Erdős-Ko-Rado Theorems for a Family of Trees

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Abstract

A family of sets is intersecting if any two sets in the family intersect. Given a graph G and an integer $r \geq 1$, let $\mathcal{I}^{(r)}(G)$ denote the family of independent sets of size r of G. For a vertex v of G, let $\mathcal{I}_v^{(r)}(G)$ denote the family of independent sets of size r that contain v. This family is called an r-star and v is its centre. Then G is said to be r-EKR if no intersecting subfamily of $\mathcal{I}^{(r)}(G)$ is bigger than the largest r-star, and if every maximum size intersecting subfamily of $\mathcal{I}^{(r)}(G)$ is an r-star, then G is said to be strictly r-EKR. Let $\mu(G)$ denote the minimum size of a maximal independent set of G. Holroyd and Talbot conjectured that if $2r \leq \mu(G)$, then G is r-EKR, and it is strictly r-EKR if $2r < \mu(G)$.

This conjecture has been investigated for several graph classes, but not trees (except paths). In this note, we present a result for a family of trees. A depth-two claw is a tree in which every vertex other than the root has degree 1 or 2 and every vertex of degree 1 is at distance 2 from the root. We show that if G is a depth-two claw, then G is strictly r-EKR if $2r \leq \mu(G)+1$, confirming the conjecture of Holroyd and Talbot for this family.

Hurlbert and Kamat had conjectured that one can always find a largest r-star of a tree whose centre is a leaf. Baber and Borg have independently shown this to be false. We show that, moreover, for all integers $n \geq 2$ and $d \geq 3$, there exists a positive integer r such that there is a tree where the centre of the largest r-star is a vertex of degree n at distance d from every leaf.

Keywords: EKR Theorem, trees, independent sets, elongated claws

1. Introduction

In this paper, we consider graph-theoretic versions of the following famous result of Erdős, Ko and Rado [8]; the extremal case was characterized by Hilton and Milner [11]. A family of sets is said to be *intersecting* if any two sets in the family intersect.

EKR Theorem (Erdős, Ko, Rado [8]; Hilton, Milner [11]) *Let n and r be positive integers, $n \geq r$, let S be a set of size n and let $\mathcal{A}$ be an intersecting family of subsets of S each of size r. If $n \geq 2r$, then*

$$|\mathcal{A}| \leq \binom{n-1}{r-1}.$$

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Moreover, if $n > 2r$ the upper bound is attained only if the sets in $\mathcal{A}$ contain a fixed element of S.

Throughout this paper, graphs are simple and undirected. Let K_n denote the complete graph on n vertices, and let $K_{1,n}$ denote a *claw*. An independent set in a graph is a set of pairwise non-adjacent vertices. Let $\mu(G)$ denote the minimum size of a maximal independent set of G.

Given a graph G and an integer $r \geq 1$, let $\mathcal{I}^{(r)}(G)$ denote the family of independent sets of G of cardinality r. For a vertex v of G, let $\mathcal{I}_v^{(r)}(G)$ be the subset of $\mathcal{I}^{(r)}(G)$ containing all sets that contain v. This is called an *r-star* (or just star) and v is its *centre*. We say that G is r-EKR if no intersecting family $\mathcal{A} \subseteq \mathcal{I}^{(r)}(G)$ is larger than the biggest r-star, and strictly r-EKR if every intersecting family that is not an r-star is smaller than the largest r-star of $\mathcal{I}^{(r)}(G)$.

The EKR Theorem can be seen as a statement about the maximum size of an intersecting family of independent sets of size r in the empty graph on n vertices. We quickly obtain another formulation of the EKR Theorem by noting that an independent set of the claw that contains more than one vertex contains only leaves.

Theorem 1.1. *Let n and r be positive integers, $n \geq r$. The claw $K_{1,n}$ is r-EKR if $n \geq 2r$ and strictly r-EKR if $n > 2r$.*

There exist EKR results for several graph classes. The reader is referred to [3] and the references therein. The following was conjectured by Holroyd and Talbot [14].

Conjecture 1.1 (Holroyd, Talbot [14]). *Let r be a positive integer and let G be a graph. Then G is r-EKR if $\mu(G) \geq 2r$ and strictly r-EKR if $\mu(G) > 2r$.*

This conjecture appears difficult to prove or disprove. The most important breakthrough is a a result of Borg [2] that addresses a uniform version of Chvátal's conjecture [6] and confirms Conjecture 1.1 for every graph G satisfying $\mu(G) \geq \frac{3}{2}(r-1)^2(3r-4) + r$. The conjecture is also known to be true for many graph classes such as the disjoint union of complete graphs each of order at least two, powers of paths [13] and powers of cycles [18]. See [5, 10, 12, 14, 15] for further examples.

A usual technique to prove results of this kind is to find the centre of the largest r-star of a graph and this will prove useful to us. In this vein, Hurlbert and Kamat [15] conjectured the following for the class of trees.

Conjecture 1.2 (Hurlbert, Kamat [15]). *Let n and r be positive integers, $n \geq r$. If T is a tree on n vertices, then there is a largest r-star of T whose centre is a leaf.*

They proved Conjecture 1.2 for $1 \leq r \leq 4$ [15]. The conjecture does not, however, hold for any $r \geq 5$. This was shown independently by Baber [1] and Borg [4] who gave counterexamples in which the largest r-star is centred at a vertex whose degree is 2. Moreover, mindful of Conjecture 1.1, we remark that in the counterexample G in [1], the value of r does not exceed $\mu(G)/2$, while in the counterexample G in [5], $5 \leq r \leq \mu(G)$.

1.1. Results

We consider a family of trees called *depth-two claws*. A depth-two claw has one vertex that is its *root*. Every other vertex has degree 1 or 2 and every leaf is at distance 2 from the root. We are now ready to state our first result.

Theorem 1.2. *Let r be a positive integer and let G be a depth-two claw. Then G is strictly r-EKR if $\mu(G) \geq 2r - 1$.*

Theorem 1.2 confirms (and is stronger than) Conjecture 1.1 for depth-two claws.

Our second result concerns the problem of trying to find the centre of a largest r-star of a tree. We show that it can, in some sense, be located anywhere within a tree.

Theorem 1.3. *Let n and d be positive integers, $n \geq 2$, $d \geq 3$. Then there exists a positive integer r such that there is a tree where the centre of the largest r-star is a vertex of degree n and at distance d from every leaf.*

In the remaining sections we prove Theorems 1.2 and 1.3.

2. Depth-two Claws

In this section, we prove Theorem 1.2. We shall need two auxiliary results.

Theorem 2.1 (Meyer [17]; Deza and Frankl [7])**.** *Let n, r and t be positive integers, $n \geq r$, $t \geq 2$, and let G be the disjoint union of n copies of K_t. Then G is r-EKR and strictly r-EKR unless $r = n$ and $t = 2$.*

For a family $\mathcal{A}$ of sets and a non-negative integer s, the s-shadow of $\mathcal{A}$, denoted $\partial_s \mathcal{A}$, is the family $\partial_s \mathcal{A} = \{S : |S| = s, \exists A \in \mathcal{A}, S \subseteq A\}$.

Lemma 2.1 (Katona [16])**.** *Let a and b be non-negative integers and let $\mathcal{A}$ be a family of sets of size a such that $|A \cap A'| \geq b \geq 0$ for all $A, A' \in \mathcal{A}$. Then $|\mathcal{A}| \leq |\partial_{a-b} \mathcal{A}|$.*

The proof of Theorem 1.2 is inspired by a proof of the EKR theorem [9]. To the best of our knowledge, the proof is the first to make use of shadows in the context of graphs.

Proof of Theorem 1.2. Let c be the root of G and let n be the number of leaves of G. Note that $n = \mu(G)$ so $n \geq 2r - 1$. Let $\mathcal{A} \subseteq \mathcal{I}^{(r)}(G)$ be any intersecting family. Define a partition $\mathcal{B} = \{A \in \mathcal{A} : c \notin A\}$ and $\mathcal{C} = \{A \in \mathcal{A} : c \in A\}$.

Notice that each vertex in each member of $\mathcal{B}$ is either a leaf or the neighbour of a leaf. For $B \in \mathcal{B}$, let M_B be the set of r leaves that each either belongs to B or is adjacent to a vertex in B. We say that M_B *represents* B. Let $\mathcal{M} = \{M_B : B \in \mathcal{B}\}$. Note that each member of $\mathcal{M}$ might represent many different members of $\mathcal{B}$. In fact, consider $M \in \mathcal{M}$. It can represent any independent set that, for each leaf $\ell \in M$, contains either ℓ or its unique neighbour. There are 2^r such sets but they can be partitioned into complementary pairs so, as $\mathcal{B}$ is intersecting, the number s_M of members of $\mathcal{B}$ that M represents is at most 2^{r-1}. We also note that $\mathcal{M}$ is intersecting (since $\mathcal{B}$ is intersecting). We have that

$$|\mathcal{B}| = \sum_{M \in \mathcal{M}} s_M \leq \binom{n-1}{r-1} 2^{r-1}, \tag{1}$$

where the inequality follows from Theorem 2.1.

For $B \in \mathcal{B}$, let N_B be the set of $n - r$ leaves that neither belong to B nor are adjacent to a vertex in B. Notice that M_B and N_B partition the set of leaves. Let $\mathcal{N} = \{N_B : B \in \mathcal{B}\}$. For any pair $B_1, B_2 \in \mathcal{B}$, we know that M_{B_1} and M_{B_2} intersect, so $|M_{B_1} \cup M_{B_2}| \leq 2r - 1$. The leaves not in this union are members of both N_{B_1} and N_{B_2}

and there are at least $n-(2r-1) \geq 0$ of them. Thus we can apply Lemma 2.1 to $\mathcal{N}$ with $a = n-r$, $b = n-(2r-1)$ to obtain

$$|\mathcal{N}| \leq |\partial_{r-1}\mathcal{N}|. \tag{2}$$

Notice that, by definition, $\partial_{r-1}\mathcal{N}$ is a collection of sets of $r-1$ leaves each of which is, for some $B \in \mathcal{B}$, a subset of N_B, and so is disjoint to M_B and certainly does not intersect B.

Let us try to bound the size of $\mathcal{C}$. Each $C \in \mathcal{C}$ contains a distinct set of $r-1$ leaves. We know this set must intersect every member of $\mathcal{B}$ so it cannot be a member of $\partial_{r-1}\mathcal{N}$. Thus we find

$$|\mathcal{C}| \leq \binom{n}{r-1} - |\partial_{r-1}\mathcal{N}|. \tag{3}$$

We apply (2) to (3) and note that $|\mathcal{N}| = |\mathcal{M}|$ to obtain

$$|\mathcal{C}| \leq \binom{n}{r-1} - |\mathcal{M}|,$$

whence

$$|\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}| \leq \sum_{M \in \mathcal{M}} s_M + \binom{n}{r-1} - |\mathcal{M}|. \tag{4}$$

If $|\mathcal{M}| > \binom{n-1}{r-1}$, then combining (1) and (4) gives us

$$\begin{aligned} |\mathcal{A}| &\leq \binom{n-1}{r-1}2^{r-1} + \binom{n}{r-1} - |\mathcal{M}| \\ &< \binom{n-1}{r-1}2^{r-1} + \binom{n}{r-1} - \binom{n-1}{r-1} = \binom{n-1}{r-1}2^{r-1} + \binom{n-1}{r-2}. \end{aligned} \tag{5}$$

However, if $|\mathcal{M}| \leq \binom{n-1}{r-1}$, then the upper bound in (5) can possibly be attained since combining again (1) and (4) and recalling that $s_M \leq 2^{r-1}$ for each $M \in \mathcal{M}$ gives us

$$|\mathcal{A}| \leq |\mathcal{M}|2^{r-1} + \binom{n}{r-1} - |\mathcal{M}| \leq \binom{n-1}{r-1}2^{r-1} + \binom{n-1}{r-2}. \tag{6}$$

We are now ready to show that G is strictly r-EKR. If $r = n$ then $r = 1$ so the result trivially holds. Suppose $r < n$. Then, by Theorem 2.1, equality holds in (1) and therefore also in (6) only if $\mathcal{B}$ is an r-star centred at a leaf x or a neighbour y of a leaf. It follows easily that $\mathcal{C} = \emptyset$ if $\mathcal{A} = \mathcal{I}_y^{(r)}(G)$; thus $\mathcal{A} = \mathcal{I}_x^{(r)}(G)$ as desired. □

Remark. We demonstrate that if G is a depth-two claw with n leaves, then G is not n-EKR by describing an intersecting family that is larger than the largest n-star. Let c be the root of G and let $G' = G - c$, a graph containing n copies of K_2 each of which contains one leaf of G. Clearly G' contains 2^n independent sets of size n which can be partitioned into complementary pairs. Let $\mathcal{B}$ be a family of 2^{n-1} independent sets of size n formed by considering each complementary pair and choosing either the one that contains the greater number of leaves of G, or, if they each contain half the leaves, choosing one arbitrarily. Notice that $\mathcal{B}$ is intersecting but is not a star. Let $\mathcal{C} = \{C \in \mathcal{I}^{(n)}(G) : c \in C\}$. Clearly, $|\mathcal{C}| = \binom{n}{n-1} = n$ and for each pair $C \in \mathcal{C}$, $B \in \mathcal{B}$, we have that $C \cap B \neq \emptyset$. Thus if $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$, then $\mathcal{A}$ is intersecting, maximal and $|\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}| = 2^{n-1} + n$. Thus $\mathcal{A}$ has one more element than the largest n-star in G.

The above remark together with Theorem 1.2 motivates the following conjecture.

Conjecture 2.1. *Let n and r be positive integers, $n > r$, and let G be a depth-two claw with n leaves. Then G is r-EKR.*

3. Centres of Largest r-stars in Trees

An *elongated claw* has one vertex that is its *root*. Every other vertex has degree 1 or 2 (it is possible that the root also has degree 1 or 2). A vertex of degree 1 is called a *leaf*. A path from the root to a leaf is a *limb*.

In this section, we prove Theorem 1.3. To do this, we need to define a family of trees. Let n, k and a be positive integers. A (k, a)-claw is an elongated claw with k limbs each of length a. The tree $T^{n,k,a}$ contains, as induced subgraphs, n disjoint (k, a)-claws, and one further vertex, the *root* of $T^{n,k,a}$, that is joined by an edge to the root of each (k, a)-claw. Figure 1 shows $T^{5,2,3}$ as an example. We note that Baber [1] and Borg [4] showed that Conjecture 1.2 is false by considering $T^{2,k,2}$.

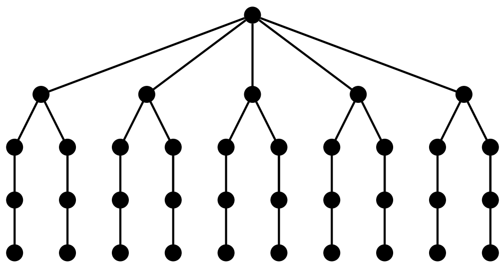

Figure 1: $T^{5,2,3}$

The key to proving Theorem 1.3 is to show that, for certain values, the largest r-star of $T^{n,k,a}$ is centred at its root. Let $\mathcal{I}_{\text{root}}(T^{n,k,a})$ be the family of independent sets of $T^{n,k,a}$ that contain its root, and let $\mathcal{I}_{\text{leaf}}(T^{n,k,a})$ be the number of independent sets of $T^{n,k,a}$ that contain a particular leaf (note that, by symmetry, the size of this family does not depend on which leaf we choose). Notice that in these definitions, we are considering independent sets of all possible sizes. In Lemma 3.3, we will think of $|\mathcal{I}_{\text{root}}(T^{n,k,a})|$ and $|\mathcal{I}_{\text{leaf}}(T^{n,k,a})|$ as sequences indexed by k with fixed n and a.

First we need some further definitions and lemmas. Let a be a non-negative integer. Let P_a denote the path on a vertices. Let $F(a)$ denote the number of independent sets in P_a (notice that the empty set is an independent set of any graph). We state without proof two simple observations.

Lemma 3.1. $F(0) = 1$, $F(1) = 2$ *and, for* $a \geq 2$, $F(a) = F(a-1)+F(a-2)$. *Moreover, for* $a \geq 3$, *each vertex of degree 1 in* P_a *belongs to more independent sets than each vertex of degree 2.*

We notice that (F_a) is, of course, the Fibonacci sequence (without the initial term).

We now prove a simple result about (k,a)-claws that we will use later.

Lemma 3.2. *Let n, k and a be positive integers, let b be a non-negative integer and let G be a graph that contains $n-1$ disjoint (k,a)-claws. The number of independent sets of G that each contain the roots of at least b of the (k,a)-claws is*

$$\sum_{i=b}^{n-1} \binom{n-1}{i} F(a-1)^{ik} F(a)^{(n-1-i)k}.$$

Proof. Note that a (k,a)-claw with its root removed is k disjoint copies of P_a and so the claw contains $F(a)^k$ independent sets that do not include the root. Similarly it contains $F(a-1)^k$ independent sets that do include the root (in this case one considers the graph obtained when the root and its neighbours are removed). This kind of argument recurs many times in this section; we use it first to complete the proof of the lemma.

Each summand is the number of independent sets that contain *exactly* i of the roots: the three factors count the number of ways of choosing i claws (whose roots will be in the independent set), the number of independent sets in those chosen claws (given that their roots are included) and the number of independent sets in the unchosen claws (given that their roots are not included). Then the sum is over the possible values of i. $\square$

For a vertex v in a graph, we denote by $N(v)$ the set of vertices that are adjacent to v. For a graph G and a subset V of vertices in G, let $G - V$ denote the graph obtained from G by deleting the vertices in V together with their incident edges.

Lemma 3.3. *Let a and n be positive integers, $n \geq 2$, $a \geq 2$. Then*

$$\frac{|\mathcal{I}_{root}(T^{n,k,a})|}{|\mathcal{I}_{leaf}(T^{n,k,a})|} \to \frac{F(a-1)+F(a-2)}{2F(a-2)} \quad \textit{as } k \to \infty.$$

Proof. Let x be the root of $T^{n,k,a}$ and let y be one of its leaves. We note that as k is not fixed, we are concerned with finding properties not of a specific graph, but of the family of graphs $T^{n,k,a}$ for fixed n and a. So we might have written x_k and y_k to indicate that when we choose a particular vertex, we must first fix which graph in the family we are looking at. For simplicity, we avoid this explicit notation throughout.

Some notation to improve readability: let $I(x) = |\mathcal{I}_{\mathrm{root}}(T^{n,k,a})|$ and $I(y) = |\mathcal{I}_{\mathrm{leaf}}(T^{n,k,a})|$, and let $I(x,y)$ be the number of independent sets that contain both x and y.

We can say immediately that

$$I(x,y) = F(a-2)F(a)^{nk-1} \tag{7}$$

as we just need to count the independent sets in the graph obtained from $T^{n,k,a}$ when x and y and their neighbours are removed and this graph contains $nk-1$ copies of P_a and one copy of P_{a-2}. Let $I'(x)$ and $I'(y)$ be the number of independent sets that contain x but not y, and y but not x, respectively; that is $I(x) = I'(x) + I(x,y)$ and $I(y) = I'(y) + I(x,y)$. Let $T_x = T^{n,k,a} - (N(x) \cup \{x,y\})$ and $T_y = T^{n,k,a} - (N(y) \cup \{x,y\})$. So $I'(x)$ is the number of independent sets in T_x and $I'(y)$ is the number of independent sets in T_y. As T_x consists of $nk-1$ disjoint copies of P_a and one copy of P_{a-1}, we have

$$I'(x) = F(a-1)F(a)^{nk-1}. \tag{8}$$

Evaluating $I'(y)$ will require a little more work. Notice that T_y contains $n-1$ disjoint (k,a)-claws and one elongated claw C that has $k-1$ limbs of length a and one limb of length $a-2$. Let R be the set of roots of the (k,a)-claws and let c denote the root of C. We define a partition of the independent sets of T_y:

- S_1 is the family of independent sets that do not contain any member of R nor c.
- S_2 is the family of independent sets that contain c.
- S_3 is the family of independent sets that do not contain c but intersect R.

So S_1 contains independent sets of $T_y - (R \cup \{c\})$, a graph that consists of $nk-1$ disjoint copies of P_a and one copy of P_{a-2}. Thus we have, using also (7),

$$|S_1| = F(a-2)F(a)^{nk-1} = I(x,y). \tag{9}$$

We will need the following observation:

$$\frac{|S_1|}{I'(x)} = \frac{F(a-2)}{F(a-1)}. \tag{10}$$

Next to find the size of S_2 we must count the number of independent sets in $T_y - (N(c) \cup \{c\})$, a graph that contains $k-1$ copies of P_{a-1}, $n-1$ disjoint (k,a)-claws and one copy of P_{a-3} (if $a \geq 3$), or one copy of P_{a-2}, the null graph, (if $a=2$). Thus, noting that $F(a-3) \leq F(a-2)$, we have

$$|S_2| \leq F(a-2)F(a-1)^{k-1} \sum_{i=0}^{n-1} \binom{n-1}{i} F(a-1)^{ik} F(a)^{(n-1-i)k}$$

where the sum is the number of independent sets in $n-1$ disjoint (k,a)-claws (by Lemma 3.2 with $b=0$). Noting as before that $F(a-2) < F(a-1)$ and that, for all i, $\binom{n-1}{i} \leq \binom{n-1}{\lfloor (n-1)/2 \rfloor}$, we obtain

$$\begin{aligned} |S_2| &\leq F(a-1)^k \sum_{i=0}^{n-1} \binom{n-1}{\lfloor (n-1)/2 \rfloor} F(a-1)^{ik} F(a)^{(n-1-i)k} \\ &= F(a-1)^k F(a)^{(n-1)k} \sum_{i=0}^{n-1} \binom{n-1}{\lfloor (n-1)/2 \rfloor} \left(\frac{F(a-1)}{F(a)} \right)^{ik} \\ &\leq F(a-1)^k F(a)^{(n-1)k} \sum_{i=0}^{n-1} \binom{n-1}{\lfloor (n-1)/2 \rfloor}. \end{aligned}$$

So we can write

$$|S_2| \leq c_2 F(a-1)^k F(a)^{(n-1)k} \tag{11}$$

where c_2 is a constant that does not depend on k.

Let us note now that, using (8) and (11), we have

$$\frac{|S_2|}{I'(x)} \leq c_2 \frac{F(a)}{F(a-1)} \left(\frac{F(a-1)}{F(a)} \right)^k.$$

Hence, since for $a \geq 2$, $F(a-1) < F(a)$,

$$\frac{|S_2|}{I'(x)} \to 0 \text{ as } k \to \infty. \tag{12}$$

And from (9) and (11), we have

$$\frac{|S_2|}{|S_1|} \leq c_2 \frac{F(a)}{F(a-2)} \left(\frac{F(a-1)}{F(a)}\right)^k.$$

Hence

$$\frac{|S_2|}{|S_1|} \to 0 \text{ as } k \to \infty. \tag{13}$$

To determine the size of S_3, we must count the number of independent sets in $T_y - \{c\}$, a graph that contains one copy of P_{a-2}, $k-1$ copies of P_a and $n-1$ disjoint (k,a)-claws that contain the root of at least one of the (k,a)-claws. Thus we have

$$|S_3| = F(a-2)F(a)^{k-1} \sum_{i=1}^{n-1} \binom{n-1}{i} F(a-1)^{ik} F(a)^{(n-1-i)k}$$

where the sum is the number of independent sets in $n-1$ disjoint (k,a)-claws that include at least one of the roots (by Lemma 3.2 with $b=1$). Reasoning as before, we find

$$\begin{aligned}
|S_3| &\leq F(a)^k \sum_{i=1}^{n-1} \binom{n-1}{\lfloor (n-1)/2 \rfloor} F(a-1)^{ik} F(a)^{(n-1-i)k} \\
&= F(a)^{nk} \sum_{i=1}^{n-1} \binom{n-1}{\lfloor (n-1)/2 \rfloor} \left(\frac{F(a-1)}{F(a)}\right)^{ik} \\
&\leq F(a)^{nk} \sum_{i=1}^{n-1} \binom{n-1}{\lfloor (n-1)/2 \rfloor} \left(\frac{F(a-1)}{F(a)}\right)^{k}.
\end{aligned}$$

Thus we obtain

$$|S_3| \leq c_3 F(a-1)^k F(a)^{(n-1)k} \tag{14}$$

where c_3 is a constant that does not depend on k. Comparing (11) and (14), we see that the same arguments used to obtain (12) and (13) give us

$$\frac{|S_3|}{I'(x)} \to 0 \quad \text{as } k \to \infty, \tag{15}$$

$$\frac{|S_3|}{|S_1|} \to 0 \quad \text{as } k \to \infty. \tag{16}$$

We combine (10), (12) and (15) to find

$$\frac{I'(y) + I(x,y)}{I'(x)} = \frac{2|S_1|}{I'(x)} + \frac{|S_2|}{I'(x)} + \frac{|S_3|}{I'(x)} \to \frac{2F(a-2)}{F(a-1)} \text{ as } k \to \infty.$$

And from (13) and (16), we have

$$\frac{I'(y)+I(x,y)}{I(x,y)} = \frac{2|S_1|}{|S_1|} + \frac{|S_2|}{|S_1|} + \frac{|S_3|}{|S_1|} \to 2 \text{ as } k \to \infty.$$

Using these last two observations, we can complete the proof:

$$\begin{aligned} \frac{I(x)}{I(y)} &= \frac{I'(x)}{I'(y)+I(x,y)} + \frac{I(x,y)}{I'(y)+I(x,y)} \\ &\to \frac{F(a-1)}{2F(a-2)} + \frac{1}{2} = \frac{F(a-1)+F(a-2)}{2F(a-2)} \text{ as } k \to \infty. \end{aligned}$$

□

The next lemma will also be used in the proof of Theorem 1.3.

Lemma 3.4. *Let r be a positive integer, and let G be an elongated claw. Then there is a largest r-star of G whose centre is a leaf.*

Proof. Let v be a vertex of G that is not a leaf, and let L be the limb of G that contains v (if v is the root, then L can be any limb). Let x be the leaf of L. We find an injection f from $\mathcal{I}_v^{(r)}(G)$ to $\mathcal{I}_x^{(r)}(G)$ which proves that $|\mathcal{I}_x^{(r)}(G)| \geq |\mathcal{I}_v^{(r)}(G)|$ and the lemma immediately follows.

Let w be the unique neighbour of x. Let $A \in \mathcal{I}_v^{(r)}(G)$.

1. If $x \in A$, then let $f(A) = A$.
2. If $x \notin A$ and $w \notin A$, then let $f(A) = A \backslash \{v\} \cup \{x\}$.
3. If $x \notin A$ and $w \in A$, then let $X = \{x = x_1, x_2, \ldots, x_m = v\}$ be the set of vertices in L from x towards v. Let $A \cap X = \{x_{i_1}, \ldots, x_{i_j}\} = Y$ for some $m > j \geq 1$. Let $Z = \{x_{i_1-1}, \ldots, x_{i_j-1}\}$. Observe that $|Y| = |Z|$ and $x \in Z$ since $w \in Y$. Then let $f(A) = (A \cup Z)\backslash Y$.

To prove that f is injective we consider distinct $A_1, A_2 \in \mathcal{I}_v^{(r)}(G)$. If $f(A_1)$ and $f(A_2)$ are defined by the same case (of the three above), then it is clear that $f(A_1)$ and $f(A_2)$ are distinct. When they are defined by different cases, we simply note that in the first $f(A)$ always contains v, in the second $f(A)$ contains neither v nor any of its neighbours, and in the third $f(A)$ contains a neighbour of v. □

We note that Lemma 3.4 confirms Conjecture 1.2 for elongated claws.

Remark. The property of Lemma 3.4 is a much weaker version of the *degree sort property*; graphs have this property if the size of an r-star centred at u is at least the size of an r-star centred at v whenever the degree of u is less than that of v. Hurlbert and Kamat [15] observed that depth-two claws have this property. We note that not all elongated claws possess it. For example, consider an elongated claw with three limbs of lengths 1, 2 and 3. Then the 4-star centred at the neighbour of the root in the limb of length 3 has size 2, but the 4-star centred at the leaf of the limb of length 2 has size 1. It remains to determine which elongated claws — or, more generally, which trees — have the degree sort property. We might also ask which trees have the following weaker property: if $i < j$, then the size of the largest r-star of all those stars centred at vertices of degree i is at least the size of the largest r-star of all those centred at vertices of degree j.

Proof of Theorem 1.3. In the family of trees $T^{n,k,d-1}$, the root vertex has degree n and is at distance d from every leaf. By Lemma 3.3, for sufficiently large k,

$$|\mathcal{I}_{\text{root}}(T^{n,k,d-1})| > |\mathcal{I}_{\text{leaf}}(T^{n,k,d-1})|.$$

As $\mathcal{I}_{\text{root}}(T^{n,k,d-1})$ and $\mathcal{I}_{\text{leaf}}(T^{n,k,d-1})$ are each the (disjoint) union, over all positive integers r, of r-stars centred at, respectively, the root and the leaf, there must be some r for which the r-star centred at the root is strictly larger than that centred at the leaf.

The theorem will follow if we can show that for any positive integer r, for any tree $T^{n,k,d-1}$, and for any vertex w that is neither the root nor a leaf, the r-star centred at w is no larger than an r-star centred at a leaf.

Let x be the root of $T^{n,k,d-1}$. Let C be the component of $T^{n,k,d-1} - \{x\}$ that contains w and let D be the union of the other components. Noting that C is an elongated claw, let y be the leaf of the limb that contains w (or any limb if w is the root of C). Let $R(w)$ and $R(y)$ be the number of independent sets of $T^{n,k,d-1}$ of size r that include x and contain, respectively, w and y. Similarly let $S(w)$ and $S(y)$ be the number of independent sets that contain, respectively, w and y, but that do not include x. For $v \in \{w, y\}$, we can write

$$S(v) = \sum_{i=0}^{r} |\mathcal{I}_v^{(i)}(C)| \times \text{number of independent sets of size } r-i \text{ in } D.$$

By Lemma 3.4, $|\mathcal{I}_y^{(i)}(C)| \geq |\mathcal{I}_w^{(i)}(C)|$ for all i, and, as the second term in the product does not depend on v, we have that $S(y) \geq S(w)$.

Now we consider independent sets of size r that contain x. These can be bijectively matched with independent sets of size $r-1$ in $T^{n,k,d-1} - (N(x) \cup \{x\})$; this graph contains nk copies of P_{d-1}. If w is the root of C it is not in this graph, and in this case $R(w) = 0$ and we are done. In all other cases, w and y belong to the same copy of P_{d-1} which we denote P. Let Q denote the union of the other paths. For $v \in \{w, y\}$,

$$R(v) = \sum_{i=0}^{r} |\mathcal{I}_v^{(i)}(P)| \times \text{number of independent sets of size } r-i \text{ in } Q.$$

As y is a vertex of degree 1 in P, by Lemma 3.1, $v = y$ maximises $|\mathcal{I}_v^{(i)}(P)|$. Again, the second term does not depend on v so $R(y) \geq R(w)$ and the proof is complete. $\square$

3.1. Further Counterexamples

Let us finally remark that one can define a much broader class of trees with the property that the largest r-stars are not centred at leaves (which therefore provides further counterexamples to Conjecture 1.2) by, for example, taking copies of $T^{n,k,a}$ and adding an additional root vertex joined to the root of each $T^{n,k,a}$ — and this process of duplicating and joining (via a new root) can be repeated ad infinitum. Moreover, it does not, in fact, matter which trees are used to initialize this process: if the number of copies made is large enough a graph where the largest r-stars are not centred at leaves is obtained. This does not ultimately add anything to the result stated in Theorem 1.3 so we omit further details.

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