

# On the Timing of Production Decisions in Monetary Economies\*

Nejat Anbarci<sup>†</sup>  
Durham University

Richard Dutu<sup>‡</sup>  
Université Paris-Dauphine,  
PSL Research University, LEDa.

Ching-Jen Sun<sup>§</sup>  
Deakin University

May 22, 2018

## Abstract

In most macroeconomic models inflation tends to be harmful. In this paper we show that by simply changing the timing of production decisions by firms from 'on demand' to 'in advance', some inflation can boost welfare as long as goods are sufficiently perishable. The main conclusion from this research is that by effectively hiding the strategic interaction between supply and demand, assuming production on demand is not without loss of generality.

**Keywords:** Timing, Perishability, Production, Money, Inflation, Search.

**JEL Classification:** C7, D2, E4.

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\*We would like to thank the editor Guido Menzio as well as two anonymous referees for insightful comments. We also thank Pedro Gomis-Porqueras, Antoine Goujard, Nicolas Jacquet, Ian King, Yiting Li, Aaron Nicholas, Xuan Nguyen, Pasquale Sgro, Bertrand Villeneuve, Neil Wallace, three anonymous referees and the Editor for many interesting comments and suggestions. We also thank participants at the University of Cergy-Pontoise, Melbourne, Paris Dauphine, New South Wales, Deakin University, Monash University, Academia Sinica, National Taiwan University, National Taipei University, GRIPS in Tokyo, Singapore Management University as well as various conferences. All errors are our own.

<sup>†</sup>Durham University Business School, Durham University, Durham DH1 3LB, UK. E-mail: nejat.anbarci@durham.ac.uk.

<sup>‡</sup>Corresponding author. Economics Department, Université Paris-Dauphine, PSL Research University, LEDa, 75016 Paris, France. Tel: +33 1 44 05 44 46. Email: richard.dutu@dauphine.fr.

<sup>§</sup>Department of Economics, Deakin University, 70 Elgar Road, Burwood Victoria 3125, Melbourne, Australia. E-mail: ching-jen.sun@deakin.edu.au.

“All production is for the purpose of ultimately satisfying a consumer. Time usually elapses, however—and sometimes much time—between the incurring of costs by the producer and the purchase of the output by the ultimate consumer. Meanwhile the entrepreneur has to form the best expectations he can as to what the consumer will be prepared to pay when he is ready to supply them after the elapse of what may be a lengthy period; and he has no choice but to be guided by these expectations, if he is to produce at all by processes which occupy time.”

John M. Keynes, *The General Theory of Employment, Interest and Money*.

## 1 Introduction

We investigate search-based models of monetary exchange along the lines of Lagos and Wright (2005), henceforth LW, but in contrast to the majority of papers in this literature, we assume that sellers produce *ex ante*, i.e. in advance, rather than *ex post*, i.e. on demand.

In LW’s model, buyers choose cash holdings first and then sellers produce on demand. Their economy can then be described as a sequential game. In that economy buyers face a holdup problem due to their up-front investment in cash, resulting in lower real balances and output (unless the buyer has all bargaining power). If, in that environment, firms are to produce in advance, both sides of the market now move simultaneously and independently: firms chose production at the same time households chose money holdings. This turns the economy into a simultaneous game, with a double holdup problem due to sellers investing in output and buyers investing in cash prior to any meeting. An equilibrium is then given by the intersection between two best-response functions, that of firms taking households’ spending plans as given and that of households taking firms’ supply decisions as given. This opens the door to strategic interactions between supply and demand, and to multiple equilibria and thus strategic uncertainty.<sup>1</sup>

Assuming production in advance, rather than on demand, raises the issue of unsold output. One avenue, followed in the DSGE literature, is to keep track of inventories over time. Their effect on the business cycle can be significant (e.g. Bils and Kahn 2000) as was illustrated in the first few months of the last financial crisis (see "The inventory cycle: Stocking filler", *The Economist*, July 8th 2010). Here, firms also hold inventories, but only for a limited time during which the goods produced depreciate more or less quickly. To do so, we follow Berentsen, Menzio and Wright (2011) and assume that a fraction of any unsold output can be sold next period. If that fraction is high, the good is said to be rather durable (e.g. household appliances). If the fraction is low, the good is said to be rather perishable (e.g. many food items). If the fraction is zero, then goods are fully perishable, as in LW.

Our main contribution is to show that different timings of production have different impli-

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<sup>1</sup>Production on demand is not specific to money-search models. The canonical New Keynesian model also has firms producing on demand for instance.

cations for the nature of the equilibrium, the effect of inflation, and optimal monetary policy. In particular we show that, everything else equal, an economy producing durable goods on demand does not need the same level of inflation as an economy producing perishable goods in advance. The former needs the Friedman rule, the later needs some inflation. Moreover, the more perishable the goods, the higher the optimal inflation rate in that economy.

How can inflation be beneficial? For that to happen, the goods must be produced in advance. This implies that sellers cannot adjust output according to the amount of money brought by the buyer. Second, the goods must be perishable. This implies that leftover output cannot have much resale value to producers. In particular the cost of not meeting a buyer can be fairly high for the seller since he would then lose most of his output. In this environment sellers play a mixed strategy randomizing between two levels of output: a small output sold entirely to the buyer, or a larger output a significant part of which is sold to the buyer. While the seller is indifferent between the two options (since higher production costs in the second option are compensated with leftovers he can sell or consume in the next market), the buyer prefers the second where he consumes more. When inflation goes up in that environment, if the buyer cannot find a trading partner, he is left with rapidly depreciating money. In LW this translates into sellers being willing to produce less, which lowers buyers' demand for real balances. Here, on the other hand, sellers have already figured out their two optimal levels of output, and the only way for sellers to prevent buyers from walking away is to increase the probability with which they pick the high output. It follows that, as inflation rises, both output and welfare increase *in expectation*.

Given the rather dramatic effect of a change in the timing of production decisions by firms, our next step will be to explore its effects quantitatively. In particular, we would like to know how the figures for the costs of inflation found in LW change when firms shift from production on demand, as in their model, to production in advance, as in here. To do so, we calibrate a version of our model with production in advance to the US economy and compute the welfare effect of 10% inflation relative to 0% inflation. We then reuse the parameters obtained from this calibration to compute the welfare effect of inflation in Lagos and Wright (2005)'s economy which only differs with regard to the timing of production decisions. Importantly we set perishability very high in the production-in-advance economy, at 95%. By doing so we make sure that the only difference between the two economies is the timing of production decision since goods are (nearly) fully perishable in the production-in advance economy, thereby approximating LW where they fully are. While 10% inflation reduces consumption by 2.81% in LW, consumption increases by 2.49% in the same economy when goods are produced in advance instead (and using the same parameters).

Our paper is not the first one to find that the Friedman rule is not always optimal. Nominal rigidities in the New Keynesian framework make price stability preferable to deflation. Inflation itself can boost GDP by inducing agents to search more (Benabou 1988, 1992, Head and Kumar

2005), by reducing the negative externality coming from one side of the market being too large (Shi 1997, Rocheteau and Wright 2005), or by forcing buyers to be less choosy—the hot potato effect as in Li (1994), Ennis (2008) and Nosal (2011). Inflation can also increase welfare by indirectly taxing monopolies’ rents (Schmitt-Grohe and Uribe 2004, Chugh 2006), a literature initiated by Phelps (1973), or by providing partial insurance to cash-poor agents (Levine 1991, Molicco 2006). To our knowledge, however, the channel unveiled here has not been studied before.

The key to unveiling this effect is to re-visit the timing of production decisions by firms. While production planning has become a field of its own in the business literature (known as *Supply Chain Management*), it has received little attention in economics. A small group of papers in game theory and experimental economics allows suppliers to choose between production on demand and production in advance (Maskin, 1986, Philips et al., 2001, Tasnádi, 2004), but they do not consider possible macroeconomic or policy implications of such change. The few papers that do so, i.e. Jafarey and Masters (2003) and Dutu and Julien (2008), use search-theoretic models of the second generation with indivisible money as in Shi (1995) and Trejos and Wright (1995), which limits their applicability. Production in advance was recently studied by Masters (2013) in a model of (imperfectly) directed search with divisible money where buyers’ preferences are match-specific and private information. He shows in particular that, when the upper bound on the number of participating sellers binds, moderate levels of inflation can increase welfare by making buyers less choosy. While production in advance does play a role in his result, it is not due to strategic interaction but to a more classic ‘hot potato’ effect. The strategic interaction we highlight, which is central to the non-optimality of the Friedman rule, comes from the random matching and bargaining with prior production environment that we use. Masters (2013) works with a price posting model where sellers, even though they produce ahead of the market, can post complete contracts. Having worked out a price posting version of our model, we find no role for strategic interaction in such environment, and then no role for inflation.

The paper is organized as follows. In Section 2 we lay out the general production-in-advance environment. In Section 3 we characterize the equilibria, efficiency and optimal monetary policy. In Section 4 we calibrate the model to measure the costs and benefits of inflation and contrast our findings with those in LW using the same parameter values. Section 5 uses lotteries as a way to circumvent the indivisibility of goods at the trading stage. Section 6 concludes.

## 2 The Environment

The backbone of this work is the search and matching model of money developed by LW. Time is discrete. Every period is divided into two trading subperiods, each with its own market: a frictional market in the first subperiod in which agents trade a first type of good called the

search good, and a Walrasian (centralized) market in the second subperiod where agents trade a different good called the general good.

There is a  $[0, 1]$  continuum of infinitely-lived agents who discount at rate  $\beta$  between periods. In the Walrasian market all agents can produce any quantity  $\bar{x}$  of the general good at cost  $\varphi(\bar{x}) = \bar{x}$ . They can also consume any quantity  $\hat{x}$ , which yields a utility  $v(\hat{x})$  with  $v' > 0$  and  $v'' < 0$ . In the frictional market, some agents called buyers can only consume the search good, and some agents called sellers can only produce the search good. Consuming  $\hat{q}$  units yields buyers a utility  $u(\hat{q}) > 0$  but 0 to sellers, with  $u' > 0$ ,  $u'' < 0$  and  $u(0) = 0$ . Similarly, producing  $\bar{q}$  units of the search good in the frictional market costs  $c(\bar{q}) < \infty$  to sellers with  $c' > 0$  and  $c'' > 0$ , but  $\infty$  to buyers. The two-subperiod utility function of a buyer is then  $U^b = v(\hat{x}) - \bar{x} + \beta u(\hat{q})$  and that of a seller is  $U^s = v(\hat{x}) - \bar{x} - \beta c(\bar{q})$ . We denote  $\hat{x}^*$  such that  $v'(\hat{x}^*) = \varphi'(\hat{x}^*) = 1$ .<sup>2</sup>

In Walrasian markets production occurs once equilibrium is reached. As for the frictional market, we assume that production takes place in advance, that is sellers produce at the beginning of the frictional market without knowing whether they will meet a buyer or what demand will be. We denote  $\alpha \in (0, 1)$  the probability with which a buyer meets a seller and there is a single coincidence of wants. Similarly we denote  $\sigma \in (0, 1)$  the probability with which a seller meets a buyer and there is a single coincidence of wants.

As in LW, we assume that the general good does not survive beyond its market, i.e. all unsold general good output fully perishes at the end of the Walrasian market, which is also the last market of the period. However, we amend their model by following Berentsen, Menzio and Wright (2011) and assuming that the search good output produced during the frictional market is partially durable in the following sense: for every unit of unsold search good at the end of the frictional market, a fraction  $1 - \delta$  of it is transformed into the general good and carried forward to the centralized market. For instance, if a seller produces  $\bar{q}$  of the search good and sells  $\hat{q} < \bar{q}$ , he will be able to bring  $y = (1 - \delta)(\bar{q} - \hat{q})$  in the form of general good to the Walrasian market where he can sell it. If  $\delta = 1$ , any unsold search good output is lost. If  $\delta = 0$ , all unsold search good output is transformed into the general good. Parameter  $\delta \in (0, 1)$  is then meant to capture the durability (or use value) of the output produced by sellers: the higher  $\delta$ , the more perishable the good.<sup>3</sup>

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<sup>2</sup>The numbers of buyers and sellers are fixed in this paper. As shown by Rocheteau and Wright (2005), participation decisions can be important. For instance, in the bargaining model free entry by sellers produces strategic interaction with money demand from buyers. This translates into multiple equilibria which, interestingly enough, does not require increasing returns as is the case in most search models going back to Diamond (1982). Here we consider a different type of strategic interaction: between money demand by buyers and supply decisions by sellers.

<sup>3</sup>Another interpretation of  $\delta$  is possible. Sellers possess a production technology which transforms the search good into the general good. Such technology  $f(q)$  uses the search good as a productive (intermediate) input to produce the general good, the cost of which is denoted  $C(q)$ . The cost function is linear with constant marginal cost  $\delta$ , as is the production technology with a marginal product of 1. In that case  $\delta$  corresponds to the cost of converting the search good into the general good. More or less general forms of these technology and cost

Money is a perfectly divisible and storable object whose value relies on its use as a medium of exchange. This comes from the double-coincidence-of-wants problem between buyers and sellers in the frictional market, which rules out barter. We also assume imperfect commitment ruling out credit, and imperfect memory ruling out trigger strategies as a way to support cooperation. These assumptions make money essential for trade (Kocherlakota 1998, Wallace 2001, Lagos and Wright 2007). By analogy with output where  $\bar{q}$  is the quantity produced and  $\hat{q}$  is the quantity consumed, we denote  $\bar{m}$  the quantity of money held by a buyer when entering the frictional market and  $\hat{m}$  the quantity spent. Money is available in quantity  $M_t$  at time  $t$  and each period new money is injected or withdrawn via lump-sum transfers to buyers by the central bank at rate  $\tau$  according to  $M_{t+1} = (1 + \tau) M_t$ . Denoting  $r$  the real interest rate, since  $\beta = 1/(1 + r)$ , the Fisher equation  $(1 + i_t) = (1 + r)(1 + \pi_t)$  produces a nominal interest rate  $i_t = (1 - \beta + \tau_t)/\beta$  where  $\pi_t = \tau_t$  is inflation (fully anticipated) at time  $t$ . The price of the general good in the centralized market is normalized to 1 and the clearing price of money in terms of the general good is denoted by  $\phi_t$ . In the paper we will focus on steady-state equilibria where the aggregate real money supply is constant. Thus,  $\phi M = \phi_{+1} (1 + \tau) M$  is equivalent to  $\phi = \phi_{+1} (1 + \tau)$  where the subscript  $+1$  denotes the value of a variable (or value function) in the next period.

### 3 Production in Advance

In LW, buyers move first by investing in money holdings. In the second stage of the game they bargain over terms of trade with a seller if they meet one. Buyers are then able to infer in the first stage how much sellers will produce in the second stage via the outcome of the Nash bargaining game. This corresponds to production on demand, or 'late' production. Their economy then corresponds to a two-stage sequential-move game.

Replacing production on demand by production in advance (or early production) changes the scene. First, it turns the game into a simultaneous-move game since each side of the market moves without knowing what the other side is up to.<sup>4</sup> Second, there is now a two-sided holdup problem between households and firms: buyers invest in money but do not get the full return on their investment unless they have all the bargaining power; sellers incur production expenses ex ante that are sunk. Only the former is present in LW. Third, changing the timing of production brings the seller's objective back into the picture, in contrast with LW where sellers passively respond to demand.

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functions may alter the equilibrium regions. Some of those changes will be discussed as we expose the model.

<sup>4</sup>As will be clear later, it is the commitment inherent in the decision to produce before meeting that matters, more than simultaneity per se.

### 3.1 Sellers

With production in advance, sellers produce at the beginning of the frictional market. Denoting  $\bar{q}$  such output, let  $V^s(\bar{q})$  be the value function of a seller holding output  $\bar{q}$  in the frictional market. In the following equations  $\hat{q}(\bar{q}, \bar{m})$  and  $\hat{m}(\bar{q}, \bar{m})$  emphasize that, in general, both the quantity traded  $\hat{q}$  and the price  $\hat{m}$  depend on the bounds in the Nash bargaining problem. Yet we will simply use  $\hat{q}$  and  $\hat{m}$  when there is no ambiguity.<sup>5</sup>

In the Walrasian market a seller's problem is

$$\begin{aligned} W^s(q, m) &= \max_{\bar{x}, \hat{x}, \bar{q}} \{v(\hat{x}) - \bar{x} + \beta [-c(\bar{q}) + V^s(\bar{q})]\}, \\ \text{s.t. } \hat{x} &= \phi m + (1 - \delta) q + \bar{x}. \end{aligned} \quad (1)$$

Substituting out for  $\bar{x}$  yields

$$W^s(q, m) = \max_{\hat{x}, \bar{q}} \{v(\hat{x}) - \hat{x} + \phi m + (1 - \delta) q + \beta [-c(\bar{q}) + V^s(\bar{q})]\}, \quad (2)$$

with

$$V^s(\bar{q}) = \sigma W_{+1}^s[\bar{q} - \hat{q}(\bar{q}, \bar{m}), \hat{m}(\bar{q}, \bar{m})] + (1 - \sigma) W_{+1}^s(\bar{q}, 0). \quad (3)$$

From (3), with probability  $\sigma$ , a seller trades with a buyer in which case the seller receives  $\hat{m}$  units of money in exchange for providing  $\hat{q}$  units of the search good and proceeds with  $\bar{q} - \hat{q} \geq 0$  units of unsold output. With probability  $1 - \sigma$ , the seller does not trade and proceeds with no money and all her output  $\bar{q}$ .

The seller's program simplifies into

$$\max_{\bar{q} \geq 0} \Phi(\bar{q}) = -c(\bar{q}) + \sigma [\phi_{+1} \hat{m}(\bar{q}, \bar{m}) + (1 - \delta) (\bar{q} - \hat{q}(\bar{q}, \bar{m}))] + (1 - \sigma) (1 - \delta) \bar{q}. \quad (4)$$

When deciding on her output for the frictional market, the seller maximizes the difference between production costs, which are sunk, and the expected return from selling part of it with probability  $\sigma$ , or selling none of it with probability  $1 - \sigma$ . In both cases only a fraction  $1 - \delta$  of the leftover is carried forward to the centralized market as inventories.

### 3.2 Buyers

Let  $W^b(m)$  be Bellman's value function for a buyer holding  $m$  units of money in the centralized market. It is given by

$$\begin{aligned} W^b(m) &= \max_{\bar{x}, \hat{x}, \bar{m}} \{v(\hat{x}) - \bar{x} + \beta V^b(\bar{m})\}, \\ \text{s.t. } \phi \bar{m} + \hat{x} &= \phi(m + T) + \bar{x}. \end{aligned} \quad (5)$$

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<sup>5</sup>We use Nash bargaining all along to facilitate qualitative and quantitative comparison with the seminal Lagos and Wright (2005) paper. Note that more papers in the literature are now using Kalai/Smorodinsky bargaining (see, e.g. Aruoba, Rocheteau and Waller (2007)).

where  $V^b(\bar{m})$  is Bellman's value function for a buyer bringing  $\bar{m}$  units of money into the frictional market. In other words, a buyer chooses how much to produce and consume of the general good,  $\bar{x}$  and  $\hat{x}$  respectively, and how much money to bring to the frictional market,  $\bar{m}$ , in order to buy the special good. His budget constraint equalizes resources,  $\phi(m+T) + \bar{x}$ , to demand,  $\phi\bar{m} + \hat{x}$ . Substituting out for  $\bar{x}$  yields

$$W^b(m) = \max_{\hat{x}, \bar{m}} \left\{ v(\hat{x}) - \hat{x} + \phi(m+T) - \phi\bar{m} + \beta V^b(\bar{m}) \right\}. \quad (7)$$

Bellman's equation for a buyer in the frictional market is given by

$$V^b(\bar{m}) = \alpha \left\{ u(\hat{q}) + W_{+1}^b(\bar{m} - \hat{m}) \right\} + (1 - \alpha) W_{+1}^b(\bar{m}). \quad (8)$$

This equation says that, in this market, a buyer trades with probability  $\alpha$ , in which case he pays  $\hat{m}$  to buy  $\hat{q}$  units of the search good and proceeds with  $\bar{m} - \hat{m}$  units of money. With probability  $1 - \alpha$  he does not trade and moves on to the centralized market with the same amount of money.

To derive the buyer's choice of money, note that next period's value function for a buyer who trades  $\hat{m}$  for  $\hat{q}$  this period is given by

$$W_{+1}^b(\bar{m} - \hat{m}) = v(\hat{x}^*) - \hat{x}^* + \phi_{+1}(\bar{m} - \hat{m} + T) + \max_{\tilde{m}} \left\{ -\phi_{+1}\tilde{m} + \beta V^b(\tilde{m}) \right\}, \quad (9)$$

where  $\tilde{m}$  represents the choice of money for the next period given that  $\bar{m}$  was chosen for this one. Similarly, next period's value function for a buyer who does not trade this period is given by

$$W_{+1}^b(\bar{m}) = v(\hat{x}^*) - \hat{x}^* + \phi_{+1}(\bar{m} + T) + \max_{\tilde{m}} \left\{ -\phi_{+1}\tilde{m} + \beta V^b(\tilde{m}) \right\}. \quad (10)$$

Inserting (9) and (10) into (8) one obtains

$$\begin{aligned} V^b(\bar{m}) &= v(\hat{x}^*) - \hat{x}^* + \phi_{+1}T + \alpha \left\{ u(\hat{q}) + \phi_{+1}(\bar{m} - \hat{m}) \right\} \\ &\quad + (1 - \alpha) \phi_{+1}\bar{m} + \max_{\tilde{m}} \left\{ -\phi_{+1}\tilde{m} + \beta V^b(\tilde{m}) \right\}. \end{aligned} \quad (11)$$

By inserting (11) into (7) and getting rid of constant terms, the buyer's program simplifies into

$$\max_{\bar{m} \geq 0} \Psi(\bar{m}) = -\phi\bar{m} + \beta \left\{ \alpha \left\{ u[\hat{q}(\bar{m})] + \phi_{+1}(\bar{m} - \hat{m}) \right\} + (1 - \alpha) \phi_{+1}\bar{m} \right\}. \quad (12)$$

When choosing money holdings, buyers maximize the difference between the opportunity cost of money and the discounted expected return from spending part of it with probability  $\alpha$ , or spending none of it with probability  $1 - \alpha$ .

**Assumption:**  $u'(0) > 1 - \delta > c'(0)$ .

The left-hand side,  $u'(0) > 1 - \delta$ , allows for positive gains from trade in the frictional market. The right-hand side,  $1 - \delta > c'(0)$ , is due to the convexity of the production function. Because



the marginal cost of producing ahead of the market,  $c'(0)$ , is smaller than the marginal gain in terms of leftovers,  $1 - \delta$ , sellers enjoy lower general good production costs in the frictional market than in the Walrasian market, up to a certain point (to be characterized below). A possible interpretation is that sellers have access to their capital, i.e. plants and machineries, in the frictional market rather than in the Walrasian market. Outside those producing hours, sellers do not have any particular cost advantage over other agents, especially buyers. It has an important implication: under conditions to be characterized later, firms voluntarily accumulate inventories during the frictional market with the prospect of the Walrasian market, offering them an outside option in the bargaining game, as in Berentsen, Menzio and Wright (2011). We discuss later the implications of relaxing this assumption.

### 3.3 Terms of trade

The generalized Nash solution to the bargaining between a buyer and a seller is

$$\arg \max_{\hat{q} \leq \bar{q}, \hat{m} \leq \bar{m}} \mathcal{B}(\hat{q}, \hat{m}) = \left[ u(\hat{q}) + W_{+1}^b[\bar{m} - \hat{m}] - W_{+1}^b(\bar{m}) \right]^\theta \left[ W_{+1}^s[\bar{q} - \hat{q}, \hat{m}] - W_{+1}^s(\bar{q}, 0) \right]^{1-\theta} \quad (13)$$

in which  $W_{+1}^b(\bar{m})$  and  $W_{+1}^s(\bar{q}, 0)$  are the buyer's and seller's disagreement payoffs, respectively.

For the sake of exposition, let us define the following functions:

$$g(x) \equiv \frac{(1 - \delta)(1 - \theta)}{\theta u'(x) + (1 - \theta)(1 - \delta)} u(x) + \frac{\theta u'(x)}{\theta u'(x) + (1 - \theta)(1 - \delta)} (1 - \delta)x \quad (14)$$

$$h(x) \equiv (1 - \theta)u(x) + \theta(1 - \delta)x \quad (15)$$

$$q_N \equiv u'^{-1}(1 - \delta) \quad (16)$$

$$m_N \equiv h(q_N)/\phi_{+1} = g(q_N)/\phi_{+1} \quad (17)$$

The functions  $g$  and  $h$  (derived from the first-order conditions of the bargaining problem with respect to  $\hat{q}$  and  $\hat{m}$  respectively) settle terms of trade in Nash bargaining. The intersection of  $g(q)$  and  $h(q)$  yields  $(q_N, m_N)$ , the unconstrained Nash bargaining solution (see Figures 3 and 4 in the Appendix).

Let us first characterize the solutions to the Nash bargaining problem. We denote  $(\hat{q}(\bar{q}, \bar{m}), \hat{m}(\bar{q}, \bar{m})) \equiv \arg \max_{\hat{q} \leq \bar{q}, \hat{m} \leq \bar{m}} \mathcal{B}(\hat{q}, \hat{m})$ . Given  $(\bar{q}, \bar{m}) \geq 0$ , the Nash bargaining problem simplifies into:

$$\max_{\hat{q} \leq \bar{q}, \hat{m} \leq \bar{m}} \mathcal{B}(\hat{q}, \hat{m}) = \left[ u(\hat{q}) - \phi_{+1}\hat{m} \right]^\theta \left[ \phi_{+1}\hat{m} - (1 - \delta)\hat{q} \right]^{1-\theta}. \quad (18)$$

We show in the online appendix that the maximization problem is well-defined and that the Nash axioms are applicable to the problem at hand.<sup>6</sup>

<sup>6</sup>The online appendix is available at <https://sites.google.com/site/econcjs/research>.

**Lemma 1** *Solutions to the Nash bargaining problem:*

$$(\hat{q}(\bar{q}, \bar{m}), \hat{m}(\bar{q}, \bar{m})) = (\min\{g^{-1}(\phi_{+1}\bar{m}), q_N, \bar{q}\}, \min\{h(\bar{q})/\phi_{+1}, m_N, \bar{m}\}).$$

**Proof.** See the Appendix. ■

Lemma 1 states that by bringing  $\bar{m}$  units of money to the frictional market, the buyer can expect to exchange them for  $g^{-1}(\phi_{+1}\bar{m})$  units of good, provided that  $g^{-1}(\phi_{+1}\bar{m})$  does not exceed the unconstrained Nash bargaining outcome  $q_N$  and the capacity constraint  $\bar{q}$  set by the seller. Similarly, by bringing  $\bar{q}$  to the frictional market, the seller can expect to exchange it for  $h(\bar{q})/\phi_{+1}$  units of money, provided that  $h(\bar{q})/\phi_{+1}$  does not exceed the unconstrained Nash bargaining outcome  $m_N$  and the capacity constraint  $\bar{m}$  set by the buyer.

Let us finally define  $\bar{q}_L$  and  $\bar{q}_H$  such that

$$c'(\bar{q}_L) = (1 - \sigma)(1 - \delta), \quad (19)$$

$$c'(\bar{q}_H) = 1 - \delta. \quad (20)$$

A second lemma characterizes the seller's best response:

**Lemma 2** *The seller's best response:* For any  $\bar{m} < m_N$ , in equilibrium the seller's best response is either  $\max\{\bar{q}_L, h^{-1}(\phi_{+1}\bar{m})\}$  or  $\bar{q}_H$ .

**Proof.** See the Appendix. ■

To understand Lemma 2, fix some  $\bar{m} < m_N$  and let us denote  $\bar{q}_U$  such that  $c'(\bar{q}_U) = \sigma h'(\bar{q}_U) + (1 - \sigma)(1 - \delta)$ . If producing the good in the frictional market is very costly such that  $h(\bar{q}_U) < \phi_{+1}\bar{m}$ , then the seller will produce  $\bar{q}_U$ . However, it also implies that the buyer brings more money than he intends to spend, so  $\bar{q}_U$  cannot be part of an equilibrium. When producing the good in the frictional market is not that costly, that is  $h(\bar{q}_U) \geq \phi_{+1}\bar{m}$ , the seller weighs two options:  $\max\{\bar{q}_L, h^{-1}(\phi_{+1}\bar{m})\}$  and  $\bar{q}_H$ . The first one is the optimal amount of output when the seller intends to sell it all in exchange for  $\bar{m}$ , and  $\bar{q}_H$  is the optimal amount of output when the seller intends to sell only some of it in exchange for  $\bar{m}$  and bring the rest to the centralized market. Which one is better depends on the amount of money brought by the buyer. For instance, if the seller expects the buyer to bring a small enough amount of money, which will presumably be the case when inflation is high, it is best for him to produce a large amount of output and sell only a fraction to the buyer.

### 3.4 Equilibria

We start by characterizing the types of equilibria that exist and the corresponding conditions on the parameters. The two main parameters are durability  $\delta$  and the nominal interest rate  $i$ . In the following, we assume that  $u'(x)/g'(x)$  is strictly decreasing on  $[0, q_N]$ .<sup>7</sup> This assumption guarantees that the pure-strategy equilibrium is unique, whenever it exists.

<sup>7</sup>The online appendix provides sufficient conditions for strictly decreasing  $u'(x)/g'(x)$  on  $[0, q_N]$ .

### 3.4.1 Non-monetary equilibrium

**Proposition 1** (*Type I Equilibrium*) *If  $\frac{u'(0)}{g'(0)} \leq 1 + \frac{i}{\alpha}$ , then  $(\bar{q}^*, \bar{m}^*) = (\bar{q}_H, 0)$  is the unique equilibrium.*

**Proof.** See the Appendix. ■

Type I Equilibrium is a pure-strategy non-monetary equilibrium. If marginal utility  $u'(0)$  is small and the nominal interest rate is high, then agents simply do not use money and all economic activity is limited to the Walrasian market.

### 3.4.2 Pure-strategy monetary equilibrium

Define  $\bar{m}_C = \inf\{\bar{m} \mid \Phi(\max\{\bar{q}_L, h^{-1}(\phi_{+1}\bar{m})\}|\bar{m}) = \Phi(\bar{q}_H|\bar{m})\}$  whenever it exists, otherwise let  $\bar{m}_C = m_N$ . That is,  $\bar{m}_C$  is the lowest amount of money that leaves the seller indifferent between producing  $\max\{\bar{q}_L, h^{-1}(\phi_{+1}\bar{m})\}$  and selling it altogether or producing  $\bar{q}_H$  and selling some of it (cf. Lemma 2).

**Proposition 2** (*Type II Equilibrium*) *Let  $\zeta$  be defined by (21). If  $\frac{u'(0)}{g'(0)} > 1 + \frac{i}{\alpha}$  and the demand for real balances  $g(\zeta) \leq \phi_{+1}\bar{m}_C$ , then  $(\bar{q}^*, \bar{m}^*)$  specified below is the unique equilibrium*

$$\begin{cases} \bar{q}^* = \bar{q}_H \\ \frac{u'(\hat{q}(\bar{q}^*, \bar{m}^*))}{g'(\hat{q}(\bar{q}^*, \bar{m}^*))} = \frac{u'(g^{-1}(\phi_{+1}\bar{m}^*))}{g'(g^{-1}(\phi_{+1}\bar{m}^*))} = \frac{u'(\zeta)}{g'(\zeta)} = 1 + \frac{i}{\alpha} \end{cases} \quad (21)$$

**Proof.** See the Appendix. ■

Type II Equilibrium is a pure-strategy monetary equilibrium in which the marginal utility of  $q$  is sufficiently high for the buyer, and the demand for real balances  $g(\zeta)$  is smaller than the  $\phi_{+1}\bar{m}_C$  threshold. In this equilibrium sellers produce  $\bar{q}_H$  and sell  $\zeta < \bar{q}_H$  for  $\bar{m}^*$  upon a meeting.

### 3.4.3 Mixed-strategy monetary equilibria

Let us now consider what happens when marginal utility is high enough but, by contrast to Proposition 2, the demand for real balances  $g(\zeta)$  is greater than  $\phi_{+1}\bar{m}_C$ . It turns out that a pure-strategy equilibrium does not exist in this case, and we need to consider agents playing mixed strategies. To do that, let us formally define  $\mu_b : B_{\mathbb{R}_+} \rightarrow [0, 1]$  as the buyer's mixed strategy on money holdings, and  $\mu_s : B_{\mathbb{R}_+} \rightarrow [0, 1]$  as the seller's mixed strategy on search good production, where  $B_{\mathbb{R}_+}$  stands for the Borel  $\sigma$ -algebra in  $\mathbb{R}_+$ . Given a mixed strategy  $\mu$ , let  $\text{supp } \mu$  denote the support of  $\mu$ . Lemma 3 below shows that in equilibrium the seller will produce at least  $\bar{q}_L$  but no more than  $\bar{q}_H$  units of search good, and the buyer will bring at least  $g(\bar{q}_L)/\phi_{+1}$  but no more than  $g(\bar{q}_H)/\phi_{+1}$  units of money to the frictional market.

**Lemma 3** *Suppose  $(\mu_s, \mu_b)$  constitutes a Nash equilibrium. Then  $\text{supp } \mu_s \subset [\bar{q}_L, \bar{q}_H]$  and  $\text{supp } \mu_b \subset [g(\bar{q}_L)/\phi_{+1}, g(\bar{q}_H)/\phi_{+1}]$ .*

**Proof.** First notice that any  $\bar{q} < \bar{q}_L$  will not be chosen by the seller with a positive probability, as it is strictly dominated by  $\bar{q}_L$ .<sup>8</sup> Hence  $\inf \text{supp } \mu_s \geq \bar{q}_L$ . Next observe that for any  $\bar{m} \leq g(\bar{q}_L)/\phi_{+1}$ ,  $\Phi(\bar{q}_H) > \Phi(\max\{\bar{q}_L, h^{-1}(\phi_{+1}\bar{m})\})$ . Therefore  $\bar{m}_C > g(\bar{q}_L)/\phi_{+1}$ .  $g(\zeta)/\phi_{+1} > \bar{m}_C$  also implies that  $\frac{u'(\bar{q})}{g'(\bar{q})} > 1 + \frac{i}{\alpha}$  for any  $\bar{q} \leq \bar{q}_L$ . Given any output level  $\bar{q} \geq \bar{q}_L$ , the buyer's payoff at  $\bar{m} < g(\bar{q}_L)/\phi_{+1}$  is strictly lower than that at  $g(\bar{q}_L)/\phi_{+1}$ . Hence  $\inf \text{supp } \mu_s \geq \bar{q}_L$  implies that  $\inf \text{supp } \mu_b \geq g(\bar{q}_L)/\phi_{+1}$ .

Next we argue that  $\sup \text{supp } \mu_s \leq \bar{q}_H$ . Given any  $\bar{m} \geq 0$ , any  $\bar{q} > \bar{q}_U$  is strictly dominated by  $\bar{q}_U$ , as  $\Phi_{\bar{q}}$  is always negative for all  $\bar{q} > \bar{q}_U$  (see Lemma 2). Therefore  $\sup \text{supp } \mu_s \leq \bar{q}_U$ . Given  $\sup \text{supp } \mu_s \leq \bar{q}_U$ , the buyer will not pick any  $\bar{m} > g(\bar{q}_U)/\phi_{+1}$ , as any such  $\bar{m}$  is strictly dominated by  $g(\bar{q}_U)/\phi_{+1}$ . Consequently  $\sup \text{supp } \mu_b \leq g(\bar{q}_U)/\phi_{+1}$ . Given  $\sup \text{supp } \mu_b \leq g(\bar{q}_U)/\phi_{+1}$ , consider two subcases: (i)  $h^{-1}(g(\bar{q}_U)) \leq \bar{q}_H$ . Fix any  $\bar{m} \leq g(\bar{q}_U)/\phi_{+1}$ . It can be readily seen that  $\Phi_{\bar{q}}(\bar{q}) < 0$  for any  $\bar{q} > \bar{q}_H$ . Therefore any  $\bar{q} > \bar{q}_H$  is strictly dominated by  $\bar{q}_H$ , and we have  $\sup \text{supp } \mu_s \leq \bar{q}_H$ . (ii)  $h^{-1}(g(\bar{q}_U)) > \bar{q}_H$ . Given any  $\bar{m} \leq g(\bar{q}_U)/\phi_{+1}$ , it can be shown that  $\Phi_{\bar{q}}(\bar{q}) < 0$  for any  $\bar{q} > h^{-1}(g(\bar{q}_U))$ , and hence any  $\bar{q} > h^{-1}(g(\bar{q}_U))$  is strictly dominated by  $h^{-1}(g(\bar{q}_U))$ . Therefore  $\sup \text{supp } \mu_b \leq g(\bar{q}_U)/\phi_{+1}$  implies  $\sup \text{supp } \mu_s \leq h^{-1}(g(\bar{q}_U))$ . Given  $u(\mathcal{F}_s) \leq h^{-1}(g(\bar{q}_U))$ , by the same token we have  $\sup \text{supp } \mu_b \leq g(h^{-1}(g(\bar{q}_U)))/\phi_{+1}$ . Continuing in this fashion, we can find a finite sequence of the form  $\{\bar{q}_U, h^{-1}(g(\bar{q}_U)), h^{-1}(g(h^{-1}(g(\bar{q}_U))))\dots\}$  in which the last term is no greater than  $\bar{q}_H$ . Applying the result in subcase (i) gives us  $\sup \text{supp } \mu_s \leq \bar{q}_H$ .  $\sup \text{supp } \mu_s \leq \bar{q}_H$  directly implies that  $\sup \text{supp } \mu_b \leq g(\bar{q}_H)/\phi_{+1}$ . ■

We are now in a position to characterize Type III equilibrium. The main comments and economic intuition behind this proposition and the following are deferred to Section 3.5. Let  $\hat{m} \in (g(\bar{q}_L)/\phi_{+1}, g(\bar{q}_H)/\phi_{+1})$  be the money holding characterized in equation (54) in the Appendix, where  $\bar{m}_C = \hat{m}$  under the conditions of Type III equilibrium below.

**Proposition 3** (*Type III Equilibrium*) *Suppose  $g(\zeta) > \phi_{+1}\bar{m}_C$  and  $\bar{q}_L \geq \min\{h^{-1}(g(\bar{q}_H)), h^{-1}(g(\zeta))\}$ . Then the pair  $(\mu_s, \mu_b)$  constructed below constitutes a unique equilibrium:*

$$\mu_b(\bar{m}) = \begin{cases} 1 & \bar{m} = \hat{m} \\ 0 & \bar{m} \neq \hat{m} \end{cases} \quad (22)$$

$$\mu_s(\bar{q}) = \begin{cases} (1 + \frac{i}{\alpha}) \frac{g'(g^{-1}(\phi_{+1}\hat{m}))}{u'(g^{-1}(\phi_{+1}\hat{m}))} & \bar{q} = \bar{q}_H \\ 1 - (1 + \frac{i}{\alpha}) \frac{g'(g^{-1}(\phi_{+1}\hat{m}))}{u'(g^{-1}(\phi_{+1}\hat{m}))} & \bar{q} = \bar{q}_L \end{cases} \quad (23)$$

**Proof.** See the Appendix. ■

<sup>8</sup>See also Claim A1 in the proof of Proposition 2.

Note that the condition  $g(\zeta)/\phi_{+1} > \bar{m}_C$  automatically implies that  $\frac{u'(0)}{g'(0)} > 1 + \frac{i}{\alpha(n)}$ . Furthermore,  $g(\zeta)/\phi_{+1} > \bar{m}_C$  combined with the fact that  $g(\bar{q}_L)/\phi_{+1} < \bar{m}_C$  implies that  $\zeta > \bar{q}_L$ . As for terms of trade  $(\hat{q}, \hat{m})$ , using Lemma 1, note that if  $\bar{q} = \bar{q}_L$  then  $(\hat{q}, \hat{m}) = (\bar{q}_L, \hat{m})$ . And if  $\bar{q} = \bar{q}_H$  then  $(\hat{q}, \hat{m}) = (g^{-1}(\phi_{+1}\hat{m}), \hat{m})$  with  $\hat{m}$  defined by equation (54). Finally, note also that  $\bar{q}_L < g^{-1}(\phi_{+1}\hat{m}) < \bar{q}_H$ .<sup>9</sup>

Finally, when  $\bar{q}_L < \min\{h^{-1}(g(\bar{q}_H)), h^{-1}(g(\zeta))\}$ , that is when  $\bar{q}_L$  and  $\bar{q}_H$  are sufficiently apart (by contrast to Type III equilibrium), we have the following proposition.

**Proposition 4** (*Type IV Equilibrium*) Suppose  $g(\zeta) > \phi_{+1}\bar{m}_C$  and  $\bar{q}_L < \min\{h^{-1}(g(\bar{q}_H)), h^{-1}(g(\zeta))\}$ . There are multiple equilibria, all of which satisfy the following properties:

- (a)  $\text{supp } \mu_s \subset [\bar{q}_L, \bar{q}_H]$  and  $\text{supp } \mu_b \subset [g(\bar{q}_L)/\phi_{+1}, g(\bar{q}_H)/\phi_{+1}]$ .
- (b)  $\mu_b$  is at most ternary, and  $\mu_s$  is at most quaternary.

**Proof.** The proof, which contains many repeats from the proof of Proposition 3, can be found in the Online Appendix. ■

In Type III Equilibrium, buyers bring a fixed amount of real balances, regardless of inflation. At the same time, sellers randomize over two levels of output,  $\bar{q}_L$  and  $\bar{q}_H$ . In Type IV equilibrium, on the other hand, the buyer brings multiple amounts of money with positive probabilities and likewise the seller brings multiple amounts of output with positive probabilities. This implies that with some probability this equilibrium gives rise to the buyer bringing more money to the frictional market than he hands over to the seller.

The frontiers between each equilibrium are represented on Figure 1. The frontier between Type II and Type III-IV equilibria is given by the pairs  $(i, \delta)$  such that  $g(\zeta)/\phi_{+1} = \bar{m}_C$ . The horizontal portion of the frontier between Type III and Type IV equilibrium is given by the pairs  $(i, \delta)$  such that  $\bar{q}_L = h^{-1}(g(\bar{q}_H))$ , which boils down to a unique  $\delta$  denoted by  $\check{\delta}$  since  $i$  enters neither  $\bar{q}_L$  nor  $h^{-1}(g(\bar{q}_H))$ ; and the curved portion of the frontier between Type III and Type IV equilibrium is given by the pairs  $(i, \delta)$  such that  $\bar{q}_L = h^{-1}(g(\zeta))$ .

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<sup>9</sup>At  $\hat{m}$  the seller is indifferent between  $\max\{\bar{q}_L, h^{-1}(\phi_{+1}\hat{m})\}$  and  $\bar{q}_H$ . It can be shown that the condition  $\bar{q}_L \geq \min\{h^{-1}(g(\bar{q}_H)), h^{-1}(g(\zeta))\}$  implies  $\bar{q}_L > h^{-1}(\phi_{+1}\hat{m})$ . Hence, the seller is indifferent between  $\bar{q}_L$  and  $\bar{q}_H$ .

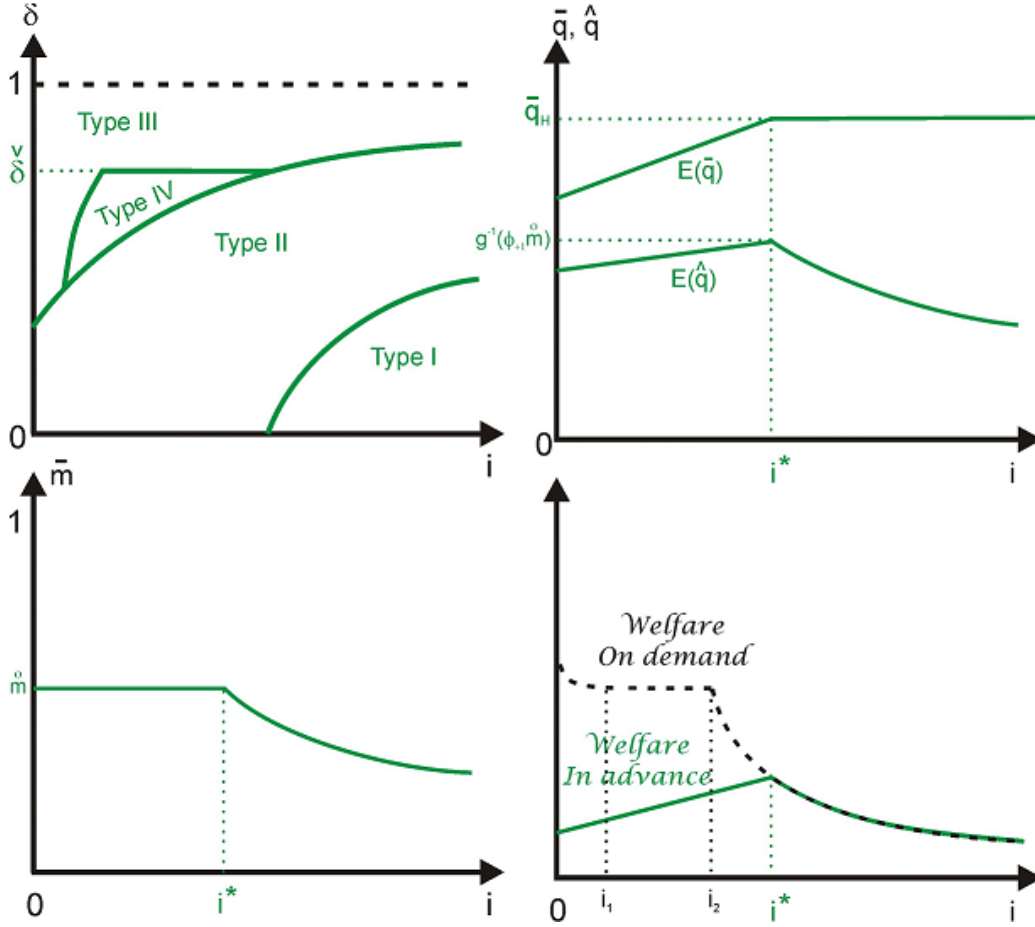


Figure 1: Partition, and comparative statics for a given  $\delta < \check{\delta}$  and  $\theta < 1$ .

### 3.5 Comparative Statics and Welfare

First note that as  $i$  decreases the equilibrium shifts from Type I to Type II, and then from Type II to Type III or IV depending on the value of  $\delta$ . Since Equilibrium IV features further multiplicity, we concentrate on the shift from Equilibrium II to Equilibrium III.

Welfare in the production-in-advance economy is given by

$$\mathcal{W}_{PIA} = \int \{-c(\bar{q}) + \sigma \{u[\hat{q}(\bar{q})] + (1 - \delta) [\bar{q} - \hat{q}(\bar{q})]\} + (1 - \sigma)(1 - \delta)\bar{q}\} d\mathcal{F}_s(\bar{q}) + v(\hat{x}^*) - \hat{x}^*. \quad (24)$$

**Proposition 5** (*Social Planner's Solution*) *Let  $q_N$  be such that  $u'(q_N) = 1 - \delta$ . A social planner would pick  $\bar{q} = \bar{q}_H$  and  $\hat{q} = q_N$  when  $q_N < \bar{q}_H$ , and  $\bar{q} = \hat{q} = \bar{q}_C$  with  $\bar{q}_C$  given by  $c'(\bar{q}_C) = \sigma u'(\bar{q}_C) + (1 - \sigma)(1 - \delta)$  when  $q_N > \bar{q}_H$ .*

**Proof.** The central planner solves  $\max_{\hat{q}, \bar{q}, \lambda} [-c(\bar{q}) + \sigma [u(\hat{q}) + (1 - \delta)(\bar{q} - \hat{q})] + (1 - \sigma)(1 - \delta)\bar{q}] + \lambda(\bar{q} - \hat{q})$  where  $\lambda$  is the Lagrange multiplier on the  $\hat{q} \leq \bar{q}$  constraint. ■

Equilibrium III welfare simplifies into

$$\begin{aligned} \mathcal{W}_{PIA-III} = & \mu_s(\bar{q}_H) \{-c(\bar{q}_H) + \sigma [u(\hat{q}) + (1 - \delta)(\bar{q}_H - \hat{q})] + (1 - \sigma)(1 - \delta)\bar{q}_H\} \\ & + \mu_s(\bar{q}_L) \{-c(\bar{q}_L) + \sigma u(\bar{q}_L) + (1 - \sigma)(1 - \delta)\bar{q}_L\} + v(\hat{x}^*) - \hat{x}^*, \end{aligned} \quad (25)$$

where  $\hat{q} = g^{-1}(\phi_{+1}\hat{m})$  and  $\mu_s(\bar{q}_H)$  and  $\mu_s(\bar{q}_L)$  are given by Equation (23).

**Proposition 6** (*Inflation and Social Welfare*) *In Type III equilibria, welfare increases as inflation rises, i.e.  $\frac{\partial \mathcal{W}_{PIA-III}}{\partial i} > 0$ .*

**Proof.** See the Appendix. ■

We now explain the intuition behind Equilibrium Type III (Proposition 3) and why inflation raises welfare (Proposition 6). In order to do so, let us start by tracking the changes in the economy as the interest rate recedes from high levels and approaches 0.

When the interest rate is high, the demand for money is low and so is buyers' demand for the good,  $\zeta$ . Sellers produce more than what they expect to sell,  $\bar{q}_H$ , and wait till the Walrasian market to sell whatever fraction  $\bar{q}_H - \zeta$  remains [Type II equilibrium].

As inflation recedes, buyers start carrying more money and buying more goods, leaving sellers with less and less leftovers. At some point, a new strategy emerges for the seller: given the shrinking utility he derives from the leftover, *for the same amount of real balances  $\phi_{+1}\hat{m}$*  he can now obtain the same payoff by producing a smaller amount of goods,  $\bar{q}_L$ , and selling it altogether to the buyer. That is, rather than compensating higher production costs with leftovers, a seller may simply decide to produce and sell a smaller amount of goods with no leftovers. Both options (high output  $\bar{q}_H$  selling  $\hat{q} > \bar{q}_L$  and keeping  $\bar{q}_H - \hat{q}$ , or low output  $\bar{q}_L$  with no leftover) yield the same payoff – see Figure 5.

However, the low output option  $\bar{q}_L$  becomes a threat to the buyer. If the seller chooses it, the buyer strictly prefers bringing less money than  $\hat{m}$ , and therefore the seller is strictly better off producing the high output  $\bar{q}_H$ , sell a fraction that corresponds to the buyer's money, and keep the rest. But if the seller chooses this high output option, the buyer strictly prefers to bring more money than  $\hat{m}$ , which implies that the seller is now strictly better off producing the low output and sell it altogether. Lower inflation creates strategic uncertainty between buyers and sellers. In response, sellers randomize between different levels of output and buyers randomize between different amounts of money. Mixed strategies emerge because low inflation makes producing for the frictional market only, i.e.  $\bar{q}_L$ , a viable alternative for sellers. By contrast, when inflation is high, sellers forecast that demand will be low and produce  $\bar{q}_H$  which is always enough to satisfy demand.

What role does goods' perishability play in this story? When goods are highly perishable ( $\delta > \check{\delta}$  defined in the last paragraph of Section 3.4), sellers produce very little due to the heavy

loss incurred if they cannot find a customer. It follows that  $\bar{q}_L$  and  $\bar{q}_H$  are not too distant from each other since both reflect the risk of losing most of it, if no buyer is found (see Equations (19) and (20)). Thus, if goods are highly perishable, an equilibrium is characterized by buyers bringing a unique amount of money  $\bar{m}_C = \hat{m}$  and sellers randomizing between two levels of output,  $\bar{q}_L$  and  $\bar{q}_H$  [Type III equilibrium]. Importantly, that amount of money is unaffected by inflation as long as inflation is not too high. However, when inflation rises, because buyers' outside option deteriorates, bargaining forces sellers to choose the high output  $\bar{q}_H$  with higher probability. And by producing  $\bar{q}_H$  more often they also sell  $\hat{q} > \bar{q}_L$  more often. As a result, when inflation increases, buyers buy more goods *on average*, which increases welfare (Proposition 6). It follows that when goods are highly perishable, the optimal inflation rate is positive. Note that in this equilibrium all real variables ( $\bar{q}_L, \bar{q}_H, \hat{m}, \hat{q}$ ) are unaffected by inflation. Only the probabilities with which sellers choose between  $\bar{q}_L$  and  $\bar{q}_H$  change as inflation rises or falls.

When goods have intermediate durability (intermediate in the sense that the economy is in the Type IV region), the difference between  $\bar{q}_L$  and  $\bar{q}_H$  is now too great for buyers to stick to a unique amount of real balances. As a result, buyers start randomizing which in turn changes the shape of the seller's objective function inducing them to enlarge the set of output over which they randomize themselves (Type IV equilibrium). In this context, multiplicity arises as a result of the indeterminacy in buyers' and sellers' beliefs. Assume for instance that if sellers believe that buyers will randomize over two amounts of money, their best response is to randomize over three levels of output. Then, if buyers anticipate sellers to randomize over three levels of output their best response is to randomize over two levels of real balances. This is one equilibrium, but there can be others such as Type III equilibrium (buyers bring  $\hat{m}$  and sellers randomize between  $\bar{q}_L$  and  $\bar{q}_H$ ). Due to non-concave objective functions, it is impossible to fully characterize Type IV equilibria.<sup>10</sup>

Finally, let us denote  $i^*$  the optimal interest rate. Note from Figure 1 that  $i^*$  is such that the buyer is indifferent between buying  $\zeta$  for sure or receiving  $\bar{q}_L$  with probability  $\mu_s(\bar{q}_L)$  or  $\hat{q} < \bar{q}_H$  with probability  $1 - \mu_s(\bar{q}_L)$ . We have the following Proposition:

**Proposition 7** (*Optimal Inflation in Type III Equilibria*) *In Type III equilibria, the optimal inflation rate increases as goods become more perishable.*

**Proof.** See the Appendix. ■

Proposition 7 shows that the more perishable the goods (higher  $\delta$ ), the higher the optimal rate of inflation. As a matter of fact, although sellers still randomize between  $\bar{q}_L$  and  $\bar{q}_H$ , they tend to choose  $\bar{q}_L$  more often now due to the very high perishability of anything they produce in the frictional market. It then requires a substantial amount of inflation to induce sellers to

<sup>10</sup>Multiplicity in monetary economies can arise due to a variety of reasons, such as the interaction between the real value of money balances and agents' choices of search intensity (Johri 1999), between money demand and entry (Rocheteau and Wright 2005), or coming from coordination (Jean, Rabinovich and Wright 2011).



opt for the high output frequently enough to make the buyer indifferent between the  $(\bar{q}_L, \bar{q}_H)$  lottery (Type III) and purchasing  $\zeta$  with certainty (Type II). One should keep in mind, however, that output (and then welfare) is low when  $\delta$  is high since sellers do not produce much. But this low output is not due to high inflation, only to high perishability.

How important is the  $c'(0) < 1 - \delta$  assumption for our results? For instance, what if  $c'(0) > 1 - \delta$  as with  $c(q) = q$ ? In this case, no monetary equilibrium exists. It indeed implies that  $c'(0) > 1 - \delta > (1 - \sigma)(1 - \delta)$ . From the seller's best response (Equation (40)) we see that the seller has then no incentive to produce unless  $h(\bar{q}) \leq \phi_{+1}\bar{m}$ , in which case he produces  $\bar{q}_U$ . But  $h(\bar{q}) \leq \phi_{+1}\bar{m}$  means that the buyer brings more money  $\phi_{+1}\bar{m}$  than he intends to spend  $h(\bar{q})$ , to which the seller reacts by producing less than  $\bar{q}_U$ . This cannot be an equilibrium. Similarly, no monetary equilibrium exists with production in advance when  $\delta = 1$ , because the seller's outside option is zero (note that by contrast such monetary equilibrium exists in LW's production-on-demand economy). In both cases economic activity is non monetary and limited to the Walrasian market.

Another assumption is worth discussing. Note that two distinct kinds of agents are assumed, along the lines of Rocheteau and Wright (2005). If instead the identity of buyers and sellers are decided by random matching, as in LW, the strategic interaction between a buyer and seller in a match will be altered. Since traders are ex-ante identical, all will produce a given quantity of the search good. Hence, both buyers and sellers may choose to carry inventories over to the centralized market. This will, of course, change the mixed strategies played by buyers and sellers.

## 4 Quantitative Assessment

In this section we measure how a change in the timing of production decisions by firms impacts on the welfare effect of inflation. To do so, we come back to Lagos and Wright (2005)'s calculations for the welfare costs of inflation and track how they are impacted when firms shift from production on demand, as in their model, to production in advance, as in Section 3 above. To make the two economies comparable on every other dimension, we will use the same functional forms and parameters across the two economies. We also set goods' perishability very high in the production-in-advance economy ( $\delta = 0.95$ ) thereby approximating LW's full perishability (i.e.  $\delta = 1$ ).<sup>11</sup> By doing so, we ensure that any difference between our welfare measure and theirs is (almost) entirely attributable to the difference in the timing of production decisions by firms.

The calibration procedure closely follows LW. First, we take the production-in-advance model from Section 3 and normalize  $\alpha = 1$  and set  $\sigma = 0.5$  as in LW. Second, we set  $c(q) = q^2$  to satisfy our  $c'(0) < 1 - \delta$  assumption. Third, we calibrate two parameters of the model, the

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<sup>11</sup>As shown previously, there is no monetary equilibrium when  $\delta = 1$  in the production in advance economy.

curvature  $\eta$  of the utility function  $q^{1-\eta}/1-\eta$  and output on the centralized market  $B$  (which is left undetermined by the model) by fitting the theoretical money demand to the data. Money demand data are taken and updated from Craig and Rocheteau (2008) where the interest rate is the short-term commercial paper rate, and money demand is M1. We use  $L(i)$  to denote money demand, i.e. real balances as a function of the nominal interest rate. It is given by  $\frac{M}{PY}$  where  $M$  is the nominal stock of money and  $PY$  is total nominal output. Denoting  $z(q) = M/P$  this simplifies into

$$L(i) = \frac{z(q)}{B + \sigma z(q)}. \quad (26)$$

Regarding  $\theta$ , we choose a value that ensures the economy remains within the Type III region (cf. Figure 1). Finally, denoting  $\Theta = (\alpha, \sigma, \eta, B, \delta, \theta)$  we calculate a compensated measure for 10% inflation relative to 0% inflation in the production-in-advance economy and in LW using the same  $\Theta$ , and compare the two. The compensated measure corresponds to the amount of consumption agents would be willing to give up (or receive) to have 0% inflation instead of 10%.

Results are reported in Table 1 below where a  $(-)$  means a welfare loss whereas a  $(+)$  means a welfare gain.

Production in advance (Type III with $\delta = 0.95$ )	+2.49%
Production on demand (LW with $\delta = 1$ )	-2.81%

When goods are close to being fully perishable ( $\delta = 0.95$ ) as in LW, the gain associated with 10% inflation in a production in advance economy is 2.49%. If we use the same functional forms and parameters as the ones we used for this last calculation but applied to LW's production-on-demand economy, in which  $\delta = 1$  by definition, we find that inflation reduces consumption by 2.81%. A change in the timing of production hence dramatically alters the effect of inflation.

## 5 Lotteries

The emergence of mixed strategies suggests that buyers and sellers should use lotteries. In this section we introduce lotteries along the lines of Berentsen, Molico and Wright (2002) to see if the welfare-improving role of money holds. Because output is no longer divisible at the trading stage, we allow buyers and sellers to bargain over a quantity of money and the probability with which the good already produced changes hands. This is in contrast with Berentsen, Molico and Wright (2002) where agents bargain over the quantity of goods and the probability with which the indivisible unit of money changes hands. The latter model was indeed constructed as an extension to the so-called "second generation" of monetary search models (Trejos and Wright 1995; Shi 1995) in which money is indivisible by assumption. Here the goods are divisible at the production stage. But they are not at the trading stage when production takes place ahead of the market.

Given  $(\bar{q}, \bar{m}) \geq 0$ , the Nash bargaining problem is as follows:

$$\max_{\tau \leq 1, \hat{m} \leq \bar{m}} \mathcal{B}(\tau, \hat{m} | \bar{q}, \bar{m}) = [\tau u(\bar{q}) - \phi_{+1} \hat{m}]^\theta [\phi_{+1} \hat{m} - \tau (1 - \delta) \bar{q}]^{1-\theta}. \quad (27)$$

The domain of  $(\tau, \hat{m})$  is restricted to

$$\mathcal{A} = \{(\tau, \hat{m}) \in [0, 1] \times [0, \bar{m}] \mid \tau u(\bar{q}) - \phi_{+1} \hat{m} \geq 0, \phi_{+1} \hat{m} - \tau (1 - \delta) \bar{q} \geq 0\}. \quad (28)$$

$\mathcal{A}$  is non-empty, compact and  $\mathcal{B}(\tau, \hat{m} | \bar{q}, \bar{m})$  is continuous. Therefore the maximization problem is well-defined.

If either  $\bar{q} = 0$  or  $\bar{m} = 0$ , then  $(\tau, \hat{m}) = (0, 0)$  solves the problem uniquely. Consider now  $\bar{q} > 0$  and  $\bar{m} > 0$ . Recall from Lemma 1 that  $\tilde{q}$  is such that  $u(\tilde{q}) = (1 - \delta) \tilde{q}$ . If  $\bar{q} \geq \tilde{q}$ , then  $u(\bar{q}) \leq (1 - \delta) \bar{q}$  and the bargaining outcome is  $(\tau, \hat{m}) = (0, 0)$  and there is no trade. Assume now  $\bar{q} < \tilde{q}$ . In this case,  $u(\bar{q}) > (1 - \delta) \bar{q}$  and we can always find some  $(\tau, \hat{m}) \in \mathcal{A}$  such that  $\mathcal{B}(\tau, \hat{m} | \bar{q}, \bar{m}) > 0$ , and hence the Nash bargaining outcome must give agents strictly positive trade surplus.

**Proposition 8** (*Nash Bargaining Outcome with Lottery*) *The terms of trade are given by  $(\tau(\bar{q}, \bar{m}), \hat{m}(\bar{q}, \bar{m})) = \arg \max_{\tau \leq 1, \hat{m} \leq \bar{m}} \mathcal{B}(\tau, \hat{m} | \bar{q}, \bar{m}) = (\min\{g^{-1}(\phi_{+1} \bar{m}), 1\}, \min\{\frac{h(1)}{\phi_{+1}}, \bar{m}\})$ , where  $g$  and  $h$  are defined in the proof below.*

**Proof.** We divide the proof of this lemma into three steps: the first-order effect of  $\tau$  [Step 1], the first-order effect of  $\hat{m}$  [Step 2], and characterize the solution [Step 3].

**Step 1.** Taking a derivative of  $\mathcal{B}(\tau, \hat{m} | \bar{q}, \bar{m})$  w.r.t.  $\tau$ , we have

$$\mathcal{B}_\tau(\tau, \hat{m} | \bar{q}, \bar{m}) = \frac{\theta u(\bar{q})(\phi_{+1} \hat{m} - \tau (1 - \delta) \bar{q}) - (1 - \theta) (1 - \delta) \bar{q} (\tau u(\bar{q}) - \phi_{+1} \hat{m})}{[\tau u(\bar{q}) - \phi_{+1} \hat{m}]^{1-\theta} [\phi_{+1} \hat{m} - \tau (1 - \delta) \bar{q}]^\theta}. \quad (29)$$

Therefore  $\mathcal{B}_\tau(\tau, \hat{m} | \bar{q}, \bar{m}) = 0$  iff

$$\phi_{+1} \hat{m} = g(\tau) \equiv \frac{u(\bar{q}) (1 - \delta) \bar{q}}{\theta u(\bar{q}) + (1 - \theta) (1 - \delta) \bar{q}} \tau.$$

$g(\tau)$  is strictly increasing in  $[0, 1]$ . The first-order effect of  $\tau$  for any given  $\hat{m}$  can be summarized as:

$$\text{sign } \mathcal{B}_\tau(\tau, \hat{m}) = \text{sign } (g^{-1}(\phi_{+1} \hat{m}) - \tau). \quad (30)$$

**Step 2.** Taking a derivative of  $\mathcal{B}(\tau, \hat{m} | \bar{q}, \bar{m})$  w.r.t.  $\hat{m}$ , we have

$$\mathcal{B}_{\hat{m}}(\tau, \hat{m} | \bar{q}, \bar{m}) = \frac{-\theta \phi_{+1} (\phi_{+1} \hat{m} - \tau (1 - \delta) \bar{q}) + (1 - \theta) \phi_{+1} (\tau u(\bar{q}) - \phi_{+1} \hat{m})}{[\tau u(\bar{q}) - \phi_{+1} \hat{m}]^{1-\theta} [\phi_{+1} \hat{m} - \tau (1 - \delta) \bar{q}]^\theta}. \quad (31)$$

Therefore  $\mathcal{B}_{\hat{m}}(\tau, \hat{m} | \bar{q}, \bar{m}) = 0$  iff

$$\phi_{+1} \hat{m} = h(\tau) \equiv [(1 - \theta) u(\bar{q}) + \theta (1 - \delta) \bar{q}] \tau. \quad (32)$$

Accordingly, the first-order effect of  $\hat{m}$  can be summarized as:

$$\text{sign } \mathcal{B}_{\hat{m}}(\hat{q}, \hat{m}) = \text{sign} \left( \frac{h(\tau)}{\phi_{+1}} - \hat{m} \right). \quad (33)$$

**Step 3.** It is straightforward to show that  $\mathcal{B}_{\hat{q}}(\hat{q}, \hat{m}) = \mathcal{B}_{\hat{m}}(\hat{q}, \hat{m}) = 0$  iff  $(\tau, \hat{m}) = (0, 0)$ . However,  $(\tau, \hat{m}) = (0, 0)$  is not a bargaining outcome as the trade surplus is zero. So the bargaining outcome must be a corner solution: either  $\tau = 1$  or  $\hat{m} = \bar{m}$ . It is straightforward to verify that  $h(\tau) > g(\tau)$  for every  $\tau > 0$ . Consider three cases. **Case 1**  $\bar{m} \geq \frac{h(1)}{\phi_{+1}}$ . Based on the first-order effects, we conclude that the bargaining outcome is  $(\tau, \hat{m}) = (1, \frac{h(1)}{\phi_{+1}})$ . **Case 2**  $\frac{h(1)}{\phi_{+1}} > \bar{m} \geq \frac{g(1)}{\phi_{+1}}$ . Based on the first-order effects, we conclude that the bargaining outcome is  $(\tau, \hat{m}) = (1, \bar{m})$ . **Case 3**  $\frac{g(1)}{\phi_{+1}} > \bar{m}$ . Based on the first-order effects, we conclude that the bargaining outcome is  $(\tau, \hat{m}) = (g^{-1}(\phi_{+1}\bar{m}), \bar{m})$ . ■

As anticipated, higher inflation does not raise welfare. As in more standard monetary models, higher inflation simply decreases real balances and the probability with which goods change hands (cf. Figure 2).

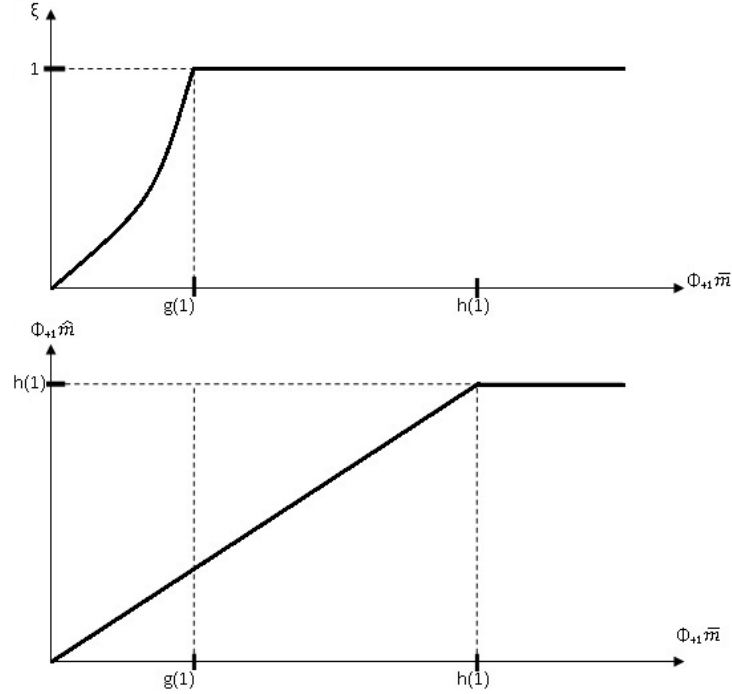


Figure 2: Lotteries.

## 6 Conclusion

We have explored the relationship between the timing of production decisions by firms, the type of goods produced (durable versus perishable), and inflation. This study was conducted within the search-theoretic model of money developed by Lagos and Wright (2005), which is explicit about market transactions and timing. Our main finding is that shifting from production on demand (the standard assumption in most macroeconomic models) to production in advance is not without loss of generality. If the economy produces mostly perishable goods, it simply leads to a reversal of monetary policy recommendations. Production on demand may then increase tractability, but such assumption is not without consequences as it effectively hides the strategic interaction between buyers and sellers.

Several extensions to the model look promising. Adding unanticipated real and nominal shocks is one of them. Also, making  $\delta$  a function of the capital stock  $k$  such that  $\delta'(k) < 0$  would make it possible for buyers to allocate their savings between money and capital, thereby creating an interesting role for monetary policy in the transition from a developing economy with a low capital stock producing mostly perishable goods to a developed economy with a larger stock of capital producing mostly durable goods. This would also address two of the main weaknesses of the model, namely that money is the only asset and that labour is the only input to sellers' production function.

# Appendix

## Proof of Lemma 1

We divide the proof of this lemma into four steps: the first-order effect of  $\hat{q}$  [Step 1], the first-order effect of  $\hat{m}$  [Step 2], characterize the unconstrained solution [Step 3] and the constrained solution [Step 4]. Figures 3 and 4 illustrate the essential features in the analysis.

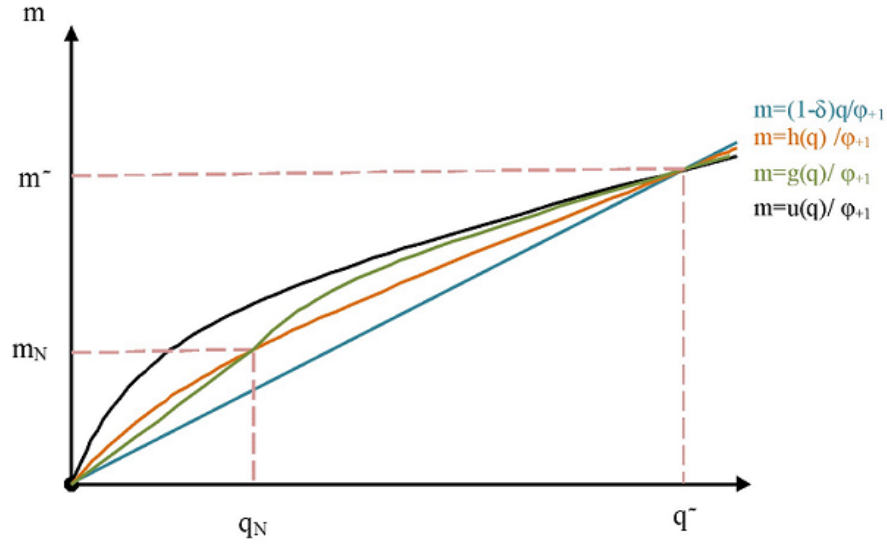


Figure 3: First-order conditions, the unconstrained Nash solution  $(m_N, q_N)$ .

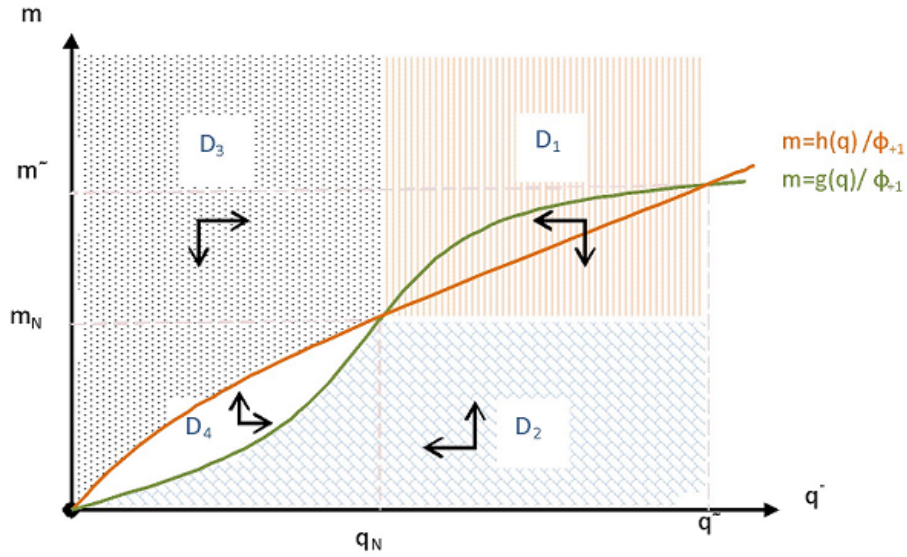


Figure 4: The bargaining solution.

**Step 1.** Taking a derivative of  $\mathcal{B}(\hat{q}, \hat{m})$  w.r.t.  $\hat{q}$ , we have  $\mathcal{B}_{\hat{q}}(\hat{q}, \hat{m}) = 0$  iff

$$\phi_{+1}\hat{m} = g(\hat{q}) \equiv \frac{(1-\delta)(1-\theta)u(\hat{q})}{\theta u'(\hat{q}) + (1-\theta)(1-\delta)} + \frac{\theta u'(\hat{q})(1-\delta)\hat{q}}{\theta u'(\hat{q}) + (1-\theta)(1-\delta)}.$$

Let  $(\tilde{q}, \tilde{m})$  be the unique positive solution of the system of equations  $u(\hat{q}) - \phi_{+1}\hat{m} = 0$  and  $\phi_{+1}\hat{m} - (1-\delta)\hat{q} = 0$ .  $(\tilde{q}, \tilde{m})$  is the northeast point of  $\mathcal{A}$  (See Figure 3). It can be shown that  $g(\hat{q})$  is strictly increasing in  $[0, \tilde{q}]$ . Thus, the inverse of  $g(\cdot)$ ,  $g^{-1}(\cdot)$ , exists in  $[0, \tilde{q}]$ . Accordingly, the first-order effect of  $\hat{q}$  for any given  $\hat{m} \in (0, g(\tilde{q}))$  can be summarized as:

$$\text{sign } \mathcal{B}_{\hat{q}}(\hat{q}, \hat{m}) = \text{sign } (g^{-1}(\phi_{+1}\hat{m}) - \hat{q}). \quad (34)$$

**Step 2.** Taking a derivative of  $\mathcal{B}(\hat{q}, \hat{m})$  w.r.t.  $\hat{m}$ , we have  $\mathcal{B}_{\hat{m}}(\hat{q}, \hat{m}) = 0$  iff

$$\phi_{+1}\hat{m} = h(\hat{q}) \equiv (1-\theta)u(\hat{q}) + \theta(1-\delta)\hat{q}. \quad (35)$$

Accordingly, the first-order effect of  $\hat{m}$  for any given  $\hat{q} \in (0, \tilde{q})$  can be summarized as:

$$\text{sign } \mathcal{B}_{\hat{m}}(\hat{q}, \hat{m}) = \text{sign } \left( \frac{h(\hat{q})}{\phi_{+1}} - \hat{m} \right). \quad (36)$$

**Step 3.** It is straightforward to show that  $\mathcal{B}_{\hat{q}}(\hat{q}, \hat{m}) = \mathcal{B}_{\hat{m}}(\hat{q}, \hat{m}) = 0$  if either  $u'(\hat{q}) = 1 - \delta$  or  $u(\hat{q}) = (1 - \delta)\hat{q}$ . The solutions that solve  $u(\hat{q}) = (1 - \delta)\hat{q}$  are ruled out as a maximizer, as  $\mathcal{B}(\hat{q}, \hat{m}) = 0$  in this case. The only candidate is then  $(q_N, m_N) \in \mathcal{A}$ , where

$$u'(q_N) = 1 - \delta \quad (37)$$

and

$$m_N = h(q_N)/\phi_{+1} = g(q_N)/\phi_{+1}. \quad (38)$$

It can be verified that  $(q_N, m_N)$  is the unique maximizer for the unconstrained Nash bargaining problem.

**Step 4.** Consider now the constrained Nash bargaining problem. Pick any  $\bar{q} > 0$  and  $\bar{m} > 0$ . We first make the following observations: (i) both  $g(\hat{q})/\phi_{+1}$  and  $h(\hat{q})/\phi_{+1}$  are convex combinations of  $u(\hat{q})/\phi_{+1}$  and  $(1-\delta)\hat{q}/\phi_{+1}$ , (ii)  $g(\hat{q}) = h(\hat{q}) = u(\hat{q}) = (1-\delta)\hat{q}$  in  $[0, \tilde{q}]$  iff  $\hat{q} = 0$  or  $\hat{q} = \tilde{q}$ , (iii)  $g(\hat{q}) = h(\hat{q})$  in  $(0, \tilde{q})$  iff  $\hat{q} = q_N$ , and (iv) it is straightforward to show that  $g(\hat{q}) < h(\hat{q})$  when  $\hat{q} \in (0, q_N)$ , and  $g(\hat{q}) > h(\hat{q})$  when  $\hat{q} \in (q_N, \tilde{q})$ . All these features are depicted in Figure 3.

To determine the bargaining solution, we partition the domain of  $(\bar{q}, \bar{m})$  into four areas (see Figure 4 for the partition). The arrows in each area indicate the trajectory towards maximization based on the first-order effects of  $\hat{q}$  and  $\hat{m}$  on  $\mathcal{B}(\hat{q}, \hat{m})$  derived in Steps 1 and 2. For example, for all  $(\hat{q}, \hat{m}) \in D_2$ , we have  $\mathcal{B}_{\hat{q}}(\hat{q}, \hat{m}) < 0$  and  $\mathcal{B}_{\hat{m}}(\hat{q}, \hat{m}) > 0$ . Hence, in order to maximize  $\mathcal{B}(\hat{q}, \hat{m})$ , one should increase  $\hat{m}$  (whenever possible) and decrease  $\hat{q}$ .

The bargaining solution can be determined as follows:

**Case 1**  $(\bar{q}, \bar{m}) \in D_1 \equiv \{(x_1, x_2) \in \mathbb{R}_+^2 | x_1 > q_N \text{ and } x_2 > m_N\}$ . As the unconstrained Nash bargaining solution  $(q_N, m_N)$  is a feasible option in this case,  $(\hat{q}, \hat{m}) = (q_N, m_N)$  is the bargaining outcome.

**Case 2**  $(\bar{q}, \bar{m}) \in D_2 \equiv \{(x_1, x_2) \in \mathbb{R}_+^2 | x_2 \leq \min\{g(x_1)/\phi_{+1}, m_N\}\}$ . Based on Step 1 and Step 2, it can be shown that  $(\hat{q}, \hat{m}) = (g^{-1}(\phi_{+1}\bar{m}), \bar{m})$  is the maximizer.

**Case 3**  $(\bar{q}, \bar{m}) \in D_3 \equiv \{(x_1, x_2) \in \mathbb{R}_+^2 | x_1 \leq \min\{h^{-1}(x_2)/\phi_{+1}, q_N\}\}$ . Based on Step 1 and Step 2, it can be shown that  $(\hat{q}, \hat{m}) = (\bar{q}, h(\bar{q})/\phi_{+1})$  is the maximizer.

**Case 4**  $(\bar{q}, \bar{m}) \in D_4 \equiv \{(x_1, x_2) \in \mathbb{R}_+^2 | x_1 < q_N, x_2 < m_N, \text{ and } h(x_1) < x_2 < g(x_1)\}$ . Based on Step 1 and Step 2, it can be shown that  $(\hat{q}, \hat{m}) = (\bar{q}, \bar{m})$  is the maximizer.

In sum, we have the following result:

$$(\hat{q}(\bar{q}, \bar{m}), \hat{m}(\bar{q}, \bar{m})) \equiv \arg \max_{\hat{q} \leq \bar{q}, \hat{m} \leq \bar{m}} \mathcal{B}(\hat{q}, \hat{m}) = (\min\{g^{-1}(\phi_{+1}\bar{m}), q_N, \bar{q}\}, \min\{h(\bar{q})/\phi_{+1}, m_N, \bar{m}\}). \quad (39)$$

### Proof of Lemma 2

First, let us denote  $\bar{q}_U$  such that  $c'(\bar{q}_U) = \sigma h'(\bar{q}_U) + (1 - \sigma)(1 - \delta)$ . Clearly  $\bar{q}_L < \bar{q}_H$ . Furthermore,  $\bar{q}_H < \bar{q}_U$  when  $\bar{q}_U < q_N$ . To see this, note from (35) that  $c'(\bar{q}_U) = \sigma h'(\bar{q}_U) + (1 - \sigma)(1 - \delta) = \sigma[(1 - \theta)u'(\bar{q}_U) + \theta(1 - \delta)] + (1 - \sigma)(1 - \delta)$ , which is greater than  $\sigma[(1 - \theta)u'(q_N) + \theta(1 - \delta)] + (1 - \sigma)(1 - \delta) = 1 - \delta = c'(\bar{q}_H)$ .

Now pick any  $\bar{m} < m_N$ . By Lemma 1, the seller's objective function can be written as:

$$\Phi(\bar{q}) = \begin{cases} -c(\bar{q}) + \sigma h(\bar{q}) + (1 - \sigma)(1 - \delta)\bar{q} & 0 \leq h(\bar{q}) \leq \phi_{+1}\bar{m} \\ -c(\bar{q}) + \sigma\phi_{+1}\bar{m} + (1 - \sigma)(1 - \delta)\bar{q} & \phi_{+1}\bar{m} \leq h(\bar{q}) \leq h[g^{-1}(\phi_{+1}\bar{m})] \\ \sigma[\phi_{+1}\bar{m} + (1 - \delta)(\bar{q} - g^{-1}(\phi_{+1}\bar{m}))] + (1 - \sigma)(1 - \delta)\bar{q} - c(\bar{q}) & h(\bar{q}) \geq h[g^{-1}(\phi_{+1}\bar{m})] \end{cases}$$

and

$$\Phi'(\bar{q}) = \begin{cases} -c'(\bar{q}) + \sigma h'(\bar{q}) + (1 - \sigma)(1 - \delta) & 0 \leq \bar{q} \leq h^{-1}(\phi_{+1}\bar{m}) \\ -c'(\bar{q}) + (1 - \sigma)(1 - \delta) & h^{-1}(\phi_{+1}\bar{m}) \leq \bar{q} \leq g^{-1}(\phi_{+1}\bar{m}) \\ -c'(\bar{q}) + (1 - \delta) & \bar{q} \geq g^{-1}(\phi_{+1}\bar{m}) \end{cases} \quad (40)$$

Hence  $\Phi'(\bar{q}) = 0$  at  $\bar{q}_U$  on  $[0, h^{-1}(\phi_{+1}\bar{m})]$ , at  $\bar{q}_L$  on  $[h^{-1}(\phi_{+1}\bar{m}), g^{-1}(\phi_{+1}\bar{m})]$ , and at  $\bar{q}_H$  on  $[g^{-1}(\phi_{+1}\bar{m}), \infty)$ , where  $\bar{q}_L < \bar{q}_H < \bar{q}_U$ . If  $\bar{q}_U < h^{-1}(\phi_{+1}\bar{m})$ , then it can be readily seen that  $\bar{q}_U$  is the unique maximizer. If  $\bar{q}_U \geq h^{-1}(\phi_{+1}\bar{m})$ , then it can be shown that the maximizer is either  $\max\{\bar{q}_L, h^{-1}(\phi_{+1}\bar{m})\}$  or  $\bar{q}_H$ .



### Proof of Proposition 1

First we show that when  $\frac{u'(0)}{g'(0)} \leq 1 + \frac{i}{\alpha}$ , a buyer has no incentive to bring money to the frictional market. If  $\bar{q} = 0$ , then it can be readily seen that  $\bar{m} = 0$  is the best response for the buyer. Now pick any  $\bar{q} > 0$ , the first order effect of  $\bar{m}$  at  $\bar{m} = 0$  on  $\Psi(\bar{m})$  is

$$\Psi_{\bar{m}}(\bar{m})|_{\bar{m}=0} = \phi_{+1}\beta\alpha\left[\frac{u'(0)}{g'(0)} - \left(1 + \frac{i}{\alpha}\right)\right], \quad (41)$$

where

$$g'(0) = \frac{(1 - \delta)u'(0)}{\theta u'(0) + (1 - \theta)(1 - \delta)}. \quad (42)$$

Hence  $\Psi_{\bar{m}}(\bar{m})|_{\bar{m}=0} \leq 0$  if  $\frac{u'(0)}{g'(0)} \leq 1 + \frac{i}{\alpha}$ . Moreover, as  $\frac{u'(x)}{g'(x)}$  is strictly decreasing on  $[0, q_N]$ , given any  $\bar{q} > 0$ ,  $\Psi_{\bar{m}}(\bar{m}) < 0$ . Thus,  $\bar{m} = 0$  is the best response for the buyer regardless of the output level  $\bar{q}$ .

Given  $\bar{m} = 0$ , the seller will pick  $\bar{q} = \bar{q}_H \equiv c'^{-1}(1 - \delta)$ . Hence  $(\bar{q}^*, \bar{m}^*) = (\bar{q}_H, 0)$  is the unique Nash equilibrium.

### Proof of Proposition 2

We first establish a sequence of claims regarding the properties and existence of a pure strategy Nash equilibrium (Claims A1-A8). Then we show that it is the unique Nash equilibrium (Claim A9).

Let  $(\bar{q}^*, \bar{m}^*)$  be a pure strategy Nash equilibrium.

**Claim A1.**  $\bar{q}^* \geq \bar{q}_L$ , where  $c'(\bar{q}_L) \equiv (1 - \sigma)(1 - \delta)$ .

**Proof.** Given any  $\bar{m} \geq 0$ ,  $\hat{m}(\bar{q}, \bar{m})$  and  $(\bar{q} - \hat{q}(\bar{q}, \bar{m}))$  are nondecreasing in  $\bar{q}$ . Then for all  $\bar{q} \in [0, \bar{q}_L)$ ,  $\Phi_{\bar{q}}(\bar{q}) \geq -c'(\bar{q}) + (1 - \sigma)(1 - \delta) > 0$ . Hence  $\bar{q} < \bar{q}_L$  cannot be chosen by the seller. ■

**Claim A2.**  $\bar{m}^* \leq \min\{g(\bar{q}^*)/\phi_{+1}, m_N\}$ , i.e.,  $(\bar{q}^*, \bar{m}^*) \in D_2$  (cf. Figure 4).

**Proof.** It suffices to show that for any  $\bar{q} \geq 0$ ,  $\bar{m} > \min\{g(\bar{q})/\phi_{+1}, m_N\}$  yields the buyer strictly lower payoff than  $\min\{g(\bar{q}), m_N\}$  does. Distinguish two cases: (i)  $\bar{q} \geq q_N$ . In this case,  $\min\{g(\bar{q})/\phi_{+1}, m_N\} = m_N$ . For any  $\bar{m} > \min\{g(\bar{q})/\phi_{+1}, m_N\} = m_N$ ,  $(\bar{q}, \bar{m}) \in D_1$  and by Lemma 1,  $\hat{q}(\bar{q}, \bar{m}) = q_N = \hat{q}(\bar{q}, m_N)$  and  $\hat{m}(\bar{q}, \bar{m}) = m_N = \hat{m}(\bar{q}, m_N)$ . Therefore  $\Psi(m_N) - \Psi(\bar{m}) = -\phi(m_N - \bar{m}) + \beta\phi_{+1}(m_N - \bar{m}) > 0$ . (ii)  $\bar{q} < q_N$ . For any  $\bar{m} > \min\{g(\bar{q})/\phi_{+1}, m_N\} = g(\bar{q})/\phi_{+1}$ ,  $(\bar{q}, \bar{m}) \in D_3 \cup D_4$  and by Lemma 1 again,  $\hat{q}(\bar{q}, \bar{m}) = \bar{q} = \hat{q}(\bar{q}, g(\bar{q})/\phi_{+1})$  and  $\hat{m}(\bar{q}, \bar{m}) = \min\{\bar{m}, h(\bar{q})/\phi_{+1}\} > \hat{m}(\bar{q}, g(\bar{q})/\phi_{+1}) = g(\bar{q})/\phi_{+1}$ . Therefore  $\Psi(g(\bar{q})/\phi_{+1}) - \Psi(\bar{m}) = (\beta\phi_{+1} - \phi)(g(\bar{q})/\phi_{+1} - \bar{m}) - \beta\alpha\phi_{+1}(g(\bar{q})/\phi_{+1} - \min\{\bar{m}, h(\bar{q})/\phi_{+1}\}) > 0$ . ■

**Claim A3.**  $(\hat{q}(\bar{q}^*, \bar{m}^*), \hat{m}(\bar{q}^*, \bar{m}^*)) = (g^{-1}(\phi_{+1}\bar{m}^*), \bar{m}^*)$ .

**Proof.** From Claim A2,  $(\bar{q}^*, \bar{m}^*) \in D_2$ , and  $(\hat{q}(\bar{q}, \bar{m}), \hat{m}(\bar{q}, \bar{m})) = (g^{-1}(\phi_{+1}\bar{m}), \bar{m})$  for all  $(\bar{q}, \bar{m})$  in  $D_2$  by Lemma 1. ■

**Claim A4.**  $\hat{q}(\bar{q}^*, \bar{m}^*) = \bar{q}^*$  only if  $\bar{q}^* = q_N$ .

**Proof.** Suppose to the contrary that  $\hat{q}(\bar{q}^*, \bar{m}^*) = \bar{q}^*$  and  $\bar{q}^* \neq q_N$ . By Claim A3,  $g^{-1}(\phi_{+1}\bar{m}^*) = \bar{q}^*$ . Thus,  $(\bar{q}^*, \bar{m}^*) = (g^{-1}(\phi_{+1}\bar{m}^*), \bar{m}^*)$  is on the boundary between  $D_2$  and  $D_4$ . Since  $\bar{q}^* \neq q_N$ ,  $\bar{q}^* < q_N$ . As  $\bar{q}^* = g^{-1}(\phi_{+1}\bar{m}^*)$  maximizes the seller's payoff given  $\bar{m}^*$ , the left hand derivative  $\Phi'_-$  at  $\bar{q}^*$  is non-negative, and the right hand derivative at  $\bar{q}^*$  is non-positive. The left hand derivative  $\Phi'_-$  at  $\bar{q}^* = g^{-1}(\phi_{+1}\bar{m}^*)$  is

$$\Phi'_-(\bar{q})|_{\bar{q}=\bar{q}^*} = -c'(\bar{q}^*) + (1 - \sigma)(1 - \delta),$$

and the right hand derivative  $\Phi'_+$  at  $\bar{q}^* = g^{-1}(\phi_{+1}\bar{m}^*)$  is

$$\Phi'_+(\bar{q})|_{\bar{q}=\bar{q}^*} = -c'(\bar{q}^*) + (1 - \delta).$$

Obviously  $\Phi'_-(\bar{q})|_{\bar{q}=\bar{q}^*} \geq 0$  implies  $\Phi'_+(\bar{q})|_{\bar{q}=\bar{q}^*} > 0$ , a contradiction. Hence we must have  $\bar{q}^* = q_N$ . ■

**Claim A5.**  $0 < \bar{m}^* < m_N$ .

**Proof.** By Claim A2,  $0 \leq \bar{m}^* \leq m_N$ . By Claim A1,  $\bar{q}^* \geq \bar{q}_L > 0$ . Then the condition  $\frac{u'(0)}{g'(0)} > 1 + \frac{i}{\alpha}$  implies that  $\bar{m}^* > 0$ . Next we show  $\bar{m}^* < m_N$  by contradiction. Suppose to the contrary that buyer's equilibrium money holding  $\bar{m}^* = m_N$ . By Claim A2,  $\bar{q}^* \geq q_N$ . Then the left derivative on  $\Psi(\bar{m})$  at  $\bar{m} = m_N$  is

$$\Psi_{\bar{m}}^-(\bar{m})|_{\bar{m}=m_N} = \phi_{+1}\beta\alpha\left[\frac{u'(q_N)}{g'(q_N)} - \left(1 + \frac{i}{\alpha}\right)\right],$$

where  $u'(q_N)/g'(q_N)$  can be calculated as

$$\frac{u'(q_N)}{g'(q_N)} = \frac{1}{1 + \frac{\theta(1-\theta)u''(q_N)[(1-\delta)q_N - u(q_N)]}{(1-\delta)^2}} < 1. \quad (43)$$

Therefore  $\Psi_{\bar{m}}^-(\bar{m})|_{\bar{m}=m_N} < 0$ , a contradiction. Hence we conclude that  $\bar{m}^* < m_N$ . ■

**Claim A6.**  $\hat{q}(\bar{q}^*, \bar{m}^*) < \bar{q}^*$ .

**Proof.** Suppose to the contrary that  $\hat{q}(\bar{q}^*, \bar{m}^*) = \bar{q}^*$ . By Claim A4,  $\bar{q}^* = q_N$ . By Claim A3,  $g^{-1}(\phi_{+1}\bar{m}^*) = \hat{q}(\bar{q}^*, \bar{m}^*) = q_N$ , which in turn implies  $\bar{m}^* = m_N$ , contradicting the fact that  $\bar{m}^* < m_N$  (Claim A5). Hence we must have  $\hat{q}(\bar{q}^*, \bar{m}^*) < \bar{q}^*$ . ■

**Claim A7.**  $\frac{u'(\hat{q}(\bar{q}^*, \bar{m}^*))}{g'(\hat{q}(\bar{q}^*, \bar{m}^*))} = \frac{u'(g^{-1}(\phi_{+1}\bar{m}^*))}{g'(g^{-1}(\phi_{+1}\bar{m}^*))} = \frac{u'(\zeta)}{g'(\zeta)} = 1 + \frac{i}{\alpha}$ .

**Proof.** By Claim A5, as  $0 < \bar{m}^* < m_N$  is an interior solution that solves the buyer's maximization problem,  $\Psi_{\bar{m}}(\bar{m})|_{\bar{m}=\bar{m}^*} = 0$ . By Claims A3 and A6,  $\hat{q}(\bar{q}^*, \bar{m}^*) = g^{-1}(\phi_{+1}\bar{m}^*) < \bar{q}^*$  is not binding. We then have

$$\Psi_{\bar{m}}(\bar{m})|_{\bar{m}=\bar{m}^*} = \phi_{+1}\beta\alpha\left[\frac{u'(\hat{q}(\bar{q}^*, \bar{m}^*))}{g'(\hat{q}(\bar{q}^*, \bar{m}^*))} - \left(1 + \frac{i}{\alpha}\right)\right]. \quad (44)$$

Hence

$$\frac{u'(\hat{q}(\bar{q}^*, \bar{m}^*))}{g'(\hat{q}(\bar{q}^*, \bar{m}^*))} = \frac{u'(g^{-1}(\phi_{+1}\bar{m}^*))}{g'(g^{-1}(\phi_{+1}\bar{m}^*))} = \frac{u'(\zeta)}{g'(\zeta)} = 1 + \frac{i}{\alpha}. \quad (45)$$

■

**Claim A8.**  $\bar{q}^* = \bar{q}_H \equiv c'^{-1}(1 - \delta)$ .

**Proof.** Since  $\bar{q}^* > \hat{q}(\bar{q}^*, \bar{m}^*)$  maximizes the seller's payoff, the first-order effect vanishes at  $\bar{q}^*$ . Using Claim A3, we have

$$\Phi_{\bar{q}}(\bar{q})|_{\bar{q}=\bar{q}^*} = -c'(\bar{q}^*) + (1 - \delta) = 0. \quad (46)$$

Hence  $\bar{q}_H$  is the unique local maximizer on  $[\zeta, \infty)$ . The condition  $g(\zeta)/\phi_{+1} \leq \bar{m}_C$  guarantees that  $\bar{q}_H$  is indeed a global maximizer given buyer's money holding  $\bar{m}^* = g(\zeta)/\phi_{+1}$ . Hence  $\bar{q}^* = \bar{q}_H$ . ■

**Claim A9.**  $(\bar{q}^*, \bar{m}^*) = (\bar{q}_H, g(\zeta)/\phi_{+1})$  is the unique Nash equilibrium.

**Proof.** Consider two cases: (i)  $g(\zeta)/\phi_{+1} < \bar{m}_C$ . First we observe that the buyer will not bring more than  $g(\zeta)/\phi_{+1}$  of money holding to the frictional market, and hence the seller's best response is always to produce  $\bar{q}_H$ . Given  $\bar{q} = \bar{q}_H$ , the buyer's best response is then to bring  $g(\zeta)/\phi_{+1}$  to the market. (ii)  $g(\zeta)/\phi_{+1} = \bar{m}_C$ . In this case, the seller may randomize between  $\bar{q}_L$  and  $\bar{q}_H$ , as the seller is indifferent between these two options when  $\bar{m} = \bar{m}_C$ . Suppose there is a mixed-strategy equilibrium in which the seller randomizes between  $\bar{q}_L$  and  $\bar{q}_H$ . As the seller chooses  $\bar{q}_H$  with a probability less than one, the buyer's best response is to bring some  $\bar{m} < g(\zeta)/\phi_{+1} = \bar{m}_C$ . But given  $\bar{m} < \bar{m}_C$ , the seller will simply bring  $\bar{q}_H$  to the market, a contradiction. Hence  $(\bar{q}_H, g(\zeta)/\phi_{+1})$  is the unique Nash equilibrium. ■

### Proof of Proposition 3

Given a mixed strategy  $\mu : B_{\mathbb{R}_+} \rightarrow [0, 1]$ , let  $\mathcal{F}(x) = \mu([0, x])$  denote the distribution function of  $\mu$ .  $\mathcal{F}$  is increasing, right continuous, and differentiable almost everywhere. Let  $\{\omega_i\}$  with  $\omega_i < \omega_{i+1}$  be the collection of points in  $[0, \infty)$  at which  $\mathcal{F}$  is not differentiable. Denote by  $f(x) \equiv \mathcal{F}'(x)$  whenever  $\mathcal{F}'(x)$  exists, and assume  $\sup\{f(x)|x \in \mathbb{R}_+ \setminus \{\omega_i\}\} < \infty$ . Then  $\mathcal{F}$  is absolutely continuous on  $[0, \infty) \setminus \{\omega_i\}$ , and  $\mathcal{F}(b) - \mathcal{F}(a) = \int_a^b f(x)dx$  for any  $[a, b] \subset (\omega_i, \omega_{i+1})$ .

Now denote by  $\mathcal{F}_b : \mathbb{R}_+ \rightarrow [0, 1]$  as the distribution function induced by the buyer's mixed strategy  $\mu_b$ , and  $\mathcal{F}_s : \mathbb{R}_+ \rightarrow [0, 1]$  as the distribution function induced by the seller's mixed strategy  $\mu_s$ .

**Claim B1.**  $\mu_b$  is degenerate, i.e.,  $\mu_b(\bar{m}^*) = 1$  for some  $\bar{m}^* \in [g(\bar{q}_L)/\phi_{+1}, g(\bar{q}_H)/\phi_{+1}]$ .

**Proof.** Recall that  $\text{supp } \mu_s \subset [\bar{q}_L, \bar{q}_H]$ . Let  $\{\omega_i\}$  with  $\omega_i < \omega_{i+1}$  be the collection of points in  $[\bar{q}_L, \bar{q}_H]$  at which  $\mathcal{F}_s$  is not differentiable. Denote by  $\int_{\omega_i+}^{\omega_{i+1}-} f(x)dx \equiv \lim_{y \downarrow \omega_i} \lim_{z \uparrow \omega_{i+1}} \int_y^z f(x)dx = \mathcal{F}(\omega_{i+1}-) - \mathcal{F}(\omega_i)$ . We show the following:

(i)  $\Psi_{\bar{m}}$  ( $\Psi_{\bar{m}}^+$  when it is not differentiable) is strictly decreasing on  $[g(\bar{q}_L)/\phi_{+1}, g(\bar{q}_H)/\phi_{+1}]$ . Pick any  $j \in \mathbb{Z}$ . The buyer's payoff at  $\bar{m} \in (g(\omega_j)/\phi_{+1}, g(\omega_{j+1})/\phi_{+1})$  is

$$\Psi(\bar{m}) = -\phi\bar{m} + \beta \left\{ \alpha \left\{ \begin{aligned} &\sum_{i=-\infty}^{j-1} \int_{\omega_i+}^{\omega_{i+1}-} u(x) d\mathcal{F}_s + \sum_{i=-\infty}^j u(\omega_i) [\mathcal{F}_s(\omega_i) - \mathcal{F}_s(\omega_i-)] \\ &+ \int_{\omega_j+}^{g^{-1}(\phi_{+1}\bar{m})} u(x) d\mathcal{F}_s + [1 - \mathcal{F}_s(g^{-1}(\phi_{+1}\bar{m}))] u(g^{-1}(\phi_{+1}\bar{m})) \end{aligned} \right\} \right\} + [1 - \alpha] \phi_{+1} \bar{m},$$

hence

$$\Psi_{\bar{m}}(\bar{m}) = \phi_{+1} \beta \alpha \left\{ [1 - \mathcal{F}_s(g^{-1}(\phi_{+1}\bar{m}))] \frac{u'(g^{-1}(\phi_{+1}\bar{m}))}{g'(g^{-1}(\phi_{+1}\bar{m}))} - (1 + \frac{i}{\alpha}) \right\}.$$

Since  $\mathcal{F}_s$  is non-decreasing and  $u'/g'$  is strictly decreasing,  $\Psi_{\bar{m}}$  is strictly decreasing on  $(g(\omega_j)/\phi_{+1}, g(\omega_{j+1})/\phi_{+1})$  for every  $j \in \mathbb{Z}$ .<sup>12</sup> Furthermore,  $\mathcal{F}_s$  is right continuous implies  $\Psi_{\bar{m}}$  is right continuous. The right derivative of  $\Psi$  at  $g(\omega_j)/\phi_{+1}$  is no greater than the left derivative of  $\Psi$  at  $g(\omega_j)/\phi_{+1}$ :

$$\Psi_{\bar{m}}^+(\frac{g(\omega_j)}{\phi_{+1}}) = \phi_{+1} \beta \alpha \left\{ [1 - \mathcal{F}_s(\omega_j)] \frac{u'(\omega_j)}{g'(\omega_j)} - (1 + \frac{i}{\alpha}) \right\} \quad (47)$$

$$\leq \phi_{+1} \beta \alpha \left\{ [1 - \mathcal{F}_s(\omega_{j-})] \frac{u'(\omega_j)}{g'(\omega_j)} - (1 + \frac{i}{\alpha}) \right\} \quad (48)$$

$$= \lim_{\bar{m} \rightarrow \frac{g(\omega_j)}{\phi_{+1}}-} \phi_{+1} \beta \alpha \left\{ [1 - \mathcal{F}_s(g^{-1}(\phi_{+1}\bar{m}))] \frac{u'(g^{-1}(\phi_{+1}\bar{m}))}{g'(g^{-1}(\phi_{+1}\bar{m}))} - (1 + \frac{i}{\alpha}) \right\} \quad (49)$$

$$= \Psi_{\bar{m}}^-(\frac{g(\omega_j)}{\phi_{+1}}). \quad (50)$$

As this holds for any  $j \in \mathbb{Z}$ ,  $\Psi_{\bar{m}}$  ( $\Psi_{\bar{m}}^+$  when it is not differentiable) is strictly decreasing on  $[g(\bar{q}_L)/\phi_{+1}, g(\bar{q}_H)/\phi_{+1}]$ .

(ii)  $\Psi(\bar{m})$  is continuous. It suffices to check the continuity at  $g(\omega_{j+1})/\phi_{+1}$  for each  $j$ . Pick any  $j \in \mathbb{Z}$ . Then  $\Psi(g(\omega_{j+1})/\phi_{+1})$

$$= -\phi g(\omega_{j+1})/\phi_{+1} + \beta \left\{ \alpha \left\{ \begin{aligned} &\sum_{i=-\infty}^j \int_{\omega_i+}^{\omega_{i+1}-} u(x) d\mathcal{F}_s + \sum_{i=-\infty}^{j+1} u(\omega_i) [\mathcal{F}_s(\omega_i) - \mathcal{F}_s(\omega_i-)] \\ &+ [1 - \mathcal{F}_s(\omega_{j+1})] u(\omega_{j+1}) \\ &+ (1 - \alpha) \phi_{+1} g(\omega_{j+1})/\phi_{+1} \end{aligned} \right\} \right\} \quad (51)$$

$$= g(\omega_{j+1})/\phi_{+1} [-\phi + \beta(1 - \alpha)\phi_{+1}] + \beta \alpha \left\{ \sum_{i=-\infty}^j \int_{\omega_i+}^{\omega_{i+1}-} u(x) d\mathcal{F}_s + \sum_{i=-\infty}^j u(\omega_i) [\mathcal{F}_s(\omega_i) - \mathcal{F}_s(\omega_i-)] + [1 - \mathcal{F}_s(\omega_{j+1}-)] u(\omega_{j+1}) \right\} \quad (52)$$

$$= \lim_{\bar{m} \rightarrow g(\omega_{j+1})/\phi_{+1}-} \Psi(\bar{m});$$

hence  $\Psi(\bar{m})$  is continuous.

<sup>12</sup>The condition  $\bar{q}_L > h^{-1}(g(\bar{q}_H))$  implies that  $\hat{m} = \bar{m}$  for any  $\bar{m}$  in the supp  $\mu_b$  (Lemma 1).

Combining (i) and (ii), we conclude that given  $\mu_s$ , the optimal level of money holding is unique. Therefore  $\mu_b$  is degenerate. ■

**Claim B2.**  $\mu_s$  is binary. More specifically,  $\mu_s(\bar{q}_L) > 0$ ,  $\mu_s(\bar{q}_H) > 0$ , and  $\mu_s(\bar{q}_L) + \mu_s(\bar{q}_H) = 1$ . **Proof.** Recall that the condition  $\bar{q}_L \geq h^{-1}(g(\bar{q}_H))$  implies that  $\bar{q}_H < q_N$ . When  $\bar{m} = g(\bar{q}_H)/\phi_{+1}$ , it can be readily verified that  $\Phi(\bar{q}_H) < \Phi(\bar{q}_L)$  (as  $\Phi_{\bar{q}}$  is always negative for all  $\bar{q} > \bar{q}_L$ ). On the other hand, when  $\bar{m} = g(\bar{q}_L)/\phi_{+1}$ ,  $\Phi(\bar{q}_H) > \Phi(\bar{q}_L)$ . Let  $\hat{m} \in (g(\bar{q}_L)/\phi_{+1}, g(\bar{q}_H)/\phi_{+1})$  be such that the seller is indifferent between choosing  $\bar{q}_L$  and  $\bar{q}_H$  when  $\bar{m} = \hat{m}$  (see Figure 5), that is,  $\hat{m}$  solves

$$\begin{aligned} & -c(\bar{q}_L) + \sigma\phi_{+1}\hat{m} + (1-\sigma)(1-\delta)\bar{q}_L \\ = & -c(\bar{q}_H) + \sigma[\phi_{+1}\hat{m} + (1-\delta)(\bar{q}_H - g^{-1}(\phi_{+1}\hat{m}))] + (1-\sigma)(1-\delta)\bar{q}_H; \end{aligned} \quad (53)$$

consequently

$$\hat{m} = \frac{1}{\phi_{+1}}g(\bar{q}_H - \frac{\int_{\bar{q}_L}^{\bar{q}_H} [c'(x) - (1-\sigma)(1-\delta)]dx}{\sigma(1-\delta)}). \quad (54)$$

Since  $(1-\sigma)(1-\delta) < c'(x) < (1-\delta)$  for  $x \in (\bar{q}_L, \bar{q}_H)$ ,  $\bar{q}_L < \int_{\bar{q}_L}^{\bar{q}_H} [c'(x) - (1-\sigma)(1-\delta)]dx / \sigma(1-\delta) < \bar{q}_H$ . Therefore  $\hat{m}$  is indeed in  $(g(\bar{q}_L)/\phi_{+1}, g(\bar{q}_H)/\phi_{+1})$ . The seller's best response on  $[g(\bar{q}_L)/\phi_{+1}, g(\bar{q}_H)/\phi_{+1}]$  can be summarized as follows:

$$\bar{q}(\bar{m}) = \begin{cases} \bar{q}_H & \bar{m} < \hat{m} \\ \bar{q}_L \text{ or } \bar{q}_H & \bar{m} = \hat{m} \\ \bar{q}_L & \bar{m} > \hat{m} \end{cases}. \quad (55)$$

By Claim B1, in equilibrium  $\mu_b(\bar{m}^*) = 1$  for some  $\bar{m}^* \in [g(\bar{q}_L)/\phi_{+1}, g(\bar{q}_H)/\phi_{+1}]$ . As there exists no pure strategy equilibria, in equilibrium we must have  $\bar{m}^* = \hat{m}$ , and the seller randomizes between  $\bar{q}_L$  and  $\bar{q}_H$  with  $\mu_s(\bar{q}_L) > 0$ ,  $\mu_s(\bar{q}_H) > 0$ , and  $\mu_s(\bar{q}_L) + \mu_s(\bar{q}_H) = 1$ . ■

Combining Claims B1 and B2, we can now construct the equilibrium. Based on Claim B2, the buyer's objective function for any  $\bar{m} \in [g(\bar{q}_L)/\phi_{+1}, g(\bar{q}_H)/\phi_{+1}]$  can be written as:

$$\Psi(\bar{m}) = \mu_s(\bar{q}_H) \{-\phi\bar{m} + \beta[\alpha u(g^{-1}(\phi_{+1}\bar{m})) + (1-\alpha)\phi_{+1}\bar{m}]\} \quad (56)$$

$$\begin{aligned} & + \mu_s(\bar{q}_L) \{-\phi\bar{m} + \beta[\alpha u(\bar{q}_L) + (1-\alpha)\phi_{+1}\bar{m}]\} \\ = & [\beta(1-\alpha)\phi_{+1} - \phi]\bar{m} + \beta\alpha[\mu_s(\bar{q}_H)u(g^{-1}(\phi_{+1}\bar{m})) + \mu_s(\bar{q}_L)u(\bar{q}_L)]; \end{aligned} \quad (57)$$

therefore

$$\Psi_{\bar{m}}(\bar{m}) = \phi_{+1}\beta\alpha \left\{ \mu_s(\bar{q}_H) \frac{u'(g^{-1}(\phi_{+1}\bar{m}))}{g'(g^{-1}(\phi_{+1}\bar{m}))} - (1 + \frac{i}{\alpha}) \right\}. \quad (58)$$

As  $\bar{m}^* = \hat{m}$  in equilibrium, we must have  $\Psi_{\bar{m}}(\hat{m}) = 0$ . Accordingly,

$$\mu_s(\bar{q}_H) = (1 + \frac{i}{\alpha}) \frac{g'(g^{-1}(\phi_{+1}\hat{m}))}{u'(g^{-1}(\phi_{+1}\hat{m}))}. \quad (59)$$

To sum up, the pair  $(\mu_s, \mu_b)$  constructed below constitutes a unique Nash equilibrium:

$$\mu_b(\bar{m}) = \begin{cases} 1 & \bar{m} = \hat{m} \\ 0 & \bar{m} \neq \hat{m} \end{cases} \quad (60)$$

$$\mu_s(\bar{q}) = \begin{cases} (1 + \frac{i}{\alpha}) \frac{g'(g^{-1}(\phi_{+1}\hat{m}))}{u'(g^{-1}(\phi_{+1}\hat{m}))} & \bar{q} = \bar{q}_H \\ 1 - (1 + \frac{i}{\alpha}) \frac{g'(g^{-1}(\phi_{+1}\hat{m}))}{u'(g^{-1}(\phi_{+1}\hat{m}))} & \bar{q} = \bar{q}_L \end{cases} . \quad (61)$$

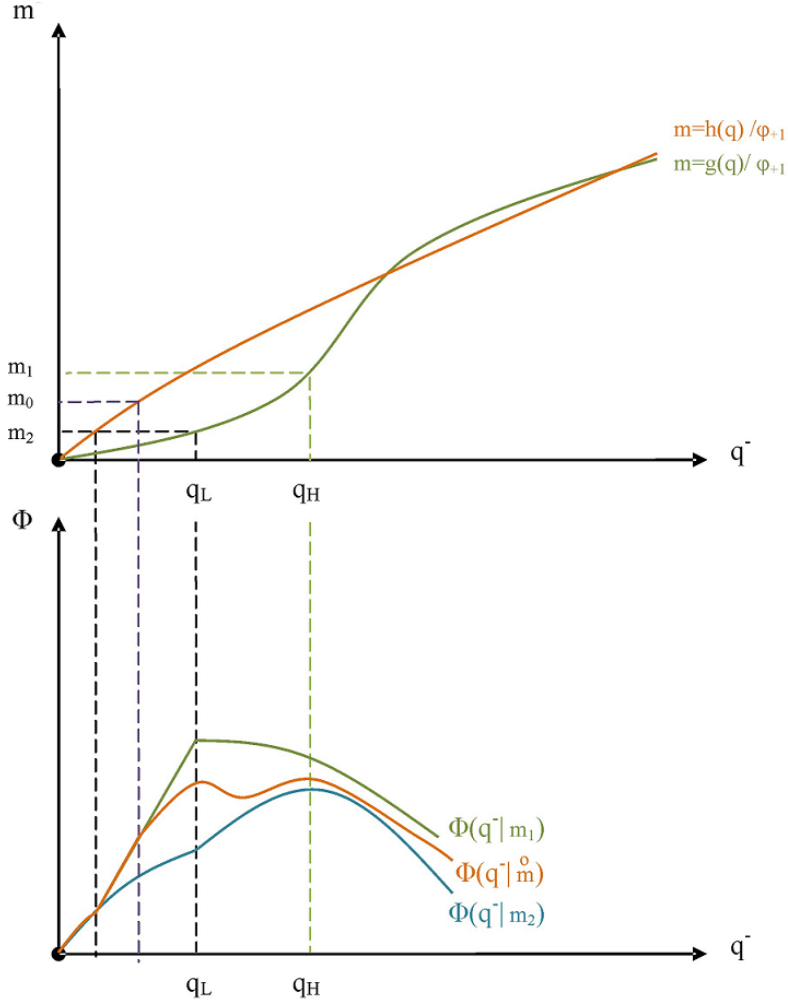


Figure 5: Type III Equilibrium.

### Proof of Proposition 6

First note from (53) that  $-c(\bar{q}_H) + \sigma(1 - \delta)(\bar{q}_H - \hat{q}) + (1 - \sigma)(1 - \delta)\bar{q}_H = -c(\bar{q}_L) + (1 - \sigma)(1 - \delta)\bar{q}_L$ . It follows that  $A = -c(\bar{q}_H) + \sigma[u(\hat{q}) + (1 - \delta)(\bar{q}_H - \hat{q})] + (1 - \sigma)(1 - \delta)\bar{q}_H > -c(\bar{q}_L) + \sigma u(\bar{q}_L) + (1 - \sigma)(1 - \delta)\bar{q}_L = B$ . Second, since neither  $A$  or  $B$  is a function of  $i$  and

$\mu_s(\bar{q}_L) = 1 - \mu_s(\bar{q}_H)$  we have  $\frac{\partial \mathcal{W}_{PIA-III}}{\partial i} = \frac{\partial \mu_s(\bar{q}_H)}{\partial i} (A - B) = \frac{g'(g^{-1}(\phi_{+1}\hat{m}))}{\alpha u'(g^{-1}(\phi_{+1}\hat{m}))} (A - B) > 0$ .

### Proof of Proposition 7

From Propositions 2 and 3 the optimal interest rate  $i^*$  is such that  $g[\zeta(i^*, \delta)] = \phi_{+1}\hat{m}$ , or equivalently  $\zeta(i^*, \delta) = g^{-1}(\phi_{+1}\hat{m})$ , where  $\hat{m}$  is given by (54) and  $\zeta$  is given by (21). Note that  $\zeta$  is also a function of  $\delta$  via the  $g$  function. Multiplying both sides by  $\sigma(1 - \delta)$  we have

$$\sigma(1 - \delta)\zeta(i^*, \delta) = \sigma(1 - \delta)\bar{q}_H - \int_{\bar{q}_L}^{\bar{q}_H} [c'(x) - (1 - \sigma)(1 - \delta)]dx.$$

Totally differentiating the equality we extract

$$\frac{di^*}{d\delta} = \frac{\bar{q}_H - [(1 - \sigma)\bar{q}_L + \sigma\zeta] - \frac{d\bar{q}_H}{d\delta} [1 - \delta - c'(\bar{q}_H)] + \frac{d\bar{q}_L}{d\delta} [(1 - \sigma)(1 - \delta) - c'(\bar{q}_L)] + \sigma \frac{\partial \zeta}{\partial \delta} (1 - \delta)}{-(1 - \delta)\sigma \frac{\partial \zeta}{\partial i^*}}.$$

Using (19) and (20) this simplifies into

$$\frac{di^*}{d\delta} = \frac{\bar{q}_H - [(1 - \sigma)\bar{q}_L + \sigma\zeta] + \sigma \frac{\partial \zeta}{\partial \delta} (1 - \delta)}{-(1 - \delta)\sigma \frac{\partial \zeta}{\partial i^*}}.$$

Since  $\bar{q}_H > [(1 - \sigma)\bar{q}_L + \sigma\zeta]$ ,  $\frac{\partial \zeta}{\partial \delta} > 0$  and  $\frac{\partial \zeta}{\partial i^*} < 0$ , we have  $\frac{di^*}{d\delta} > 0$ .

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