On a conjecture of Mohar concerning Kempe equivalence of regular graphs

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Abstract

Let G be a graph with a vertex colouring α. Let a and b be two colours. Then a connected component of the subgraph induced by those vertices coloured either a or b is known as a Kempe chain. A colouring of G obtained from α by swapping the colours on the vertices of a Kempe chain is said to have been obtained by a Kempe change. Two colourings of G are Kempe equivalent if one can be obtained from the other by a sequence of Kempe changes.

A conjecture of Mohar (2007) asserts that, for $k \geq 3$, all k-colourings of a k-regular graph that is not complete are Kempe equivalent. It was later shown that all 3-colourings of a cubic graph that is neither K_4 nor the triangular prism are Kempe equivalent. In this paper, we prove that the conjecture holds for each $k \geq 4$. We also report the implications of this result on the validity of the Wang-Swendsen-Kotecký algorithm for the antiferromagnetic Potts model at zero-temperature.

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1   Introduction

Let $G = (V, E)$ denote a simple undirected graph. Let k be a positive integer. Then a *k-colouring* of G is a function $\alpha : V \to \{1, \dots, k\}$ such that for each edge $uv \in E$, $\alpha(u) \neq \alpha(v)$. The chromatic number of G, denoted $\chi(G)$, is the least k such that G has a k-colouring. We call $\{1, \dots, k\}$ the set of colours and refer to $\alpha(u)$ as the colour of the vertex u. For a colouring α, and colours a and b, $G_\alpha(a, b)$ is the subgraph of G induced by vertices with colour a or b. A connected component of $G_\alpha(a, b)$ is known as an (a, b)-*component* of G and α. These components are also referred to as *Kempe chains*. If a colouring β is obtained from a colouring α by exchanging the colours a and b on the vertices of an (a, b)-component of G and α, then β is said to have been obtained from α by a *Kempe change*. A pair of colourings are *Kempe equivalent* if each can be obtained from the other by a sequence of Kempe changes. A set of Kempe equivalent colourings is called a *Kempe class*.

   A graph is *k-regular* if every vertex has degree k. By Brooks' Theorem [9], a k-regular connected graph has a k-colouring unless it is a complete graph or a cycle on an odd number of vertices. Mohar conjectured that for all other k-regular graphs, the set of k-colourings form a Kempe class [25]; that is, any possible colouring can be obtained from an initial colouring by a series of Kempe changes. The first non-trivial case is when $k = 3$ and van den Heuvel [18] showed that there is a counterexample to the conjecture, the triangular prism (the graph obtained by joining the vertices of two vertex-disjoint triangles by a perfect matching, see Figure 1 below and the discussion at the end of this section).
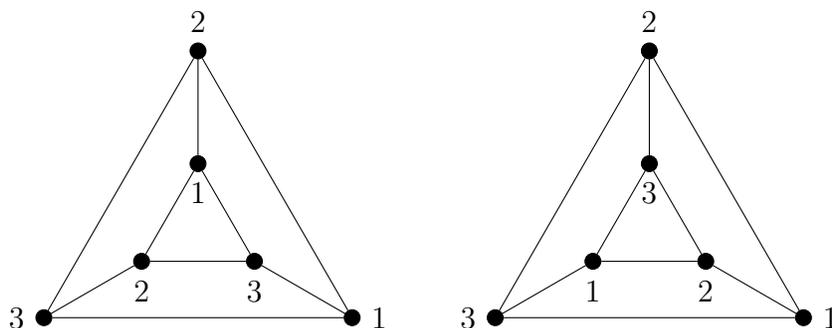
Figure 1: The triangular prism with two non-Kempe-equivalent 3-colourings.

Recently Feghali et al. [16] showed that, for $k = 3$, this is the only counterexample: for a non-complete 3-regular connected graph G, the 3-colourings of G form a Kempe class *unless* G is the triangular prism. In this paper, we affirm the conjecture for larger k.

Theorem 1.1. *Let $k \geq 4$ be an integer. If G is a connected non-complete k-regular graph, then the set of k-colourings of G is a Kempe class.*

We consider only connected graphs, as other graphs can be considered componentwise. Notice that we need not have included the condition that G is not complete, since one can say that if a graph has no k-colourings, then its set of colourings (the empty set) is a Kempe class, but it is neater to exclude this case. We will note in Corollary 5.1 that this result about k-regular graphs can easily be extended to an analogous result on graphs of maximum degree k.

Let us describe another way to think of Theorem 1.1. Let C_G^k be the set of k-colourings of a graph G. Let $\mathcal{K}_k(G)$ be the graph that has vertex set C_G^k, and an edge between two vertices α and β whenever the colouring β can be obtained from α by a single Kempe change. Theorem 1.1 states that, for $k \geq 4$, for any connected non-complete k-regular graph G, $\mathcal{K}_k(G)$ is connected.

We might call $\mathcal{K}_k(G)$ a *solution graph*; it represents all possible solutions to the problem of finding a k-colouring of G. Or we can call it the *reconfiguration graph* of k-colourings of G, and refer to Kempe changes as reconfiguration steps. In fact, reconfiguration graphs of k-colourings have been much studied when the edge relation is defined by the alternative reconfiguration step of *trivial* Kempe changes. A Kempe chain is trivial if it contains a single vertex v and the corresponding Kempe change alters the colour of only v, and so pairs of colourings are connected in these reconfiguration graphs if they disagree on only one vertex. These graphs were introduced in [11]. Much work on these graphs has focussed on (the computational complexity of) deciding whether or not the reconfiguration graph is connected [12], on deciding whether a given pair of colourings belong to the same connected component [5, 6, 13, 19], and on the diameter of the reconfiguration graph or its components [3, 4, 8, 15]. Similar work has been done for reconfiguration graphs for search problems other than graph colouring; see, for example, the survey of van den Heuvel [18].

Reconfiguration graphs defined by Kempe changes have received less attention. Kempe changes were introduced in 1879 by Kempe in his attempted

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proof of the Four Colour Theorem [20]. Though this proof was fallacious, the Kempe change technique has proved useful in, for example, the proof of the Five Colour Theorem and a short proof of Brooks' Theorem [23]. We briefly review the purely graph theoretical studies of Kempe equivalence. Fisk [17] showed in 1977 that the 4-colourings of a Eulerian triangulation of the plane are a Kempe class. This was generalized both by Meyniel [24], who showed that the 5-colourings of a planar graph are a Kempe class, and by Mohar [25], who proved that the k-colourings of a planar graph G are a Kempe class if $k > \chi(G)$. The former result was further extended by Las Vergnas and Meyniel [21] who showed that the 5-colourings of a K_5-minor free graph are a Kempe class. Bertschi [2] proved that the k-colourings of a perfectly contractile graph are a Kempe class. (A graph is perfectly contractile if, from each of its induced subgraphs, a complete graph can be obtained by repeatedly contracting pairs of vertices that have no odd-length induced path between them.) The Kempe equivalence of edge-colourings has also been investigated [1, 22, 25].

Applications of the Kempe change technique can be found in theoretical physics [26, 27, 30, 32, 33], and in the study of Markov chains [31] and timetables [28]. In particular, our result implies that the Wang-Swendsen-Kotecký (WSK) algorithm used to sample from the Gibbs distribution of the zero-temperature antiferromagnetic q-state Potts model is valid on a triangular lattice with periodic boundary conditions (that is, embedded on the torus) for $q = 6$, and on a Kagomé lattice with periodic boundary conditions for $q = 4$, thus answering some of the questions raised in [26, 27]. We discuss this in Section 2.

In Section 3 we introduce some useful lemmas. In Section 4, we prove Theorem 1.1. In Section 5, we present a corollary of Theorem 1.1 and some closing remarks. We conclude this section with some final comments on our investigations towards proving Theorem 1.1. We now know that, for $k \geq 3$, the only non-complete connected k-regular graph whose k-colourings are not a Kempe class is the triangular prism. Let us explain why its 3-colourings are not all Kempe equivalent. Notice from Figure 1 that no Kempe change modifies the colour partition. Thus the two 3-colourings illustrated are not Kempe equivalent, as they are not the same up to colour permutation.

So one might have hoped to find a counterexample to Theorem 1.1 by finding, for some $k \geq 4$, a connected non-complete k-regular graph with a k-colouring such that all Kempe changes maintain the colour partition. However, it is not hard to convince oneself that such a graph does not exist.

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Indeed, let us consider such a graph G and k-colouring α, and obtain a contradiction. As G has more than k vertices some colour a appears on more than one vertex. If a colour b does not appear on any vertex, then changing the colour of one vertex from a to b gives a colouring with a different partition. And, if b appears on only one vertex u, then changing the colour of a vertex not adjacent to u to b again changes the partition. So every colour appears on at least two vertices. If, for any vertex u, there is a colour other than $\alpha(u)$ that does not appear in its neighbourhood, then another trivial Kempe change gives a colouring with a different partition; so on the k vertices in the neighbourhood of u, one colour appears twice and every other colour but $\alpha(u)$ appears once. For every pair of colours a and b, $G_\alpha(a, b)$ is connected (else a Kempe change of one component gives a different partition). Since $k \geq 4$, there are at least four distinct colours a, b, c and d. As every vertex in $G_\alpha(a, b)$ has degree 1 or 2, $G_\alpha(a, b)$ is either a path or a cycle. As there are at least two vertices coloured a, there is a vertex u coloured c that has degree 2 in $G_\alpha(a, c)$. Similarly, there is a vertex v coloured c that has degree 2 in $G_\alpha(b, c)$; clearly $u \neq v$. Notice that u and v must both have degree 1 in $G_\alpha(c, d)$; that is, $G_\alpha(c, d)$ is a path whose endvertices are both coloured c. But, by the same argument, $G_\alpha(c, d)$ is a path whose endvertices are both coloured d. This contradiction proves that such a G and α cannot exist.

2 Validity of the WSK algorithm

The q-state Potts models are among the simplest and most studied models in statistical mechanics with various applications from the theory of critical phenomena to condensed-matter systems. We will give a very brief overview of how our results relate to this model. See [26, 27, 29, 34, 35] for further details.

The model uses a finite graph $G = (V, E)$ where each vertex $v \in V$ is assigned a *spin* $\sigma(v) \in \{1, \ldots, q\}$. There exist two main Potts models. In the *ferromagnetic Potts model*, the state of a spin is attracted to the spin states of adjacent vertices. (Roughly speaking, it is likely that every vertex attains the same spin.) We are concerned with the *antiferromagnetic Potts model* where the state of a spin is repelled by the spin states of adjacent vertices, which means that every spin "tries" to achieve a state that is distinct from its neighbours. (Roughly speaking, it is likely that the vertices attain spins

such that the spins provide a q-colouring of the vertices.) Spins become more repulsive as the temperature decreases. At zero temperature, spins are repulsive so that adjacent vertices *must* receive distinct spins, and then the *only* possible assignments of spin values correspond to q-colourings of the graph. Associated with this model is a Markov chain, which we will denote $\mathcal{M}_q(G)$, whose states are the set of q-colourings of G. The initial state X_0 is an arbitrary q-colouring. We describe a procedure that determines the probabilities for transitions $X_t \mapsto X_{t+1}$: from X_t choose a vertex v and a colour a uniformly at random, and perform a Kempe change on the $(\sigma(v), a)$-component of G containing v to obtain a new q-colouring that is set to be X_{t+1}.

The Wang-Swendsen-Kotecký (WSK) non-local cluster dynamics [32, 33] is an algorithm that seeks to sample from the set of all q-colourings of a graph G by simulating a random walk on $\mathcal{M}_q(G)$. The aim is to take a sufficient number of steps such that the total variation distance between the probability distribution of the colouring reached at the end of the walk and the uniform distribution is small. A chain such as $\mathcal{M}_q(G)$ is said to be *rapidly mixing* if it can be shown that, for any $\epsilon > 0$, this total variation distance is at most ϵ for a number of steps that is polynomial in the order of G, q and $\log(1/\epsilon)$. Moreover, an algorithm that simulates a random walk on such a chain is also said to be rapidly mixing.

One requirement for the WSK algorithm to be rapidly mixing is that $\mathcal{M}_q(G)$ must be *ergodic*; that is, it must be possible to reach any state from any other by some sequence of transitions. We will say the WSK algorithm is *valid* if the chain $\mathcal{M}_q(G)$ is ergodic. Since the ergodicity of $\mathcal{M}_q(G)$ does not hold in general, the statistical mechanics community has investigated special graphs, especially highly structured graphs that can be embedded on surfaces. Amongst them, triangular lattices and Kagomé lattices with boundary conditions have received considerable attention. Both these graph classes can be defined in terms of toroidal grid graphs (graphs that are the Cartesian product of two cycles). A triangular lattice with boundary conditions (abbreviated to triangular lattice) is formed from a toroidal grid graph by adding a single "diagonal" edge to each face of the grid in such a way that all the diagonals are parallel. Thus any triangular lattice is 6-regular, and every face is a triangle. A Kagomé lattice with boundary conditions (abbreviated to Kagomé lattice) is obtained from a toroidal grid graph by first subdividing each edge. Each face of the grid now contains four midpoints (vertices of degree two) on its, let us say, north, east, south and west
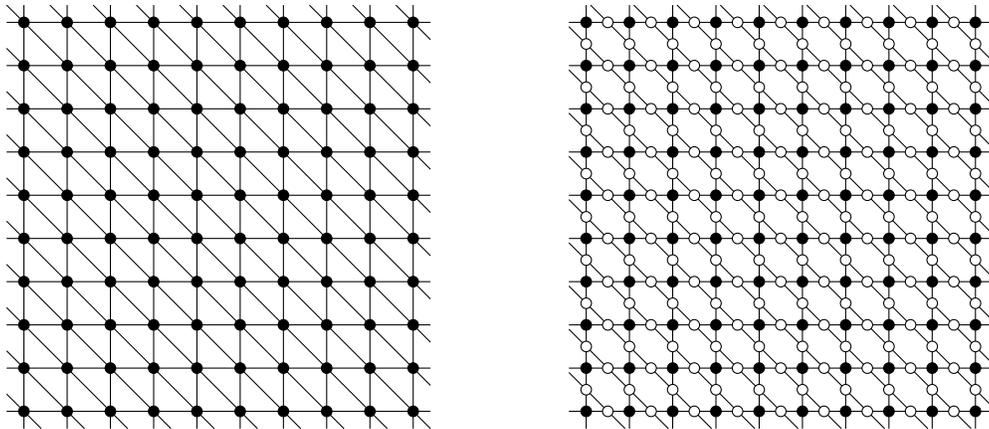
Figure 2: Portions of a triangular lattice and a Kagomé lattice

sides. Then the north and east midpoints, and the south and west midpoints, are joined by an edge. Thus a Kagomé lattice is 4-regular, every face is a triangle or a hexagon, and every edge belongs to exactly one triangle and one hexagon. (One can also think of a Kagomé lattice as the line graph of a honeycomb embedded on the torus.) The two lattices are illustrated in Figure 2.

In Theorem 2.2, we summarise what is currently known about the validity of the WSK algorithm for these lattices. As defined above, the WSK algorithm is valid if and only if $\mathcal{M}_q(G)$ is ergodic, and this is equivalent to the set of q-colourings of G forming a Kempe class. Thus, we are able to include new results implied by Theorem 1.1. Note that in this work we do not exhibit any bound on the length of the random walk required (which is called the mixing time or, informally speaking, the speed of convergence) by the WSK algorithm on these instances.

We require a definition. Let k be a positive integer. Then a *k-elimination ordering* of the vertices of G is an ordering such that each vertex is adjacent to at most k vertices later in the ordering. A graph is *k-degenerate* if there is a k-elimination ordering of its vertices, or, equivalently, if every induced subgraph has a vertex of degree at most k. We state a lemma of Las Vergnas and Meyniel [21] that we will need to prove Theorem 2.2.

Lemma 2.1 ([21]). *Let d and k be positive integers such that $d \geq k + 1$. If G is a k-degenerate graph, then C_G^d is a Kempe class.*

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Theorem 2.2. *For all $q \geq 6$, the WSK algorithm for q-colourings of any triangular lattice is valid. For all $q \leq 4$, there is a triangular lattice for which the WSK algorithm for q-colourings is not valid. For all $q \geq 4$, the WSK algorithm for q-colourings of any Kagomé lattice is valid. For all $q \leq 3$, there is a Kagomé lattice for which the WSK algorithm for q-colourings is not valid.*

Proof. Mohar and Salas showed in [26] that the WSK algorithm is not valid for some triangular lattices when $q \leq 4$. Theorem 1.1 ensures that when $q = 6$, as triangular lattices are 6-regular, the WSK algorithm is valid. For larger values of q, Lemma 2.1 ensures that the WSK algorithm is also valid.

Mohar and Salas proved in [27] that WSK algorithm is not valid when $q \leq 3$ for some Kagomé lattices. Theorem 1.1 ensures that when $q = 4$, as Kagomé lattices are 4-regular, the WSK algorithm is valid And, again, for larger values of q, we use Lemma 2.1. ∎

We observe that this leaves the single open case of triangular lattices with $q = 5$.

We note that in this section, we have discussed the validity of the WSK algorithm on lattices with boundary conditions. The validity of the WSK algorithm on lattices with free conditions (that is, lattices embedded in the plane) has already been studied by Mohar and Salas [26, 27].

3 Preliminaries

Let S be a subset of the vertex set of a graph G. Then $G[S]$ denotes the subgraph of G induced by S. Let k be a positive integer. We say that a *k-elimination ordering* of the vertices of G *ends in S* if the vertices of S are later in the ordering than all other vertices.

Lemma 3.1. *Let k be a positive integer. Let G be a graph, and let S be a subset of the vertex set of G. If G admits a k-elimination ordering that ends in S, then any $(k+1)$-colouring of $G[S]$ can be extended to a $(k+1)$-colouring of G.*

Let us refine this in a way that will prove useful.

Lemma 3.2. *Let k be a positive integer. Let $G = (V, E)$ be a graph, and let $S \subseteq V$, $|S| \leq k$, be a subset of the vertex set of G. Suppose that $G[V \setminus S]$*

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is connected, that the vertices of $V \setminus S$ each have degree at most k in G, and that there is a vertex $x \in V \setminus S$ of degree at most $k-1$ in G. Then any k-colouring of $G[S]$ can be extended to a k-colouring of G.

Proof. Let the vertices of $V \setminus S$ be ordered according to the order in which they are found by a breadth-first search from x. (Certainly the search finds all these vertices, as $G[V \setminus S]$ is connected.) Append the vertices of S to this ordering. This is certainly a $(k-1)$-elimination ordering of G, since x has at most $k-1$ neighbours in total, every other vertex in $V \setminus S$ has at most k neighbours, but at least one — the vertex from which it was discovered during the breadth-first search — is earlier in the ordering, and each vertex of S is followed in the ordering only by other vertices of S of which there are at most $k-1$. So, by Lemma 3.1 with $d = k-1$, the k-colouring of S can be extended to G. □

We need some known results.

Lemma 3.3 ([21]). *Let k be a positive integer. Let G_1, G_2 be two graphs such that $G_1 \cap G_2$ is a complete graph. If each of $C_{G_1}^k$ and $C_{G_2}^k$ is a Kempe class, then $C_{G_1 \cup G_2}^k$ is a Kempe class.*

We *identify* two non-adjacent vertices u and v in a graph G if we replace them by a new vertex adjacent to all neighbours of either u or v (or both) in G. The graph obtained is denoted G_{u+v}. In the proof of Theorem 1.1, we will often think about G_{u+v} when reasoning about the colourings of G. Let $C_G^k(u,v)$ denote the colourings of G for which u and v are coloured alike. We note that there is an obvious bijection between $C_G^k(u,v)$ and $C_{G_{u+v}}^k$.

Lemma 3.4 ([16]). *Let $k \geq 3$ be an integer. Let G be a 3-connected graph. Let u and v be non-adjacent vertices of G with a common neighbour. Then, if $C_G^k(u,v)$ is not empty, it is a Kempe class.*

We observe that Lemma 3.4 is [16, Lemma 9]. (In fact, [16, Lemma 9] states, equivalently, that any two colourings of $C_G^k(u,v)$ are Kempe equivalent.) The proof in [16] first establishes the following statement which is useful to state explicitly.

Lemma 3.5 ([16]). *Let $k \geq 3$ be an integer. Let G be a 3-connected graph of maximum degree k. Let u and v be non-adjacent vertices of G with a common neighbour. Then G_{u+v} is $(k-1)$-degenerate.*

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A list assignment of a graph $G = (V, E)$ is a function L with domain V such that, for each vertex $u \in V$, $L(u)$ is a set of colours. We say that G is *L-colourable* if there is a colouring of G where every vertex u is coloured with a colour of $L(u)$, and G is *degree-choosable* if it is L-colourable for any list assignment L where, for each vertex u in G, the length of the list $L(u)$ is equal to the degree of u. The *blocks* of a graph are its maximal 2-connected subgraphs. The following well-known fact is a special case of the characterization of degree-choosable connected graphs of Borodin [7] and Erdős et al. [14].

Lemma 3.6 ([7, 14]). *Let G be a connected graph. Then G is degree-choosable unless each block of G is a complete graph or an odd cycle.*

We require two more definitions. Given two sets S_1 and S_2 of vertices of G, we say that S_1 *dominates* S_2 if every vertex in S_2 is adjacent to at least one vertex in S_1. Additionally, S_1 *weakly dominates* S_2 if every vertex in S_2 is adjacent to exactly one vertex in S_1.

4   Proof of Theorem 1.1

We must show that, for $k \geq 4$, if G is a connected non-complete k-regular graph, then the set of k-colourings of G is a Kempe class. In Propositions 4.1, 4.7 and 4.8, we show that this claim holds, respectively, whenever G is not 3-connected, 3-connected with diameter at least 3, and 3-connected with diameter exactly 2. It is clear that taken together the propositions imply Theorem 1.1.

4.1   Graphs that are not 3-connected

We first prove that Theorem 1.1 holds when G is not 3-connected.

Proposition 4.1. *Let $k \geq 4$ be an integer. Let G be a connected k-regular graph that is not 3-connected. Then C_G^k is a Kempe class.*

Proof. Let S be a minimum size vertex cut of G that separates a connected component C_1 of $G - S$ from the rest of the graph C_2. Let $G_1 = G[C_1 \cup S]$ and $G_2 = G[C_2 \cup S]$. Note that both G_1 and G_2 are $(k-1)$-degenerate. Thus $C_{G_1}^k$ and $C_{G_2}^k$ are Kempe classes by Lemma 2.1.

If $G[S]$ is a clique, then, by Lemma 3.3, C_G^k is a Kempe class.

As G is not 3-connected, $|S| \leq 2$. So if $G[S]$ is not a clique, then $S = \{x, y\}$ where x and y are a pair of non-adjacent vertices. We can assume that one vertex of S has more than one neighbour in G_1, and the other has more than one neighbour in G_2. Suppose instead that, for example, x and y both have only one neighbour in G_1 (and so at least three neighbours in G_2). Note that we can instead, in this case, let S be the cut of size 2 containing y and the unique neighbour u of x in G_1. Now S does have the desired property if we consider how it separates $C_1 - \{u\}$ from $C_2 + \{x\}$: u has at least two neighbours distinct from x and y which are in $C_1 - \{u\}$, and y has at least three neighbours in $C_2 + \{x\}$. So we can assume, without loss of generality, that x has at least two neighbours in G_1, and y has at least two neighbours in G_2.

Let G'_1, G'_2 and G' be the graphs obtained from, respectively, G_1, G_2 and G by adding the edge xy. As x has degree at least 2 in G_1, it has degree at most $k - 2$ in G_2, and thus degree at most $k - 1$ in G'_2. Similarly y has degree at most $k - 1$ in G'_1. Hence G'_1 and G'_2 are $(k - 1)$-degenerate, and, by Lemma 2.1, $C^k_{G'_1}$ and $C^k_{G'_2}$ are Kempe classes. By Lemma 3.3, $C^k_{G'}$ is a Kempe class.

So the set of k-colourings of G in which x and y have distinct colours are all Kempe equivalent, since this is the set of k-colourings of G'. To prove that C^k_G is a Kempe class, it is enough to show that every k-colouring α of G such that $\alpha(x) = \alpha(y)$ is Kempe equivalent to a k-colouring where x and y are coloured differently. We will describe how to find a series of Kempe changes that, starting from α, give us a colouring in which x and y are not coloured alike.

We can assume that $\alpha(x) = \alpha(y) = 1$. If, for either x or y, there is a colour that does not appear on any vertex in its neighbourhood, then we can apply a trivial Kempe change to obtain the required colouring. So we assume that, under α, x and y each have a neighbour of each colour. Thus, for each of x and y, exactly one colour appears on two neighbours. We consider two cases.

Case 1: *Either x or y has at least two neighbours in each of G_1 and G_2.*

Let us assume that it is x that has at least two neighbours in both G_1 and G_2. There exist two colours — let us say 2 and 3 — such that no neighbour of x in G_1 is coloured 3, and no neighbour of x in G_2 is coloured 2. Consider the $(2, 3)$-components of G that include the neighbours of x coloured 2. Since they are wholly in G_1, they do not contain any neighbour of x coloured 3.

So in the colouring obtained by a Kempe change of these components, the vertex x has no neighbour coloured 2. Thus by one further trivial Kempe change of x, the required colouring is obtained.

Case 2: *Neither x nor y has at least two neighbours in each of G_1 and G_2.*

Since S is minimal, x has exactly one neighbour w in G_2, and y has exactly one neighbour z in G_1. Let $\alpha(w) = 2$. If $\alpha(z) \neq 2$, then consider the $(2, \alpha(z))$-component that contains z. From the Kempe change of this component (which does not contain x, y or w), we obtain a colouring where z is coloured 2. Thus we can as well assume that $\alpha(z) = 2$. Consider the $(1,3)$-component that contains x. As x has no neighbour coloured 3 in G_2, and y has no neighbour coloured 3 in G_1, this component does not contain y. Thus from the Kempe change of this component we obtain the required colouring. $\qquad\square$

4.2 3-connected graphs with diameter at least 3

We present a number of lemmas that will allow us to show that Theorem 1.1 is true for 3-connected graphs with diameter at least 3.

If two neighbours t_1 and t_2 of a vertex u are not adjacent, we say that (t_1, t_2) is an *eligible* pair of neighbours of u. Let $P(u)$ denote the set of eligible pairs of neighbours of u. We observe that in a regular connected non-complete graph, every vertex has an eligible pair of neighbours.

The next lemma follows from Lemma 3.4. (In fact, it is just a special case of Lemma 3.4, but it is helpful to have it as a separate statement.)

Lemma 4.2. *Let $k \geq 3$ be an integer. Let G be a 3-connected k-regular graph G. Let u be a vertex in G, and let (t_1, t_2) be an eligible pair in $P(u)$. Then $C_G^k(t_1, t_2)$ is a Kempe class.*

It is worth noting that as $G_{t_1 + t_2}$ is $(k-1)$-degenerate, by Lemma 3.5, it has a k-colouring. Thus $C_G^k(t_1, t_2)$ is non-empty.

Lemma 4.3. *Let $k \geq 4$ be an integer. Let G be a 3-connected k-regular graph. Let u and v be two vertices of G, and let (w_1, w_2) be an eligible pair in $P(v)$. If, for every eligible pair (t_1, t_2) in $P(u)$, there is a k-colouring of G such that t_1 and t_2 are coloured alike, and w_1 and w_2 are coloured alike, then C_G^k is a Kempe class.*

Proof. In a k-colouring of G at most $k - 1$ colours appear on the neighbours of u. Thus at least two of its neighbours, which must be an eligible pair, are coloured alike. That is, for every colouring α of G, there is an eligible pair (t_1, t_2) in $P(u)$ such that α belongs to $C_G^k(t_1, t_2)$. So

$$C_G^k = \bigcup_{(t_1,t_2) \in P(u)} C_G^k(t_1, t_2),$$

and, as each $C_G^k(t_1, t_2)$ is, by Lemma 4.2, a Kempe class, we have that C_G^k is a Kempe class if it contains a subset that is a Kempe class and intersects $C_G^k(t_1, t_2)$ for each $(t_1, t_2) \in P(u)$. The premise of the lemma is that $C_G^k(w_1, w_2)$ intersects each $C_G^k(t_1, t_2)$, and it is, by Lemma 4.2, a Kempe class. $\qquad\square$

So Lemma 4.3 suggests an approach to proving that Theorem 1.1 holds for 3-connected graphs. We note first that it will be easier to apply if we know that G has diameter at least 3, since then we can choose u and v such that their eligible pairs of neighbours are distinct. We just need to prove that we can find the types of k-colourings that the premise of the lemma requires. To do this we need a number of rather technical lemmas.

Lemma 4.4. *Let $k \geq 4$ be an integer. Let G be a k-regular 3-connected graph with a vertex cut S of size 3 such that one connected component C of $G - S$ is a clique on k vertices. Then C_G^k is a Kempe class.*

Proof. Every vertex of C has exactly one neighbour that is not in C, and so must therefore be in S. Thus S weakly dominates $V(C)$. As C has at least four vertices each adjacent to exactly one of the three vertices of S, we can assume that there is a vertex in S with at least two neighbours in C. Let this vertex be u. Let w_1 be a neighbour of u in C. Let w_2 be a neighbour of u not in $S \cup V(C)$. (If u does not have such neighbours, then $S \setminus \{u\}$ is a vertex cut, and G is not 3-connected.)

By Lemma 4.2, $C_G^k(w_1, w_2)$ is a Kempe class. Let α be a k-colouring of G. The lemma follows if we can show that α is Kempe equivalent to a colouring in $C_G^k(w_1, w_2)$; that is, if by performing a number of Kempe changes we can reach a colouring where w_1 and w_2 are coloured alike.

Let us assume that $\alpha(w_1) = 1$. If $\alpha(w_2) = 1$, we are done so assume that $\alpha(w_2) = 2$. Let w_3 be the vertex in C for which $\alpha(w_3) = 2$. (As C is a clique on k vertices every colour appears on exactly one vertex.)

If w_3 is a neighbour of u, then $\{w_1, w_3\}$ is a Kempe chain, and, by a single Kempe change, we obtain the required colouring. Otherwise suppose that the neighbour of w_3 in S is $v \neq u$. As u has at least two (distinctly coloured) neighbours in C, we can assume that there is a neighbour w_4 of u in C such that $\alpha(w_4) \neq \alpha(v)$ (possibly $w_4 = w_1$). Then $\{w_3, w_4\}$ is a Kempe chain. If we exchange the colours of this chain, then either $w_4 = w_1$ and we are done or, as before, we have two neighbours of u coloured 1 and 2 which form a Kempe chain, and one more Kempe change is needed to obtain the required colouring. □

At various points in the proofs of the following lemmas we will have defined a graph G with vertices u and v, and eligible pairs $(t_1, t_2) \in P(u)$ and $(w_1, w_2) \in P(v)$. Whenever this is the case, we will use the following definitions. Let G^+ be the graph obtained from G by identifying t_1 and t_2, and then identifying w_1 and w_2, and label the two vertices created t and w respectively. Let G^- be the graph obtained from G^+ by deleting t and w. (So G^- is the graph obtained from G by deleting t_1, t_2, w_1 and w_2.) The blocks of a graph G that contain at most one cut vertex of G are called *end blocks*, and all other blocks are called *intermediate blocks*.

Lemma 4.5. *Let $k \geq 4$ be an integer. Let G be a 3-connected k-regular graph. Let u and v be two vertices of G, and let (w_1, w_2) be an eligible pair in $P(v)$ such that neither of w_1, w_2 is adjacent to u. Suppose that C_G^k is not a Kempe class. Then there is an eligible pair (t_1, t_2) in $P(u)$ such that G contains a subgraph weakly dominated by both $\{t_1, t_2\}$ and $\{w_1, w_2\}$ that is isomorphic to K_{k-1}.*

Proof. As C_G^k is not a Kempe class, by Lemma 4.3 we know that we can choose (t_1, t_2) as an eligible pair in $P(u)$ such that there is no k-colouring of G where t_1 and t_2 are coloured alike, and w_1 and w_2 are coloured alike. We note that t_1, t_2, w_1 and w_2 are distinct, as the latter two are not adjacent to u. So here G^+ is well-defined, and, by our choice of t_1 and t_2, does not have a k-colouring. To prove the lemma, we attempt to construct a k-colouring of G^+, and use the fact that we know that we cannot succeed to lead us to the conclusion.

For a component C of G^-, let G_C^* be $G^+[C \cup \{t, w\}]$. For each C, we shall show that one of the following holds:

(1) the structure of G_C^* implies that G^+ has a k-colouring, or

(2) there is a k-colouring of G_C^* where t and w are coloured 1 and 2 respectively, or

(3) $G[C]$ contains a subgraph weakly dominated, in G, by both $\{t_1, t_2\}$ and $\{w_1, w_2\}$ that is isomorphic to K_{k-1}.

By the assumption that G^+ has no k-colouring, there cannot be any component that satisfies (1), and it cannot be the case that every component satisfies (2). Thus there must be at least one component that satisfies (3), and the lemma follows.

Case 1: There is a vertex x in C that has degree less than k in G_C^*.

We can find a k-colouring of G_C^* with t and w coloured with 1 and 2 by applying Lemma 3.2 to G_C^* and x with $S = \{t, w\}$. So C satisfies (2).

Case 2: Every vertex in C has degree k in G_C^*, and $G[C]$ is degree-chooseable.

We create a list assignment L for $G[C]$. For each vertex x in C, let

$$
L(x) = \begin{cases}
\{1, \ldots, k\} & \text{if } x \text{ is not adjacent to } t \text{ or } w, \\
\{2, \ldots, k\} & \text{if } x \text{ is adjacent to } t \text{ but not } w, \\
\{1, 3 \ldots, k\} & \text{if } x \text{ is adjacent to } w \text{ but not } t, \\
\{3 \ldots, k\} & \text{if } x \text{ is adjacent to both } t \text{ and } w.
\end{cases}
$$

Note that $|L(x)|$ is equal to the degree of x in $G[C]$, since $|L(x)| = k - |N_{G^+}(x) \cap \{t, w\}|$. As $G[C]$ is degree-chooseable, there is a colouring of $G[C]$ that respects L, and, as $1 \notin L(x)$ if x is adjacent to t, and $2 \notin L(x)$ if x is adjacent to w, this provides a k-colouring of G_C^* when t and w are coloured 1 and 2 respectively. Thus C satisfies (2).

Case 3: Every vertex in C has degree k in G_C^*, and $G[C]$ is not degree-chooseable.

By Lemma 3.6, each block of $G[C]$ is either a clique or an odd cycle. For an end block B of $G[C]$, let B^- be the vertices of B that are not cutvertices in $G[C]$. (So B^- contains one fewer vertex than B unless $G[C]$ contains only one block, and then $B^- = B$.) The degree of each vertex of B^- in G_C^* is k, and this is the sum of the number of neighbours it has in C and the number of neighbours it has in $\{t, w\}$. As the former is the same for each vertex (as they belong to just one block that is a cycle or a clique), the latter must also be the same for each vertex. So let $d_B \in \{0, 1, 2\}$ be the number of neighbours in $\{t, w\}$ of each vertex of B^-.

Case 3.1: There is an end block B of C with $d_B = 0$.

This implies that each vertex of B^- is joined to k vertices in C which, as $k \geq 4$, implies that B is a clique rather than a cycle, and so B is isomorphic to K_{k+1}, contradicting that G is connected and non-complete.

Case 3.2: There is an end block B of C with $d_B = 1$.

Note that B must be a clique; if it were an odd cycle the degree of each vertex of B^- in G_C^* would be $3 \neq k$.

Suppose that every vertex in B^- is adjacent to t. (The case where they are all adjacent to w is equivalent.) We cannot have $B = B^-$, since then t is a cutvertex, and so $\{t_1, t_2\}$ is a cutset in G, contradicting that it is 3-connected. So let x be the cutvertex of $G[C]$ in B. Then x has exactly one neighbour s in $C \setminus B^-$. Thus $\{s, t_1, t_2\}$ is a cutset of G that weakly dominates B which is a clique on k vertices. Therefore C_G^k is a Kempe class by Lemma 4.4; a contradiction.

So there must be vertices y and z in B^- such that y is adjacent to t (but not w), and z is adjacent to w (but not t). We show that we can colour t and w with 1 and 2, and extend this to a k-colouring of G_C^*. First colour z with 1. Note that a vertex h other than y and z in B^- has degree less than k in $G_C^* \setminus \{y\}$. So we can apply Lemma 3.2 to $G_C^* \setminus \{y\}$ with $S = \{t, w, z\}$ and $x = h$. (If B^- does not contain three vertices, then the degree of y and z in G_C^* is at most $3 < k$.) Finally colour y, which is possible as two of its neighbours are coloured alike. Thus C satisfies (2).

For the remaining cases, we will need the following claim.

Claim 1. *If u and v are not in C, then*

A. *each of t and w is adjacent to at most $2k - 2$ vertices in C,*
B. *one of t and w is adjacent to at most $2k - 3$ vertices in C,*
C. *if each of t and w has at least $2k - 3$ neighbours in C, then t is not adjacent to w, and*
D. *if the sum of the number of neighbours of t and w in C is at least $4k - 6$, then $G^+[V \setminus C]$ has a k-colouring in which t and w are coloured alike.*

We note that this claim can be applied within Case 3; we know that every vertex in C has degree k in G_C^*, and u and v have degree less than k, since a pair of neighbours — t_1 and t_2 or w_1 or w_2 — were identified when G^+ was formed from G. We prove each part of the claim. (We give a proof only for the statement about t when the argument for w is equivalent.) We keep in mind that, for each edge incident with t in G^+, there is a corresponding edge or edges incident with t_1 or t_2 in G.

A. The total number of edges incident with t_1 and t_2 in G is $2k$, but two of these edges are incident with u which is not in C.

B. If t and w each have $2k - 2$ neighbours in C, then in G, each of t_1, t_2, w_1 and w_2 only has neighbours in $C \cup \{u, v\}$. As u has at least four neighbours and is not adjacent to any vertex in $C \cup \{w_1, w_2\}$, it has at least one neighbour in $G^\dagger = G \backslash (C \cup \{u, v, t_1, t_2, w_1, w_2\})$. Thus $\{u, v\}$ is a cutset, because it separates $G^\dagger$ from $C \cup \{t_1, t_2, w_1, w_2\}$. This contradicts that G is 3-connected.

C. If t and w both have $2k - 3$ neighbours in C, and are adjacent, then, in G, each of t_1, t_2, w_1 and w_2 only has neighbours in $C \cup \{u, v, t_1, t_2, w_1, w_2\}$. As in the previous part, this implies that $\{u, v\}$ is a cutset.

D. We can say that t and w are not adjacent: either one of t and w has $2k - 2$ neighbours in C so its only other neighbour is either u or v, or they both have $2k - 3$ neighbours in C, and so we can apply the previous part of the claim. In G, there are at least $4k - 6$ edges from $\{t_1, t_2, w_1, w_2\}$ to the vertices of C so at most 6 other incident edges. And, as t_1 and t_2 are both adjacent to u, and w_1 and w_2 are both adjacent to v, in $G^+[V \backslash C]$ the sum of the degrees of t and w is at most 4. Let $G^\dagger$ be the graph formed from $G^+[V \backslash C]$ by identifying t and w to form a new vertex with degree at most 4. Thus every vertex in $G^\dagger$ has degree at most k, and the graph is not isomorphic to K_{k+1}, since u, for example, has degree less than k. So, by Brooks' Theorem, $G^\dagger$ has a k-colouring. From this colouring, we can obtain a colouring of $G^+[V \backslash C]$ in which t and w are coloured alike. This completes the proof of the claim.

Case 3.3: For every end block B of C, $d_B = 2$, and there is one end block B_1 that is not a clique.

So B_1 is an odd cycle on at least five vertices. In G_C^*, each vertex of B_1^- has degree k, and is adjacent to two vertices in B_1 and t and w. Hence we have $k = 4$. If either B_1 has more than five vertices or C has more than one end block, then there are at least six vertices in end blocks that are not cutvertices of C. So these vertices are adjacent to both t and w, which therefore each have at least $6 = 2k - 2$ neighbours in C, contradicting Claim 1.B. So $C = B_1$ is a 5-cycle, and the sum of the number of neighbours of t and w in C is $10 = 4k - 6$. Thus, by Claim 1.D, $G^+[V \backslash C]$ has a 4-colouring in which t and w are coloured alike. We can extend this colouring to the whole of G^+ by using the other 3 colours on B_1. So C satisfies (1).

Case 3.4: For every end block B of C, $d_B = 2$, and B is a clique.

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Notice that each end block is isomorphic to K_{k-1}. If there is only one end block, then, as it is weakly dominated by $\{t_1, t_2\}$ and $\{w_1, w_2\}$ in G, C satisfies (3). If there are at least three end blocks, then there are at least $3(k-2)$ vertices in C adjacent to both t and w. As, for $k \geq 4$, we have $3k - 6 \geq 2k - 2$, this contradicts Claim 1.B.

So we can assume that C has exactly two end blocks each isomorphic to K_{k-1}. Note that an intermediate block B of C that is a clique on more than two vertices has vertices (the ones that are not cutvertices in $G[C]$) whose k neighbours are each either in B or in $\{t, w\}$. In fact, at least one neighbour must be in $\{t, w\}$, else B is isomorphic to K_{k+1}, and not connected to $V \setminus B$. Therefore B is isomorphic to either K_{k-1} or K_k.

Case 3.4.1: $k \geq 5$.

No block is an odd cycle, since the vertices that are not cutvertices in the cycle would have degree at most 4 in G_C^*. So the blocks of C are each isomorphic to K_2, K_{k-1} or K_k, and, for each cutvertex, one of the two blocks it belongs to must be K_2 else it would have degree at least $2(k-2) > k$. Thus the cutvertex of each end block is also adjacent to one of t and w. So there are $4k - 6$ edges from t and w to vertices of the two end blocks. If there is an intermediate block that is isomorphic to K_{k-1} or K_k, then it contains at least two vertices that are not cutvertices, and these are also joined to at least one of t and w. So the sum of the number of neighbours of t and w in C is at least $4k - 4$; a contradiction to the first two parts of Claim 1. Therefore the only intermediate block is K_2, and there is exactly one of these. (If there are none, the two end blocks intersect, and the cutvertex has degree more than k; if there is more than 1, there are vertices that in $G[C]$ have degree 2 so have degree at most 4 in G_C^*.) So $G[C]$ contains two disjoint cliques each isomorphic to K_{k-1} joined by a single edge. Thus the sum of the number of neighbours of t and w in C is exactly $4k - 6$, and we can assume, by Claim 1.D, that $G^+[V \setminus C]$ has a k-colouring in which t and w are coloured alike. This can be extended to a colouring of G^+, because $G[C]$ is easily seen to be $(k-1)$-colourable. So C satisfies (1).

Case 3.4.2: $k = 4$.

Let the two end blocks be B_1 and B_2; both are isomorphic to K_3. If they intersect in a vertex, then we can colour t and w with 1 and 2, colour the vertex in both B_1 and B_2 with 1, and the other vertices with 3 and 4. So C satisfies (2).

For the remaining cases, we note that Claim 1.D says that if there are at

least 10 edges joining t and w to C we can assume they are coloured alike in a 4-colouring of $G^+[V \setminus C]$. And Claim 1.A and 1.B say that there cannot be more than 11 edges from t and w to C.

If $G[C]$ is B_1 and B_2 plus an edge between them, then there are 10 edges from t and w to C, and clearly $G[C]$ is 3-colourable so C satisfies (1) by Claim 1.D.

Suppose that $G[C]$ contains more blocks than B_1, B_2 and an additional K_2. If C does not contain a K_4, then either

- there is a block isomorphic to K_3 or a longer odd cycle, and this block contains a vertex x that is not a cutvertex, or

- there is a cutvertex x that belongs to two blocks both isomorphic to K_2.

In both cases, x must be joined to both t and w, which are therefore again joined by at least 10 edges to C, and, as there is no K_4, $G[C]$ is 3-colourable, and, by Claim 1.D, C again satisfies (1).

If C does contain a K_4, then the two vertices that are not cutvertices are both incident to one of t and w. The cutvertices in B_1 and B_2 are each either adjacent to one of t or w, or belong to a K_3 or a longer odd cycle that contains a vertex adjacent to both t and w. In any case, t and w are incident to at least 12 edges joining them to C, and we have a contradiction. $\qquad\square$

Lemma 4.6. *Let $k \geq 4$ be an integer. Let G be a 3-connected k-regular graph. Let u and v be two vertices of G that are not adjacent. Let (w_1, w_2) be an eligible pair in $P(v)$ neither of which is adjacent to u. Then C_G^k is a Kempe class.*

Proof. If C_G^k is not a Kempe class, then, by Lemma 4.5, there is an eligible pair (t_1, t_2) in $P(u)$ such that G contains an induced subgraph isomorphic to K_{k-1} that is weakly dominated by both $\{t_1, t_2\}$ and $\{w_1, w_2\}$. Let C be the vertex set of this induced subgraph. Note that each vertex in C is adjacent to the other $k-2$ vertices of C, to one of $\{t_1, t_2\}$, and to one of $\{w_1, w_2\}$, and so is not adjacent to u or v. Neither of u and v can be in C, as they are each adjacent to both of the vertices in either $\{t_1, t_2\}$ or $\{w_1, w_2\}$. We know that at least three of the vertices of $\{t_1, t_2, w_1, w_2\}$ have a neighbour in C, since otherwise G is not 3-connected. In fact, we claim that every vertex in $\{t_1, t_2, w_1, w_2\}$ has a neighbour in C. Suppose instead that exactly one of them, say t_1, has no neighbour in C. Then $C \cup t_2$ would induce a clique

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on k vertices that is weakly dominated by $\{u, w_1, w_2\}$, since every vertex in C is adjacent to one of $\{w_1, w_2\}$ but not to u, and t_2 is adjacent to u but, considering its degree, to neither of $\{w_1, w_2\}$. Hence Lemma 4.4 with $S = \{u, w_1, w_2\}$ is contradicted.

Assume, without loss of generality, that w_1 has at least as many neighbours in C as w_2. Let x be a neighbour of w_1 in C, and assume, again without loss of generality, that x is also a neighbour of t_1. Then (x, v) is an eligible pair in $P(w_1)$. We apply Lemma 4.5 to u, w_1 and (x, v). So, under the assumption that C_G^k is not a Kempe class, there is a pair (t_3, t_4) (not necessarily distinct from (t_1, t_2)) of eligible neighbours in $P(u)$ such that G contains an induced subgraph isomorphic to K_{k-1} that is weakly dominated by both $\{t_3, t_4\}$ and $\{x, v\}$. Let C' be the vertex set of this induced subgraph. Arguing as we did for C, we can assume that each of $\{t_3, t_4, x, v\}$ is adjacent to C'.

Suppose that neither t_1 nor t_2 belongs to C'. The k neighbours of x are $C \setminus \{x\} \cup \{t_1, w_1\}$, and we know at least one of these vertices is in C'. By definition, w_1 is not in C'. By assumption, nor is t_1 in C'. So there is a vertex $y \neq x$ that belongs to both C and C'. As C' induces a clique, the other $k-2$ vertices of C' are neighbours of y. But, as none of $\{t_1, t_2, w_1, x\}$ are in C', we must have that C' is $C \setminus \{x\} \cup \{w_2\}$. So w_2 is adjacent to every vertex of C except x. By our assumption that w_1 has at least as many neighbours as w_2 in C, we have that C has only two vertices, and so $k = 3$. This contradiction tells us that, in fact, at least one of t_1 and t_2 belongs to C'; let us assume it is t_1.

So t_1 has $k-2$ neighbours in C'. It has two more neighbours: we know it must be adjacent to one of $\{v, x\}$ by the definition of C', and we know that it is also adjacent to u. But neither of t_3 and t_4 belongs to $C' \cup \{u, v, x\}$ so t_1 is not adjacent to either of them. This contradicts the definition of C', and completes the proof. □

We can now conclude this subsection on graphs of diameter at least 3.

Proposition 4.7. *Let $k \geq 4$ be an integer. Let G be a 3-connected k-regular graph with diameter at least 3. Then $C_G^{\prime k}$ is a Kempe class.*

Proof. Let u and v be two vertices in G at distance at least 3. As G has diameter 3, it is not a clique, and so, as it is regular, v has a pair of non-adjacent neighbours, an eligible pair. As no neighbour of v is adjacent to u (they must be at distance at least 2), the result follows from Lemma 4.6. □

4.3   3-connected graphs with diameter 2

To complete the proof of Theorem 1.1, it only remains to consider 3-connected graphs of diameter 2.

First we need two definitions. For a vertex v in a graph G, we denote by $N(v)$ the *neighbourhood* of v, that is, the set of vertices adjacent to v. The *second neighbourhood* of v is the set of vertices at distance 2 from v in G.

Proposition 4.8. *Let $k \geq 4$ be an integer. Let G be a 3-connected k-regular graph of diameter 2. Then C_G^k is a Kempe class.*

Proof. If there are three vertices in the second neighbourhood of a vertex v that induce a path, then the proposition follows immediately from Lemma 4.6. Therefore we can assume that the second neighbourhood of each vertex induces a disjoint union of cliques.

Assume that there is a vertex v whose second neighbourhood includes two sets of vertices C_1 and C_2 that each induce a clique, and that there are no edges between the two sets. Let x and y be vertices of C_1 and C_2 respectively. If x is adjacent to a neighbour z of v that is not adjacent to y, then the second neighbourhood of y contains an induced path on v, z and x, and, again, we are done by Lemma 4.6. Thus, by symmetry, the intersections of each of the neighbourhoods of x and y with $N(v)$ are the same, and, repeating the argument, we must have that every vertex of C_1 and C_2 has the same set of neighbours within $N(v)$. Let α be a k-colouring of G. Suppose that $\alpha(x) = 1$, and $\alpha(y) = 2$. Note that the $(1, 2)$-component that contains x contains only vertices of C_1. Exchange the colours on this $(1, 2)$-component, and let β be the resulting colouring. So $\beta(x) = \beta(y) = 2$. Thus, from any k-colouring, we can obtain by a single Kempe change a colouring in $C_G^k(x, y)$. The proposition follows from Lemma 3.4.

Therefore we can assume that the second neighbourhood of each vertex induces a single clique. Let α and β be two k-colourings of G. Let v be a vertex, and let us denote by C the second neighbourhood of v. Up to a Kempe change, we can assume that $\alpha(v) = \beta(v) = 1$. To complete the proof, we assume that α and β are not Kempe equivalent, and show that this leads to a contradiction.

Claim 2. *Neither α nor β is Kempe equivalent to a colouring γ such that $\gamma(v) = 1$, and the colour 1 is not used on C.*

Suppose that there is such a colouring γ that is Kempe equivalent to, say, α. Let x be the vertex in C with $\beta(x) = 1$ if such a vertex exists;

otherwise let x be any vertex in C. In γ, v is the only vertex in G coloured 1, since certainly there is no vertex in $N(v)$ coloured 1. So we can apply a trivial Kempe change to x from γ to obtain a colouring γ' where $\gamma'(x) = 1$. If no vertex in β is coloured 1, then we can use the same argument; that is, apply a trivial Kempe change to x to obtain a colouring where x is coloured 1. So we may as well assume that $\beta(x) = 1$, and thus, as v and x are coloured 1 in both γ' and β, we have, by Lemma 3.4, that γ' and β, and so also α and β, are Kempe equivalent; a contradiction that proves the claim.

One thing that Claim 2 tells us is that α and β are colourings where the colour 1 is used on C. So let u and w be vertices in C such that $\alpha(u) = 1$ and $\beta(w) = 1$. If $u = w$, then Lemma 3.4 implies that α and β are Kempe equivalent. So, by assumption, we have that $u \neq w$.

One more definition is required: given a colouring γ, a vertex x is *locked* if all the colours distinct from $\gamma(x)$ appear in its neighbourhood. Notice that if x is not locked, then we can apply a trivial Kempe change to x from γ.

Claim 3. *Each vertex in $u \cup N(u) \setminus w$ is locked in α. Moreover, only colour $\alpha(w)$ appears twice in the neighbourhood of u.*

First consider the $(1, \alpha(w))$-component of α containing u and w. If this component does not contain v, then the Kempe change of this component from α gives us a colouring in which w and v are both coloured 1. By Lemma 3.4, this colouring is Kempe equivalent to β, a contradiction. Thus v must be in the $(1, \alpha(w))$-component. Since no other neighbour of w distinct from u is coloured 1 (every vertex in G is a neighbour of v or u), another neighbour y of u must be coloured with $\alpha(w)$. If u is not locked, then a trivial Kempe change of u gives us a colouring in which 1 is not used on C, contradicting Claim 2. Thus all the colours appear exactly once on the neighbourhood of u except colour $\alpha(w)$ which appears twice. If y is not locked, then a trivial Kempe change of y returns us to the case where the $(1, \alpha(w))$-component of α containing u and w does not contain v. And if a neighbour z of u not in $\{w, y\}$ is not locked, then a trivial Kempe change of z returns us to the case where u is not locked. The claim is proved.

Case 1: $|C| \geq 3$.

Let $z \in C \setminus \{u, w\}$. Clearly u is the unique neighbour of z coloured with 1 in α, since, again, every in G is a neighbour of v or u. Similarly w is the unique neighbour of z coloured with 1 in β. By Claim 3, z is the unique neighbour of u coloured $\alpha(z)$, and so $\{u, z\}$ is a Kempe chain in α. Similarly, noting

that Claim 3 also holds for β with the roles of u and w interchanged, $\{w, z\}$ is a Kempe chain in β. By exchanging the colours on these Kempe chains, we obtain two colourings where v and z are each coloured 1. Lemma 3.4 then implies that α and β are Kempe equivalent, a contradiction.

Case 2: $|C| = 2$.

So G contains v, its k neighbours, and u and w. Each of u and w are adjacent to all but one of the neighbours of v, so at least $k - 2$ of the neighbours of v are adjacent to both u and w; let this set of neighbours be denoted S. By Claim 3, in α a common neighbour z of u and v is coloured $\alpha(w)$, so it follows that S contains exactly $k - 2$ vertices, because it cannot contain z. As each vertex of S is locked in α by Claim 3 and has two neighbours, u and v, coloured 1, they each have exactly one neighbour of each other colour. Thus, as w and z are coloured alike, and every vertex in S is adjacent to w, no vertex in S is adjacent to z. But then the only vertices that can be adjacent to z are u, v and the other neighbour of v that is not in S, which contradicts that $k \geq 4$. This completes Case 2, and the proof of the proposition. $\square$

5   Final Remarks

Let us note an immediate corollary of the results on regular graphs.

Corollary 5.1. *Let d and k be integers, $d \geq k \geq 3$, and let G be a connected graph with maximum degree k. If G has a d-colouring, then C_G^d is a Kempe class unless $d = k = 3$, and G is the triangular prism.*

Proof. A connected graph with maximum degree k that is k-colourable is either k-regular, or $(k - 1)$-degenerate but not complete. The corollary thus follows from Theorem 1.1, [16, Theorem 1] and Lemma 2.1, which prove the result for, respectively, regular graphs of degree greater than 3, 3-regular graphs, and $(k - 1)$-degenerate graphs. $\square$

It would also be interesting to find an upper bound on the diameter of $\mathcal{K}_k(G)$ for graphs of bounded maximum degree. In the case where the edge relation of the reconfiguration graph represents trivial Kempe changes, this question has been well-studied; for example, see [3, 4, 8, 13, 15]. Notice that from the proofs of Theorem 1.1 and [16, Theorem 1], it suffices to determine an upper bound on the diameter of $\mathcal{K}_{k+1}(H)$, where $k \geq 2$ and H is a k-degenerate graph, and we note that a trivial upper bound of $O(k^{|V(H)|})$ can be

easily derived from the proof of Lemma 2.1. We make the following conjecture in light of Cereceda's conjecture [10] on the diameter of reconfiguration graphs of colourings of graphs with bounded degeneracy.

Conjecture 5.1. *Let d and k be positive integers such that $d \geq k+1$, and let G be a k-degenerate graph on n vertices. Then $\mathcal{K}_d(G)$ has diameter $O(n^2)$.*

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