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Energy spectra and passive tracer cascades in turbulent flows

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We study the influence of the energy spectrum on the extent of the cascade range of a passive tracer in turbulent flows. The interesting cases are when there are two different spectra over the potential range of the tracer cascade (in 2D when the tracer source is in the inverse energy cascade range and in 3D when the Schmidt number Sc is large). The extent of the tracer cascade range is then limited by the width of the range for the shallower of the two energy spectra. Nevertheless, we show that in dimension $d = 2, 3$, the tracer cascade range extends (up to a logarithm) to $\kappa_{d\mathrm{D}}^{p}$, where $\kappa_{d\mathrm{D}}$ is the wavenumber beyond which diffusion should dominate and p is arbitrarily close to 1, provided Sc is larger than a certain power (depending on p) of the Grashof number. We also derive estimates which suggest that in 2D, for Sc ~ 1, a wide tracer cascade can coexist with a significant inverse energy cascade at Grashof numbers large enough to produce a turbulent flow. *Published by AIP Publishing.*
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I. INTRODUCTION

Passive tracers play an important role in the study of fluid motion. On the one hand, the experimental and observational studies of fluid flows rely heavily on passive tracers to deduce the advecting velocity field. On the other hand, knowledge of the underlying fluid flows is essential to predict the future dispersion of tracers (particularly, but not exclusively, harmful ones).

It is natural to believe that if the advecting fluid flow is turbulent (however this is defined), the evolution of the tracer will be turbulent as well. Following the pioneering work by Kolmogorov, Obukhov[14] and Corrsin[4] argued that if the energy spectrum of the fluid is $\mathcal{E}(\kappa) = K\kappa^{-n}$, a passive tracer whose dissipation rate is χ should have the spectrum $\mathcal{T}(\kappa) \sim \chi K^{-1/2}\kappa^{(n-5)/2}$ between the injection and dissipation scales (see also Ref. 1). Thus, in the inertial range in 3D, both the energy and tracer spectra scale as $\kappa^{-5/3}$. Following Kraichnan,[13] over the direct enstrophy cascade range in 2D, the energy spectrum should scale as κ^{-3}, giving a κ^{-1} tracer spectrum. Although these scaling arguments were derived with little reference to the governing equations, they have been supported to a surprising extent by experimental and numerical studies (cf. Refs. 7 and 17), primarily in 3D, slightly less so for 2D and still less so for tracers.

In 3D and 2D, respectively, dissipative effects are expected to dominate beyond the Kolmogorov and Kraichnan wavenumbers κ_ϵ and κ_η. The corresponding scales for our tracer depend in addition on the Schmidt number Sc, i.e., the ratio of the viscosity to the tracer dissipativity. Another lengthscale of great importance is the Taylor microscale κ_τ^{-1}. Initially (and to this day among experimentalists) defined using the velocity correlation, mathematicians prefer to use an alternate definition for κ_τ in terms of the energy and its dissipation rate (7); the two definitions can be shown to be nearly (formally) equivalent under some assumptions (Ref. 7, 6.44b). Assuming that κ_τ is much greater than the forcing scale, it has been proved rigorously that a direct energy cascade exists for solutions

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of 3D Navier-Stokes equations (NSE).[9] Similarly, in 2D, one defines in terms of the enstrophy and its dissipation rate a wavenumber κ_σ, which if sufficiently larger than the forcing scale rigorously implies the existence of the direct enstrophy cascade.[11] In Sec. IV B, we derive an analogous result for tracers in terms of a corresponding wavenumber κ_θ.

While it is plausible that κ_τ, κ_σ, and κ_θ are large for turbulent solutions of the NSE and the advected tracers, these remain unproved (directly from the NS and the tracer equations) to this day. If one were to assume the expected spectra, namely, $\epsilon^{2/3}\kappa^{-5/3}$ and $\eta^{2/3}\kappa^{-3}$, however, it has been shown that $\kappa_\tau \sim \kappa_\epsilon^{2/3}\kappa_0^{1/3}$ in 3D[6] and $\kappa_\sigma \sim \kappa_\eta$ up to a logarithm in 2D,[5] where $\kappa_0 = 2\pi/L$, in a periodic domain of length L in each direction. Following this approach, we prove the tracer analogs in Secs. V and VI. There are a number of qualitatively distinct cases here, depending on the viscosity ν and tracer dissipativity μ, as well as on the injection scales of energy κ_f and of the tracer κ_g. When $\nu \gg \mu$, it is possible for κ_θ to asymptotically approach (up to constants and logarithms) its largest possible value, in that $\kappa_\theta \sim \kappa_{d\mathrm{D}}^{1-r}\kappa_0^r$ for any $r < 1$, both for $d = 2$ (Sec. V) and $d = 3$ (Sec. VI). When $\nu \sim \mu$, the situation is more complicated as discussed in detail below. In the case $\mu \gg \nu$, the velocity stirring scale is smaller than the dissipativity scale, giving rise to non-turbulent dynamics; we believe a different approach would be needed here.

The rest of this paper is structured as follows. After some mathematical setups in Sec. II, we recall the heuristic argument for the tracer spectra in Sec. III. Earlier NSE estimates for the enstrophy and energy transfer rates in terms of κ_σ and κ_τ in 2D and 3D are gathered in Sec. IV, along with the implications that the expected energy spectra have on these wavenumbers, vis-à-vis κ_η, κ_ϵ, respectively. An analogous estimate for the tracer transfer rate in terms of κ_θ is also derived in Sec. IV. We treat 2D tracer flow in Sec. V and 3D tracer flow in Sec. VI.

II. PRELIMINARIES

We consider the evolution of a passive scalar θ under a prescribed velocity field $u(x, t)$ and a time-independent source $g = g(x)$,

$$\begin{aligned} &\partial_t\theta - \mu\Delta\theta + u\cdot\nabla\theta = g,\\ &\int_\Omega \theta\,dx = 0, \qquad \int_\Omega g\,dx = 0, \end{aligned} \tag{1}$$

with periodic boundary conditions in $\Omega = [0, L]^d$ for $d = 2, 3$. We focus on the case where u satisfies the incompressible Navier-Stokes equations

$$\begin{aligned} &\partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = F,\\ &\nabla\cdot u = 0,\\ &\int_\Omega u\,dx = 0, \qquad \int_\Omega F\,dx = 0,\\ &u(x, t_0) = u_0(x). \end{aligned} \tag{2}$$

We write (2) as a differential equation in a certain Hilbert space H (see Refs. 2 and 16),

$$\begin{aligned} &\frac{d}{dt}u(t) + \nu Au(t) + B(u(t), u(t)) = f,\\ &u(t)\in H, \quad t\ge t_0, \qquad \text{and} \quad u(t_0) = u_0. \end{aligned} \tag{3}$$

The phase space H is the closure in $L^2(\Omega)^d$ of all $\mathbb{R}^d$-valued trigonometric polynomials u such that

$$\nabla\cdot u = 0 \qquad \text{and} \qquad \int_\Omega u(x)\,dx = 0.$$

The bilinear operator B is defined as

$$B(u, v) = \mathcal{P}((u\cdot\nabla)v),$$

where $\mathcal{P}$ is the Helmholtz–Leray orthogonal projector of $L^2(\Omega)^d$ onto H and $f = \mathcal{P}F$. The scalar product in H is taken to be

$$(u, v) = \int_\Omega u(x) \cdot v(x)\, dx,$$

with the associated norm

$$|u| = (u, u)^{1/2} = \Big(\int_\Omega u(x) \cdot u(x)\, dx \Big)^{1/2}.$$

The operator $A = -\Delta$ is self-adjoint with compact inverse and a complete set of eigenfunctions associated with eigenvalues of the form

$$(2\pi/L)^2 k \cdot k, \quad \text{where } k \in \mathbb{Z}^d \setminus \{0\}.$$

We denote these eigenvalues by

$$0 < \lambda_0 = (2\pi/L)^2 \le \lambda_1 \le \lambda_2 \le \cdots$$

arranged in non-decreasing order (counting multiplicities) and write w_0, w_1, w_2, ..., for the corresponding normalized eigenvectors (i.e., $|w_j| = 1$ and $Aw_J = \lambda_j w_j$ for $j = 0, 1, 2, \ldots$).

For $\alpha \in \mathbb{R}$, the positive roots of A are defined by linearity from

$$A^\alpha w_j = \lambda_j^\alpha w_j, \quad \text{for } j = 0, 1, 2, \ldots$$

on the domain

$$D(A^\alpha) = \Big\{ u \in H : \sum_{j=0}^{\infty} \lambda_j^{2\alpha} (u, w_j)^2 < \infty \Big\}.$$

We take the natural norm on $V = D(A^{1/2})$ to be

$$\|u\| = |A^{1/2} u| = \left(\int_\Omega \sum_{j=1}^{d} \frac{\partial}{\partial x_j} u(x) \cdot \frac{\partial}{\partial x_j} u(x)\, dx \right)^{1/2} = \left(\sum_{j=0}^{\infty} \lambda_j (u, w_j)^2 \right)^2.$$

Since the boundary conditions are periodic, we may express an element in H as a Fourier series

$$u(x) = \sum_{k \in \mathbb{Z}^d} \hat{u}_k e^{i\kappa_0 k \cdot x}, \tag{4}$$

where

$$\kappa_0 = \lambda_0^{1/2} = \frac{2\pi}{L}, \quad \hat{u}_0 = 0, \quad \hat{u}_k^* = \hat{u}_{-k},$$

and due to incompressibility, $k \cdot \hat{u}_k = 0$. We associate to each term in (4) a *wavenumber* $\kappa_0 |k|$. Parseval's identity reads as

$$|u|^2 = L^d \sum_{k \in \mathbb{Z}^d} \hat{u}_k \cdot \hat{u}_{-k} = L^d \sum_{k \in \mathbb{Z}^d} |\hat{u}_k|^2.$$

Two important dimensionless parameters are the Grashof and Schmidt numbers,

$$G := \frac{|f|}{\nu^2 \kappa_0^{3-d/2}} \quad \text{and} \quad \mathrm{Sc} := \frac{\nu}{\mu}.$$

The former indicates the complexity of the (velocity) flow, and the latter indicates the importance of (momentum) viscosity relative to tracer dissipativity.

Since the infinite time limit is not known to exist, for each solution $u(t)$ of the 2D NSE (Leray–Hopf weak solution in the 3D case), we work with the average

$$\langle \Phi \rangle = \operatorname*{Lim}_{T \to \infty} \frac{1}{T} \int_0^T \Phi(u(t))\, dt \quad \text{for any } \Phi \text{ weakly continuous in } H,$$

where Lim is a Hahn–Banach extension of the classical time limit. The average $\langle \cdot \rangle$ is the mathematical equivalent of the ensemble average in the statistical theory of turbulence; see Refs. 11 and 10 for more details. Using this, we define the average *energy*, *enstrophy,* and *tracer variance as*

$$\mathsf{e} = \frac{1}{L^d}\langle |u|^2\rangle, \qquad \mathsf{E} = \frac{1}{L^d}\langle \|u\|^2\rangle, \quad \text{and} \quad \frac{1}{L^d}\langle |\theta|^2\rangle, \tag{5}$$

as well as their dissipation (diffusion) rates

$$\epsilon := \frac{\nu}{L^d}\langle \|u\|^2\rangle, \qquad \eta := \frac{\nu}{L^d}\langle |Au|^2\rangle, \quad \text{and} \quad \chi := \frac{\mu}{L^d}\langle |\nabla\theta|^2\rangle. \tag{6}$$

By classical dimensional arguments, the dissipation range is expected to start at

$$\kappa_\epsilon = \left(\frac{\epsilon}{\nu^3}\right)^{1/4} \quad \text{and} \quad \kappa_\eta = \left(\frac{\eta}{\nu^3}\right)^{1/6}$$

in 3D and 2D, respectively; these are sometimes known as the Kolmogorov and Kraichnan wavenumbers. Their analogs for the tracer cascade are more complicated and depend on the advecting velocity; see κ_{2D} and κ_{3D} in Secs. V and VI below. Another set of important wavenumbers are

$$\kappa_\tau^2 := \frac{\langle \|u\|^2\rangle}{\langle |u|^2\rangle}, \quad \kappa_\sigma^2 := \frac{\langle |\Delta u|^2\rangle}{\langle \|u\|^2\rangle}, \quad \text{and} \quad \kappa_\theta^2 := \frac{\langle \|\theta\|^2\rangle}{\langle |\theta|^2\rangle}. \tag{7}$$

In 3D turbulence, κ_τ is closely related to the Taylor wavenumber, the scale at which the velocity correlation is lost; it has been shown that direct energy cascade takes place within the range $(\bar\kappa, \kappa_\tau)$. Its analogs in 2D and tracer turbulence are κ_σ and κ_θ, with corresponding results on enstrophy[11] and tracer [(25) below] cascades.

We make use of the following notation: $a \lesssim b$ means $a \le cb$ for a nondimensional universal constant c, independent of G and Sc (as well as κ_0, ν, and μ), *under the condition that* $G \ge G_*$, where G_* may be different for each inequality and similarly for $\gtrsim$. By $a \sim b$, we mean that both $a \lesssim b$ and $b \lesssim a$ hold. We write $a \ll b$ if $a/b < \delta$ for some small $\delta \in (0, 1)$, and a/b is nondimensional provided the ranges of a, b are *a priori* specified (e.g., for large values of a, b). The value of δ shall remain unspecified and may vary from one statement involving $\ll$ to the next.

III. INFLUENCE OF ENERGY SPECTRUM

A. Classical theory

We recall briefly from Ref. 17, Ch. 8 some elements of the Kolmogorov–Obukhov theory for 3D turbulence in a form suitable for its extension to passive tracers. Suppose that a parcel ("eddy") of size $1/\kappa$ has velocity $U_\kappa \sim [\kappa\mathcal{E}(\kappa)]^{1/2}$. Assuming that such an eddy breaks up in the time τ_κ, it takes to travel its own size, i.e.,

$$\tau_\kappa U_\kappa = 1/\kappa \quad \text{so that} \quad \tau_\kappa \sim [\kappa^3\mathcal{E}(\kappa)]^{-1/2}, \tag{8}$$

and the resulting downscale energy transfer rate is

$$\frac{U_\kappa^2}{\tau_\kappa} \sim \frac{\kappa\mathcal{E}(\kappa)}{\tau_\kappa}.$$

Assuming that this transfer rate is a constant ϵ for κ in the so-called inertial range and solving for $\mathcal{E}$, we arrive at the Kolmogorov spectrum

$$\mathcal{E}_{3D}(\kappa) \sim \epsilon^{2/3}\kappa^{-5/3}.$$

The situation in 2D is more complicated in that, for scales smaller than the forcing, we expect the enstrophy to undergo a direct cascade to smaller scales, while energy is mainly transferred to larger scales in an inverse cascade for scales larger than the forcing. Yet a similar dimensional argument in the enstrophy inertial range leads to the Kraichnan spectrum

$$\mathcal{E}_{2D}(\kappa) \sim \eta^{2/3}\kappa^{-3}.$$

An analogous cascade mechanism for the tracer suggests a connection between its spectrum $\mathcal{T}(\kappa)$ and the energy spectrum. Taking the amount of tracer (variance) at wavenumber κ to be $\kappa\mathcal{T}(\kappa)$, assuming that it is transferred to wavenumber 2κ by the advecting velocity over a time τ_κ given by (8), and setting the transfer rate to a constant χ, we find

$$\chi \sim \frac{\kappa \mathcal{T}(\kappa)}{\tau_\kappa}. \tag{9}$$

If we take $\mathcal{E}(\kappa) \sim K\kappa^{-n}$ in (8) and solve for $\mathcal{T}$ in (9), we have

$$\mathcal{T}(\kappa) \sim \chi K^{-1/2} \kappa^{(n-5)/2}.$$

B. Mathematical formulation

These spectral relations can be reformulated in terms of partial sums

$$\mathsf{e}_{\kappa,2\kappa} := \frac{1}{L^d} \sum_{\kappa \le \kappa_0 |k| < 2\kappa} \langle |\hat{u}_k|^2 \rangle \quad \text{and} \quad \vartheta_{\kappa,2\kappa} := \frac{1}{L^d} \sum_{\kappa \le \kappa_0 |k| < 2\kappa} \langle |\hat{\theta}_k|^2 \rangle. \tag{10}$$

As L increases (so κ_0 decreases), each quantity in (10) can be viewed as a Riemann sum approximation of the integral of the corresponding spectrum (this assumes smoothness of the summands, but below we will use this approximation only for explicit functions of κ). For instance, for the energy in 3D, we have

$$\int_\kappa^{2\kappa} \mathcal{E}_{3\mathrm{D}}(\tilde{\kappa})\, d\tilde{\kappa} \sim \int_\kappa^{2\kappa} \epsilon^{2/3} \tilde{\kappa}^{-5/3}\, d\tilde{\kappa} = \frac{3}{2} \epsilon^{2/3} (1 - 2^{-2/3}) \kappa^{-2/3} \sim \epsilon^{2/3} \kappa^{-2/3}.$$

In the inertial range, this leads to the energy power law

$$\mathsf{e}_{\kappa,2\kappa} \sim \epsilon^{2/3} \kappa^{-2/3} \quad \text{in 3D}$$

and similarly

$$\mathsf{e}_{\kappa,2\kappa} \sim \eta^{2/3} \kappa^{-2} \quad \text{in 2D}.$$

We gather the expected spectra according to classical theory in Table I.

We conclude this section with a brief calculation regarding the summation of the tracer variance over the relevant wavenumber range assuming that a certain power law holds. It will be used repeatedly.

Lemma 1. Suppose $\vartheta_{\kappa,2\kappa} \sim \alpha \kappa^{-p}$ *for* $\kappa_1 \le \kappa \le \kappa_2$, *with* $4\kappa_1 \le \kappa_2$ *and* $p \ge 0$. *Then*

$$\vartheta_{\kappa 1,\kappa 2} \sim \begin{cases} \alpha \left(\kappa_1^{-p} - \kappa_2^{-p} \right), & \text{if } p > 0, \qquad (11a) \\ \alpha \ln(\kappa_2/\kappa_1), & \text{if } p = 0. \qquad (11b) \end{cases}$$

Proof. As in Refs. 5 and 6, let $J = \lfloor \log_2(\kappa_2/\kappa_1) \rfloor - 1$. If $p > 0$, then

$$\vartheta_{\kappa_1,\kappa_2} \sim \sum_{\kappa = 2^j \kappa_1, j=0}^{J} \vartheta_{\kappa,2\kappa} \sim \frac{\alpha}{\kappa_1^p} \sum_{j=0}^{J} (2^p)^{-j} = \frac{\alpha}{\kappa_1^p} \frac{1}{1 - 2^{-p}} \left[1 - (2^{-J})^{-p} \right],$$

$$\sim \frac{\alpha}{\kappa_1^p} \left[1 - \left(\frac{\kappa_1}{\kappa_2} \right)^p \right].$$

If $p = 0$,

$$\vartheta_{\kappa 1,\kappa 2} \sim \alpha \sum_{j=0}^{J} 1 = \alpha \log_2(\kappa_2/\kappa_1) \sim \alpha \ln(\kappa_2/\kappa_1).$$

□

TABLE I. Spectra according to classical theory.

dir.	d	$\mathcal{E}(\kappa)$	$\mathsf{e}_{\kappa,2\kappa}$	$\mathcal{T}(\kappa)$	$\vartheta_{\kappa,2\kappa}$
fwd	3	$\epsilon^{2/3}\kappa^{-5/3}$	$\epsilon^{2/3}\kappa^{-2/3}$	$\chi\epsilon^{-1/3}\kappa^{-5/3}$	$\chi\epsilon^{-1/3}\kappa^{-2/3}$
fwd	2	$\eta^{2/3}\kappa^{-3}$	$\eta^{2/3}\kappa^{-2}$	$\chi\eta^{-1/3}\kappa^{-1}$	$\chi\eta^{-1/3}$
bkwd	2	$\epsilon^{2/3}\kappa^{-5/3}$	$\epsilon^{2/3}\kappa^{-2/3}$	$\chi\epsilon^{-1/3}\kappa^{-5/3}$	$\chi\epsilon^{-1/3}\kappa^{-2/3}$

IV. INDICATORS FOR CASCADES

Returning to the Navier–Stokes (3) and tracer equations (1), we henceforth assume that the forcing F and source g are spectrally-bounded, i.e., there exist $\kappa_0 < \kappa_g < \infty$ and $\kappa_0 \le \underline{\kappa} \le \bar{\kappa} < \infty$ such that

$$g = g_{\kappa_0,\kappa_g} \quad \text{and} \quad f = f_{\underline{\kappa},\bar{\kappa}}.$$

Given a fixed κ, we define

$$u^< := u_{\kappa_0,\kappa}, \quad u^> := u_{\kappa,\infty} \quad \text{and} \quad \theta^< := \theta_{\kappa_0,\kappa}, \quad \theta^> := \theta_{\kappa,\infty}.$$

The notation here, unlike in (10), does not involve the average and factor of L^d, e.g.,

$$u_{\kappa_1,\kappa_2} = \sum_{\kappa \le \kappa_0 |k| < 2\kappa} \hat{u}_k e^{i\kappa_0 k\cdot x}.$$

A. Navier–Stokes equations

We start by giving sufficient conditions for enstrophy and energy cascades. In terms of the solution of the 2D NSE, the *net rate of enstrophy transfer (flux)* is given by $\mathfrak{E}_\kappa = \mathfrak{E}_\kappa^{\rightarrow} - \mathfrak{E}_\kappa^{\leftarrow}$, where

$$\mathfrak{E}_\kappa^{\rightarrow}(u) = -\frac{1}{L^2}(B(u^<,u^<),Au^>) \quad \text{and} \quad \mathfrak{E}_\kappa^{\leftarrow}(u) = -\frac{1}{L^2}(B(u^>,u^>),Au^<)$$

are the *rates of enstrophy transfer* (low to high) and (high to low), respectively. It was shown in Ref. 11 that

$$1 - \Big(\frac{\kappa}{\kappa_\sigma}\Big)^2 \le \frac{\langle \mathfrak{E}_\kappa \rangle}{\eta} \le 1 \quad \text{if} \quad \bar{\kappa} \le \kappa \le \kappa_\sigma. \tag{12}$$

It follows that if

$$\kappa_\sigma \gg \bar{\kappa}, \tag{13}$$

then there exists an *enstrophy cascade*

$$\langle \mathfrak{E}_\kappa \rangle \approx \eta \quad \text{for} \quad \bar{\kappa} \le \kappa \ll \kappa_\sigma.$$

Similarly, the transfer of energy $\mathfrak{e}_\kappa = \mathfrak{e}_\kappa^{\rightarrow} - \mathfrak{e}_\kappa^{\leftarrow}$ is shown in Refs. 11 and 9 to satisfy

$$1 - \Big(\frac{\kappa}{\kappa_\tau}\Big)^2 \le \frac{\langle \mathfrak{e}_\kappa \rangle}{\epsilon} \le 1 \quad \text{for} \quad \bar{\kappa} \le \kappa \le \kappa_\tau, \tag{14}$$

where

$$\mathfrak{e}_\kappa^{\rightarrow}(u) = -\frac{1}{L^d}(B(u^<,u^<),u^>) \quad \text{and} \quad \mathfrak{e}_\kappa^{\leftarrow}(u) = -\frac{1}{L^d}(B(u^>,u^>),u^<).$$

It is shown in Ref. 9 that (14) holds as well in 3D for sufficiently regular solutions and for weak solutions with $\mathfrak{e}_\kappa$ replaced by

$$\mathfrak{e}_\kappa^* = \mathfrak{e}_\kappa - \lim_{\kappa\to\infty} \langle \mathfrak{e}_\kappa \rangle \tag{15}$$

to account for a possible loss of energy. Thus if

$$\kappa_\tau \gg \bar{\kappa}, \tag{16}$$

there is a direct energy cascade

$$\langle \mathfrak{e}_\kappa \rangle \approx \epsilon \quad \text{for} \quad \bar{\kappa} \le \kappa \ll \kappa_\tau.$$

It is easy to show that $\kappa_\tau \le \kappa_\sigma$, which is consistent with the expectation that for a 2D flow, a direct enstrophy cascade be more pronounced than a direct energy cascade.

We note a couple of useful bounds for κ_η and κ_ϵ. For the 2D NSE (regardless of whether the flow is turbulent), it was shown in Ref. 8 that

$$G^{1/6} \lesssim \kappa_\eta/\kappa_0 \le G^{1/3}. \tag{17}$$

While for the 3D NSE, Ref. 6 showed that

$$(\kappa_0/\bar{\kappa})^{5/8} G^{1/4} \lesssim \frac{\kappa_\epsilon}{\kappa_0}. \tag{18}$$

If, however, one assumes the power spectrum (which *a priori* says nothing about energy transfer) one does obtain lower bounds on κ_σ and κ_η, or equivalently by (13) and (16), one obtains sufficient conditions for the enstrophy and energy cascades. In 2D, we have the following estimate from Ref. 5.

Theorem 1. *If for the 2D NSE we have*

$$\mathsf{e}_{\kappa,2\kappa} \sim \eta^{2/3}\kappa^{-2} \quad \textit{for} \quad \underline{\kappa}_\mathrm{i} \le \kappa \le \kappa_\eta, \tag{19}$$

with $4\underline{\kappa}_\mathrm{i} \le \kappa_\eta$ *and*

$$\langle \|u_{\kappa_0,\underline{\kappa}_\mathrm{i}}\|^2 \rangle \lesssim \langle \|u_{\underline{\kappa}_\mathrm{i},\infty}\|^2 \rangle,$$

then

$$\kappa_\sigma^2 \sim \kappa_\eta^2 / \ln(\kappa_\eta/\underline{\kappa}_\mathrm{i}). \tag{20}$$

The wavenumber $\underline{\kappa}_\mathrm{i}$ marks the start of the inertial range. Based on (12) and (14), we expect that $\underline{\kappa}_\mathrm{i} \sim \bar{\kappa}$.

Thanks to (17), the dissipation wavenumber κ_η can be controlled by the Grashof number. Thus, under (19), κ_σ can indeed be made large by increasing G. It is shown in Ref. 12 that if conversely (20) holds, then one side of the power law holds (up to a log)

$$\mathsf{e}_{\kappa,2\kappa} \lesssim \eta^{2/3}\kappa^{-2}\ln(\kappa_\eta/\underline{\kappa}_\mathrm{i}) \quad \text{for} \quad \underline{\kappa}_\mathrm{i} \le \kappa \le \kappa_\eta.$$

Moreover, under (20), it is shown in Ref. 5 that (17) is sharpened to

$$\left(\frac{\kappa_0}{\bar{\kappa}}\right)^{1/4} \frac{G^{1/4}}{(\ln G)^{1/4}} \lesssim \frac{\kappa_\eta}{\kappa_0} \lesssim \left(\frac{\bar{\kappa}}{\kappa_0}\right)^{1/4} G^{1/4}(\ln G)^{1/8}. \tag{21}$$

The following 3D analog of Theorem 1 is proved in Ref. 6.

Theorem 2. *If for a Leray–Hopf solution to the 3D NSE we have*

$$\mathsf{e}_{\kappa,2\kappa} \sim \epsilon^{2/3}\kappa^{-2/3} \quad \textit{for} \quad \bar{\kappa} \le \kappa \le \kappa_\epsilon,$$

with $4\bar{\kappa} \le \kappa_\epsilon$ *and*

$$\langle |u|^2 \rangle \sim \langle |u_{\bar{\kappa},\kappa_\epsilon}|^2 \rangle,$$

then

$$\kappa_\tau^3 \sim \kappa_\epsilon^2 \bar{\kappa}. \tag{22}$$

Assuming (22), the bound (18) can be sharpened to

$$\left(\frac{\kappa_0}{\bar{\kappa}}\right)^{11/16} G^{3/8} \lesssim \frac{\kappa_\epsilon}{\kappa_0} \lesssim \left(\frac{\kappa_0}{\bar{\kappa}}\right)^{1/8} G^{3/8} \quad \text{for all } G \gtrsim \left(\frac{\bar{\kappa}}{\kappa_0}\right)^{3/2}. \tag{23}$$

The powers in (20) and (22) are suggestive of the extent to which the corresponding fluxes are constant over a given range, or alternatively, the width of the inertial range in each case.

B. Passive tracer

A condition for a cascade of the tracer is derived just as those for the NSE. Let κ and κ_g be fixed with $\kappa > \kappa_g$. Multiply (1) by $\theta^>$ in L^2 to get (the inequality is to account for possible lack of regularity)

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\theta^>|^2 + \mu|\nabla\theta^>|^2 &\le -(u\cdot\nabla\theta^<,\theta^>) + (g^>,\theta^>)\\ &= -(u^<\cdot\nabla\theta^<,\theta^>) + (u^>\cdot\nabla\theta^>,\theta^<) + (g^>,\theta^>)\\ &= L^d\Theta_\kappa + (g^>,\theta^>),\end{aligned}$$

where

$$\Theta_\kappa := \frac{1}{L^d}\left[-(u^<\cdot\nabla\theta^<,\theta^>) + (u^>\cdot\nabla\theta^>,\theta^<)\right]$$

is the downscale (i.e., toward larger $|k|$) flux of θ through wavenumber κ. Now $g^> = 0$ since $\kappa > \kappa_g$, so upon taking average, the time derivative disappears and we get

$$\mu\langle|\nabla\theta^>|^2\rangle = L^d\langle\Theta_\kappa\rangle. \tag{24}$$

If θ is not (known to be) sufficiently regular, we replace Θ_κ by

$$\Theta^*_\kappa := \Theta_\kappa - \lim_{\kappa\to\infty}\langle\Theta_\kappa\rangle$$

in analogy with (15).

The tracer "energy" cascade mechanism requires that $\langle\Theta_\kappa\rangle$ is (nearly) constant for $\kappa \in [\kappa_*, \kappa^*] \subset [\bar\kappa, \kappa_\theta]$. Noting that

$$\begin{aligned}
\chi \geq \langle\Theta_\kappa\rangle = \frac{\mu}{L^d}\langle|\nabla\theta^>|^2\rangle &= \frac{\mu}{L^d}\langle|\nabla\theta|^2\rangle - \frac{\mu}{L^d}\langle|\nabla\theta^<|^2\rangle \\
\geq \chi - \kappa^2\frac{\mu}{L^d}\langle|\theta^<|^2\rangle &\geq \chi - \kappa^2\frac{\mu}{L^d}\langle|\theta|^2\rangle \\
= \chi - \frac{\kappa^2}{\kappa_\theta^2}\frac{\mu}{L^d}\langle|\nabla\theta|^2\rangle &= \chi\Big[1 - \Big(\frac{\kappa}{\kappa_\theta}\Big)^2\Big],
\end{aligned}$$

we obtain the tracer analog of (12) and (14),

$$1 - \Big(\frac{\kappa}{\kappa_\theta}\Big)^2 \leq \frac{\langle\Theta_\kappa\rangle}{\chi} \leq 1 \quad \text{for} \quad \kappa_g \leq \kappa \leq \kappa_\theta. \tag{25}$$

The relations (12), (14), and (25) all imply cascades (more precisely, constancy of fluxes) provided that the indicator wavenumbers κ_σ, κ_τ, and κ_θ are sufficiently large. Criteria on the forcing f and source g that would give these conditions, directly from the NSE without further assumptions, so far remain elusive.

V. 2D CASE EFFECT OF ENERGY SPECTRUM ON κ_θ

In this section, we prove tracer analogs of Theorem 1, relating the indicator wavenumber κ_θ to κ_η. The interesting cases are where there are two spectra for the tracer, which in 2D is expected when the injection wavenumbers for tracer are below those for the fluid.

A. Large Schmidt number

For large Schmidt number $\mathrm{Sc} = \nu/\mu$, there is a range $[\kappa_\eta, \kappa_{2\mathrm{D}}]$ where the tracer is advected by a viscous fluid flow (Fig. 1). According to the classical theory (Ref. 17, pp. 367–369), here we expect a κ^{-1} tracer spectrum over the full range $[\bar\kappa, \kappa_{2\mathrm{D}}]$: First, the time scale for this range is determined by substituting κ_η into (8), which gives

$$\tau_{\kappa_\eta} = \eta^{-1/3}. \tag{26}$$

One then sets τ_{κ_η} equal to the diffusive time scale $(\mu\kappa^2)^{-1}$ to find

$$\kappa_{2\mathrm{D}} := (\eta/\mu^3)^{1/6} = \mathrm{Sc}^{1/2}\kappa_\eta, \tag{27}$$

so $\kappa_{2\mathrm{D}} \gg \kappa_\eta$. Using (26) in (9) and solving for $\mathcal{T}(\kappa)$ gives

$$\mathcal{T}(\kappa) \sim \chi\eta^{-1/3}\kappa^{-1} \quad \text{for} \quad \kappa_\eta \leq \kappa \leq \kappa_{2\mathrm{D}}. \tag{28}$$

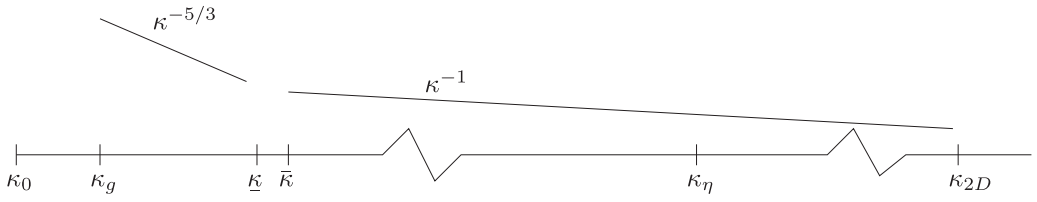

FIG. 1. Expected tracer spectra for the case of inverse cascade with a large Schmidt number.

Note that in the enstrophy cascade range $\bar\kappa \le \kappa \lesssim \kappa_\eta$, we expect $\mathcal{E}(\kappa) \sim \eta^{2/3}\kappa^{-3}$, giving rise to the same tracer spectrum, so in fact, (28) should hold for the extended range $\bar\kappa \le \kappa \le \kappa_{2\mathrm{D}}$.

Assuming power laws corresponding to the tracer spectra, we relate κ_θ to κ_η and show that asymptotically $\kappa_\theta \sim \kappa_{2\mathrm{D}}$ for large Sc:

Theorem 3. *Suppose that $\kappa_g < \underline{\kappa}$ holds along with*

$$\kappa_\sigma^2 \sim \kappa_\eta^2/\ln(\kappa_\eta/\bar\kappa), \tag{29}$$

$$\langle|\theta|^2\rangle \sim \langle|\theta_{\kappa_g,\underline{\kappa}}|^2\rangle + \langle|\theta_{\bar\kappa,\kappa_{2\mathrm{D}}}|^2\rangle, \tag{30}$$

and

$$\vartheta_{\kappa,2\kappa} \sim \begin{cases} \chi\epsilon^{-1/3}\kappa^{-2/3} & \text{for } \kappa_g \le \kappa \le \underline{\kappa}, \quad (31a)\\ \chi\eta^{-1/3} & \text{for } \bar\kappa \le \kappa \lesssim \kappa_{2D}\,. \quad (31b)\end{cases}$$

We then have

$$\kappa_\theta^2 \sim \frac{1}{a+b},$$

where

$$a = \kappa_{2D}^{-4/3}\mathrm{Sc}^{-1/3}(\kappa_g^{-2/3} - \underline{\kappa}^{-2/3})\ln(\kappa_\eta/\bar\kappa)^{-1/3} \quad \text{and} \quad b = \kappa_{2D}^{-2}\ln(\kappa_{2D}/\bar\kappa).$$

If, moreover,

$$\kappa_g \sim \kappa_0 \text{ and } \underline{\kappa} \sim \bar\kappa \tag{32}$$

along with

$$\bar\kappa/\kappa_0 \le (G(\ln G)^{1/2}/e)^{1/5} \tag{33}$$

and

$$\mathrm{Sc} \gtrsim (G\ln G\,\bar\kappa/\kappa_0)^{(3r-4)/(12-6r)}, \tag{34}$$

for some $r \in [4/3, 2)$, we have

$$\kappa_{2D}^r\kappa_0^{2-r}/\ln(\kappa_{2D}/\bar\kappa) \lesssim \kappa_\theta^2 \lesssim \kappa_{2D}^2/\ln(\kappa_{2D}/\bar\kappa). \tag{35}$$

Note that by Theorem 1, condition (29) could be replaced by the more natural (e.g., from the computational point of view) but stronger assumptions

$$\begin{aligned}&\mathsf{e}_{\kappa,2\kappa} \sim \eta^{2/3}\kappa^{-2} \quad \text{for} \quad \bar\kappa \le \kappa \lesssim \kappa_\eta,\\ &\langle\|u_{\kappa_0,\bar\kappa}\|^2\rangle \lesssim \langle\|u_{\bar\kappa,\infty}\|^2\rangle,\\ &4\bar\kappa \le \kappa_\eta,\end{aligned}$$

which are consistent with the discrete tracer spectrum (31b). Note also that if $\kappa_g \sim \bar\kappa$, one can neglect the contribution of a so that

$$\kappa_\theta^2 \sim \kappa_{2\mathrm{D}}^2/\ln(\kappa_{2\mathrm{D}}/\bar\kappa).$$

Proof. First we estimate over the inverse cascade as follows:

$$\begin{aligned}\vartheta_{\kappa_g,\underline{\kappa}} &\sim \frac{\chi}{\mu}\Big(\frac{\mu^3}{\epsilon}\Big)^{1/3}(\kappa_g^{-2/3} - \underline{\kappa}^{-2/3}) && \text{by (11a) and (27)},\\ &= \frac{\chi}{\mu}\kappa_{2\mathrm{D}}^{-2}\kappa_\sigma^{2/3}(\kappa_g^{-2/3} - \underline{\kappa}^{-2/3}),\\ &\sim \frac{\chi}{\mu}\kappa_{2\mathrm{D}}^{-2}\kappa_\eta^{2/3}(\kappa_g^{-2/3} - \underline{\kappa}^{-2/3})\ln(\kappa_\eta/\bar\kappa)^{-1/3} && \text{by (29)},\\ &= \frac{\chi}{\mu}\kappa_{2\mathrm{D}}^{-4/3}\mathrm{Sc}^{-1/3}(\kappa_g^{-2/3} - \underline{\kappa}^{-2/3})\ln(\kappa_\eta/\bar\kappa)^{-1/3} && \text{by (27)}.\end{aligned}$$

Then, over the range beyond $\bar\kappa$, we find

$$\vartheta_{\bar\kappa,\kappa_{2\mathrm{D}}} \sim \frac{\chi}{\mu\kappa_{2\mathrm{D}}^2}\ln(\kappa_{2\mathrm{D}}/\bar\kappa). \tag{36}$$

It follows from (30) that

$$\vartheta_{\kappa_0,\infty} \sim \vartheta_{\kappa_g,\underline{\kappa}} + \vartheta_{\bar{\kappa},\kappa_{2D}} \sim \frac{\chi}{\mu}(a+b)$$

and hence

$$\kappa_\theta^2 = \frac{(\mu/L^2)\langle|\nabla\theta|^2\rangle}{(\mu/L^2)\langle|\theta|^2\rangle} = \frac{\chi}{\mu\vartheta_{\kappa_0,\infty}} \sim \frac{1}{a+b}.$$

For the second part of the theorem, we seek to majorise a as

$$a \lesssim \kappa_{2D}^{-r}\kappa_0^{r-2}\ln(\kappa_{2D}/\bar{\kappa}),$$

which, by (27), is equivalent to

$$\left(\frac{\kappa_{2D}}{\kappa_0}\right)^{r-4/3}\left[\left(\frac{\kappa_g}{\kappa_0}\right)^{-2/3} - \left(\frac{\underline{\kappa}}{\kappa_0}\right)^{-2/3}\right] \lesssim \mathrm{Sc}^{1/3}\left(\ln\frac{\kappa_\eta}{\bar{\kappa}}\right)^{1/3}\ln\left(\frac{\kappa_\eta}{\bar{\kappa}}\mathrm{Sc}^{1/2}\right).$$

Since $\underline{\kappa} > \kappa_g$ and $\kappa_g \sim \kappa_0$ (but $\kappa_g > \kappa_0$), we have by (32),

$$(\kappa_g/\kappa_0)^{-2/3} - (\underline{\kappa}/\kappa_0)^{-2/3} \sim (\kappa_g/\kappa_0)^{-2/3} \sim 1,$$

so the last inequality is in turn, by (27), equivalent to

$$(\kappa_\eta/\kappa_0)^{r-4/3} \lesssim \mathrm{Sc}^{1-r/2}\left(\ln\frac{\kappa_\eta}{\bar{\kappa}}\right)^{1/3}\left(\ln\frac{\kappa_\eta}{\bar{\kappa}} + \ln\mathrm{Sc}\right). \tag{37}$$

From the upper bound in (21), we have, with $\zeta := \bar{\kappa}/\kappa_0$,

$$\kappa_\eta/\kappa_0 \lesssim (\zeta G)^{1/4}(\ln G)^{1/8}.$$

Using this to bound the left-hand side of (37), we have

$$(\kappa_\eta/\kappa_0)^{r-4/3} \lesssim (\zeta G)^{r/4-1/3}(\ln G)^{r/8-1/6}. \tag{38}$$

Now the lower bound in (21) implies

$$\zeta^{-5/4}(G/\ln G)^{1/4} \lesssim \kappa_\eta/\bar{\kappa},$$

which we then apply to the right-hand side of (37) to obtain

$$\begin{aligned}\left(\ln\frac{\kappa_\eta}{\bar{\kappa}}\right)^{1/3}\left(\ln\frac{\kappa_\eta}{\bar{\kappa}} + \ln\mathrm{Sc}\right) &\gtrsim \left(\ln\frac{G}{\zeta^5\ln G}\right)^{1/3}\left[\ln\left(\frac{G}{\zeta^5\ln G}\right) + \ln\mathrm{Sc}\right]\\ &\gtrsim \left(\ln\frac{G}{\zeta^5\ln G}\right)^{4/3}.\end{aligned}$$

Putting this together with (38), we find that (37) is implied by

$$(\zeta G(\ln G)^{1/2})^{r/4-1/3} \lesssim \mathrm{Sc}^{1-r/2}\left(\ln\frac{G}{\zeta^5\ln G}\right)^{4/3}. \tag{39}$$

Now for $G \geq 1$, we have

$$(G(\ln G)^{1/2})^{1/2} \leq G/\ln G,$$

so assuming this and writing $\gamma := G(\ln G)^{1/2}$, (39) is implied by

$$(\zeta\gamma)^{r/4-1/3} \lesssim \mathrm{Sc}^{1-r/2}(\ln\zeta^{-5}\gamma)^{4/3}.$$

Applying (33), we see that (34) implies (35). □

B. Moderate Schmidt number

For moderate Schmidt numbers, i.e., $\nu/\mu \sim 1$, we have from (27) that $\kappa_{2D} \sim \kappa_\eta$ (Fig. 2). In the simplest case, where $\kappa_g = \bar{\kappa}$, the tracer cascade occurs in the enstrophy cascade range, viz,

$$\langle|\theta|^2\rangle \sim \langle|\theta_{\bar{\kappa},\kappa_\eta}|^2\rangle, \tag{40}$$

$$\vartheta_{\kappa,2\kappa} \sim \chi\eta^{-1/3} \qquad \text{for } \bar{\kappa} \leq \kappa \lesssim \kappa_\eta. \tag{41}$$

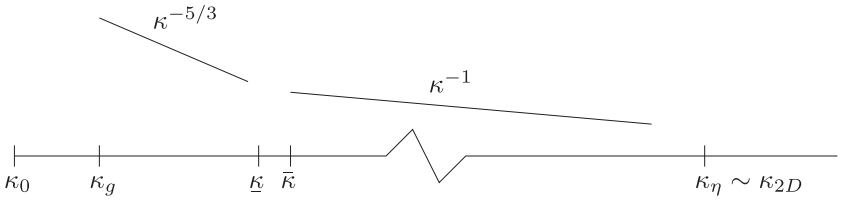

FIG. 2. Expected tracer spectra for the case of inverse cascade with a moderate Schmidt number.

We then have

$$\begin{aligned}\vartheta_{\bar{\kappa},\kappa_\eta} &\sim \chi\eta^{-1/3}\ln(\kappa_\eta/\bar{\kappa}) && \text{by (41) and (11b)},\\ &= \frac{\chi}{\mu}\Big(\frac{\mu^3}{\eta}\Big)^{1/3}\ln(\kappa_\eta/\bar{\kappa}) && \sim \frac{\chi}{\mu\kappa_\eta^2}\ln(\kappa_\eta/\bar{\kappa}),\end{aligned}$$

which by (40) implies

$$\vartheta_{\kappa_0,\infty} \sim \vartheta_{\kappa_g,\kappa_\eta} \sim \frac{\chi}{\mu\kappa_\eta^2}\ln(\kappa_\eta/\bar{\kappa}).$$

Thus, $\kappa_\theta \sim \kappa_\eta \sim \kappa_{2D}$ up to logarithm,

$$\kappa_\theta^2 = \frac{\langle|\nabla\theta|^2\rangle}{\langle|\theta|^2\rangle} = \frac{\chi}{\mu\vartheta_{\kappa_0,\infty}} \sim \kappa_\eta^2/\ln(\kappa_\eta/\bar{\kappa}) \sim \kappa_{2D}^2/\ln(\kappa_{2D}/\bar{\kappa}).$$

If the energy injection scale is small compared to the tracer injection scale, i.e., $\kappa_g \ll \underline{\kappa}$, we again expect to have two tracer cascade ranges (both downscale). In the gap between κ_g and $\underline{\kappa}$, the energy spectrum is expected to take the form $\mathcal{E}(\kappa) \sim \epsilon^{2/3}\kappa^{-5/3}$ so that the tracer spectrum should be $\mathcal{T}(\kappa) \sim \chi\epsilon^{-1/3}\kappa^{-5/3}$. Recall that in the enstrophy cascade range $\bar{\kappa} \le \kappa \lesssim \kappa_\eta$, we expect $\mathcal{E}(\kappa) \sim \eta^{2/3}\kappa^{-3}$ so that $\mathcal{T}(\kappa) \sim \chi\eta^{-1/3}\kappa^{-1}$. This case is virtually identical to that treated in Theorem 3, except since $\mathrm{Sc} = \nu/\mu \sim 1$, we have

$$\kappa_\theta^2 \lesssim \kappa_{2D}^{4/3}\kappa_g^{2/3}\big[\ln(\kappa_{2D}/\bar{\kappa})\big]^{-1/3}.$$

Under the assumptions of Theorem 3, we have $\kappa_\theta \lesssim \kappa_{2D}$ up to a logarithm. If those assumptions are dropped, though it is not expected that κ_θ exceeds κ_{2D}, one may ask if $\kappa_\theta \sim \kappa_{2D}$ implies $\vartheta_{\kappa,2\kappa} \sim \chi\eta^{1/3}$. The following is then a partial converse to Theorem 3.

Theorem 4. *If* $\kappa_\theta \gtrsim \kappa_{2D}$, *then* $\vartheta_{\kappa,2\kappa} \lesssim \chi\eta^{1/3}$.

Proof. We can rewrite the assumption as

$$\kappa_\theta^2 = \frac{\langle|\nabla\theta|^2\rangle}{\langle|\theta|^2\rangle} \gtrsim \left(\frac{\eta}{\mu^3}\right)^{1/3} = \kappa_{2D}^2$$

or as

$$\frac{\mu}{L^2}\langle|\nabla\theta|^2\rangle \gtrsim \frac{\eta^{1/3}}{L^2}\langle|\theta|^2\rangle$$

so that

$$\chi\eta^{-1/3} \gtrsim \frac{1}{L^2}\langle|\theta|^2\rangle \ge \frac{1}{L^2}\langle|\theta_{\kappa,2\kappa}|^2\rangle = \vartheta_{\kappa,2\kappa}.$$

□

Theorem 3 (with $\kappa_{2D} \sim \kappa_\eta$) imposes a restriction on the ranges of the forcing/source terms and the Grashof number. The indicator κ_θ would achieve its maximum value, $\kappa_\eta \sim \kappa_{2D}$ (up to a log), if one could choose κ_g, $\underline{\kappa}$, and $\bar{\kappa}$ in such a way that $a \sim b$. To investigate this, we seek an r such that

$$a \le c_0\kappa_\eta^{-r}\kappa_0^{r-2}\ln\frac{\kappa_\eta}{\bar{\kappa}}, \quad \text{where} \quad \frac{4}{3} \le r \le 2,$$

for some c_0, which is equivalent to

$$\left(\frac{\kappa_\eta}{\kappa_0}\right)^{r-4/3}\left[\left(\frac{\kappa_g}{\kappa_0}\right)^{-2/3} - \left(\frac{\underline{\kappa}}{\kappa_0}\right)^{-2/3}\right] \le c_0\left(\ln\frac{\kappa_\eta}{\bar{\kappa}}\right)^{4/3}. \tag{42}$$

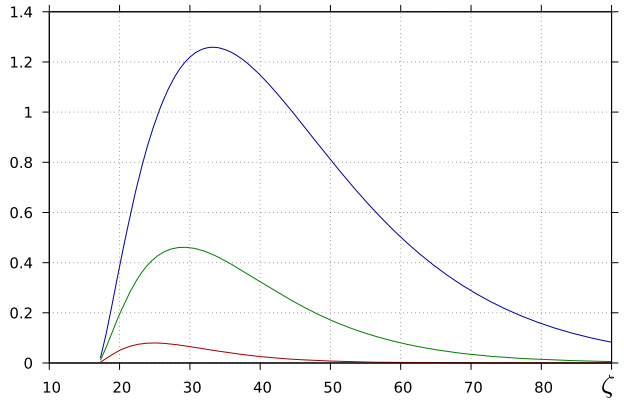

FIG. 3. From bottom to top: $\varphi_{1/6}$, $\varphi_{1/9}$, and $\varphi_{1/12}$.

We now derive a sufficient condition for (42). Rewriting the lower bound in (21) as

$$c_1\Big(\frac{\kappa_0}{\bar\kappa}\Big)^{5/4}\frac{G^{1/4}}{(\ln G)^{1/4}} \le \frac{\kappa_\eta}{\bar\kappa} \tag{43}$$

and the upper bound in (21) as

$$\frac{\kappa_\eta}{\kappa_0} \le c_2\Big(\frac{\bar\kappa}{\kappa_0}\,G\,(\ln G)^{1/2}\Big)^{1/4}, \tag{44}$$

we use (43) on the right and (44) on the left in (42), and we obtain a sufficient condition for (42), with $p \coloneqq (3r-4)/12 \in [0,\,1/6]$, $\zeta \coloneqq \bar\kappa/\kappa_0$, and $c_3 = c_0c_1/c_2^p$,

$$\begin{aligned} &[\zeta G(\ln G)^{1/2}]^p(1-\zeta^{-2/3}) \le c_3\ \ln(\zeta^{-5}G/\ln G)^{4/3} \\ \Leftrightarrow\qquad &\frac{1}{c_3} \le \frac{\ln(\zeta^{-5}G/\ln G)^{4/3}}{[\zeta G(\ln G)^{1/2}]^p(1-\zeta^{-2/3})}. \end{aligned} \tag{45}$$

Putting $G = e^\zeta$, this in turn is equivalent to

$$\frac{1}{c_3} \le \frac{(\zeta - 6\ln\zeta)^{4/3}}{\zeta^{3p/2}e^{p\zeta}(1-\zeta^{-2/3})} =: \varphi_p(\zeta). \tag{46}$$

In Fig. 3, we plot $\varphi_{1/6}$, $\varphi_{1/9}$, and $\varphi_{1/12}$ against ζ. It is clear that, at least for these values of p, there is a range of $\zeta = \bar\kappa/\kappa_0$ such that (46), and thus (42), is satisfied, *provided* that c_3 is sufficiently large. (Since we are seeking a sufficient condition for (42), we can take c_3 smaller but not larger.) While a good estimate for c_3 is not known, this plot suggests that even in the presence of a significant inverse cascade ($10 \lesssim \zeta \lesssim 20$), a wide tracer cascade range can be achieved,

$$\kappa_\theta^2 \sim \kappa_{2\mathrm{D}}^r\kappa_0^{2-r}/\ln(\kappa_{2\mathrm{D}}/\bar\kappa), \tag{47}$$

with $r = 2$, 16/9, and 5/3 for $p = 1/6$, 1/9, and 1/12 respectively, for large enough Grashof number ($G \sim e^\zeta$) to sustain turbulent fluid flow.

C. Effect of log corrected energy spectrum

In order to enforce constant enstrophy flux, Kraichnan[13] proposed a log correction to the energy spectrum in the inertial range for 2D turbulence

$$\mathcal{E}(\kappa) \sim \eta^{2/3}\kappa^{-3}(\ln\kappa/\underline{\kappa}_\mathrm{i})^{-1/3},$$

which leads to a turnover time of

$$\tau_\kappa \sim \eta^{-1/3}(\ln\kappa/\underline{\kappa}_\mathrm{i})^{-1/3}. \tag{48}$$

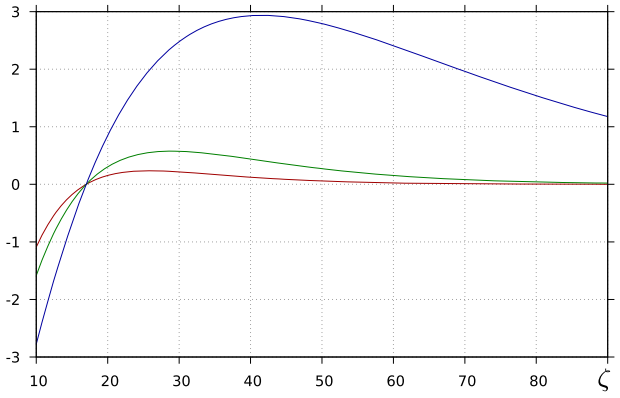

FIG. 4. From bottom to top: $\tilde{\varphi}_{1/9}$, $\tilde{\varphi}_{1/12}$ and $\tilde{\varphi}_{1/24}$.

This correction was shown in Ref. 15 to be consistent with an upper bound on the dimension of the global attractor in Ref. 3.

If (48) is used in (9) and the lower end of the inertial range is $\underline{\kappa}_{\mathrm{i}} = \bar{\kappa}$, the tracer spectrum takes the form

$$\mathcal{T}(\kappa) \sim \chi\eta^{-1/3}\kappa^{-1}(\ln \kappa/\bar{\kappa})^{-1/3} \quad \text{for } \bar{\kappa} \le \kappa \le \kappa_\eta$$

and

$$\vartheta_{\kappa,2\kappa} \sim \int_\kappa^{2\kappa} \mathcal{T}(s)\, ds \sim \chi\eta^{-1/3}\left[(\ln 2\kappa/\bar{\kappa})^{2/3} - (\ln \kappa/\bar{\kappa})^{2/3}\right].$$

Summing as in Lemma 1, the terms telescope so that

$$\vartheta_{\bar{\kappa},\kappa_\eta} \sim \frac{\chi}{\mu}\left(\frac{\mu^3}{\eta}\right)^{1/3}(\ln \kappa_\eta/\bar{\kappa})^{2/3} \sim \frac{\chi}{\mu\kappa_\eta^2}(\ln \kappa_\eta/\bar{\kappa})^{2/3}.$$

Using this instead of (36) in the proof of Theorem 3 yields

$$\kappa_\theta^2 \sim \frac{1}{a + b'}, \quad \text{where} \quad b' = \kappa_\eta^{-2}(\ln \kappa_\eta/\bar{\kappa})^{2/3}.$$

We now seek r such that

$$a \lesssim \kappa_\eta^{-r}\kappa_0^{2-r}(\ln \frac{\kappa_\eta}{\bar{\kappa}})^{2/3}, \quad \text{where} \quad \frac{4}{3} \le r \le 2,$$

which is equivalent to the analog of (42),

$$\left(\frac{\kappa_\eta}{\kappa_0}\right)^{r-4/3}\left[\left(\frac{\kappa_g}{\kappa_0}\right)^{-2/3} - \left(\frac{\underline{\kappa}}{\kappa_0}\right)^{-2/3}\right] \lesssim \ln \frac{\kappa_\eta}{\bar{\kappa}},$$

the only change being the power of the log on the right.

Proceeding as before, with $\zeta = \bar{\kappa}/\kappa_0$ and $p = (3r - 4)/12$, and putting $G = e^\zeta$, this is implied by

$$\frac{1}{c_4} \le \frac{\zeta - 6\ln\zeta}{\zeta^{3p/2}e^{p\zeta}(1 - \zeta^{-2/3})} =: \tilde{\varphi}_p(\zeta). \tag{49}$$

In Fig. 4, we plot $\tilde{\varphi}_{1/9}$, $\tilde{\varphi}_{1/12}$ and $\tilde{\varphi}_{1/24}$ against ζ. Again, we need c_4 sufficiently large for (49) to hold.

D. Tracer injection scales below energy injection scales

In case the injection scales are reversed so that $\bar{\kappa} < \kappa_g$, then the analysis for both moderate and large Schmidt number proceeds as before, except the term a is dropped in both cases, so the conclusion is that $\kappa_\theta \sim \kappa_{2\mathrm{D}}$ (up to a log).

VI. 3D CASE

The large Schmidt number case is also interesting in 3D, as then we expect two ranges with distinct tracer spectra (Ref. 17, p. 368): For $\kappa \in (\bar{\kappa}, \kappa_\epsilon)$, we have the classical spectrum $\mathcal{T}(\kappa) \sim \kappa^{-5/3}$. For κ beyond κ_ϵ, substituting $\kappa = \kappa_\epsilon$ in (8) gives a turnover time of

$$\tau_{\kappa_\epsilon} = (\nu/\epsilon)^{1/2}. \tag{50}$$

Putting this equal to the diffusive time scale $(\mu\kappa^2)^{-1}$ then yields

$$\kappa_{3\mathrm{D}} = \left(\frac{\epsilon}{\nu\mu^2}\right)^{1/4} = \mathrm{Sc}^{1/2}\kappa_\epsilon,$$

the wavenumber where diffusion becomes important. Using (50) in (9) and solving for $\mathcal{T}(\kappa)$ gives

$$\mathcal{T}(\kappa) \sim \chi\left(\frac{\nu}{\epsilon}\right)^{1/2}\kappa^{-1} \quad \text{for} \quad \kappa_\epsilon \le \kappa \le \kappa_{3\mathrm{D}}.$$

We have the following analog of Theorem 3:

Theorem 5. *Suppose that (22) holds along with* $4\kappa_g \le \kappa_\epsilon$, $\mathrm{Sc} > 2$,

$$\langle|\theta|^2\rangle \sim \langle|\theta_{\kappa_g,\kappa_{3\mathrm{D}}}|^2\rangle, \tag{51}$$

and

$$\vartheta_{\kappa,2\kappa} \sim \begin{cases} \chi\epsilon^{-1/3}\kappa^{-2/3} & \text{for} \quad \kappa_g \le \kappa \le \kappa_\epsilon, \\ \chi(\nu/\epsilon)^{1/2} & \text{for} \quad \kappa_\epsilon \le \kappa \lesssim \kappa_{3D}. \end{cases}$$

We then have

$$\kappa_\theta^2 \sim \frac{1}{a+b}, \tag{52}$$

where

$$a = \kappa_{3D}^{-4/3}\mathrm{Sc}^{-1/3}(\kappa_g^{-2/3} - \kappa_\epsilon^{-2/3}) \quad \text{and} \quad b = \kappa_{3D}^{-2}\ln(\mathrm{Sc}).$$

If, moreover, $\kappa_g \sim \kappa_0$ *and* $\underline{\kappa} \sim \bar{\kappa}$, *along with*

$$\mathrm{Sc} \gtrsim G^{(3r-4)/(8-4r)},$$

then

$$\kappa_\theta^2 \sim \kappa_{3D}^r\kappa_0^{2-r}/\ln(\kappa_{3D}/\kappa_\epsilon) \quad \text{for} \quad 4/3 \le r < 2. \tag{53}$$

Proof. As in the 2D case, we first compute

$$\vartheta_{\kappa_g,\kappa_\epsilon} \sim \frac{\chi}{\mu}\left(\frac{\mu^3}{\epsilon}\right)^{1/3}(\kappa_g^{-2/3} - \kappa_\epsilon^{-2/3}) = \frac{\chi}{\mu}\kappa_{3\mathrm{D}}^{-4/3}\mathrm{Sc}^{-1/3}(\kappa_g^{-2/3} - \kappa_\epsilon^{-2/3}),$$

$$\vartheta_{\kappa_\epsilon,\kappa_{3\mathrm{D}}} \sim \frac{\chi}{\mu}\left(\frac{\nu\mu^2}{\epsilon}\right)^{1/2}\ln(\kappa_{3\mathrm{D}}/\kappa_\epsilon) \sim \frac{\chi}{\mu}\kappa_{3\mathrm{D}}^{-2}\ln(\mathrm{Sc}).$$

By hypothesis, $\langle|\theta|^2\rangle \sim \vartheta_{\kappa_g,\kappa_\epsilon} + \vartheta_{\kappa_\epsilon,\kappa_{2\mathrm{D}}}$, giving us (52),

$$\kappa_\theta^2 = \frac{\chi}{\mu\vartheta_{\kappa_0,\infty}} = \frac{1}{a+b}.$$

For the second part of the theorem, we note that

$$a \lesssim \kappa_{2\mathrm{D}}^{-r}\kappa_0^{r-2}\ln(\mathrm{Sc})$$

is equivalent to

$$(\kappa_{3\mathrm{D}}/\kappa_0)^{r-4/3}\left[(\kappa_0/\kappa_g)^{2/3} - (\kappa_0/\bar{\kappa})^{2/3}\right] \lesssim \mathrm{Sc}^{1/3}\ln(\mathrm{Sc}),$$

$$\Leftrightarrow \qquad (\kappa_\epsilon/\kappa_0)^{r-4/3} \lesssim \mathrm{Sc}^{1-r/2}\ln(\mathrm{Sc}).$$

Arguing as in the 2D case, we bound the left-hand side by the upper bound in (23) and using $\ln(\mathrm{Sc}) > 1$ on the right-hand side, this is implied by

$$G^{(3r-4)/8} \lesssim \mathrm{Sc}^{1-r/2},$$

which gives us (53). □

Remark 1. The decay rate of the energy spectrum in the $(\kappa_g, \kappa_\epsilon)$-inertial range is not crucial here. It is the prefactor in the tracer spectrum that produces the helpful Schmidt number effect in the estimate in Theorem 5. In fact, we would achieve the same estimate for κ_θ if we consider a dimensionally correct energy spectrum with a different decay rate

$$\mathcal{E}_{3\mathrm{D}}(\kappa) \sim \epsilon^{2/3}\kappa_0^{p-5/3}\kappa^{-p} \quad \text{for any} \quad p \in (1,3).$$

Note that this would violate Kolmogorov's assumption that $\mathcal{E}_{3\mathrm{D}}$ depend on only ϵ and κ, as it would now also depend on L. Nevertheless, an energy spectrum of this form would result in a tracer spectrum [Ref. 17 (8.94)],

$$\mathcal{T}(\kappa) \sim \chi\epsilon^{-1/3}\kappa_0^q\kappa^{q'-1} \qquad \text{with } q = (p-3)/2 \text{ and } q' = (5-3p)/6,$$

corresponding to a discrete dyadic tracer spectrum

$$\vartheta_{\kappa,2\kappa} \sim \chi\epsilon^{-1/3}\kappa_0^{q'}\kappa^q \quad \text{for} \quad \kappa_g \le \kappa \le \kappa_\epsilon.$$

Assuming again, that $\kappa_0 \sim \kappa_g \ll \kappa_\epsilon$, we have

$$\begin{aligned}\vartheta_{\kappa_g,\kappa_\epsilon} \sim \chi\epsilon^{-1/3}\kappa_0^{q'}\left(\kappa_g^q - \kappa_\epsilon^q\right) &\sim \chi\epsilon^{-1/3}\kappa_0^{-2/3},\\ &= \frac{\chi}{\mu}\left(\frac{\nu\mu^2}{\epsilon}\right)^{1/3}\left(\frac{\mu}{\nu}\right)^{1/3}\kappa_0^{-2/3},\\ &= \frac{\chi}{\mu}\kappa_{3\mathrm{D}}^{-4/3}\mathrm{Sc}^{-1/3}\kappa_0^{-2/3}.\end{aligned}$$

The rest of the estimate for κ_θ follows as in the proof of Theorem 5.

A. Moderate Schmidt number case, 3D

If in 3D, $\mathrm{Sc} \sim 1$, we have just the single steeper tracer spectrum and $\kappa_\theta^2 \sim 1/a$ with a as in Theorem 5. This can be expressed as $\kappa_\theta \sim \kappa_{3\mathrm{D}}^{2/3}\kappa_0^{1/3}$, which gives the same fractional power for the tracer cascade range width as for the energy cascade in Proposition 2.

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