Steady states of lattice population models with immigration

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Abstract

In a lattice population model where individuals evolve as subcritical branching random walks subject to external immigration, the cumulants are estimated and the existence of the steady state is proved. The resulting dynamics are Lyapunov stable in that their qualitative behavior does not change under suitable perturbations of the main parameters of the model. An explicit formula of the limit distribution is derived in the solvable case of no birth. Monte Carlo simulation shows the limit distribution in the solvable case.

Keywords: spatial population dynamics; branching random walk; immigration; correlation functions; steady state; Lyapunov stability

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Short title:

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1 Introduction

The Galton–Watson process is a simple branching process (Watson and Galton, 1875), devoid of spatial dynamic. Models where particles can move randomly are called branching random walks. Branching random walks can be used, for example, in the modeling of viral epidemics (Ermakova et al., 2019). To determine the regime, towards which an epidemic is tending, one computes a limit distribution, which corresponds to a steady state, associated with the model. The question, which we address here, of the existence of such a limit distribution, is therefore fundamental.

Molchanov and Whitmeyer (2017) proved the existence of a steady state for the critical branching process with binary splitting and transient underlying random walk on the lattice $\mathbb{Z}^d$. Chernousova and Molchanov (2018) extended Molchanov and Whitmeyer (2017) by considering an arbitrary total number of offspring which spread randomly in space around the parental particle. These authors proved the existence of a limit distribution of the population under the following additional conditions: 1) the tail of the distribution of the total number of offspring decreases at least geometrically; 2) the sum of the generator associated with migration of each particle and the generator associated with the spreading of offspring is a generator of a transient random walk. Critical branching processes are unstable with respect to small perturbations of the birth and death rates. Han, Molchanov, and Whitmeyer (2017) and Han et al. (2017) introduced immigration, which can stabilize the population size when the birth rate is less than the mortality rate (subcritical case). These authors proved the existence of limits for the first two moments, but that does not prove the existence of a steady state. We extend their analysis of a subcritical random walk with immigration in proving the existence of a steady state and its stability in the Lyapunov sense. Our proof is based on Molchanov and Whitmeyer (2017), who estimated limits for all moments of the total population and used Carleman conditions (Feller, 1971, Sect. VII.3) to establish a unique limit distribution. For simplicity, we consider binary splitting as in Han, Molchanov, and Whitmeyer (2017). Based on Chernousova and Molchanov (2018), we prove a unique limit distribution in the model of Han et al. (2017) with arbitrary total number of offspring under the additional condition that the tail of the distribution of the total number of offspring decreases at least geometrically.

Instead of $\mathbb{Z}^d$, Chernousova et al. (2019) explored the continuous-time and continuous-space subcritical branching process subject to immigration in $\mathbb{R}^d$ and proved the existence of a steady state and its stability. The methods used in the proof are different in a lattice model and in a continuous-space model.

Yarovaya (2013) analyzed the limit behavior of all moments for the total population in a branching random walk with a finite total number of branching sources of different types. Khristolyubov and Yarovaya (2019) did the same for supercritical branching random walks.

Individuals move on the lattice $\mathbb{Z}^d$ as independent random walks (Han, Molchanov, and Whitmeyer, 2017), subject to splitting or duplication at rate $\beta > 0$ and mortality at rate $\mu > 0$. The critical case corresponds to $\beta = \mu$. The random walk X on $\mathbb{Z}^d$ is governed by the generator

$$\mathcal{L}_a f(x) = \kappa \sum_{z \in \mathbb{Z}^d \setminus \{0\}} \big(f(x+z) - f(x)\big) a(z)\,,$$
$$a(z) \geq 0\,, \qquad \sum_{z \in \mathbb{Z}^d \setminus \{0\}} a(z) = 1\,, \qquad \sum_{z \in \mathbb{Z}^d \setminus \{0\}} z\, a(z) = 0\,, \tag{1}$$

where $a(\,\cdot\,)$ is a suitable zero-mean probability kernel. The population size at site $y \in \mathbb{Z}^d$ at time $t \geq 0$ is $N(t, y)$.

For $\beta = \mu$, if X is a transient Markov process, then, as $t \to \infty$, the particle field $N(t, y)$ converges in law to a limit field $N^*(y)$, which is a steady state (Han, Molchanov, and Whitmeyer, 2017). If X is recurrent, no steady state exists and, as $t \to \infty$, the field $N(t, y)$ clusterizes: as time goes on, particles form larger and larger clusters farther and farther away from each other.

Although steady states may exist in the transient case, such critical processes are unstable under arbitrarily small random perturbations affecting its parameters. Namely, a statistical equilibrium disappears once the previously constant rates are replaced by $\beta(x, \omega) = \beta_0 + \varepsilon\xi(x, \omega)$ and $\mu(x, \omega) = \mu_0 + \varepsilon\eta(x, \omega)$, where $\beta_0 = \mu_0$,

$\varepsilon > 0$ is a small parameter, and the random pairs $\big(\xi(x,\omega), \eta(x,\omega)\big)$ are independent of one another for different locations x and have a symmetric distribution (say, on $[-1,1]^2$). This phenomenon is related to individual localization theorems for random Schrödinger operators (Molchanov, 1994; Molchanov and Whitmeyer, 2017).

We address a class of lattice population models with immigration, for which the steady state exists and is stable in the Lyapunov sense, which means for sufficiently small (in L_∞-norm) perturbations affecting the parameters. Unlike the continuous-time continuous-space model in Chernousova et al. (2019), here in the lattice case, several individuals can successively occupy the same location, which leads to more complex combinations.

After presenting the model in section 2, we solve a case without splitting mechanism ($\beta = 0$) in section 3. For the general case $\beta \geq 0$ in section 4, we rely on the connection between moments and cumulants. Together with Carleman type bounds, this connection provides the uniqueness of the limit state. In section 5, we extend these results to space-dependent bounded rates $\beta(x)$ and $\mu(x)$ satisfying $0 < \Delta_1 \leq \mu(x) - \beta(x) \leq \Delta_2 < \infty$ for all $x \in \mathbb{Z}^d$, where Δ_1 and Δ_2 are constants. Thus the steady state is stable in the strongest Lyapunov sense, which means that the stochastic equilibrium survives under sufficiently small perturbations of the rates.

2 Model

We consider the population as a particle field $\big(N(t,y)\big)_{t\geq 0, y\in\mathbb{Z}^d}$. Individual particles independently of one another die at rate μ or split into two at rate β, and, between these events, move around as random walks with generator $\mathcal{L}_a$ in Eq. (1) with a suitable kernel $a(\,\cdot\,)$. The system is subcritical ($\mu > \beta$) and is subject to external immigration at rate $\gamma > 0$.

The random walk X describes independent movements of individual particles between death or splitting events. Its generator is $\mathcal{L}_a$ in Eq. (1), where the kernel $a(\,\cdot\,)$ is symmetric: $a(z) = a(-z)$ for all $z \in \mathbb{Z}^d \setminus \{0\}$. X is supported on the whole lattice, which is equivalent to positivity of the transition probability:

$$p(t,x,y) = \mathsf{P}_x\left(X(t) = y\right) > 0\,, \tag{2}$$

for all x, $y \in \mathbb{Z}^d$ and $t \geq 0$.

In terms of Fourier expansion,

$$\widehat{\mathcal{L}}_a(k) = \kappa\Big(\sum_{z\in\mathbb{Z}^d} \cos(kz)a(z) - 1\Big)\,, \tag{3}$$

where $k \in [-\pi,\pi]^d =: T^d$. The transition probability is the inverse Fourier transform

$$p(t,x,y) = \frac{1}{(2\pi)^d} \int_{T^d} e^{t\widehat{\mathcal{L}}_a(k)} e^{-ik(y-x)}\, dk. \tag{4}$$

It satisfies

$$\begin{aligned} p(t,x,y) &= p(t,y,x) = p(t,y-x,0) = p(t,x-y,0)\,, \\ p(t,x,y) &\le p(t,x,x) = p(t,0,0) = \frac{1}{(2\pi)^d} \int_{T^d} e^{t\widehat{\mathcal{L}}_a(k)}\, dk\,, \\ \sum_{x\in\mathbb{Z}^d} p(t,x,y) &= \sum_{x\in\mathbb{Z}^d} p(t,y,x) = 1\,. \end{aligned} \tag{5}$$

The inequality $p(t,x,y) \le p(t,x,x)$ results from the fact that $\widehat{\mathcal{L}}_a(k)$ in Eq. (3) is real and $|\widehat{\mathcal{L}}_a(k)e^{-ik(y-x)}| = \widehat{\mathcal{L}}_a(k)$.

In the time interval $[t, t+dt)$, each particle can die independently of one another with probability $\mu\, dt$ or split into two particles with probability $\beta\, dt$ at the same site. The subcriticality assumption

$$\Delta = \mu - \beta > 0 \tag{6}$$

means that the initial configuration vanishes at a random finite future time: for each $y \in \mathbb{Z}^d$, there is a finite random time τ_y such that $N(t,y) = 0$ for $t \ge \tau_y$.

Indeed, under Eq. (6) and a constant (not random) initial population, say

$N(0,y) \equiv 1$,

$$m_1(t,y) = \mathsf{E}\big(N(t,y)\big) \tag{7}$$

solves the forward Kolmogorov equation

$$\frac{\partial m_1}{\partial t} = \mathcal{L}_a m_1 - \Delta m_1\,, \qquad m_1(0,y) = 1\,, \tag{8}$$

so that

$$m_1(t,y) = e^{-\Delta t} \to 0\,, \qquad \text{as } t \to \infty\,. \tag{9}$$

Consider $\{t_k\}_{k=1}^{\infty}$ an increasing sequence tending to infinity fast enough such that

$$\sum_{k=1}^{\infty} e^{-\Delta t_k} < \infty \tag{10}$$

and $\{A_k(\Gamma)\}_{k=1}^{\infty}$ the sequence of events $A_k(\Gamma) = \left\{\max_{y\in\Gamma} N(t_k,y) \geq 1\right\}$. From Chebyshev–Markov inequality (Feller, 1968, chap. IX) for any $t > 0$ and $y \in \mathbb{Z}^d$,

$$\mathsf{P}\left(N(t,y) \geq 1\right) \leq m_1(t,y). \tag{11}$$

Eq. (9) and (11) lead to

$$\mathsf{P}\left(A_k(\Gamma)\right) \leq |\Gamma| e^{-\Delta t_k}, \tag{12}$$

and, due to Eq. (10),

$$\sum_{k=1}^{\infty} \mathsf{P}\left(A_k(\Gamma)\right) < \infty. \tag{13}$$

Thus from the Borel-Cantelli lemma (Feller, 1968, chap. VIII.3) events $A_k(\Gamma)$ occur with probability one only in finite total number: there is a finite random time τ_Γ such that $N(t, y) = 0$ for all $y \in \Gamma$ and $t \geq \tau_\Gamma$. Equivalently, the particle field vanishes at a random finite future time.

For each $x \in \mathbb{Z}^d$, we represent external immigration as a Poissonian point field $\left\{\tau_i(x)\right\}_{i>0}$ on $\{x\} \times [0, \infty)$ with parameter γ. Given $x \in \mathbb{Z}^d$, immigrant particles arrive at times $\left\{\tau_i(x)\right\}_{i>0}$, where $0 = \tau_0(x) < \tau_1(x) < \tau_2(x) < \ldots$ and the differences $\tau_{i+1}(x) - \tau_i(x)$ are independent random variables following an $\mathsf{Exp}(\gamma)$ distribution. We write $\xi \sim \mathsf{Exp}(\gamma)$ if $\mathsf{P}(\xi > a) = e^{-\gamma a}$ for all $a \geq 0$. For different $x \in \mathbb{Z}^d$, the corresponding Poissonian point fields are assumed independent of one another.

Individual sub-populations, each one being generated by an individual existing at time $t = 0$, decay exponentially as $t \to \infty$. We thus assume that $N(0, y) \equiv 0$ for all $y \in \mathbb{Z}^d$. In the model with immigration, the first moment $m_1(t, y) = \mathsf{E}\big(N(t, y)\big)$ solves the forward Kolmogorov equation

$$\frac{\partial m_1}{\partial t} = \mathcal{L}_a m_1 - \Delta m_1 + \gamma\,, \qquad m_1(0, y) = 0\,, \tag{14}$$

and thus satisfies

$$m_1(t,y) \equiv \gamma \int_0^t e^{-\Delta s}\, ds \to \frac{\gamma}{\Delta} \qquad \text{as } t \to \infty\,. \tag{15}$$

For fixed $x \in \mathbb{Z}^d$ and $\tau_i(x) < t$, $n(t-\tau_i(x), x, y)$ is the total number of individuals at $y \in \mathbb{Z}^d$ at time t descending from the common ancestor who immigrated to x at time $\tau_i(x)$. Then, with $N_x(t,y)$ denoting the total number of individuals at y at time t, whose ancestors immigrated to x at $s \in [0,t)$:

$$N_x(t,y) := \sum_{\tau_i(x)\le t} n(t-\tau_i(x), x, y)\,. \tag{16}$$

The solution $N(t,y)$ is the independent sum

$$\begin{aligned} N(t,y) &= \sum_{x\in\mathbb{Z}^d} N_x(t,y) \\ &\overset{\text{law}}{=} \sum_{x\in\mathbb{Z}^d} \sum_{\xi_1^{(x)}+\cdots+\xi_k^{(x)}\le t} n\big(t-(\xi_1^{(x)}+\cdots+\xi_k^{(x)}), x, y\big)\,, \end{aligned} \tag{17}$$

where $\xi_i^{(x)} \sim \mathsf{Exp}(\gamma)$ are independent of one another for $i>0$ and $x \in \mathbb{Z}^d$.

For each $x \in \mathbb{Z}^d$, the sub-population size $\nu_x(t) = \sum_{y\in\mathbb{Z}^d} n(t,x,y)$ at time $t \ge 0$ is a Galton–Watson process (Sevastyanov, 1971). Its generating function

$\psi_z(t) := \mathsf{E} z^{\nu_x(t)}$ satisfies

$$\frac{\partial \psi_z}{\partial t} = \beta\psi_z^2 - (\beta+\mu)\psi_z + \mu = (\psi_z - 1)(\beta\psi_z - \mu)\,, \qquad \psi_z(0) = z, \tag{18}$$

with $\alpha = \beta/\mu < 1$. Separation of variables gives

$$\frac{\psi_z(t) - 1}{\alpha\psi_z(t) - 1} = \frac{z-1}{\alpha z - 1}\, e^{-\Delta t}\,, \tag{19}$$

so that

$$\psi_z(t) = \frac{(\alpha - e^{-\Delta t})z - (1 - e^{-\Delta t})}{\alpha(1 - e^{-\Delta t})z - (1 - \alpha e^{-\Delta t})}\,, \tag{20}$$

which is the generating function of a generalized geometric distribution. Hence

$$\mathsf{E}\,\nu_x(t) = \frac{d}{dz}\psi_z(t)\,\Big|_{z=1} = e^{-\Delta t} \to 0\,, \qquad \text{as } t \to \infty\,. \tag{21}$$

ϕ_z such that $\phi_z(t,x,y) := \mathsf{E} z^{n(t,x,y)}$ is the generating function of the subpopulation $n(t,x,y)$. It satisfies the backward Kolmogorov equation

$$\frac{\partial \phi_z}{\partial t} = \mathcal{L}_a\phi_z + \beta\phi_z^2 - (\beta+\mu)\phi_z + \mu\,, \qquad \phi_z(0,x,y) = \begin{cases} z & x = y\,, \\ 1 & x \neq y\,, \end{cases} \tag{22}$$

which is the lattice analogue of the classical Fischer–Kolmodorov–Petrovskii–Piskunov equation (Fisher, 1937; Kolmogorov et al., 1937). We solve Eq. (22) in the particular case $\mu > \beta = 0$.

3 A solvable case: $\mu > \beta = 0$

First consider the special case $\beta = 0$, $\mu > 0$, and $\gamma > 0$ (no birth).

Without random movements, for each fixed point $y \in \mathbb{Z}^d$, the process $N(t, y)$ behaves as a queueing system with an infinite number of servers, whose incoming calls arrive according to a Poisson process with parameter γ. Each call is served independently of others during exponentially distributed times of mean μ^{-1}. This queueing system is ergodic and the Poisson process of parameter $\frac{\gamma}{\mu}$ is at its steady state (Feller, 1968; Karlin and Taylor, 1975). Lemma 1 states that this steady state survives under any symmetric random walk.

Lemma 1. *If $\beta = 0$, as $t \to \infty$, $N(t, y)$ converges in distribution to that of $N(\infty, y)$, which is a Poisson random variable with parameter γ/μ. The limit random variables $\big(N(\infty, y)\big)_{y\in\mathbb{Z}^d}$ are independent of one another.*

Proof. Fix x, $y \in \mathbb{Z}^d$ and $0 \le s < t$. The random variable $n(t-s, x, y)$ at $y \in \mathbb{Z}^d$ and at time t has the same distribution as the total number of offspring, whose ancestor immigrated to x at time $s < t$. As there are no births ($\beta = 0$), the variable $n(t-s, x, y)$ is Bernoulli distributed with

$$\mathsf{P}\big(n(t-s, x, y) = 1\big) = e^{-\mu(t-s)}\, p(t-s, x, y)\,, \tag{23}$$

because the event is possible only if the ancestor particle survives during the time interval $[s,t]$ and is located at y at time t. The generating function is

$$\phi_z(t-s,x,y) = \mathsf{E} z^{n(t-s,x,y)} = 1 + (z-1)\, e^{-\mu(t-s)}\, p(t-s,x,y)\,. \tag{24}$$

The total number of ancestors who immigrated to x at t is Poisson distributed with parameter γt. If their total number is fixed, then the descendants are independent of one another and distributed uniformly over $[0,t)$ (Kingman, 1993):

$$\begin{aligned} \mathsf{E}\, z^{N_x(t,y)} &= \sum_{m=0}^{\infty} e^{-\gamma t} \frac{(\gamma t)^m}{m!} \left(\frac{1}{t} \int_0^t \phi_z(t-s,x,y)\, ds \right)^m \\ &= \exp\left(\gamma(z-1) \int_0^t e^{-\mu(t-s)} p(t-s,x,y)\, ds \right)\,. \end{aligned} \tag{25}$$

Consequently, the generating function of $N(t,y) = \sum_{x\in\mathbb{Z}^d} N_x(t,y)$ satisfies

$$\mathsf{E}\, z^{N(t,y)} \equiv \mathsf{E}\, z^{\sum_x N_x(t,y)} = \exp\left(\gamma(z-1) \int_0^t e^{-\mu(t-s)} \sum_{x\in\mathbb{Z}^d} p(t-s,x,y)\, ds \right). \tag{26}$$

By the last property in Eq. (5),

$$\mathsf{E}\, z^{N(t,y)} \equiv \exp\left(\gamma(z-1) \int_0^t e^{-\mu(t-s)}\, ds \right) = \exp\left(\frac{\gamma(z-1)}{\mu} \big(1 - e^{-\mu t}\big) \right), \tag{27}$$

and, as $t \to \infty$, the generating function of $N(t,y)$ converges to $\exp\left(\frac{\gamma}{\mu}(z-1)\right)$, which is a Poisson distribution with parameter γ/μ. Namely,

$$\mathsf{P}\big(N(\infty, y) = k\big) = \frac{\left(\frac{\gamma}{\mu}\right)^k}{k!} e^{-\frac{\gamma}{\mu}} \qquad \text{for all } k = 0, 1, 2, \ldots. \tag{28}$$

We show now that for distinct $y \in \mathbb{Z}^d$, the limit random variables $N(\infty, y)$ are independent of one another. For notation simplicity, we consider only the case of two variables; the general case is similar.

Fix $y_1 \neq y_2$. As in Eq. (24),

$$\begin{aligned} \mathsf{E}\big(z_1^{n(t-s,x,y_1)}\, z_2^{n(t-s,x,y_2)}\big) &= 1 + (z_1 - 1)\, e^{-\mu(t-s)}\, p(t-s, x, y_1) \\ &\quad + (z_2 - 1)\, e^{-\mu(t-s)}\, p(t-s, x, y_2)\,, \end{aligned} \tag{29}$$

so that, as in Eq. (25),

$$\begin{aligned} \mathsf{E}\left(z_1^{N_x(t,y_1)} z_2^{N_x(t,y_2)}\right) &= \sum_{m=0}^{\infty} e^{-\gamma t} \frac{(\gamma t)^m}{m!} \left(\frac{1}{t}\int_0^t \mathsf{E}_x\big(z_1^{n(t-s,x,y_1)} z_2^{n(t-s,x,y_2)}\big)\, ds\right)^m \\ &= \exp\left(\gamma \int_0^t e^{-\mu s}\big((z_1 - 1)p(s, x, y_1) + (z_2 - 1)p(s, x, y_2)\big)\, ds\right). \end{aligned} \tag{30}$$

As descendants are independent of one another,

$$\begin{aligned}
\mathsf{E}\left(z_1^{N(t,y_1)} z_2^{N(t,y_2)}\right) &= \prod_{x\in\mathbb{Z}^d} \mathsf{E}\left(z_1^{N_x(t,y_1)} z_2^{N_x(t,y_2)}\right) \\
&= \exp\left(\gamma \sum_{x\in\mathbb{Z}^d} \int_0^t e^{-\mu s}\big((z_1-1)p(s,x,y_1)+(z_2-1)p(s,x,y_2)\big)\,ds\right) \\
&= \exp\left(\frac{\gamma}{\mu}\big((z_1-1)+(z_2-1)\big)\big(1-e^{-\mu t}\big)\right) \\
&= \mathsf{E}\, z_1^{N(t,y_1)}\, \mathsf{E}\, z_2^{N(t,y_2)},
\end{aligned} \tag{31}$$

which, when $t\to\infty$, gives

$$\mathsf{E}\left(z_1^{N(t,y_1)} z_2^{N(t,y_2)}\right) \to \mathsf{E}\, z_1^{N(\infty,y_1)}\, \mathsf{E}\, z_2^{N(\infty,y_2)}\,. \tag{32}$$

It is straightforward to extend to any finite collection $\{y_1, y_2, \ldots, y_k\}$. $\square$

4 The general case $\mu > \beta \geq 0$

4.1 Growth of moments

The factorial moments of $n(t,x,y)$

$$\begin{aligned}
m_l(t,x,y) &:= \mathsf{E}\big(n(t,x,y)\,(n(t,x,y)-1)\ldots(n(t,x,y)-l+1)\big) \\
&\equiv \mathsf{E}\frac{n(t,x,y)!}{(n(t,x,y)-l)!}
\end{aligned} \tag{33}$$

are obtained by successively differentiating in Eq. (22) and using the fact that

$$m_l(t,x,y) = \frac{\partial^l \phi_z(t,x,y)}{\partial z^l}\Big|_{z=1} \,. \tag{34}$$

For the first moment, as $\Delta = \mu - \beta > 0$,

$$\frac{\partial m_1}{\partial t} = \mathcal{L}_a m_1 - \Delta m_1\,, \qquad m_1(0,x,y) = \delta_x(y)\,, \tag{35}$$

where $\delta_x(y) = \delta(y-x)$ is the Dirac delta function in 0:

$$\delta(z) = \begin{cases} 1 & \text{if } z = 0\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{36}$$

The solution to Eq. (35) then is

$$m_1(t,x,y) = e^{-\Delta t} p(t,x,y)\,, \tag{37}$$

where $p(t,x,y)$ is defined in Eq. (2) and is the solution of the homogeneous equation

$$\frac{\partial p}{\partial t} = \mathcal{L}_a p\,, \qquad p(0,x,y) = \delta_x(y)\,. \tag{38}$$

It follows from Eq. (5) that $m_1(t,x,y) \leq m_1(t,x,x)$ for all $y \in \mathbb{Z}^d$.

Likewise, the l-th factorial moment with $l \geq 2$ satisfies

$$\frac{\partial m_l}{\partial t} = \mathcal{L}_a m_l - \Delta m_l + \beta \sum_{i=1}^{l-1} \binom{l}{i} m_i m_{l-i} \,, \qquad m_l(0, x, y) = 0 \,. \tag{39}$$

We first introduce Duhamel's principle (Vasy, 2015):

Lemma 2. *If $f(t, x)$, $t \geq 0$, $x \in \mathbb{Z}^d$, is the fundamental solution of the homogeneous equation:*

$$\frac{\partial f}{\partial t}(t, x) = \mathcal{L} f(t, x), \qquad f(0, x) = \delta(x), \tag{40}$$

then the solution to the non-homogeneous equation

$$\frac{\partial F}{\partial t}(t, x) = \mathcal{L} f(t, x) + g(t, x), \qquad F(0, x) = 0, \tag{41}$$

is

$$F(t, x) = \int_0^t \sum_{v \in \mathbb{Z}^d} f(t - s, x - v) g(s, v) ds. \tag{42}$$

As in Molchanov and Whitmeyer (2017), we have:

Theorem 3. *There exists a finite positive constant c such that*

$$m_l(t, x, y) \leq c^l \, l! \, e^{-\Delta t} \, p(t, x, y) \tag{43}$$

for all real $t \geq 0$*, integer* $l \geq 1$*, and* $x,\, y \in \mathbb{Z}^d$.

By the spatial homogeneity of the dynamic and the first line in Eq. (5), the distribution of $n(t,x,y)$ coincides with that of either $n(t,0,y-x)$, $n(t,0,x-y)$, $n(t,y-x,0)$, or $n(t,x-y,0)$, which are the same by the first property in Eq. (5). It is thus sufficient to study the behavior of $n(t,x,0)$, that is, when $y=0$.

Proof. Because the case $l=1$ results from Eq. (37), we start with $l=2$. Differentiating Eq. (22) yields

$$\frac{\partial m_2}{\partial t} = \mathcal{L}_a m_2 - \Delta m_2 + 2\beta m_1^2\,, \qquad m_2(0,x,y) = 0\,, \tag{44}$$

whose solution we obtain using Duhamel's principle recalled in Lemma 2. We get

$$\begin{aligned} m_2(t,x,0) =& 2\beta \int_0^t \sum_{v\in\mathbb{Z}^d} p(t-s,x-v,0)\, e^{-\Delta(t-s)}\, m_1^2(s,v,0)\, ds \\ =& 2\beta \int_0^t \sum_{v\in\mathbb{Z}^d} p(t-s,x-v,0)\, e^{-\Delta(t-s)}\, e^{-2\Delta s}\, p^2(s,v,0)\, ds\,. \end{aligned} \tag{45}$$

From $p(s,v,0) \leq p(s,0,0)$, Eq. (5), and the Chapman-Kolmogorov relation

$$\sum_{v\in\mathbb{Z}^d} p(t-s,x-v,0)\, p(s,v,0) = \sum_{v\in\mathbb{Z}^d} p(t-s,x,v)\, p(s,v,0) = p(t,x,0)\,, \tag{46}$$

we get

$$m_2(t,x,0) \leq 2\beta\, p(t,x,0)\, e^{-\Delta t} \int_0^t e^{-\Delta s} p(s,0,0)\, ds\,. \tag{47}$$

Denoting $G_\Delta(x,y) := \int_0^\infty e^{-\Delta s} p(s,x,y)\, ds$ the Green function corresponding to the operator $\mathcal{L}_a$, we deduce that

$$m_2(t,x,0) \le 2\beta G_\Delta(0,0)\, e^{-\Delta t}\, p(t,x,0)\,. \tag{48}$$

From now on, we proceed by induction and show that, for all $l \ge 1$,

$$m_l(t,x,0) \le B^{l-1} D_l\, e^{-\Delta t} p(t,x,0)\,, \tag{49}$$

where $B = \max\{1, \beta G_\Delta(0,0)\}$ is a finite constant, and the sequence D_l is recurrently defined from

$$D_1 = 1\,, \qquad D_l = \sum_{i=1}^{l-1} \binom{l}{i} D_i\, D_{l-i}\,, \quad \forall\, l \ge 2\,. \tag{50}$$

Assume that Eq. (49) holds for all $l' < l$. Then, from Eq. (50), we get

$$\begin{aligned} \sum_{i=1}^{l-1} \binom{l}{i} m_i(s,v,0)\, m_{l-i}(s,v,0) &\le B^{l-2}\, e^{-2\Delta s}\, p^2(s,v,0) \sum_{i=1}^{l-1} \binom{l}{i} D_i\, D_{l-i} \\ &= B^{l-2}\, D_l\, e^{-2\Delta s}\, p^2(s,v,0) \end{aligned} \tag{51}$$

and thus, as for Eq. (46),

$$
\begin{aligned}
\sum_{v\in\mathbb{Z}^d} p(t-s,x-v,0)\sum_{i=1}^{l-1}\binom{l}{i} m_i(s,v,0)m_{l-i}(s,v,0) \\
&\le B^{l-2}\,D_l\,e^{-2\Delta s}\sum_{v\in\mathbb{Z}^d} p(t-s,x-v,0)\,p^2(s,v,0) \\
&\le B^{l-2}\,D_l\,e^{-2\Delta s}\,p(s,0,0)\,p(t,x,0)\,.
\end{aligned} \tag{52}
$$

Therefore, applying Duhamel's principle to Eq. (39), we deduce

$$
\begin{aligned}
m_l(t,x,0) &\le \beta\int_0^t e^{-\Delta(t-s)}B^{l-2}\,D_l\,e^{-2\Delta s}\,p(s,0,0)\,p(t,x,0)\,ds \\
&\le \beta e^{-\Delta t}\,B^{l-2}\,D_l\,p(t,x,0)\,G_\Delta(0,0)\,,
\end{aligned} \tag{53}
$$

so that Eq. (49) holds for all $l \ge 1$, by induction.

We finally estimate the sequence $(D_l)_{l\ge 1}$. Because, in terms of $d_l := D_l/l!$, the sum in Eq. (50) is a convolution, the generating function $D(z) := \sum_{l=1}^{\infty} d_l z^l$ satisfies the quadratic equation

$$
D(z) = z + D^2(z) \tag{54}
$$

which is similar to the generating function for Catalan's numbers (Flajolet and Sedgewick, 2009). Only the solution of Eq. (54):

$$
D(z) = \frac{1-\sqrt{1-4z}}{2} \tag{55}
$$

satisfies the condition $D(0) = 0$. The growth of a coefficient is defined by the radius of convergence which is equal to the distance from origin to the closest singularity (Flajolet and Sedgewick, 2009): here $R = \frac{1}{4}$. Then $d_l \leq \left(\frac{1}{R} + \epsilon\right)^l = (4 + \epsilon)^l$ for all $\epsilon > 0$ and thus $D_l \leq 5^l\, l!$. This, together with Eq. (49), implies Theorem 3 for $c = 5B$. $\square$

4.2 Existence of a steady state

We extend the convergence property of Lemma 1 to the general subcritical case $\mu > \beta \geq 0$. Our main result is:

Theorem 4. *There exists a unique particle field $\big(N(\infty, y)\big)_{y \in \mathbb{Z}^d}$ such that, as $t \to \infty$, the distribution of $N(t, y)$ converges in distribution to that of $N(\infty, y)$ for all $y \in \mathbb{Z}^d$.*

We prove the convergence of the moments in terms of cumulants and then use a priori bounds introduced in Theorem 3 to establish the uniqueness of the limit distribution.

Y is an integer-valued random variable of generating function ϕ_Y such that $\phi_Y(z) = \mathsf{E} z^Y$. The l-th factorial moment of Y is defined as the l-th derivative of $\phi_Y(z)$ at $z = 1$; the l-th cumulant $\chi_l(Y)$ is defined as the l-th derivative of $\ln \phi_Y(z)$. There is a one-to-one correspondence between moments and cumulants. Cumulants possess the additivity property: if Y_1 and Y_2 are independent random variables, then $\chi_l(Y_1+Y_2)=\chi_l(Y_1)+\chi_l(Y_2)$ for all $l\geq 1$.

By Eq. (17), for all $t\geq 0$ and $x,\, y\in\mathbb{Z}^d$, this additive property yields

$$\chi_l\big(N(t,y)\big)=\chi_l\Big(\sum_{x\in\mathbb{Z}^d}N_x(t,y)\Big)=\sum_{x\in\mathbb{Z}^d}\chi_l\big(N_x(t,y)\big). \tag{56}$$

Lemma 5. *For all $t\geq 0$ and $x,\, y\in\mathbb{Z}^d$,*

$$\chi_l\big(N_x(t,y)\big)=\gamma\int_0^t m_l(s,x,y)\,ds\,. \tag{57}$$

Consequently, $\chi_l\big(N(t,y)\big)=\sum_{x\in\mathbb{Z}^d}\chi_l\big(N_x(t,y)\big)$ increases with t.

Proof. As in Eq. (25), we have

$$\mathsf{E}z^{N_x(t,y)}=\sum_{m=0}^{\infty}e^{-\gamma t}\frac{(\gamma t)^m}{m!}\Big(\frac{1}{t}\int_0^t\mathsf{E}z^{n(t-s,x,y)}\,ds\Big)^m=\exp\left(\gamma\int_0^t\big(\mathsf{E}z^{n(s,x,y)}-1\big)\,ds\right) \tag{58}$$

so that

$$\ln\mathsf{E}z^{N_x(t,y)}=\gamma\int_0^t\big(\mathsf{E}z^{n(s,x,y)}-1\big)\,ds=\sum_{l\geq 1}\frac{(z-1)^l}{l!}\,\gamma\int_0^t m_l(s,x,y)\,ds\,. \tag{59}$$

By definition of cumulants,

$$\ln\mathsf{E}z^{N_x(t,y)}=\sum_{l\geq 1}\frac{(z-1)^l}{l!}\chi_l\big(N_x(t,y)\big)\,, \tag{60}$$

from which the first claim of Lemma 5 follows.

By its definition, $m_l(s,x,y) \geq 0$ for all $s \geq 0$ and x, $y \in \mathbb{Z}^d$, the integral in Eq. (57) implies that both the individual cumulants $\chi_l\big(N_x(t,y)\big)$ and their sum $\chi_l\big(N(t,y)\big)$ are increasing functions of time. □

Combining Lemma 5 with Theorem 3, we obtain

Corollary 6. *For all integer $l \geq 1$, real $t \geq 0$ and $y \in \mathbb{Z}^d$,*

$$\chi_l\big(N(t,y)\big) \leq c^l l! \gamma \int_0^t e^{-\Delta s} \sum_{x \in \mathbb{Z}^d} p(s,x,y)\, ds \leq c^l l! \frac{\gamma}{\Delta}\,. \tag{61}$$

It follows from Lemma 5 and Corollary 6 that, for each $y \in \mathbb{Z}^d$, the limit of $\chi_l\big(N(t,y)\big)$ exists and satisfies

$$\chi_l\big(N(\infty,y)\big) := \lim_{t\to\infty} \chi_l\big(N(t,y)\big) \leq c^l l! \frac{\gamma}{\Delta} \tag{62}$$

so that the function

$$\ln \mathsf{E} z^{N(\infty,y)} := \sum_{l\geq 1} \frac{(z-1)^l}{l!} \chi_l\big(N(\infty,y)\big) \tag{63}$$

is analytic in a complex neighbourhood of $z = 1$. By Feller (1971, Sect. VII.3), it corresponds to a unique probability distribution and thus identifies the limit random variable $N(\infty, y)$.

This completes the proof of Theorem 4.

A similar argument holds for all joint moments and cumulants. Indeed, fix $t \geq 0$ and lattice nodes x, y_1, y_2, and consider the joint generating function

$$\phi_{z_1,z_2}(t,x,y_1,y_2) := \mathsf{E}\big(z_1^{n(t,x,y_1)}\, z_2^{n(t,x,y_2)}\big) \tag{64}$$

and the single sub-population joint moment of orders $l_1 > 0$, $l_2 > 0$. Based on Eq. (33) and (34),

$$\begin{aligned} m_{l_1,l_2}(t,x,y_1,y_2) &:= \mathsf{E}\Big(\frac{n(t,x,y_1)!}{(n(t,x,y_1)-l_1)!}\frac{n(t,x,y_2)!}{(n(t,x,y_2)-l_2)!}\Big) \\ &\equiv \frac{\partial^{l_1}}{(\partial z_1)^{l_1}}\frac{\partial^{l_2}}{(\partial z_2)^{l_2}}\phi_{z_1,z_2}(t,x,y_1,y_2)\,\Big|_{z_1=z_2=1}\ . \end{aligned} \tag{65}$$

Then for the corresponding cumulant

$$\chi_{l_1,l_2}\big(N_x(t,y_1),N_x(t,y_2)\big) := \frac{\partial^{l_1}}{(\partial z_1)^{l_1}}\frac{\partial^{l_2}}{(\partial z_2)^{l_2}} \ln \mathsf{E}\big(z_1^{N_x(t,y_1)}\, z_2^{N_x(t,y_2)}\big)\,\Big|_{z_1=z_2=1}, \tag{66}$$

the analogue of Lemma 5 holds:

$$\chi_{l_1,l_2}\big(N_x(t,y_1),N_x(t,y_2)\big) = \gamma \int_0^t m_{l_1,l_2}(s,x,y_1,y_2)\,ds\,. \tag{67}$$

Because $m_{l_1,l_2}(s,x,y_1,y_2) \geq 0$, the cumulant $\chi_{l_1,l_2}\big(N_x(t,y_1),N_x(t,y_2)\big)$ increases with t and, as $t \to \infty$, it converges to a finite limit.

Extending this argument to all joint moments and cumulants, we deduce the convergence of all finite-dimensional distributions of the particle field $N(t,\,\cdot\,)$ to that of $N(\infty,\,\cdot\,)$ as $t \to \infty$. As in Chernousova et al. (2019), it follows that the distribution of $N(\infty,\,\cdot\,)$ is the unique steady state of the model.

5 Non-homogeneous dynamics

We extend the argument of section 4 to the case where the space is not homogeneous. The birth rate $\beta(x)$ and the mortality rate $\mu(x)$ are bounded functions of $x \in \mathbb{Z}^d$, so that the difference $\Delta(x) := \mu(x) - \beta(x)$ satisfies

$$0 < \Delta_1 \leq \Delta(x) \leq \Delta_2 < \infty\,, \qquad \forall\, x \in \mathbb{Z}^d\,,$$

for suitable constants Δ_1 and Δ_2. Eq. (35) becomes

$$\frac{\partial \bar{f}_y}{\partial t}(t,x) = \mathcal{L}_a \bar{f}_y(t,x) - \Delta(x)\bar{f}_y(t,x)\,, \qquad \bar{f}_y(0,x) = \delta(y-x)\,. \tag{68}$$

Following Chernousova et al. (2019), we can construct the random processes N_1 and N_2 on the same probability space as the random process N, where the dynamic of N_i corresponds to Δ_i, $i = 1, 2$, such that the particle field N_2 is a subset of the particle field N and the particle field N is a subset of the particle field N_1 using the coupling argument or the monotonicity properties of the solution to the parabolic equation, for each x, we have that $m_1(t,x,y) \equiv \bar{f}_y(t,x)$ is smaller than the solution to Eq. (35) with $\Delta = \Delta_1$:

$$m_1(t,x,y) \equiv \bar{f}_y(t,x) \leq e^{-\Delta_1 t} p(t,x,y). \tag{69}$$

The distribution of $N(t,y)$ is no longer shift-invariant and the factorial moments of the sub-populations $n(t,x,y)$ now depend on the pair (x,y), not just on the difference $y-x$.

In the non-homogeneous case, the second factorial moment $m_2(t,x,y)$ satisfies the analogue of Eq. (44):

$$\frac{\partial m_2}{\partial t} = \mathcal{L}_a m_2 - \Delta(x) m_2 + 2\beta m_1^2\,, \qquad m_2(0,x,y) = 0\,, \tag{70}$$

so that, thanks to the non-homogeneous version of Duhamel's principle recalled in Lemma 2, Eq. (45) becomes

$$\begin{aligned} m_2(t,x,y) &= 2\beta \int_0^t \sum_{v\in\mathbb{Z}^d} \bar{f}_v(t-s,x)\, m_1^2(s,v,y)\, ds \\ &\leq 2\beta e^{-\Delta_1 t} \int_0^t \sum_{v\in\mathbb{Z}^d} p(t-s,x,v)\, e^{-\Delta_1 s}\, p^2(s,v,y)\, ds \\ &\leq 2\beta e^{-\Delta_1 t}\, p(t,x,y) \int_0^t e^{-\Delta_1 s}\, p(s,v,v)\, ds \\ &\leq 2\beta\, G_{\Delta_1}(0,0)\, e^{-\Delta_1 t}\, p(t,x,y)\,. \end{aligned} \tag{71}$$

We extend Theorem 3 with the estimate

$$m_l(t, x, y) \le c^l\, l!\, e^{-\Delta_1 t}\, p(t, x, y)\,, \tag{72}$$

and deduce the analogue of Theorem 4 for the non-homogeneous case.

6 Monte Carlo Simulation

We present a Monte Carlo simulation. We consider a branching random walk on $\mathbb{Z}^1$. This simple setting is done to focus on the limit distribution. We set the birth rate to $\beta = 0$, the death rate to $\mu = 0.2$, and the external immigration rate to $\gamma = 0.5$. For the random walk, $\kappa = 1$, $a(1) = a(-1) = 0.5$, and for $z \in \mathbb{Z}^1 \setminus \{-1, 1\}$, $a(z) = 0$. At initial time $t = 0$, there is a single population located at the origin $x = 0$. We simulate our model in Z^1 based on Eq. (17), Eq. (18) and repeat the simulations 10,000 times so as to obtain an approximation of the population at $t \to \infty$.

Figure 1 shows the limit distribution of the population at $x = 0$ after large time t ($\frac{t}{dt}$ is ≥ 1000). The left panel in Figure 1 shows the histogram of the population size at location $x = 0$, the right panel in Figure 1 allows comparing the fitted with the theoretical distributions. As indicated in Lemma 1, the limit distribution is a Poisson distribution with parameter $\frac{\gamma}{\mu}$, which is 2.5 in our setting. The simulation is consistent with result of Eq. (28) in the solvable case $\mu > \beta = 0$. Figures 2 and 3 show the limit distributions at $x = 4$ and $x = -5$. There is no noteworthy difference between Figures 2 and 3, because the limit distribution depends only on the ratio of immigration rate γ and death rate μ and it does not depend on the location of the population x in the case $\mu > \beta = 0$.

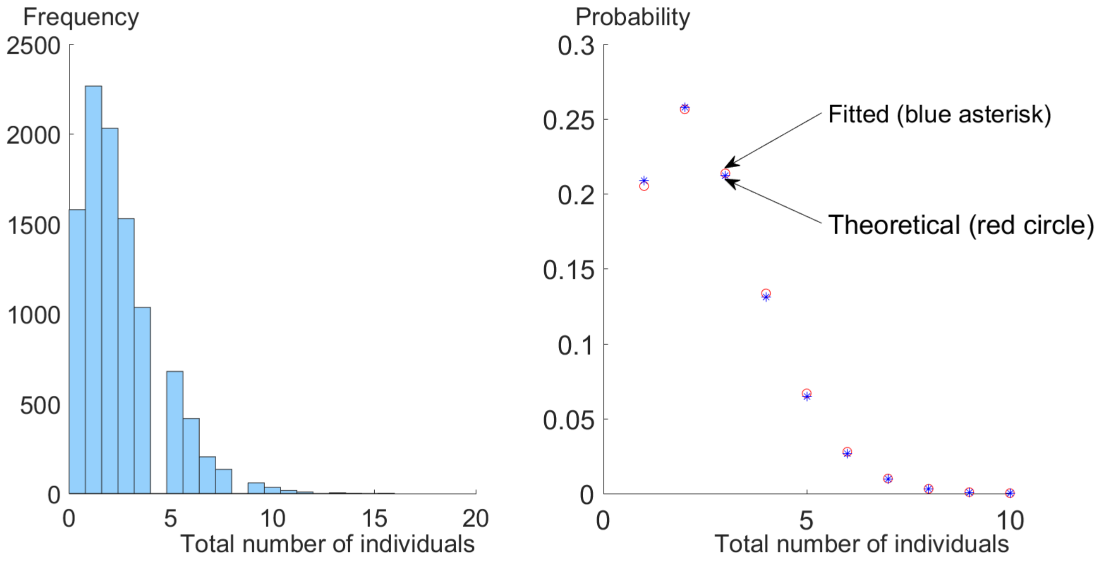

Figure 1: $\beta = 0$, $\mu = 0.2$, $\gamma = 0.5$, $\kappa = 1$. The left panel is the histogram for $N(\infty, 0)$; the right panel shows that the distribution fitted for the histogram is close to the theoretical distribution for $N(\infty, 0)$.

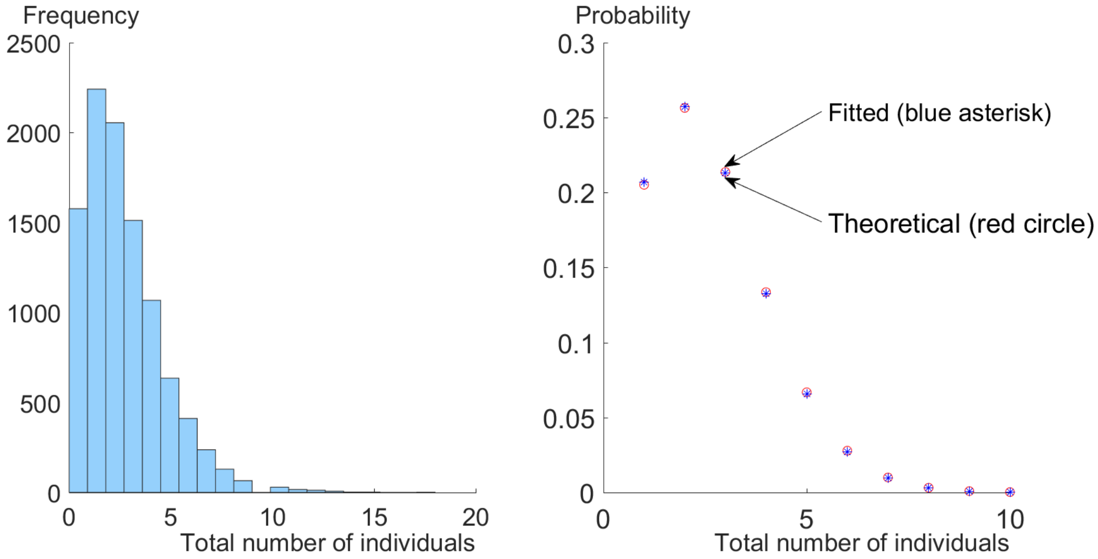

Figure 2: $\beta = 0$, $\mu = 0.2$, $\gamma = 0.5$, $\kappa = 1$. The left panel is the histogram for $N(\infty, 4)$; the right panel shows that the distribution fitted for the histogram is close to the theoretical distribution for $N(\infty, 4)$.

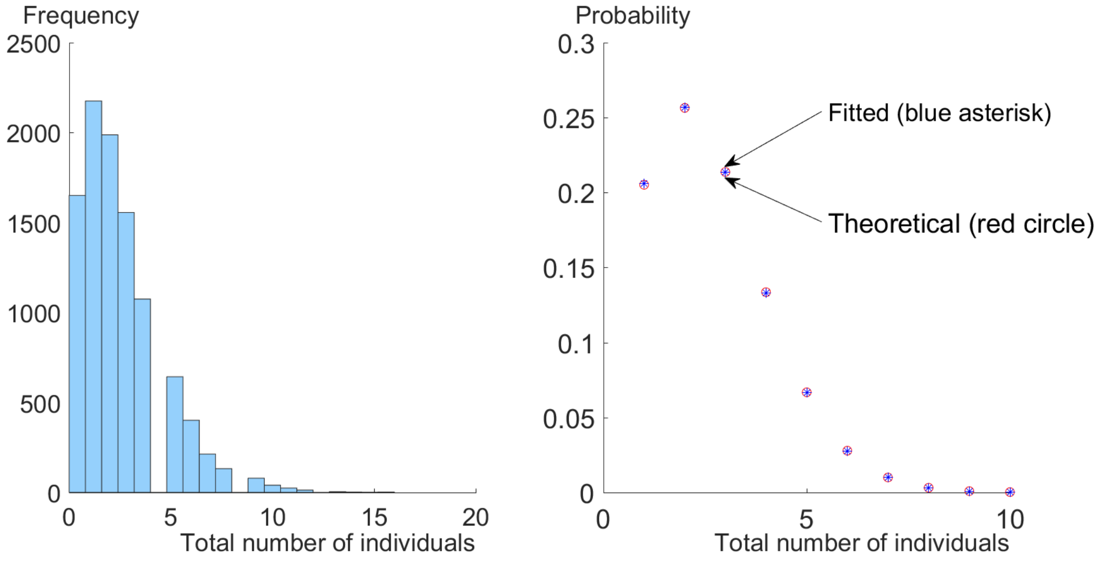

Figure 3: $\beta = 0$, $\mu = 0.2$, $\gamma = 0.5$, $\kappa = 1$. The left panel is the histogram for $N(\infty, -5)$; the right panel shows that the distribution fitted for the histogram is close to the theoretical distribution for $N(\infty, -5)$.

7 Conclusion

We have introduced immigration into a population model in $\mathbb{Z}^d$, $d \geq 1$, where individuals evolve independently as branching random walks with simple binary splitting. In the the stability region where the mortality rate μ is higher than the birth splitting rate β, for large time, the distribution of the population converges to a steady state (also called stochastic equilibrium). In the solvable case $\mu > \beta = 0$, we have identified the limit distribution as an independent Poisson point field on $\mathbb{Z}^d$ (Eq. (28) and (32)).

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