decreasing,  $\lim_{x \rightarrow \infty} F(x) = \infty$  and one of the following two assumptions holds

- function  $G$  is non-decreasing and  $\lim_{x \rightarrow \infty} G(x) = \infty$ ,
- function  $G$  is non-increasing,  $\lim_{x \rightarrow \infty} G(x) = 0$ , and either  $\lim_{x \rightarrow \infty} F(x)G(x) = 0$ , or  $\lim_{x \rightarrow \infty} F(x)G(x) = \infty$ ,

then CTMC  $\xi(t)$  is transient.

**Remark 1** It is easy to see that Theorem 1 describes the long term evolution of the Markov chain in six different cases. Firstly, if both  $F(x)$  and  $G(x)$  are non-increasing and have limit 0 at infinity (and, hence,  $F(x)G(x) \rightarrow 0$  as  $x \rightarrow \infty$ ), then the Markov chain is positive recurrent. Secondly, if both  $F(x)$  and  $G(x)$  increase to infinity (and, hence,  $F(x)G(x) \rightarrow \infty$  as  $x \rightarrow \infty$ ), then the Markov chain is transient. Approximate sketches of a vector field of mean infinitesimal jumps of the Markov chain in other four cases are shown in Figure 1 and Figure 2.

**Remark 2** It should be noted that assumptions of the theorem are mostly motivated by the case of polynomial functions, e.g.  $F(x) = (x + 1)^\alpha$ ,  $\alpha \in \mathbb{R}$ , and  $G(x) = (x + 1)^\beta$ ,  $\beta \in \mathbb{R}$ . Some of these assumptions can be slightly weakened without changing the proof. For example, in Part 2) the infinite limit of the product  $FG$  at infinity in the case of non-increasing  $G$  can be replaced by a sufficiently large limit (at least 2). Such generalizations are not of much interest. Also, some of these assumptions can be weakened provided that an additional information is available about functions  $F$  and  $G$  (e.g. see Remark 5 in Appendix).

**Remark 3** Let us also discuss assumption **(A1)**:  $\lim_{x \rightarrow \infty} F(x - 1)G(x) = 0$  in Part 1)a) of the theorem. Ideally, we would like to replace it by the following assumption **(A2)**:  $\lim_{x \rightarrow \infty} F(x)G(x) = 0$ . Assumption **(A1)** is violated, for example, by functions  $F(x) = e^{-x^2}$  and  $G(x) = e^{x^2}/x$ . Note that assumptions **(A1)** and **(A2)** are equivalent in many cases. Moreover, in many cases these assumptions are equivalent to the following stronger assumption **(A3)**:  $\lim_{x \rightarrow \infty} F(\gamma x)G(x) = 0$ , where  $\gamma \in (0, 1)$ . For example, this is the case if  $F(x)$  is a regularly varying function of index  $\alpha < 0$ . Equivalence can take place for a non-regular varying  $F$  as well, for example, if  $F(x) = e^{-\alpha x}$  and  $G(x) \leq e^{\beta x}$ , where  $\alpha, \beta > 0$  and  $\alpha > \beta$ .

**Remark 4** It should be noted that there is a certain phase transition in the long term behaviour of the Markov chain in the case of non-increasing and vanishing at infinity  $F$ . Indeed, if  $G$  is also non-increasing with zero limit at infinity, then the Markov chain is positive recurrent. If  $G$  increases, but  $F(x - 1)G(x) \rightarrow 0$  as  $x \rightarrow \infty$ , then the CTMC is still recurrent. If  $G$  increases sufficiently fast so that  $F(x)G(x) \rightarrow \infty$  as  $x \rightarrow \infty$ , then the Markov chain becomes transient and can be even explosive.

Before we formulate Theorems 2 and 3, we would like to consider an exponential case, i.e.  $F(x) = e^{\alpha x}$  and  $G(x) = e^{\beta x}$ , where  $\alpha, \beta \in \mathbb{R}$ . Note first that in this case Theorem 1 yields the following. If  $\alpha < 0$  and  $\alpha + \beta < 0$ , then CTMC  $\xi(t)$  is positive recurrent. Also, if either

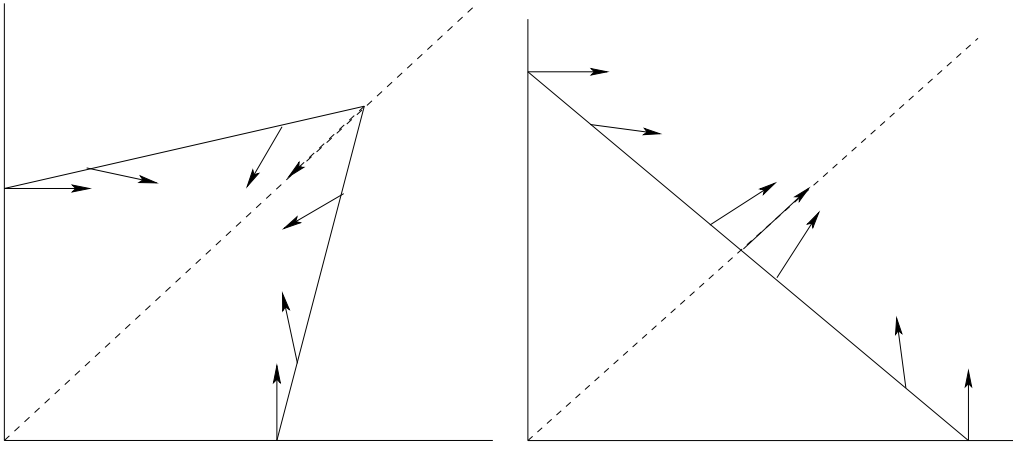


Figure 1:  $\lim_{x \rightarrow \infty} F(x) = 0$ ,  $\lim_{x \rightarrow \infty} G(x) = \infty$ ; Left:  $\lim_{x \rightarrow \infty} F(x-1)G(x) = 0$ ; Right:  $\lim_{x \rightarrow \infty} F(x)G(x) = \infty$ .

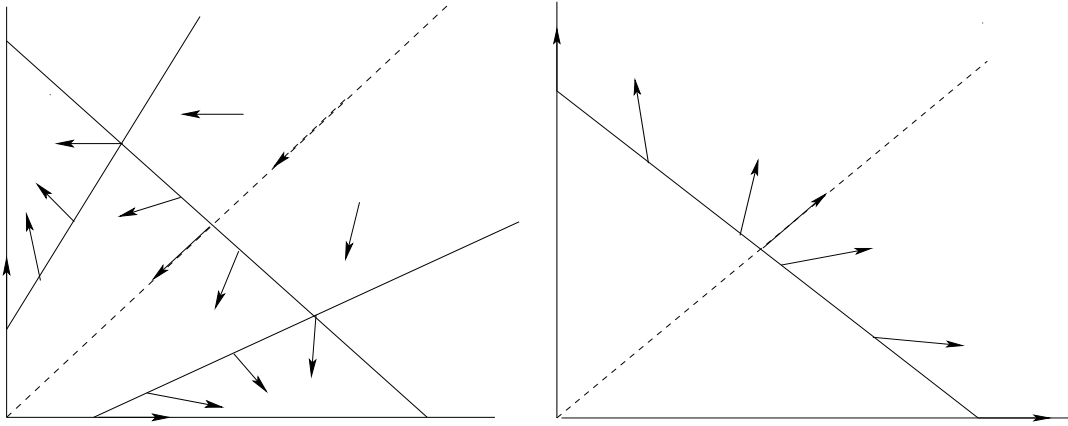


Figure 2:  $\lim_{x \rightarrow \infty} F(x) = \infty$ ,  $\lim_{x \rightarrow \infty} G(x) = 0$ ; Left:  $\lim_{x \rightarrow \infty} F(x)G(x) = 0$ ; Right:  $\lim_{x \rightarrow \infty} F(x)G(x) = \infty$ .

$\alpha < 0$ ,  $\alpha + \beta > 0$ , or  $\alpha > 0$ , then CTMC  $\xi(t)$  is transient. A direct computation gives that the CTMC is reversible with the following invariant measure  $e^{\alpha \frac{x(x-1)+y(y-1)}{2} + \beta xy}$ ,  $(x, y) \in \mathbb{Z}_+^2$ , which is summable if and only if  $\alpha < 0$ ,  $\alpha + \beta < 0$ . Thus, the sufficient condition of positive recurrence in Theorem 1 is also a necessary one in the exponential case. Note that CTMC  $\xi(t)$  in the exponential case is a particular case of a Markov chain studied in [16]. The Markov chain in [16] describes evolution of a system of locally interacting birth-and-death processes labelled by vertices of a finite connected graph. In terms of [16], CTMC  $\xi(t)$  corresponds to the simplest graph with just two vertices. The following proposition is an extract of results in [16] complementing Theorem 1 in the exponential case.

**Proposition 1** 1) *If  $\alpha < 0$  and  $\alpha + \beta = 0$ , then CTMC  $\xi(t)$  is transient and does not explode.*

2) *If either  $\alpha > 0$ , or  $\alpha + \beta > 0$ , then CTMC  $\xi(t)$  is explosive.*

- 3) If  $\alpha = 0$  and  $\beta \leq 0$ , then both CTMC  $\xi(t)$  and DTMC  $\zeta(t)$  are null recurrent.
- 4) If  $\alpha = 0$  and  $\beta > 0$ , then DTMC  $\zeta(t)$  is transient and CTMC  $\xi(t)$  is explosive.
- 5) Furthermore, (i) if  $\alpha < 0$  and  $\alpha + \beta \geq 0$ , or, if  $0 < \alpha < \beta$ , then  $\mathbb{P}(\zeta_1(t) = \zeta_2(t) \text{ infinitely often}) = 1$ , (ii) if  $\alpha > |\beta|$ , then with probability 1 eventually a single component of DTMC  $\zeta(t)$  grows while the other component stops changing at all.

Theorems 2 and 3 below are examples of statements that are similar to Proposition 1. Namely, these theorems complement Theorem 1 by providing more detailed description of the long term behaviour of the Markov chain under additional assumptions about functions  $F$  and  $G$ . Theorem 2 complements Part 1)b) of Theorem 1. Theorem 3 describes a rather unusual phenomenon in a transient case specified by polynomial functions  $F$  and  $G$ .

**Theorem 2** *Let functions  $F$  and  $G$  be positive. Suppose that function  $F$  is non-increasing and  $\lim_{x \rightarrow \infty} F(x) = 0$ , function  $G$  is non-decreasing and  $\lim_{x \rightarrow \infty} G(x) = \infty$ . Suppose also that  $\lim_{x \rightarrow \infty} F(x)G(x) = \infty$ . Then, with probability 1,*

- 1)  $\zeta_1(t) = \zeta_2(t)$  for infinitely many  $t$ ;
- 2) if, in addition,  $\lim_{x \rightarrow \infty} \frac{F(x+a)G(x+b)}{F(x)G(x)} = 1$  for any  $a, b \in \mathbb{R}$ , then given any  $\delta \in (0, 1)$   $\zeta(t) \in \{(x, y) : \delta x \leq y \leq \delta^{-1}x\}$  for all but finitely many  $t$ .

**Theorem 3** *Let  $F(x) = (x + 1)^{\lambda_1}$  and  $G(x) = (x + 1)^{-\lambda_2}$ , where  $0 < \lambda_1 < \lambda_2$ .*

- 1) If  $0 < \lambda_1 \leq 1$ , then CTMC  $\xi(t)$  is transient and non-explosive. Further, let  $k \in \mathbb{Z}_+$  be such that  $\lambda_1 + k\lambda_2 \leq 1 < \lambda_1 + (k + 1)\lambda_2$ . Then, with a positive probability  $\tilde{p}$  (depending on an initial state), CTMC  $\xi(t)$  is eventually absorbed by horizontal strip  $\{(x, y) \in \mathbb{Z}_+^2 : y \leq k\}$  and each of the following sets  $\{t \in \mathbb{R}_+ : \xi_2(t) = j\}$ ,  $j \leq k$ , is unbounded; with probability  $1 - \tilde{p}$ , CTMC  $\xi(t)$  is eventually absorbed by vertical strip  $\{(x, y) : x \leq k\}$  and each of the following sets  $\{t \in \mathbb{R}_+ : \xi_1(t) = j\}$ ,  $j \leq k$ , is unbounded.
- 2) If  $\lambda_1 > 1$ , then CTMC  $\xi(t)$  is transient and explodes with probability 1. Further, if  $\tau_{exp}$  is the time to explosion, then with probability 1 there exists a random integer  $m$  and a random time  $\tau < \tau_{exp}$  such that  $\min(\xi_1(t), \xi_2(t)) = m$  for all  $t \geq \tau$ . In other words, with probability one there exists a random integer  $m$  such that the Markov chain explodes by moving eventually along either a horizontal ray  $\{(x, y) \in \mathbb{Z}_+^2 : y = m\}$ , or along a vertical ray  $\{(x, y) \in \mathbb{Z}_+^2 : x = m\}$ .

## 3 Proofs

### 3.1 Proof of Theorem 1

*Proof of Part 1)a) of Theorem 1.* There are two cases to consider. If both functions  $F$  and  $G$  are non-increasing and tend to zero at infinity, then positive recurrence of CTMC  $\xi(t)$  is rather obvious and we omit the proof. In the second case, where  $\lim_{x \rightarrow \infty} F(x) = 0$ ,  $\lim_{x \rightarrow \infty} G(x) = \infty$  and  $\lim_{x \rightarrow \infty} F(x - 1)G(x) = 0$ , we are going to prove positive recurrence

of DTMC  $\zeta(t)$ . Positive recurrence of the DTMC will yield positive recurrence of CTMC  $\xi(t)$  as the transition rates are uniformly bounded below.

To prove positive recurrence of the DTMC  $\zeta(t)$  we are going to apply Theorem 2.2.4 from [5] which is a generalisation of the classical Foster criterion for positive recurrence of irreducible DTMC's (e.g., Theorem 2.2.3, [5]). According to this theorem, DTMC  $\zeta(t)$  is positive recurrent, if there exist positive functions  $f : \mathbb{Z}_+^2 \rightarrow (0, \infty)$  (the Lyapunov function) and  $\kappa : \mathbb{Z}_+^2 \rightarrow \mathbb{N} = \{1, 2, \dots\}$ , and  $\varepsilon > 0$ , such that  $f(x, y) \rightarrow \infty$  as  $(x, y) \rightarrow \infty$  in any reasonable sense (e.g.  $x + y \rightarrow \infty$ ), and

$$m_f(x, y, \kappa(x, y)) \leq -\varepsilon\kappa(x, y), \quad (5)$$

where  $m_f$  is defined by (1), for all  $(x, y)$  outside a bounded neighbourhood of the origin. Here we define functions  $f$  and  $\kappa$  as follows

$$f(x, y) = \begin{cases} \alpha x - y, & 0 \leq y \leq x, \\ \alpha y - x, & 0 \leq x < y, \end{cases}$$

where  $\alpha > 3$ , and

$$\kappa(x, y) = \begin{cases} 1, & y \neq x, \\ 2, & y = x. \end{cases} \quad (6)$$

It is easy to see that  $f(x, y) > 0$  on  $\mathbb{Z}_+^2$  and  $f(x, y) \rightarrow \infty$  as  $x + y \rightarrow \infty$ . Let us verify that inequality (5) is satisfied with these functions. Without loss of generality, suppose that  $0 \leq y \leq x$ . Notice that, in this case, if  $x + y$  is large, then  $x$  is also necessarily large (at least  $(x + y)/2$ ).

It is easy to see that if  $y < x$ , then inequality (5) becomes  $m_f(x, y, 1) \leq -\varepsilon$ , or, equivalently,

$$(\alpha - \varepsilon)F(x)G(y) - (1 + \varepsilon)F(y)G(x) - \alpha + \varepsilon + \mathbf{1}_{\{y>0\}}(1 + \varepsilon) \leq 0.$$

Monotonicity of both  $F$  and  $G$  imply that the left side of the preceding display can be bounded by

$$(\alpha - \varepsilon)F(x)G(x) - (1 + \varepsilon)F(0)G(x) - \alpha + 1 + 2\varepsilon,$$

where the first term vanishes and negative second and third terms dominate for large  $x$ .

Let us show that

$$m_f(x, x, 2) \leq -2\varepsilon. \quad (7)$$

Starting at  $(x, x)$  the Markov chain can reach in two steps the following states  $(x + i, x + j)$ , where integers  $i$  and  $j$  are such  $|i| + |j| = 2$ . It is easy to see that under assumptions of the theorem  $\lim_{x \rightarrow \infty} \gamma(x + a, x + b) = 2$ , and  $\lim_{x \rightarrow \infty} F(x + a)G(x + b) = 0$ . This means that in a finite vicinity of the diagonal located sufficiently far from the origin the DTMC jumps only either down or left with probabilities close to  $1/2$ , and other jumps can be neglected. This yields that starting at  $(x, x)$ , where  $x$  is sufficiently large,  $\zeta(2)$  takes values  $(x - 2, x)$ ,  $(x - 1, x - 1)$  or  $(x, x - 2)$  with probabilities converging to  $1/4, 1/2$  and  $1/4$  respectively, as  $x \rightarrow \infty$ , and probabilities of other potentially reachable in two steps states



tend to zero in the same limit. Also, the differences  $f(x+i, x+j) - f(x, x)$  are uniformly bounded in  $x$ . Therefore,

$$\begin{aligned} m_f(x, x, 2) &= \frac{f(x-2, x) - 2f(x, x) + f(x, x-2)}{4} + \frac{f(x-1, x-1) - f(x, x)}{2} + C(x) \\ &= \frac{3-\alpha}{2} + C(x), \end{aligned}$$

where  $C(x) \rightarrow 0$  as  $x \rightarrow \infty$ , which means that the left side of (7) is less than  $-2\varepsilon$  for some  $\varepsilon > 0$  for all sufficiently large  $x$  by the choice of  $\alpha$ .

*Proof of Part 1)b) of Theorem 1.* We are going to show transience of DTMC  $\zeta(t)$ . Define  $f(x, y) = x + y$  and  $D_a = \{(x, y) : x + y \geq a\} \in \mathbb{Z}_+^2$ , where  $a > 0$ . Let us show that if  $a$  is sufficiently large, then there exists  $\varepsilon > 0$  such that for all  $(x, y) \notin D_a$

$$m_f(x, y, 1) \geq \varepsilon. \quad (8)$$

Notice that if  $x + y \geq a$  and  $0 \leq y \leq x$ , then necessarily  $x \geq a/2$ . It is easy to see that if  $0 \leq y \leq x$ , then equation (8) is equivalent to the following one

$$(F(x)G(y) + F(y)G(x))(1 - \varepsilon) - (1 + 1_{\{y>0\}})(1 + \varepsilon) \geq 0,$$

and the left side of the preceding inequality can be bounded below as follows

$$\begin{aligned} (F(x)G(y) + F(y)G(x))(1 - \varepsilon) - (1 + 1_{\{y>0\}})(1 + \varepsilon) &\geq F(y)G(x)(1 - \varepsilon) - 2(1 + \varepsilon) \\ &\geq F(x)G(x)(1 - \varepsilon) - 2(1 + \varepsilon) \\ &\geq F(a/2)G(a/2)(1 - \varepsilon) - 2(1 + \varepsilon) \end{aligned}$$

It is easy to see that given  $\varepsilon \in (0, 1)$  the right side of the last inequality is positive for sufficiently large  $a$ . Thus, inequality (8) holds, which implies, by Theorem 4, that DTMC  $\zeta(t)$  is transient.

*Proof of Part 2) of Theorem 1.* Recall that in this part  $F$  is non-decreasing and tends to infinity as  $x \rightarrow \infty$ . If also  $\lim_{x \rightarrow \infty} G(x) = \infty$ , then transience of the Markov chain is obvious. In the rest of the proof we assume that  $G$  is non-increasing and  $\lim_{x \rightarrow \infty} G(x) = 0$ . As in the proof of Part 1)b), we show transience of DTMC  $\zeta(t)$ . There are two cases to consider:  $\lim_{x \rightarrow \infty} F(x)G(x) = \infty$  and  $\lim_{x \rightarrow \infty} F(x)G(x) = 0$ .

Suppose first that  $\lim_{x \rightarrow \infty} F(x)G(x) = \infty$ . We are going to show that there exists  $\varepsilon > 0$  such that for all  $(x, y) \notin D_a$ , where  $a = a(\varepsilon)$  is sufficiently large, inequality (8) holds with the same function  $f(x, y) = x + y$  as in the proof of Part 1)b). Without loss of generality, suppose that  $0 \leq y \leq x$ , in which case inequality (8) is equivalent to the following one

$$(F(x)G(y) + F(y)G(x))(1 - \varepsilon) - (1 + 1_{\{y>0\}})(1 + \varepsilon) \geq 0.$$



The left side of the preceding inequality can be bounded below as follows

$$\begin{aligned}
(F(x)G(y) + F(y)G(x))(1 - \varepsilon) - (1 + 1_{\{y>0\}})(1 + \varepsilon) \\
&\geq F(x)G(y)(1 - \varepsilon) - 2(1 + \varepsilon) \\
&\geq F(x)G(x)(1 - \varepsilon) - 2(1 + \varepsilon) \\
&\geq F(a/2)G(a/2)(1 - \varepsilon) - 2(1 + \varepsilon).
\end{aligned}$$

It is easy to see that given  $\varepsilon \in (0, 1)$  the right side of the last inequality is positive for sufficiently large  $a$ . Therefore, by Theorem 4 DTMC  $\zeta(t)$  is transient.

Suppose now that  $\lim_{x \rightarrow \infty} F(x)G(x) = 0$ . Fix  $\alpha \in (0, 1)$  and define the following function

$$f(x, y) = \begin{cases} \alpha x - y, & 0 \leq y < \alpha x, \\ \alpha y - x, & 0 \leq x < \alpha y, \\ 1, & \text{otherwise.} \end{cases} \quad (9)$$

We are going to show that if  $(x, y) \in A = \{y < \alpha x - C, x \geq a\} \cup \{x < \alpha y - C, y \geq a\}$ , where  $C > 1$  and  $a$  is sufficiently large, then  $m_f(x, y, 1) \geq \varepsilon$  for  $0 < \varepsilon < (1 - \alpha)/2$ . Due to symmetry between  $x$  and  $y$  it suffices to show this bound for  $0 \leq y < x$ , in which case inequality  $m_f(x, y, 1) \geq \varepsilon$  is equivalent to the following one

$$F(x)G(y)(\alpha - \varepsilon) - F(y)G(x)(1 + \varepsilon) + 1 - \alpha - \varepsilon(1 + 1_{\{y>0\}}) \geq 0.$$

If  $0 < y < x$ , then  $F(x)G(y) \geq F(x)G(x)$  and  $-F(y)G(x) \geq -F(x)G(x)$ , therefore the left side of the preceding display can be bounded below as follows

$$\begin{aligned}
F(x)G(y)(\alpha - \varepsilon) - F(y)G(x)(1 + \varepsilon) + 1 - \alpha - \varepsilon(1 + 1_{\{y>0\}}) \\
&\geq (\alpha - 1 - 2\varepsilon)F(x)G(x) + 1 - \alpha - 2\varepsilon \\
&\geq (\alpha - 1 - 2\varepsilon)F(a)G(a) + 1 - \alpha - 2\varepsilon,
\end{aligned}$$

and the right side of the last inequality is positive for sufficiently large  $a$ , as  $1 - \alpha - 2\varepsilon > 0$  and  $\lim_{a \rightarrow \infty} F(a)G(a) = 0$ . Now we apply again Theorem 4 with function (9) and set  $A$  to finish the proof.

## 3.2 Proof of Theorem 2

*Proof of Part 1) of Theorem 2.* Define the following function

$$f(x, y) = \begin{cases} x - y, & 0 \leq y \leq x, \\ y - x, & y > x. \end{cases}$$

If  $0 \leq y \leq x$ , then

$$m_f(x, y, 1) = \frac{F(x)G(y) - F(y)G(x) - 1 + 1_{\{y>0\}}}{\gamma(x, y)} \leq \frac{F(x)G(y) - F(y)G(x)}{\gamma(x, y)} \leq 0,$$

as  $-F(y) \leq -F(x)$  and  $G(y) \leq G(x)$ . Symmetry between  $x$  and  $y$  implies that  $m_f(x, y, 1) \leq 0$  holds in the case  $y > x$  as well. This yields that  $\eta(t) = f(\zeta_1(t \wedge \tau), \zeta_2(t \wedge \tau))$ , where  $\tau = \min\{t : \zeta_1(t) = \zeta_2(t)\}$ , is a non-negative supermartingale. Therefore,  $\eta(t)$  converges almost surely to a finite limit as  $t \rightarrow \infty$ . This necessarily implies that  $\tau = \min\{t : \zeta_1(t) = \zeta_2(t)\}$  is almost surely finite as  $|\eta(t+1) - \eta(t)| = 1$  for  $t < \tau$ , and, hence, with probability 1 DTMC  $\zeta(t)$  hits the diagonal  $y = x$  infinitely many times.

*Proof of Part 2) of Theorem 2.* Given  $\delta \in (0, 1)$  define  $K_\delta = \{(x, y) : \delta x \leq y \leq \delta^{-1}x\}$  and  $\sigma = \inf\{t : \zeta(t) \notin K_\delta\}$ .

**Proposition 2** *There exists  $\varepsilon > 0$  such that  $\inf_{(x,y) \in K_\delta} \mathbf{P}(\sigma = \infty | \zeta(0) = (x, y)) > \varepsilon$ .*

*Proof of Proposition 2.* Given  $\delta > 0$  define the following functions

$$f(x, y) = \begin{cases} y - \delta x, & y \leq x, \\ x - \delta y, & x < y, \end{cases}$$

and

$$\kappa(x, y) = \begin{cases} 1, & x \neq y, \\ n, & x = y, \end{cases}$$

where  $n = n(\delta)$  is sufficiently large and to be chosen later. We are going to show that

$$m_f(x, y, \kappa(x, y)) \geq \varepsilon', \quad (10)$$

for some  $\varepsilon' > 0$ . Indeed, if  $0 < y < x$ , then inequality (10) becomes  $m_f(x, y, 1) \geq \varepsilon'$ , which is equivalent to

$$(1 - \varepsilon')F(y)G(x) - (\delta + \varepsilon')F(x)G(y) - 1 + \delta - 2\varepsilon' \geq 0.$$

It is easy to see that the left side of the preceding display can be bounded below as follows

$$\begin{aligned} & (1 - \varepsilon')F(y)G(x) - (\delta + \varepsilon')F(x)G(y) - 1 + \delta - 2\varepsilon' \\ & \geq (1 - \delta - 2\varepsilon')F(x)G(x) - 1 - 2\varepsilon' + \delta \geq 0. \end{aligned}$$

Due to symmetry between  $x$  and  $y$  inequality (10) holds for  $0 < x < y$  as well.

If  $y = x$  then we are going to show that, given  $0 < \delta < 1$ , there exists  $n = n(\delta)$  such that  $m_f(x, x, n) \geq \varepsilon'$ , for some  $\varepsilon' > 0$ . Indeed, assumption  $\lim_{x \rightarrow \infty} \frac{F(x+a)G(x+b)}{F(x)G(x)} = 1$  implies that given integers  $n, i$  and  $j$  such that  $|i| + |j| \leq n$  the DTMC jumps from  $(x+i, x+j)$  up and right with probabilities that tend to  $1/2$  as  $x \rightarrow \infty$ . In turn, this yields that starting at  $(x, x)$ , where  $x$  is sufficiently large,  $\zeta(n)$  takes values  $(x+k, x+n-k)$ ,  $k = 0, \dots, n$  with probabilities that tend to the binomial probabilities  $\binom{n}{k}2^{-n}$ ,  $k = 0, \dots, n$  as  $x \rightarrow \infty$ , and probabilities of other states reachable in  $n$  steps tend to zero in the same limit. Therefore,

$$m_f(x, x, n) = \mathbf{E}(f(x+Y, x+n-Y)) - f(x, x) + C(x),$$

where  $Y$  is a Binomial random variable with parameters  $n$  and  $p = 1/2$ , and  $C(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Notice also, that  $f(x+a, x+b) = f(x+b, x+a)$  for any  $a, b \in \mathbb{Z}$ . Without loss of generality, assume that  $n = 2m + 1$ . A direct computation (we skip some details) gives that

$$\begin{aligned} \mathbf{E}(f(x+Y, x+n-Y)) - f(x, x) &= \frac{1}{2^{n-1}} \sum_{k=0}^m \binom{n}{k} (k - \delta(n-k)) \\ &= \frac{1+\delta}{2^{n-1}} \sum_{k=0}^m \binom{n}{k} k - \frac{\delta n}{2^{n-1}} \sum_{k=0}^m \binom{n}{k} \\ &= \frac{1+\delta}{2^{n-1}} \left( n2^{n-1} - \frac{n}{2} \binom{2m}{m} \right) - \delta n \\ &\approx \frac{1-\delta}{2} \left( n - C \frac{1+\delta}{1-\delta} \sqrt{n} \right) > \varepsilon', \end{aligned}$$

for some  $\varepsilon' > 0$ , if  $n$  is large enough. Given  $(x_0, y_0)$  define the following sequence of random times  $n_0 = 0$  and  $n_t = n_{t-1} + \kappa(\zeta(n_{t-1}))$ ,  $t \geq 1$ , and the following random process  $S(t) = f(\zeta(n_t))$ ,  $t \geq 0$ . By construction,  $S(t) \geq 0$  if and only if  $\zeta(n_t) \in K_\delta$ . Define also  $\tau_0 = \inf\{t : S(t) < 0\}$ . It is easy to see that event  $\{\tau_0 = \infty\}$  implies event  $\{\sigma = \infty\}$ . Inequality (10) yields that  $\mathbb{E}(S(t+1) - S(t)|S(t)) \geq \varepsilon'$  and, therefore, by Theorem 5, we obtain that there exists  $\varepsilon > 0$  such that  $\mathbb{P}(\tau_0 = \infty|\zeta(0) \in K_\delta) > \varepsilon$ . Consequently,  $\mathbb{P}(\sigma = \infty|\zeta(0) \in K_\delta) > \varepsilon$ . Proposition 2 is proved.

Part 1) of the theorem implies that with probability 1 DTMC  $\zeta(t)$  returns to set  $K_\delta$ . Define  $A_m = \{\text{the DTMC leaves set } K_\delta \text{ at least } m \text{ times}\}$ . By Proposition 2, we have that  $\mathbb{P}(A_m|A_{m-1}) \leq 1 - \varepsilon$ , where  $\varepsilon \in (0, 1)$ . Consequently, this yields that  $\mathbb{P}(A_m) = \mathbb{P}(A_m|A_{m-1}) \cdots \mathbb{P}(A_1) \leq (1 - \varepsilon)^m$ , so that with probability 1 DTMC  $\xi(t)$  leaves set  $K_\delta$  finitely many times. The proof of Part 2) of the theorem is finished.

### 3.3 Proof of Theorem 3

First we note that if  $0 < \lambda_1 < \lambda_2$  then  $F(x) = (x+1)^{\lambda_1} \rightarrow \infty$ ,  $G(x) = (x+1)^{-\lambda_2} \rightarrow 0$  and  $F(x)G(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Therefore transience of the CTMC  $\xi(t)$  in both parts of the theorem is implied by Theorem 1.

#### 3.3.1 Proof of Part 1 of Theorem 3

The proof is divided on steps given by Propositions 3, 4 and 5, Corollary 1, and Lemmas 1 and 2. The lemmas form the cornerstone of the proof and based on the so called Lyapunov functions approach (e.g., see [5]) widely used for study the long term behaviour of Markov processes.

We start with showing non-explosiveness of the CTMC.

**Proposition 3** *Let  $F(x) = (x+1)^{\lambda_1}$  and  $G(x) = (x+1)^{-\lambda_2}$ , where  $0 < \lambda_1 \leq 1$  and  $\lambda_2 > 0$ . Then CTMC  $\xi(t)$  is non-explosive with probability 1.*

*Proof of Proposition 3.* Let  $\gamma(x, y)$  be a total intensity of jumps of the CTMC at state  $(x, y)$ . It is easy to see that

$$\begin{aligned} \gamma(x, y) &= (x+1)^{\lambda_1}(y+1)^{-\lambda_2} + (y+1)^{\lambda_1}(x+1)^{-\lambda_2} + 1_{\{x>0\}} + 1_{\{y>0\}} \\ &\leq (x+1)^{\lambda_1} + (y+1)^{\lambda_1} + 2 \\ &\leq 2(\max(x, y) + 1)^{\lambda_1} + 2, \end{aligned}$$

and, hence,  $\gamma^{-1}(x, y) \geq [2(\max(x, y) + 1)^{\lambda_1} + 2]^{-1}$ . Let  $(x_n, y_n)$ ,  $n \in \mathbb{Z}_+$ , be a trajectory of the Markov chain, such that  $\lim_{n \rightarrow \infty} \max(x_n, y_n) = \infty$ , and consider any of its subsequences  $(x_{n_k}, y_{n_k})$ ,  $k \in \mathbb{Z}_+$ , such that  $\max(x_{n_k}, y_{n_k}) = k$ . It is easy to see that

$$\sum_{n=1}^{\infty} \frac{1}{\gamma(x_n, y_n)} \geq \sum_{k=1}^{\infty} \frac{1}{\gamma(x_{n_k}, y_{n_k})} \geq \sum_{k=1}^{\infty} \frac{1}{2((k+1)^{\lambda_1} + 1)} = \infty.$$

Thus  $\sum_{n=1}^{\infty} \gamma^{-1}(x_n, y_n) = \infty$ , and, hence, by the well-known criterion of non-explosiveness, the Markov chain is not explosive. Proposition 3 is proved.

**Proposition 4** Let  $F(x) = (x + 1)^{\lambda_1}$  and  $G(x) = (x + 1)^{-\lambda_2}$ , where  $0 < \lambda_1 \leq 1$  and  $\lambda_1 < \lambda_2$ . Let  $\tau_0 = \inf\{t : \min(\xi_1(t), \xi_2(t)) = 0\}$ . Then there exists  $\varepsilon > 0$  such that for any initial state  $(x, y)$

$$\mathbb{E}(\tau_0 | \xi(0) = (x, y)) \leq \min(x, y) / \varepsilon.$$

*Proof of Proposition 4.* Note first that by Proposition 3 CTMC  $\xi(t)$  is non-explosive. Denote  $\eta(t) = \min(\xi_1(t), \xi_2(t))$  and define  $Y_t = \eta(t \wedge \tau_0)$ . If  $(\xi_1(t), \xi_2(t)) = (x, y)$ , where  $0 \leq y \leq x$ , then  $\eta(t) = \xi_2(t) = y$  and

$$\begin{aligned} \mathbb{E}(Y(t + dt) - Y(t) | \xi(t) = (x, y)) &= ((x + 1)^{-\lambda_2} (y + 1)^{\lambda_1} - 1) dt + \bar{o}(dt) \\ &\leq ((x + 1)^{\lambda_1 - \lambda_2} - 1) dt + \bar{o}(dt) \leq -\varepsilon dt, \end{aligned}$$

on  $\{t < \tau_0\}$ , for some  $\varepsilon > 0$ , and where  $\bar{o}(dt)/dt \rightarrow 0$  as  $dt \rightarrow 0$ . By the symmetry between  $x$  and  $y$  we get that

$$\mathbb{E}(Y(t + dt) - Y(t) | \xi(t) = (x, y)) \leq \left( (\max(x, y) + 1)^{\lambda_1 - \lambda_2} - 1 \right) dt + \bar{o}(dt) \leq -\varepsilon dt,$$

for all  $(x, y) \in \mathbb{Z}_+^2$ , on  $\{t < \tau_0\}$ . Proposition 4 is now implied by Theorem 6 in Appendix.

Proposition 3 and Proposition 4 yield the following corollary.

**Corollary 1** Under assumptions of Proposition 4 set  $\{t \in \mathbb{R}_+ : \min(\xi_1(t), \xi_2(t)) = 0\}$  is unbounded with probability 1.

The next lemma states that with a positive probability the Markov chain stays forever in a strip along one of the coordinate axis.

**Lemma 1** Let  $F(x) = (x + 1)^{\lambda_1}$  and  $G(x) = (x + 1)^{-\lambda_2}$ , where  $0 < \lambda_1 \leq 1$  and  $\lambda_2 > 0$ . Let  $k \in \mathbb{Z}_+$  be such that  $\lambda_1 + (k + 1)\lambda_2 > 1$ . Given  $N \in \mathbb{Z}_+$  define  $D_{1,k,N} = \{x \geq N, y \leq k\}$  and  $\tau_{1,k,N} = \inf\{t : \xi(t) \notin D_{1,k,N}\}$ . Similar, define  $D_{2,k,N} = \{x \leq k, y \geq N\}$  and  $\tau_{2,k,N} = \inf\{t : \xi(t) \notin D_{2,k,N}\}$ . If  $N$  is sufficiently large then there exists  $\delta > 0$  such that

$$\inf_{(x,y) \in D_{1,k,N}} \mathbb{P}(\tau_{1,k,N} = \infty | \xi(0) = (x, y)) > \delta \quad (11)$$

and

$$\inf_{(x,y) \in D_{2,k,N}} \mathbb{P}(\tau_{2,k,N} = \infty | \xi(0) = (x, y)) > \delta. \quad (12)$$

Lemma 1 is proved in Section 3.4.

We are interested in the minimal  $k$  satisfying the requirement of Lemma 1. Namely, let  $k_{min}$  be such that  $\lambda_1 + \lambda_2 k_{min} \leq 1 < \lambda_1 + \lambda_2 (k_{min} + 1)$ . As the Markov chain is transient, we can assume for the rest of the proof that  $N$  is so large that i) sets  $D_{1,k_{min},N}$  and  $D_{2,k_{min},N}$  are disjoint; ii) bounds (11) and (12) hold.

**Proposition 5** With a positive probability  $\tilde{p}$ , depending on  $\xi(0)$ , CTMC  $\xi(t)$  is eventually absorbed by horizontal strip  $D_{1,k_{min},N}$ , and with probability  $1 - \tilde{p}$  CTMC  $\xi(t)$  is eventually absorbed by vertical strip  $D_{2,k_{min},N}$ .

*Proof of Proposition 5.* Note first that by Corollary 1 CTMC  $\xi(t)$  returns to set  $\{x \geq N, y \leq k_{min}\} \cup \{x \leq k_{min}, y \geq N\}$  with probability 1. Further, by Lemma 1, if the Markov chain is in either of these strips, then it remains there with a probability bounded away from zero. Consequently, with probability 1 CTMC  $\xi(t)$  is eventually absorbed by the union of these strips. This can be shown in the same way as the similar fact in the proof of Part 2) of Theorem 2 (i.e. absorption by cone  $K_\delta$ ). Finally, it is obvious that absorption by strip  $\{x \geq N, y \leq k_{min}\}$  and absorption by strip  $\{x \leq k_{min}, y \geq N\}$  are mutually exclusive events, as the strips are disjoint by assumption. Proposition 5 is proved.

**Lemma 2** Define  $\tau_{k,1} = \inf(t \geq 0 : \xi_1(t) = k)$  and  $\tau_{k,2} = \inf(t \geq 0 : \xi_2(t) = k)$ . If  $0 < \lambda_1 < 1, \lambda_2 > 0$  and integer  $k \geq 1$  are such that  $\lambda_1 + k\lambda_2 \leq 1$ , then

$$\mathbb{P}(\tau_{k,1} < \infty | \xi_1(0) = 0) = \mathbb{P}(\tau_{k,2} < \infty | \xi_2(0) = 0) = 1.$$

Lemma 2 is proved in Section 3.5. Now we use this lemma to finish the proof. Lemma 2 and Corollary 1 yield that if CTMC  $\xi(t)$  is absorbed by horizontal strip  $\{x \geq N, y \leq k_{min}\}$ , then it visits each of the following sets  $y \equiv i, i = 0, \dots, k_{min}$ , infinitely many times. Similar, if CTMC  $\xi(t)$  is absorbed by vertical strip  $\{x \leq k_{min}, y \geq N\}$ , it visits each of the following sets  $x \equiv i, i = 0, \dots, k_{min}$ , infinitely many times.

Part 1) of Theorem 3 is now proved.

### 3.3.2 Proof of Part 2) of Theorem 3

Given  $m \in \mathbb{Z}_+$  and  $0 < \nu < \lambda_1 - 1$ , define the following function

$$f(x, y) = \begin{cases} x^{-\nu}, & y = m, x > 0 \\ 1, & y \neq m \text{ or } x = 0. \end{cases} \quad (13)$$

It is easy to see that

$$\begin{aligned} \mathbb{L}f(x, m) &= \left( \frac{1}{(x+1)^\nu} - \frac{1}{x^\nu} \right) \frac{(x+1)^{\lambda_1}}{(m+1)^{\lambda_2}} + \left( \frac{1}{(x-1)^\nu} - \frac{1}{x^\nu} \right) \\ &\quad + \left( 1 - \frac{1}{x^\nu} \right) \left( \frac{(m+1)^{\lambda_1}}{(x+1)^{\lambda_2}} + 1 \right), \\ &\leq -C_1 x^{-\nu-1+\lambda_1} + \nu x^{-\nu-1} + \frac{(m+1)^{\lambda_1}}{(x+1)^{\lambda_2}} + 1 \leq -\varepsilon, \end{aligned} \quad (14)$$

for some  $\varepsilon > 0$  and for all  $x \geq N_m$ , where  $N_m$  is sufficiently large. Bound (14) implies that conditioned to stay in set  $K_{m,N}$  CTMC  $\xi(t)$  explodes, with a positive probability depending on  $m$ , by Theorem 1.12, [12]. By symmetry between  $x$  and  $y$  we immediately obtain the same for any vertical ray  $\{y \geq N_m, x = m\}$ . Let  $\tau_{exp}$  be the time to explosion,  $\tau_0 = \inf\{t : \min(\xi_1(t), \xi_2(t)) = 0\}$  (as in Proposition 4) and  $\tau = \min(\tau_{exp}, \tau_0)$ . One can show, by repeating verbatim the proof of Proposition 4, that there exists  $\varepsilon > 0$  such that  $\mathbb{E}(\tau | \xi(0) = (x, y)) \leq \min(x, y)/\varepsilon$ . This bound and conditional explosion along a horizontal and a vertical ray yield that  $\mathbb{P}(\tau_{exp} < \infty) = 1$ . Next, it is easy to see that  $\min(\xi_1(t), \xi_2(t))$  jumps with uniformly bounded rates, therefore it changes finitely many times before explosion. This yields that the Markov chain eventually explodes being absorbed by either a horizontal ray  $\{y = const\}$  or a vertical ray  $\{x = const\}$ .

### 3.4 Proof of Lemma 1

Due to symmetry between  $x$  and  $y$  it suffices to prove bound (11) only. It should be noted that the proof is reminiscent of the proof of the well known criteria for transience of a countable Markov chain (e.g., Theorem 2.2.2, [5]). In particular, it consists in constructing a *bounded* positive function  $f$  such that random process  $f(\xi(t))$  is supermartingale.

Fix an integer  $k \geq 1$  such that  $0 < \lambda_1 \leq 1 < \lambda_1 + (k + 1)\lambda_2$ . Suppose there exists a positive function  $f_k$  on  $\mathbb{Z}_+^2$  such that

1.  $\max_{y \leq k} f_k(x, y) \rightarrow 0$  as  $x \rightarrow \infty$ ,
2.  $\mathbb{L}f_k(x, y) \leq 0$  for all  $(x, y) \in \{x \geq N, y \leq k\}$ ,
3.  $\sup_{x \geq N} \max_{i=0, \dots, k} f_k(x, i) \leq N^{-\beta}$ , where  $N, \beta > 0$ , and
4.  $f_k(x, y) = 1$  for all  $(x, y) \notin \{x \geq N, y \leq k\}$ .

Define  $\tau = \inf(t : \xi(t) \notin \{x \geq N, y \leq k\})$ . The properties of  $f_k$  imply that random process  $\eta(t) = f_k(\xi_1(t \wedge \tau), \xi_2(t \wedge \tau))$  is a positive supermartingale and, hence, it almost surely converges to a finite limit  $\eta_\infty$  that can take only values 1 and 0. By Fatou's Lemma

$$\begin{aligned} \mathbb{E}(\eta_\infty | \xi(0) = (x, y)) &= \mathbb{P}(\tau < \infty | \xi(0) = (x, y)) \leq \mathbb{E}(\eta(0) | \xi(0) = (x, y)) \\ &= f_k(x, y) \leq N^{-\beta}, \end{aligned}$$

for all  $(x, y) \in \{x \geq N, y \leq k\}$ , and, hence,  $\mathbb{P}(\tau = \infty | \xi(0) = (x, y)) \geq 1 - N^{-\beta}$ , for all  $(x, y) \in \{x \geq N, y \leq k\}$ . In the rest of the proof we provide functions  $f_k$ .

*Function  $f_0$ .* Fix  $0 < \nu < \lambda_1 + \lambda_2 - 1$  and define the following function

$$f_0(x, y) = \begin{cases} 1, & y > 0 \text{ or } x = 0, \\ x^{-\nu}, & y = 0, x > 0. \end{cases} \quad (15)$$

The following bound is obvious

$$\sup_{x \geq N} f_0(x, 0) \leq N^{-\nu}. \quad (16)$$

Let us show that, if  $x \geq N$ , where  $N$  is sufficiently large, then  $\mathbb{L}f_0(x, 0) \leq 0$ . Indeed, a direct computation gives that

$$\begin{aligned} \mathbb{L}f_0(x, 0) &= \left( \frac{1}{(x+1)^\nu} - \frac{1}{x^\nu} \right) (x+1)^{\lambda_1} + \left( \frac{1}{(x-1)^\nu} - \frac{1}{x^\nu} \right) + \left( 1 - \frac{1}{x^\nu} \right) (x+1)^{-\lambda_2} \\ &\leq -C_1 x^{-\nu-1+\lambda_1} + \nu x^{-\nu-1} + C_2 x^{-\lambda_2} \leq 0, \end{aligned}$$

for all sufficiently large  $x$ , as  $\lambda_1 + \lambda_2 - 1 > \nu$ .

Functions  $f_k$ ,  $k \geq 1$ . If  $k = 1$ , then we define

$$f_1(x, y) = \begin{cases} 1, & y \geq 2 \text{ or } x = 0, \\ x^{-\nu_1}, & y = 1, x > 0, \\ x^{-\nu_1} - x^{-\nu_1-\nu_2}, & y = 0, x > 0, \end{cases} \quad (17)$$

where  $\nu_1 > 0$  and  $\nu_2 > 0$  are such that  $\nu_1 + \nu_2 < \lambda_2$  and  $\lambda_1 + \lambda_2 + \nu_2 > 1$  (it is easy to see that such numbers  $\nu_1$  and  $\nu_2$  exist). If  $k \geq 2$ , then we define

$$f_k(x, y) = \begin{cases} 1, & y \geq k + 1 \text{ or } x = 0, \\ x^{-\nu_1}, & y = k, x > 0, \\ x^{-\nu_1} - x^{-\nu_1-\nu_2}, & y = k - 1, x > 0, \\ x^{-\nu_1} - x^{-\nu_1-\nu_2} - \dots - x^{-\nu_1-\nu_2-\dots-\nu_{k+1}-y}, & y = 0, \dots, k - 2, x > 0, \end{cases} \quad (18)$$

where positive real numbers  $\nu_1, \dots, \nu_{k+1}$  satisfy the following system of inequalities

$$\begin{cases} \lambda_2 > \nu_1 + \nu_2, \\ 0 < \nu_i < \lambda_2, \quad i = 3, \dots, k + 1, \\ 1 < \lambda_1 + \lambda_2 + \nu_2 + \dots + \nu_{k+1}. \end{cases} \quad (19)$$

It is easy to see that system of inequalities (19) has many solutions and for all  $k \geq 1$  the following bound holds

$$\sup_{x \geq N} \max_{0 \leq i \leq k} f_k(x, i) \leq \sup_{x \geq N} f_k(x, k) \leq N^{-\nu_1}. \quad (20)$$

A direct computation gives that

$$\begin{aligned} \mathbb{L}f_1(x, 1) &= \left( \frac{1}{(x+1)^{\nu_1}} - \frac{1}{x^{\nu_1}} \right) \frac{(x+1)^{\lambda_1}}{2^{\lambda_2}} + \left( \frac{1}{(x-1)^{\nu_1}} - \frac{1}{x^{\nu_1}} \right) \\ &+ \left( 1 - \frac{1}{x^{\nu_1}} \right) \frac{2^{\lambda_1}}{(x+1)^{\lambda_2}} - \frac{1}{x^{\nu_1+\nu_2}}, \end{aligned}$$

and, hence,

$$\begin{aligned} \mathbb{L}f_1(x, 1) &\leq -C_1 x^{-1-\nu_1+\lambda_1} + C_2 x^{-\lambda_2} - x^{-\nu_1-\nu_2} \leq 0, \\ \mathbb{L}f_1(x, 0) &\leq x^{\lambda_1} (-C_1 x^{-1-\nu_1} + C_2 x^{-\nu_1-\nu_2-\lambda_1-\lambda_2}) \leq 0, \end{aligned}$$

for all sufficiently large  $x$ , as  $\lambda_2 > \nu_1 + \nu_2$  and  $\lambda_1 + \lambda_2 + \nu_2 > 1$ .

If  $k \geq 2$ , then a direct computation gives that

$$\begin{aligned} \mathbb{L}f_k(x, k) &\leq \left( \frac{1}{(x+1)^{\nu_1}} - \frac{1}{x^{\nu_1}} \right) \frac{(x+1)^{\lambda_1}}{(k+1)^{\lambda_2}} + \left( \frac{1}{(x-1)^{\nu_1}} - \frac{1}{x^{\nu_1}} \right) \\ &+ \left( 1 - \frac{1}{x^{\nu_1}} \right) \frac{(k+1)^{\lambda_1}}{(x+1)^{\lambda_2}} - \frac{1}{x^{\nu_1+\nu_2}} \\ &\leq -C_1 x^{-1-\nu_1+\lambda_1} + C_2 x^{-\lambda_2} - x^{-\nu_1-\nu_2} \leq 0, \end{aligned}$$



for sufficiently large  $x$ , as  $\lambda_2 > \nu_1 + \nu_2$ . Further, given  $i = 2, \dots, k$ , we get in a similar way that

$$\mathsf{L}f_k(x, k+1-i) \leq x^{\lambda_1}(-C_1x^{-1-\nu_1} + C_2x^{-\nu_1-\dots-\nu_i-\lambda_1-\lambda_2} - x^{-\nu_1-\nu_2-\dots-\nu_{i+1}-\lambda_1}).$$

Notice that the second inequality of (19) implies that

$$-\nu_1 - \dots - \nu_i - \lambda_1 - \lambda_2 < -\nu_1 - \nu_2 - \dots - \nu_{i+1} - \lambda_1,$$

and, hence,  $\mathsf{L}f_k(x, k-i+1) \leq 0$ , provided that  $x$  is sufficiently large.

Finally, the bottom inequality in (19) implies that

$$\mathsf{L}f_k(x, 0) \leq x^{\lambda_1}(-C_1x^{-1-\nu_1} + C_2x^{-\nu_1-\dots-\nu_{k+1}-\lambda_1-\lambda_2}) \leq 0,$$

for sufficiently large  $x$ .

The lemma is proved.

### 3.5 Proof of Lemma 2

Due to symmetry between  $x$  and  $y$  it suffices to prove only that  $\mathsf{P}(\tau_{k,2} < \infty | \xi_2(0) = 0) = 1$ . It should be noted that the proof is reminiscent of the proof of the well-known criteria for recurrence of a countable Markov chain (e.g., Theorem 2.2.1, [5]). In particular, it consists in constructing an *unbounded* positive function  $g$  such that random process  $g(\xi(t))$  is a supermartingale.

Given an integer  $k \geq 1$ , we are going to construct function  $g_k$  satisfying the following conditions

1.  $g_k(x, i) \rightarrow \infty$  as  $x \rightarrow \infty$  for all  $0 \leq i < k$ ,
2.  $g_k(x, y) = 1$  on  $\{x = 0\} \cup \{y \geq k\}$ ,
3.  $\mathsf{L}g_k(x, y) \leq 0$  for all  $(x, y) \in \{x \geq N, y \leq k-1\}$ , where  $N > 0$ .

Properties of such function  $g_k$  imply that the random process  $\eta_k(t) = g_k(\xi(t \wedge \tau_{k,2}))$  is a positive supermartingale and, hence, converges almost surely. If  $(x, y) \in \{x \geq N, y \leq k-1\}$ , then the Markov chain jumps to the right with a rate that is approximately equal to  $x^{\lambda_1}$  for sufficiently large  $x \rightarrow \infty$ , while rates of jumps down, up or left are uniformly bounded over states  $(x, y) \in \{x \geq N, y \leq k-1\}$ . It means that conditioned to stay in strip  $\{x \geq N, y \leq k-1\}$  component  $\xi_1(t)$  tends to infinity as  $t \rightarrow \infty$  and, by construction, so does  $\eta_k(t)$ , which contradicts its convergence, unless  $\mathsf{P}(\tau_{k,2} < \infty) = 1$ .

In the rest of the proof we construct the functions  $g_k$ ,  $k \geq 1$ . Note that in what follows we write  $\psi(x) \approx \phi(x)$  for all sufficiently large  $x$ , if  $\lim_{x \rightarrow \infty} \psi(x)/\phi(x) = 1$ .

*Function  $g_1$ .* Suppose that  $\lambda_1 + \lambda_2 \leq 1$  and define

$$g_1(x, y) = \begin{cases} 1, & y \geq 1 \text{ or } x = 0, \\ x^{\nu_1}, & x > 0, y = 0, \end{cases}$$

where  $0 < \nu_1 < 1$ . It is easy to see that

$$\mathsf{L}g_1(x, 0) \approx \nu_1 x^{\nu_1-1+\lambda_1} - x^{\nu_1-\lambda_2},$$

for all sufficiently large  $x$ . If  $\lambda_1 + \lambda_2 < 1$ , then  $\nu_1 - 1 + \lambda_1 < \nu_1 - \lambda_2$ , hence,  $\mathsf{L}g_1(x, 0) \leq 0$ . If  $\lambda_1 + \lambda_2 = 1$ , then  $\mathsf{L}g_1(x, 0) \approx (\nu_1 - 1)x^{\nu_1-\lambda_2} < 0$  for all sufficiently large  $x$ , as  $\nu_1 < 1$ .

Function  $g_2$ . If  $\lambda_1 + 2\lambda_2 \leq 1$ , then we define

$$g_2(x, y) = \begin{cases} 1, & y \geq 2 \text{ or } x = 0, \\ x^{\nu_1}, & y = 1, x > 0, \\ x^{\nu_1} + B_1 x^{\nu_1 - \nu_2}, & y = 0, x > 0, \end{cases}$$

where

$$\begin{cases} \lambda_2 < \nu_2 < \nu_1, \nu_2 < 1 - \lambda_1 - \lambda_2, B_1 = 1, & \text{if } \lambda_1 + 2\lambda_2 < 1, \\ \nu_2 = \lambda_2 < \nu_1 < B_1 < 2^{\lambda_1}, & \text{if } \lambda_1 + 2\lambda_2 = 1. \end{cases} \quad (21)$$

It is easy to see that  $\mathsf{L}g_2(x, 0) \approx \nu_1 x^{\nu_1 - 1 + \lambda_1} - B_1 x^{\nu_1 - \nu_2 - \lambda_2}$  for all sufficiently large  $x$ . If  $\lambda_1 + 2\lambda_2 = 1$ , then  $\mathsf{L}g_2(x, 0) \leq 0$ , because of the bottom line in condition (21). If  $\lambda_1 + 2\lambda_2 < 1$ , then the upper line in condition (21) yields that  $\nu_1 - 1 + \lambda_1 < \nu_1 - \nu_2 - \lambda_2$ , and, hence,  $\mathsf{L}g_2(x, 0) \leq 0$  for all sufficiently large  $x$ .

Further, it is easy to see that  $\mathsf{L}g_2(x, 1) \approx 2^{-\lambda_2} \nu_1 x^{\nu_1 - 1 + \lambda_1} + B_1 x^{\nu_1 - \nu_2} - 2^{\lambda_1} x^{\nu_1 - \lambda_2}$  for all sufficiently large  $x$ . If  $\lambda_1 + 2\lambda_2 = 1$ , then both positive terms are smaller than  $2^{\lambda_1} x^{\nu_1 - \lambda_2}$ , as  $\nu_1 - 1 + \lambda_1 < \nu_1 - \lambda_2$  and  $B_1 < 2^{\lambda_1}$  respectively. If  $\lambda_1 + 2\lambda_2 < 1$ , then the negative term dominates both positive terms because  $\nu_1 - 1 + \lambda_1 < \nu_1 - \lambda_2$  (as  $\lambda_1 + \lambda_2 < 1$ ), and  $\nu_1 - \nu_2 < \nu_1 - \lambda_2$  (as  $\lambda_2 < \nu_2$ ). Hence, we have again that  $\mathsf{L}g_2(x, 1) \leq 0$  for all sufficiently large  $x$ .

Functions  $g_k, k \geq 3$ . If  $\lambda_1 + k\lambda_2 \leq 1$ , where  $k \geq 3$ , then we define function  $g_k$  as follows

$$g_k(x, y) = \begin{cases} 1, & y \geq k \text{ or } x = 0, \\ x^{\nu_1}, & y = k - 1, x > 0, \\ x^{\nu_1} + B_1 x^{\nu_1 - \nu_2}, & y = k - 2, x > 0, \\ x^{\nu_1} + B_1 x^{\nu_1 - \nu_2} + \dots + B_{i-1} x^{\nu_1 - \nu_2 - \dots - \nu_i}, & y = k - i, i = 3, \dots, k, x > 0, \end{cases}$$

where

$$\begin{cases} B_i = 1, i = 1, \dots, k - 1, \lambda_2 < \nu_i, i = 2, \dots, k, \\ \nu_2 + \dots + \nu_k < \min(1 - \lambda_1 - \lambda_2, \nu_1), & \text{if } \lambda_1 + \lambda_2 k < 1, \end{cases} \quad (22)$$

and

$$\begin{cases} B_1 < k^{\lambda_1}, \lambda_2(k - 1) < \nu_1 < B_{k-1}, \\ B_i < (k - i + 1)^{\lambda_1} B_{i-1}, i = 2, \dots, k - 2, \nu_i = \lambda_2, i = 2, \dots, k, & \text{if } \lambda_1 + \lambda_2 k = 1. \end{cases} \quad (23)$$

A direct computation gives that

$$\begin{aligned} \mathsf{L}g_k(x, 0) &\approx \nu_1 x^{\nu_1 - 1 + \lambda_1} - B_{k-1} x^{\nu_1 - \nu_2 - \dots - \nu_k - \lambda_2} \\ &= x^{\nu_1 - 1 + \lambda_1} (\nu_1 - B_{k-1} x^{1 - \lambda_1 - \lambda_2 - \nu_2 - \dots - \nu_k}), \end{aligned}$$

for all sufficiently large  $x$ , where if  $\lambda_1 + \lambda_2 k = 1$ , then the right hand side is  $x^{\nu_1 - 1 + \lambda_1} (\nu_1 - B_{k-1}) < 0$  by condition (23), and if  $\lambda_1 + \lambda_2 k < 1$ , then the right hand side is negative by condition (22).

Further, a direct computation gives that

$$\begin{aligned} \mathsf{L}g_k(x, k-i) &\approx \nu_1(k-i+1)^{-\lambda_2} x^{\nu_1-1+\lambda_1} - (k-i+1)^{\lambda_1} B_{i-1} x^{\nu_1-\nu_2-\dots-\nu_i-\lambda_2} \\ &\quad + B_i x^{\nu_1-\nu_2-\dots-\nu_i-\nu_{i+1}}, \end{aligned}$$

for  $i = 2, \dots, k-2$ , for all sufficiently large  $x$ . As before, consider two cases. If  $\lambda_1 + \lambda_2 k = 1$ , then

$$x^{\nu_1-1+\lambda_1} (\nu_1 - (k-i+1)^{\lambda_1} B_{i-1} x^{1-\lambda_1-\lambda_2-\nu_2-\dots-\nu_i}) < 0,$$

as  $1 - \lambda_1 - \lambda_2 - \nu_2 - \dots - \nu_i = 1 - \lambda_1 - i\lambda_2 > 0$ , so that the first positive term is asymptotically dominated by the negative one. Also, comparing the negative term with the second positive one we get that

$$\begin{aligned} B_i x^{\nu_1-\nu_2-\dots-\nu_i-\nu_{i+1}} - (k-i+1)^{\lambda_1} B_{i-1} x^{\nu_1-\nu_2-\dots-\nu_i-\lambda_2} \\ = x^{\nu_1-i\lambda_2} (B_i - (k-i+1)^{\lambda_1} B_{i-1}) < 0, \end{aligned}$$

by (23). If  $\lambda_1 + \lambda_2 k < 1$ , then condition (22) implies that  $\nu_1 - 1 < \nu_1 - \nu_2 - \dots - \nu_i - \lambda_1 - \lambda_2$  and  $\nu_1 - \nu_2 - \dots - \nu_i - \lambda_1 - \lambda_2 > \nu_1 - \nu_2 - \dots - \nu_i - \nu_{i+1} - \lambda_1$ , so that  $\mathsf{L}f_k(x, k-i) \leq 0$  for all sufficiently large  $x$ .

Finally, we get that

$$\mathsf{L}g_k(x, k-1) \approx \nu_1 k^{-\lambda_2} x^{\nu_1-1+\lambda_1} + B_1 x^{\nu_1-\nu_2} - k^{\lambda_1} x^{\nu_1-\lambda_2} \leq 0,$$

for all sufficiently large  $x$ . Indeed, if  $\lambda_1 + \lambda_2 k \leq 1$ ,  $k \geq 3$ , then  $\nu_1 - 1 + \lambda_1 < \nu_1 - \lambda_2$ , so that term  $k^{\lambda_1} x^{\nu_1-\lambda_2}$  is larger (for sufficiently large  $x$ ) than  $\nu_1 x^{\nu_1-1+\lambda_1}$ . To deal with another positive term in the preceding display, we consider two cases. If  $\lambda_1 + \lambda_2 k < 1$ , then  $\nu_1 - \nu_2 < \nu_1 - \lambda_2$ , because of condition (22). If  $\lambda_1 + \lambda_2 k = 1$ , then  $\nu_1 - \nu_2 = \nu_1 - \lambda_2$ , but  $B_1 < k^{\lambda_1}$ . Thus, in both cases  $B_1 x^{\nu_1-\nu_2} < k^{\lambda_1} x^{\nu_1-\lambda_2}$  for all sufficiently large  $x$ .

The lemma is proved.

## Appendix

**Remark 5** It should be noted that our results imply transience of CTMC  $\xi(t)$  in the case of polynomial functions  $F(x) = (x+1)^{\lambda_1}$  and  $G(x) = (x+1)^{-\lambda_2}$  for *any*  $\lambda_1, \lambda_2 > 0$ . Indeed, if  $0 < \lambda_1 < \lambda_2$  then, as it is mentioned at the beginning of the proof of Theorem 3, Theorem 1 applies. If  $0 < \lambda_1 \leq 1$  and  $\lambda_2 > 0$ , then Lemma 1 implies transience. If  $\lambda_1 > 1$ , then (whatever  $\lambda_2$  is) transience is implied the criteria for transience of a countable Markov chain (e.g. Theorem 2.2.2, [5]) which applies in this case with Lyapunov function (13). Further, condition  $0 < \lambda_1 < \lambda_2$  in Theorem 3 is not necessary to show *just* transience. We essentially use this condition in both parts of Theorem 3 to describe how *exactly* the Markov chain escapes to infinity.

For the reader's convenience we provide some facts that were used in our paper. Theorem 4 is a version of Theorem 2.2.7, [5], Theorem 5 is a version of Theorem 2.1.9, [5], and Theorem 6 is Lemma 7.3.6 in [11].

**Theorem 4** (Theorem 2.2.7, [5]). Let  $\eta(t)$  be an irreducible aperiodic discrete time Markov chain on a countable space  $\mathcal{A}$ . For  $\eta(t)$  to be transient, it suffices that there exist a positive function  $f(\eta)$ ,  $\eta \in \mathcal{A}$ , a bounded positive integer valued function  $\kappa(\eta)$ ,  $\eta \in \mathcal{A}$ , and numbers  $\varepsilon, C > 0$  such that, setting  $A_C = \{\eta \in \mathcal{A} : f(\eta) \geq C\} \neq \emptyset$ , the following conditions hold:

- 1)  $\sup_{\eta \in \mathcal{A}} \kappa(\eta) < \infty$ ;
- 2)  $\mathbf{E}(f(\eta(t + \kappa(\eta)) | \eta(t) = \eta) - f(\eta) \geq \varepsilon$  for all  $\eta \in A_C$ ;
- 3) for some  $d > 0$ , the inequality  $|f(\eta') - f(\eta'')| > d$  implies that the transition probability from  $\eta'$  to  $\eta''$  is zero.

**Theorem 5** (Theorem 2.1.9, [5]). Let  $\eta(t)$ ,  $t \in \mathbb{Z}_+$ , be  $\mathbb{R}_+$ -valued process adapted to a filtration  $(\mathcal{F}_t, t \in \mathbb{Z}_+)$ . Define  $\tau_C = \min\{t \geq 1 : \eta(t) \leq C\}$ , where  $C > 0$ . Suppose that its jumps  $\eta(t + 1) - \eta(t)$ ,  $t \in \mathbb{Z}_+$ , are uniformly bounded and there exists  $\varepsilon > 0$  such that  $\mathbf{E}(\eta(t + 1) | \mathcal{F}_t) \geq \eta(t) + \varepsilon$ , on  $\{t \leq \tau_C\}$ , and  $\eta(0) > C$ . Then  $\mathbf{P}(\tau_C = \infty) > 0$ .

**Theorem 6** (Lemma 7.3.6, [11]). Let  $(\eta(t), t \in \mathbb{R}_+)$  be an  $\mathbb{R}_+$ -valued process adapted to a filtration  $(\mathcal{F}_t, t \in \mathbb{R}_+)$  and let  $\tau = \inf\{t : \eta(t) = 0\}$ . Suppose that there exists  $\varepsilon > 0$  such that  $\mathbf{E}(\eta(t + dt) - \eta(t) | \mathcal{F}_{t-}) \leq -\varepsilon dt$ , on  $\{t \leq \tau\}$ . Then  $\mathbf{E}(\tau | \mathcal{F}_0) \leq \varepsilon^{-1} \eta(0)$ .

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