

Hereditary Graph Classes: When the Complexities of Colouring and Clique Cover Coincide*

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Abstract. A graph is (H_1, H_2) -free for a pair of graphs H_1, H_2 if it contains no induced subgraph isomorphic to H_1 or H_2 . In 2001, Král', Kratochvíl, Tuza, and Woeginger initiated a study into the complexity of COLOURING for (H_1, H_2) -free graphs. Since then, others have tried to complete their study, but many cases remain open. We focus on those (H_1, H_2) -free graphs where H_2 is $\overline{H_1}$, the complement of H_1 . As these classes are closed under complementation, the computational complexities of COLOURING and CLIQUE COVER coincide. By combining new and known results, we are able to classify the complexity of COLOURING and CLIQUE COVER for (H, \overline{H}) -free graphs for all cases except when $H = sP_1 + P_3$ for $s \geq 3$ or $H = sP_1 + P_4$ for $s \geq 2$. We also classify the complexity of COLOURING on graph classes characterized by forbidding a finite number of self-complementary induced subgraphs, and we initiate a study of k -COLOURING for $(P_r, \overline{P_r})$ -free graphs.

1 Introduction

A colouring of a graph is an assignment of labels, called colours, to its vertices in such a way that no two adjacent vertices have the same label. The corresponding decision problem, COLOURING, which is that of deciding whether a given graph can be coloured with at most k colours for some given positive integer k , is a central problem in discrete optimization. Its complementary problem CLIQUE COVER is that of deciding whether the vertices of a graph can be covered with at most k cliques. As COLOURING and CLIQUE COVER are NP-complete even for $k = 3$ [28], it is natural to restrict their input to some special graph class. A classic result in this area is due to Grötschel, Lovász, and Schrijver [19], who showed that COLOURING is polynomial-time solvable for perfect graphs. However, for both problems, finding the exact borderline between tractable and computationally hard graph classes is still open.

A graph class is *hereditary* if it can be characterized by some set of forbidden induced subgraphs, or equivalently, if it is closed under vertex deletion (see e.g. [25]). The aforementioned class of perfect graphs is an example of such a class, as a graph is perfect if and only if it contains no induced odd holes and no induced odd antihole [9]. For the case where exactly one induced subgraph is forbidden, Král', Kratochvíl, Tuza, and Woeginger [27] were able to prove a complete dichotomy; see also Fig. 1 (we write $G' \subseteq_i G$ to denote that the graph G' is an *induced* subgraph of the graph G and refer to Section 2 for other notation used in this section).

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Theorem 1 ([27]). *Let H be a graph. The COLOURING problem on H -free graphs is polynomial-time solvable if $H \subseteq_i P_1 + P_3$ or $H \subseteq_i P_4$ and it is NP-complete otherwise.*



Fig. 1. The graphs H such that COLOURING can be solved in polynomial time on H -free graphs.

We study the complexity of COLOURING and CLIQUE COVER for (H_1, H_2) -free graphs, that is, when two induced subgraphs H_1 and H_2 are both forbidden. Both polynomial-time and NP-completeness results are known for (H_1, H_2) -free graphs, as shown by many teams of researchers (see, for example, [5,6,13,14,20,22,27,31,34,35,36,39]), but the complexity classification is far from complete: even if we forbid two graphs H_1 and H_2 of up to four vertices, there are still three open cases left, namely when $(H_1, H_2) \in \{(K_{1,3}, 4P_1), (K_{1,3}, 2P_1 + P_2), (C_4, 4P_1)\}$; see [31] (the graph $K_{1,3}$ is the claw and C_4 is the 4-vertex cycle). We refer to [18, Theorem 21] for a summary and to the recent paper [36], in which the number of missing cases when H_1 and H_2 are both connected graphs on at most five vertices was reduced from ten to eight.

To narrow the complexity gap between hard and easy cases and to increase our understanding of it, we initiate a systematic study into (H_1, H_2) -free graph classes that are *closed under complementation*: if a graph G belongs to the class, then so does its *complement* \overline{G} , which is the graph with vertex set $V(G)$ and an edge between two distinct vertices if and only if these two vertices are not adjacent in G . For such graph classes the complexities of COLOURING and CLIQUE COVER coincide, so we only need to consider the COLOURING problem. A graph H is *self-complementary* if $H = \overline{H}$. We observe that a class of (H_1, H_2) -free graphs with $H_1 \not\subseteq_i H_2$ and $H_2 \not\subseteq_i H_1$ is closed under complementation if and only if either H_1 and H_2 are both self-complementary, or $H_2 = \overline{H_1}$. For the first case we prove the following more general classification (namely for every integer $k \geq 1$) in Section 3.

Theorem 2. *Let H_1, \dots, H_k be self-complementary graphs. Then COLOURING is polynomial-time solvable for (H_1, \dots, H_k) -free graphs if $H_j \subseteq_i P_4$ for some j and it is NP-complete otherwise.*

Hence, we may now focus on the case when $H_2 = \overline{H_1}$ and H_1 is not self-complementary. In this case few results are known. First notice that for any integer $s \geq 1$, the class of $(sP_1, \overline{sP_1})$ -free graphs consists of graphs with no large independent set and no large clique. The number of vertices in such graphs is bounded by a constant by Ramsey's Theorem [38]. Hence, the class of $(sP_1, \overline{sP_1})$ -free graphs is finite, which means that COLOURING is constant-time solvable for this graph class. Dabrowski et al. [14] researched the effect on the complexity of changing the forbidden subgraph sP_1 by adding an extra edge, and proved that COLOURING is polynomial-time solvable on $(sP_1 + P_2, \overline{sP_1 + P_2})$ -free graphs for every integer $s \geq 1$. A result of Malyshev [34] implies that COLOURING is polynomial-time solvable for $(K_{1,3}, \overline{K_{1,3}})$ -free graphs. Hoàng and Lazzarato [22] proved the same for $(P_5, \overline{P_5})$ -free graphs. By combining results of [26] and [37], COLOURING is polynomial-time solvable on graph classes of bounded clique-width. It is known that $(P_1 + P_4, \overline{P_1 + P_4})$ -free graphs [4] and $(2P_1 + P_3, \overline{2P_1 + P_3})$ -free graphs [1] have bounded clique-width, so COLOURING is polynomial-time solvable for

these two graph classes. In light of these results, it is a natural research question to find out to what extent we may add edges to sP_1 so that COLOURING remains polynomial-time solvable on (H, \overline{H}) -free graphs for the resulting graph H .

To narrow down the large number of open cases (H, \overline{H}) , we will use the known results that COLOURING is NP-complete for graphs of girth at least p for any $p \geq 3$ [16,27,30] (the girth of a graph is the length of a shortest induced cycle in it) and for $(K_{1,3}, K_4, \overline{2P_1 + P_2})$ -free graphs [27]. Even after using these results there are still many open cases left, and to get a handle on them we need another NP-hardness result for a very restricted graph class. In Section 4.1 we will prove such a result, namely, we show that COLOURING is NP-complete even for $(P_1 + 2P_2, \overline{P_1 + 2P_2}, 2P_3, \overline{2P_3}, P_6, \overline{P_6})$ -free graphs (see Fig. 2). We will show that, by combining this result with those above, only the following open cases are left: $H = P_2 + P_3$ (see Fig. 3), $H = (s + 1)P_1 + P_3$ or $H = sP_1 + P_4$ for $s \geq 2$. In Section 4.2 we give a polynomial-time algorithm for COLOURING restricted to the class of $(P_2 + P_3, \overline{P_2 + P_3})$ -free graphs (note that this class has unbounded clique-width, as it contains the class of split graphs, which have unbounded clique-width [33]).

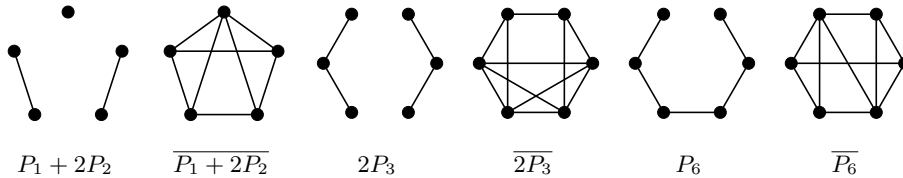


Fig. 2. The six graphs corresponding to our result that COLOURING is NP-complete for $(P_1 + 2P_2, \overline{P_1 + 2P_2}, 2P_3, \overline{2P_3}, P_6, \overline{P_6})$ -free graphs.

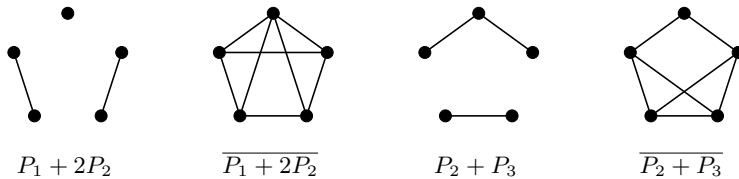


Fig. 3. The graphs H on five vertices for which the complexity of COLOURING for (H, \overline{H}) -free graphs was unknown. We show NP-completeness if $H \in \{P_1 + 2P_2, \overline{P_1 + 2P_2}\}$ and polynomial-time solvability if $H \in \{P_2 + P_3, \overline{P_2 + P_3}\}$.

In Section 4.3 we show how to combine our new hardness result and new polynomial-time algorithm together with the aforementioned known results to obtain the following result, which is the main theorem of our paper. In particular, we solved the remaining cases where H is a graph on five vertices; see also Fig. 3.

Theorem 3. *Let H be a graph with $H, \overline{H} \notin \{(s + 1)P_1 + P_3, sP_1 + P_4 \mid s \geq 2\}$. Then COLOURING is polynomial-time solvable for (H, \overline{H}) -free graphs if H or \overline{H} is an induced subgraph of $K_{1,3}$, $P_1 + P_4$, $2P_1 + P_3$, $P_2 + P_3$, P_5 , or $sP_1 + P_2$ for some $s \geq 0$ and it is NP-complete otherwise.*

From Theorem 3 we deduce that we may only add a few edges to the graph sP_1 if COLOURING stays polynomial-time solvable on (H, \overline{H}) -free graphs for the resulting graph H (even in the two missing cases where $H = (s+1)P_1 + P_3$ or $H = sP_1 + P_4$ for $s \geq 2$, the graph H is only allowed to contain at most three edges). From a more general perspective, our new hardness result has significantly narrowed the classification for (H_1, H_2) -free graphs.

Our immediate goal is to complete the complexity classification of COLOURING for (H, \overline{H}) -free graphs, thus to solve the cases when $H = (s+1)P_1 + P_3$ or $H = sP_1 + P_4$ for $s \geq 2$. We note that the class of $(3P_1 + P_2, K_4)$ -free graphs, and thus the class of $(3P_1 + P_3, \overline{3P_1 + P_3})$ -free graphs, has unbounded clique-width [14]. We emphasize that our long-term goal is to increase our understanding of the computational complexity of COLOURING for hereditary graph classes. Another natural question is whether k -COLOURING (the variant of COLOURING where the number of colours is fixed) is polynomial-time solvable for (H, \overline{H}) -free graphs when $H = (s+1)P_1 + P_3$ or $H = sP_1 + P_4$ for $s \geq 2$. This is indeed the case, as Couturier et al. [12] extended the result of [21] on k -COLOURING for P_5 -free graphs by proving that for every pair of integers $k, s \geq 1$, k -COLOURING is polynomial-time solvable even for $(sP_1 + P_5)$ -free graphs. However, for other classes of (H, \overline{H}) -free graphs, we show that k -COLOURING turns out to be NP-hard by using a construction of Huang [23], that is, we show the following results in Section 5 (for the second result we also use an observation of Chudnovsky, Huang, Spirkl and Zhong [8]).

Theorem 4. *4-COLOURING is NP-complete for $(P_7, \overline{P_8})$ -free graphs, and thus for $(P_8, \overline{P_8})$ -free graphs.*

Theorem 5. *5-COLOURING is NP-complete for $(P_2 + P_5, P_6, \overline{P_1 + P_6})$ -free graphs, and thus for $(P_1 + P_6, \overline{P_1 + P_6})$ -free graphs.*

As COLOURING is polynomial-time solvable for $(P_5, \overline{P_5})$ -free graphs [22], it would be interesting to solve the following two open problems (see also Table 1):

- is there an integer k such that k -COLOURING for $(P_6, \overline{P_6})$ -free graphs is NP-complete?
- is there an integer k such that k -COLOURING for $(P_7, \overline{P_7})$ -free graphs is NP-complete?

As 4-COLOURING is polynomial-time solvable for P_6 -free graphs [10,11], k must be at least 5 in an affirmative answer for the first question. As 3-COLOURING is polynomial-time solvable for P_7 -free graphs [2], k must be at least 4 in an affirmative answer for the second question. We refer to the end of Section 5 for some further remarks on these two open problems. Similar to summaries for k -COLOURING for P_t -free graphs (see, for example, [18]) and (C_s, P_t) -free graphs [20], we can survey the known results and the missing cases of k -COLOURING for $(P_t, \overline{P_t})$ -free graphs; see Table 1.

2 Preliminaries

Throughout our paper we consider only finite, undirected graphs without multiple edges or self-loops. The *disjoint union* $(V(G) \cup V(H), E(G) \cup E(H))$ of two vertex-disjoint graphs G and H is denoted by $G + H$ and the disjoint union of r copies of a graph G is denoted by rG . For a subset $S \subseteq V(G)$, we let $G[S]$ denote the subgraph of G induced by S , which has vertex set S and edge set $\{uv \mid u, v \in S, uv \in E(G)\}$. If $S = \{s_1, \dots, s_r\}$, then we may write $G[s_1, \dots, s_r]$ instead of $G[\{s_1, \dots, s_r\}]$. We may also write $G \setminus S$ instead of $G[V(G) \setminus S]$. Recall that for two graphs G and G' we write $G' \subseteq_i G$ to denote that G' is an induced subgraph of G .

$t \setminus k$	≤ 2	3	4	≥ 5
≤ 5	P	P	P	P
6	P	P	P	?
7	P	P	?	?
≥ 8	P	?	NP-c	NP-c

Table 1. The complexity of k -COLOURING $(P_t, \overline{P_t})$ -free graphs for fixed values of k and t . Here, P means polynomial-time solvable and NP-c means NP-complete. The entries in this table originate from Theorem 4 and the following three results: k -COLOURING is polynomial-time solvable for P_5 -free graphs for any $k \geq 1$ [21], 3-COLOURING is polynomial-time solvable for P_7 -free graphs [2], and 4-COLOURING is polynomial-time solvable for P_6 -free graphs [10,11].

For a set of graphs $\{H_1, \dots, H_p\}$, a graph G is (H_1, \dots, H_p) -free if it has no induced subgraph isomorphic to a graph in $\{H_1, \dots, H_p\}$; if $p = 1$, we may write H_1 -free instead of (H_1) -free. (Note that we may assume that each forbidden graph H_j contains at least one vertex, otherwise the class of (H_1, \dots, H_p) -free graphs is empty.) For a graph G , the set $N(u) = \{v \in V \mid uv \in E\}$ denotes the *neighbourhood* of $u \in V(G)$.

The graphs C_r , K_r , $K_{1,r-1}$ and P_r denote the cycle, complete graph, star and path on r vertices, respectively. The graph $\overline{P_1} + 2P_2$ is also known as the 5-vertex *wheel*. The graphs K_3 and $K_{1,3}$ are also called the *triangle* and *claw*, respectively. A graph G is a *linear forest* if every component of G is a path (on at least one vertex). The graph $S_{h,i,j}$, for $1 \leq h \leq i \leq j$, denotes the *subdivided claw*, that is, the tree that has only one vertex x of degree 3 and exactly three leaves, which are of distance h , i and j from x , respectively. Observe that $S_{1,1,1} = K_{1,3}$.

The *chromatic number* $\chi(G)$ of a graph G is the smallest integer k such that G is k -colourable. The *clique number* $\omega(G)$ is the size of a largest clique in G . A graph is *bipartite* if its vertex set can be partitioned into two (possibly empty) independent sets.

3 The Proof of Theorem 2

As mentioned in Section 1, if we study the complexity of COLOURING for (H_1, H_2) -free classes of graphs that are closed under complementation, it is sufficient to consider the case where H_1 and H_2 are not self-complementary and $H_2 = \overline{H_1}$. We will give a short proof for this claim. To do so, we will need the following lemma.

Lemma 1 ([16,27,30]). *For any integer $p \geq 3$, COLOURING is NP-complete for (C_3, C_4, \dots, C_p) -free graphs.*

Theorem 2 (restated). *Let H_1, \dots, H_k be self-complementary graphs. Then COLOURING is polynomial-time solvable for (H_1, \dots, H_k) -free graphs if $H_j \subseteq_i P_4$ for some j and it is NP-complete otherwise.*

Proof. Let H be a self-complementary graph on n vertices. Then H must have $\frac{1}{2} \binom{n}{2}$ edges. If $n = 1$ then $H = P_1$. Now n cannot be 2 or 3, since $\frac{1}{2} \binom{2}{2}$ and $\frac{1}{2} \binom{3}{2}$ are not integers. If $n = 4$ then $H = P_4$, by inspection. Suppose $n \geq 5$. Then $\frac{1}{2} \binom{n}{2} = \frac{n(n-1)}{4} \geq n$, so H must contain a cycle. Thus, if H is a self-complementary graph then it is either an induced subgraph of P_4 or it contains a cycle.

Let H_1, \dots, H_k be self-complementary graphs. If $H_j \subseteq_i P_4$ for some j , then $H_j = P_1$ or $H_j = P_4$ and the COLOURING problem for (H_1, \dots, H_k) -free graphs is polynomial-time solvable by Theorem 1. If $H_j \not\subseteq_i P_4$ for all j , then each H_j must contain a cycle. Let $p \geq 3$ be the number of vertices in a largest induced cycle in the graphs in $\{H_1, \dots, H_k\}$.

Then the class of (H_1, \dots, H_k) -free graphs contains the class of (C_3, \dots, C_p) -free graphs. As COLOURING is NP-complete for the latter class by Lemma 1, we find that COLOURING is NP-complete for the class of (H_1, \dots, H_k) -free graphs. \square

4 The Proof of Theorem 3

In this section we prove Theorem 3. As part of the proof we first show that COLOURING is NP-complete for $(P_1 + 2P_2, \overline{P_1 + 2P_2}, 2P_3, \overline{2P_3}, P_6, \overline{P_6})$ -free graphs in Section 4.1 and polynomial-time solvable for $(P_2 + P_3, \overline{P_2 + P_3})$ -free graphs in Section 4.2.

4.1 The Hardness Result

In this section we prove the following result.

Theorem 6. *COLOURING is NP-complete for $(P_1 + 2P_2, \overline{P_1 + 2P_2}, 2P_3, \overline{2P_3}, P_6, \overline{P_6})$ -free graphs.*

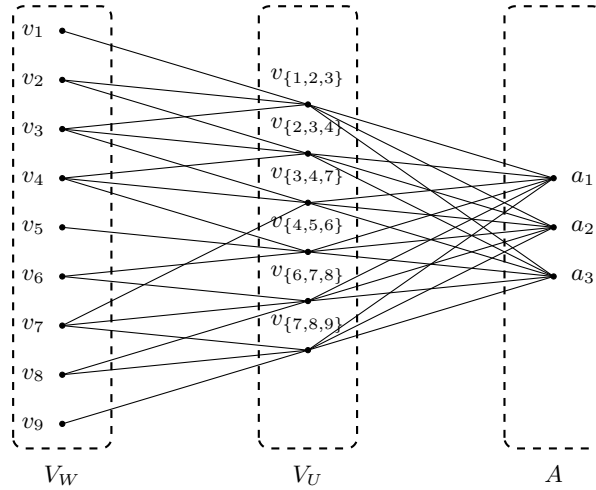


Fig. 4. The graph $G_{W,U}$ for the instance (W, U) of EXACT 3-COVER where $q = 3$, $k = 6$, $W = \{1, \dots, 9\}$ and $U = \{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 7\}, \{4, 5, 6\}, \{6, 7, 8\}, \{7, 8, 9\}\}$. The edges of the clique V_W are not shown.

Proof. Let k and q be positive integers with $k \geq q$. Let W be a set of size $3q$. Let U be a collection of k subsets of W each of size 3. An *exact 3-cover* for (W, U) is a set $U' \subseteq U$ of size q such that every member of W belongs to one of the subsets in U' . The NP-complete problem EXACT 3-COVER [17] is that of determining if such a set U' exists. To prove the theorem, we describe a reduction from EXACT 3-COVER to COLOURING for $(P_1 + 2P_2, \overline{P_1 + 2P_2}, 2P_3, \overline{2P_3}, P_6, \overline{P_6})$ -free graphs.

Given an instance (W, U) of EXACT 3-COVER, we construct the graph $G_{W,U}$ as follows (see Fig. 4 for an example):

- Introduce a set of vertices $V_W = \{v_w \mid w \in W\}$ which forms a clique in $G_{W,U}$.

- Introduce a set of vertices $V_U = \{v_u \mid u \in U\}$ which forms an independent set in $G_{W,U}$.
- Add an edge from $v_w \in V_W$ to $v_u \in V_U$ if and only if $w \in u$.
- Introduce a set of vertices $A = \{a_i \mid 1 \leq i \leq k - q\}$ which forms an independent set in $G_{W,U}$.
- Add an edge from each vertex of A to each vertex of V_U .

We now show a reduction from EXACT 3-COVER to the problem of finding k pairwise vertex-disjoint cliques that cover $V(G_{W,U})$. The latter problem is equivalent to finding a k -colouring of $\overline{G_{W,U}}$ but has a closer resemblance to EXACT 3-COVER, which makes it easier to work with (note that k is part of the input of EXACT 3-COVER). Afterwards, it will remain to show that $\overline{G_{W,U}}$ is $(P_1 + 2P_2, \overline{P_1 + 2P_2}, 2P_3, \overline{2P_3}, P_6, \overline{P_6})$ -free.

Claim 1. *The vertices of $G_{W,U}$ can be covered by at most k pairwise vertex-disjoint cliques if and only if U contains an exact 3-cover U' .*

First suppose that U contains an exact 3-cover U' . Let $V_{U'} = \{v_u \in V_U \mid u \in U'\}$. For each $u \in U'$, let K^u be the clique on four vertices containing v_u and its three neighbours in V_W ; there are q such cliques. Form a perfect matching between the vertices of $V_U \setminus V_{U'}$ and the vertices of A , that is, a collection of $k - q$ cliques on two vertices. Together, these k pairwise vertex-disjoint cliques cover $V(G_{W,U})$.

Now suppose that the vertices of $G_{W,U}$ can be covered by at most k pairwise vertex-disjoint cliques. As V_U is independent and $|V_U| = k$, we have exactly k cliques, and each of them contains exactly one vertex of V_U . Hence, as A is an independent set and each vertex in A is only adjacent to the vertices of V_U , each of the $k - q$ vertices of A must be contained in a clique of size 2 that consists of a vertex of A and a vertex of V_U . There are q other cliques, which, as we deduced, also contain exactly one vertex of V_U . Each vertex of V_W must be in one of these cliques. As each vertex in V_U has exactly three neighbours in V_W and there are $3q$ vertices in V_W , this means that each of these cliques must contain four vertices, consisting of one vertex of V_U and three vertices of V_W . Hence, if we let $u \in U$ belong to U' whenever v_u is in a clique of size 4, then U' forms an exact 3-cover. This completes the proof of Claim 1.

As mentioned above, it remains to show that $\overline{G_{W,U}}$ is $(P_1 + 2P_2, \overline{P_1 + 2P_2}, 2P_3, \overline{2P_3}, P_6, \overline{P_6})$ -free. Since a graph G is $(P_1 + 2P_2, \overline{P_1 + 2P_2}, 2P_3, \overline{2P_3}, P_6, \overline{P_6})$ -free if and only if \overline{G} is, it is sufficient to show that $G_{W,U}$ is $(P_1 + 2P_2, \overline{P_1 + 2P_2}, 2P_3, \overline{2P_3}, P_6, \overline{P_6})$ -free. We do this in Claims 3-8, but first we prove a useful observation.

Claim 2. *Let J be an induced subgraph of $G_{W,U}$ that is the complement of a linear forest on six vertices. Then J contains at least four vertices of V_W .*

As a linear forest contains no triangle, J contains no independent set on three vertices. So J cannot contain more than two vertices from either of the independent sets V_U or A . Thus it contains vertices of V_W . But then it cannot contain two vertices from A as, combined with a vertex of V_W this would again induce an independent set of size 3. Hence there are at least three vertices of V_W in J . But this implies that not even one vertex of A belongs to J (as it would have three non-neighbours and every vertex of J is adjacent to all but at most two of the others). This completes the proof of Claim 2.

Claim 3. *$G_{W,U}$ is $(P_1 + 2P_2)$ -free.*

Suppose for contradiction that there is an induced $P_1 + 2P_2$ in $G_{W,U}$. Since V_U and A are independent sets, every P_2 in $G_{W,U}$ must either have two vertices in V_W or one vertex in V_U and the other in V_W or A . Since V_W is a clique and every vertex in A is adjacent to every vertex in V_U , one of the P_2 's must have both its vertices in V_W and the other must have one vertex in V_U and the other in A . But every vertex in $G_{W,U}$ has a neighbour in such a $2P_2$, so no induced $P_1 + 2P_2$ can exist. This completes the proof of Claim 3.

Claim 4. $G_{W,U}$ is $\overline{P_1 + 2P_2}$ -free.

Suppose for contradiction that there is an induced $\overline{P_1 + 2P_2}$ in $G_{W,U}$. Note this subgraph consists of a C_4 , which we denote C , plus an additional vertex, which we denote z , that is adjacent to every vertex in C . If $z \in A$, then the vertices of C are all in V_U , and if $z \in V_U$, the vertices of the C are either all in A or all in V_W . Neither is possible, so we must have $z \in V_W$. Therefore none of the vertices of C is in A . At most two of the vertices of C are in each of V_U and V_W as C contains neither an independent set nor a clique on three vertices. So C contains exactly two vertices from each of V_U and V_W , but the pair from V_U must be non-adjacent in C and the pair from V_W must be adjacent in C . This contradiction completes the proof of Claim 4.

Claim 5. $G_{W,U}$ is $2P_3$ -free.

Suppose for contradiction that there is an induced $2P_3$ in $G_{W,U}$. Denote this graph by P . As V_W is a clique, P contains at most two of its vertices. If it contains exactly two, then P must also contain a vertex $v_u \in V_U$ such that the three vertices together induce a P_3 . But then v_u cannot be adjacent to any other vertex of P so the remaining three vertices must all belong to V_U , a contradiction as V_U is independent. So P must contain at least five vertices of A and V_U . They cannot all belong to one of these two sets as P has no independent set of size 5, but if P contains vertices of both sets then there must be a vertex of degree 3. This contradiction proves that P does not exist. This completes the proof of Claim 5.

Claim 6. $G_{W,U}$ is $\overline{2P_3}$ -free.

Suppose for contradiction that there is an induced $\overline{2P_3}$ in G . By Claim 2, it contains four vertices of V_W . We note that $\overline{2P_3}$ contains exactly one induced K_4 and that the other two vertices are adjacent to exactly two vertices in the clique (so must both be in V_U) and to each other (so cannot both be in V_U). This contradiction completes the proof of Claim 6.

Claim 7. $G_{W,U}$ is P_6 -free.

Suppose for contradiction that there is an induced P_6 in $G_{W,U}$. Denote this path by P . As V_W is a clique, P contains at most two of its vertices. We see that P cannot contain exactly four vertices of either V_U or A as they would form an independent set of size 4. P cannot contain exactly three vertices of either V_U or A as then there would be a vertex in the other of the two sets of degree 3. Finally, P cannot contain at least two vertices from each of V_U and A as then it would contain an induced C_4 . These contradictions prove that P does not exist. This completes the proof of Claim 7.

Claim 8. $G_{W,U}$ is $\overline{P_6}$ -free.

Suppose for contradiction that there is an induced $\overline{P_6}$ in G . By Claim 2, it contains four vertices of V_W . This contradiction – a clique on four vertices is not an induced subgraph of $\overline{P_6}$ – completes the proof of Claim 8. \square

4.2 A Tractable Case

We show that COLOURING is polynomial-time solvable for $(P_2 + P_3, \overline{P_2 + P_3})$ -free graphs. We first introduce some additional terminology. The *clique-width* $cw(G)$ of a graph G is the minimum number of labels needed to construct G by using the following four operations:

1. creating a new graph consisting of a single vertex v with label i ;
2. taking the disjoint union of two labelled graphs G_1 and G_2 ;
3. joining each vertex with label i to each vertex with label j ($i \neq j$);
4. renaming label i to j .

A class of graphs \mathcal{G} has *bounded* clique-width if there is a constant c such that the clique-width of every graph in \mathcal{G} is at most c ; otherwise the clique-width is *unbounded*. For an induced subgraph G' of a graph G , the *subgraph complementation* operation (acting on G with respect to G') replaces every edge present in G' by a non-edge, and vice versa. Similarly, for two disjoint vertex subsets S and T in G , the *bipartite complementation* operation with respect to S and T acts on G by replacing every edge with one end-vertex in S and the other one in T by a non-edge and vice versa. Let $k \geq 0$ be a constant and let γ be some graph operation. We say that a graph class \mathcal{G}' is (k, γ) -*obtained* from a graph class \mathcal{G} if the following two conditions hold:

1. every graph in \mathcal{G}' is obtained from a graph in \mathcal{G} by performing γ at most k times, and
2. for every $G \in \mathcal{G}$ there exists at least one graph in \mathcal{G}' that is obtained from G by performing γ at most k times.

We say that γ *preserves* boundedness of clique-width if for any finite constant k and any graph class \mathcal{G} , any graph class \mathcal{G}' that is (k, γ) -obtained from \mathcal{G} has bounded clique-width if and only if \mathcal{G} has bounded clique-width.

Fact 1. Vertex deletion preserves boundedness of clique-width [32].

Fact 2. Subgraph complementation preserves boundedness of clique-width [24].

Fact 3. Bipartite complementation preserves boundedness of clique-width [24].

Two vertices are *false twins* if they are non-adjacent and have the same neighbourhood. We will use the following two lemmas, which are readily seen.

Lemma 2. *The clique-width of a graph of maximum degree at most 2 is at most 4.*

Lemma 3. *If a vertex x in a graph G has a false twin then $\text{cw}(G) = \text{cw}(G \setminus \{x\})$.*

Recall that the graph $S_{1,2,3}$ is the tree with exactly one vertex x of degree 3 and exactly three leaves, which are of distance 1, 2 and 3 from x , respectively. The following lemma for $(P_2 + P_3)$ -free bipartite graphs follows from a result of Lozin [29], who proved that the superclass of $S_{1,2,3}$ -free bipartite graphs has bounded clique-width (see [15] for a full classification of the boundedness of clique-width of H -free bipartite graphs).

Lemma 4 ([29]). *The class of $(P_2 + P_3)$ -free bipartite graphs has bounded clique-width.*

A graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G . Besides the above lemmas we will also apply the following three well-known theorems.

Theorem 7 ([9]). *A graph is perfect if and only if it is C_r -free and $\overline{C_r}$ -free for every odd $r \geq 5$.*

Theorem 8 ([19]). *COLOURING is polynomial-time solvable on perfect graphs.*

Theorem 9 ([26,37]). *For any constant c , COLOURING is polynomial-time solvable on graphs of clique-width at most c .*

If G is a graph, then a (possibly empty) set $C \subseteq V(G)$ is a *clique separator* if C is a clique and $G \setminus C$ is disconnected. A graph G is an *atom* if it has no clique separator. If C is a clique separator in G , then we can partition $V(G) \setminus C$ into two sets A and B and we say that we can *decompose* G into the two graphs $G' = G[A \cup C]$ and $G''[B \cup C]$ that are *separated* by C . By recursively decomposing G' and G'' as far as possible, we can decompose G into a set of atoms G_1, \dots, G_r , which can fit together in a specific way (called a *binary decomposition tree*) to give G . Due to the following result of Tarjan, when considering the COLOURING problem in a hereditary class, we only need to consider atoms in the class.

Theorem 10 ([40]). *There is a polynomial-time algorithm that, given a graph G , decomposes G into atoms G_1, \dots, G_r such that $\chi(G) = \max\{\chi(G_1), \dots, \chi(G_r)\}$.*

Let X be a set of vertices in a graph $G = (V, E)$. A vertex $y \in V \setminus X$ is *complete* (resp. *anti-complete*) to X if it is adjacent (resp. non-adjacent) to every vertex of X . A set of vertices $Y \subseteq V \setminus X$ is *complete* (resp. *anti-complete*) to X if every vertex in Y is complete (resp. anti-complete) to X . If X and Y are disjoint sets of vertices in a graph, we say that the edges between these two sets form a *matching* if each vertex in X has at most one neighbour in Y and vice versa (if each vertex has exactly one such neighbour, we say that the matching is *perfect*). Similarly, the edges between these sets form a *co-matching* if each vertex in X has at most one non-neighbour in Y and vice versa.

In the proof of Theorem 11 we present an algorithm for COLOURING restricted to $(P_2 + P_3, \overline{P_2 + P_3})$ -free graphs. Our algorithm first tries to reduce to perfect graphs, in which case we can use Theorem 8. If a $(P_2 + P_3, \overline{P_2 + P_3})$ -free graph is not perfect, then we show that it must contain an induced C_5 . The clique-width of such graphs is not bounded (we can construct a $(P_2 + P_3, \overline{P_2 + P_3})$ -free graph of arbitrarily large clique-width by taking a split graph of arbitrarily large clique-width [33] with clique C and independent set I and adding five new vertices that form an induced C_5 and that are complete to C but anti-complete to I). Therefore, we do some pre-processing to simplify the graph. This enables us to bound the clique-width so that we may then apply Theorem 9. Our general scheme for bounding the clique-width is to partition the remaining vertices into sets according to their neighbourhood in the C_5 and then investigate the possible edges both inside these sets and between them. This enables us to use graph operations (see also Facts 1–3) that do not change the clique-width by “too much” to partition the input graph into disjoint pieces known to have bounded clique-width.

Theorem 11. *COLOURING is polynomial-time solvable for $(P_2 + P_3, \overline{P_2 + P_3})$ -free graphs.*

Proof. Let G be a $(P_2 + P_3, \overline{P_2 + P_3})$ -free graph. By Theorem 10 we may assume that G is an atom, that is, G has no clique separator (we will use this assumption in the proof of Claim 7). We first test in $O(|V(G)|^5)$ time whether G contains an induced C_5 . If not, then G is C_5 -free. Since $\overline{C_5} = C_5$, G is also $\overline{C_5}$ -free. Since G is $(P_2 + P_3)$ -free, G is C_k -free for all $k \geq 7$. Since G is $(\overline{P_2 + P_3})$ -free, G is $\overline{C_k}$ -free for all $k \geq 7$. By Theorem 7, this means that G is a perfect graph. Therefore, by Theorem 8, we can solve COLOURING in polynomial time in this case.

Now assume our algorithm found an induced 5-vertex cycle C with vertices v_1, v_2, v_3, v_4, v_5 , in that order. For $S \subseteq \{1, \dots, 5\}$, let V_S be the set of vertices $x \in V(G) \setminus V(C)$ such that $N(x) \cap V(C) = \{v_i \mid i \in S\}$. A set V_S is *large* if it contains at least three vertices, otherwise it is *small*.

To ease notation, in the remainder of the proof, subscripts on vertex sets should be interpreted modulo 5 and whenever possible we will write V_i instead of $V_{\{i\}}$ and $V_{i,j}$ instead of $V_{\{i,j\}}$ and so on.

We start with three structural claims.

Claim 1. V_\emptyset is an independent set.

If $x, y \in V_\emptyset$ are adjacent then $G[x, y, v_1, v_2, v_3]$ is a $P_2 + P_3$, a contradiction. Therefore V_\emptyset is an independent set. This completes the proof of Claim 1.

Claim 2. $|V_i \cup V_{i+1}| \leq 1$ and $|V_{i+1} \cup V_{i+2}| \leq 1$ for $i \in \{1, 2, 3, 4, 5\}$.

Suppose for contradiction that $x, y \in V_1 \cup V_{1,2}$. If x and y are adjacent then $G[x, y, v_3, v_4, v_5]$ is a $P_2 + P_3$ and if x and y are non-adjacent then $G[v_3, v_4, x, v_1, y]$ is a $P_2 + P_3$. This contradiction implies that $|V_1 \cup V_{1,2}| \leq 1$. Claim 2 follows by symmetry.

Claim 3. $V_{i,i+2}$ is an independent set for $i \in \{1, 2, 3, 4, 5\}$.

Suppose for contradiction that $x, y \in V_{1,3}$ are adjacent. Then $G[v_1, v_3, x, v_2, y]$ is a $\overline{P_2 + P_3}$, a contradiction. Therefore $V_{1,3}$ is an independent set. Claim 3 follows by symmetry.

Since $\overline{C_5} = C_5$, it follows that \overline{G} also contains a C_5 , namely on the vertices v_1, v_3, v_5, v_2 and v_4 in that order. Therefore \overline{G} is also a $(P_2 + P_3, \overline{P_2 + P_3})$ -free graph containing an induced C_5 . As a result, we immediately obtain the following three claims as corollaries of Claims 1–3. For instance, $V_{1,2,3,4,5}$ is exactly the set of vertices that are not adjacent to any of the vertices v_1, \dots, v_5 in \overline{G} . As v_1, \dots, v_5 form an induced C_5 in \overline{G} , by Claim 1 we find that $V_{1,2,3,4,5}$ is an independent set in \overline{G} and thus a clique in G , which proves Claim 4. Claims 5 and 6 are proven by analogous reasoning (taking into account the cycle order v_1, v_3, v_5, v_2, v_4 in \overline{G}).

Claim 4. $V_{1,2,3,4,5}$ is a clique.

Claim 5. $|V_{i,i+1,i+2,i+3} \cup V_{i,i+1,i+3}| \leq 1$ and $|V_{i,i+1,i+2,i+3} \cup V_{i,i+2,i+3}| \leq 1$ for $i \in \{1, 2, 3, 4, 5\}$.

Claim 6. $V_{i,i+1,i+2}$ is a clique for $i \in \{1, 2, 3, 4, 5\}$.

Let I be the set of vertices in V_\emptyset that have a non-neighbour in $V_{1,2,3,4,5}$.

Claim 7. G is k -colourable if and only if $G \setminus I$ is k -colourable.

The “only if” direction is trivial. Suppose $G \setminus I$ is k -colourable. Fix a k -colouring c of $G \setminus I$. We will show how to extend c to all vertices of G .

We may assume V_\emptyset and $V_{1,2,3,4,5}$ are non-empty, otherwise $I = \emptyset$ and the claim follows trivially. Note that V_\emptyset is an independent set by Claim 1 and that $V_{1,2,3,4,5}$ is a clique by Claim 4. By assumption, G is an atom and hence $V_{1,2,3,4,5}$ is not a clique separator in G . Since V_\emptyset is an independent set, every vertex of V_\emptyset must have a neighbour in $V(G) \setminus (V_\emptyset \cup V_{1,2,3,4,5} \cup V(C))$ that is adjacent to some vertex of the cycle C . Let A be the set of vertices outside $V_{1,2,3,4,5}$ that have a neighbour in V_\emptyset . Note that A contains no vertex of V_\emptyset , as V_\emptyset is an independent set. Hence, every vertex of A is adjacent to at least one vertex of C (but not to all vertices of C).

We claim that A is complete to $V_{1,2,3,4,5}$. Indeed, suppose for contradiction that $x \in A$ is non-adjacent to $y \in V_{1,2,3,4,5}$. Let s be a vertex in V_\emptyset that is a neighbour of x . Without loss of generality, assume that x is adjacent to v_1 . Then x is adjacent to v_3 or v_4 , otherwise $G[v_3, v_4, s, x, v_1]$ would be a $P_2 + P_3$, a contradiction. Without loss of generality, assume that x is adjacent to v_3 . Then x is adjacent to v_2 , otherwise $G[v_1, v_3, y, x, v_2]$ would be a $\overline{P_2 + P_3}$, a contradiction. Thus x is adjacent to v_4 or v_5 , otherwise $G[v_4, v_5, s, x, v_2]$ would be a $P_2 + P_3$, a contradiction. Without loss of generality, assume that x is adjacent to v_4 . Now x must be adjacent to v_5 , otherwise $G[v_1, v_4, y, x, v_5]$ would be a $\overline{P_2 + P_3}$, a contradiction. Therefore $x \in V_{1,2,3,4,5}$, a contradiction. It follows that A must indeed be complete to $V_{1,2,3,4,5}$.

Let $x \in I$. By definition of I , there is a vertex $y \in V_{1,2,3,4,5}$ that is non-adjacent to x . We claim that $c(y)$ can also be used to colour x . Indeed, every neighbour of x lies in $A \cup V_{1,2,3,4,5} \setminus \{y\}$. Since $V_{1,2,3,4,5}$ is a clique and A is complete to $V_{1,2,3,4,5}$, no vertex of $A \cup V_{1,2,3,4,5} \setminus \{y\}$ is coloured with the colour $c(y)$. We may therefore colour x with the colour $c(y)$. Repeating this process, we can extend the k -colouring of $G \setminus I$ to a k -colouring of G . This completes the proof of Claim 7.

By Claim 7, our algorithm may safely remove all vertices of I from G without changing the chromatic number. Hence we may assume that the following claim holds.

Claim 8. V_\emptyset is complete to $V_{1,2,3,4,5}$.

Our algorithm has finished the pre-processing. We are now ready to prove, through a series of further claims, that G has **bounded clique-width**. Hence our algorithm can

colour G in polynomial time by Theorem 9. For an overview of our whole algorithm see the pseudo-code below.

Algorithm: Colouring Algorithm for $(P_2 + P_3, \overline{P_2 + P_3})$ -free graphs

Input: a $(P_2 + P_3, \overline{P_2 + P_3})$ -free graph G

Output: $\chi(G)$

determine a decomposition of G into atoms G_1, \dots, G_r using the algorithm of Theorem 10

```

for each  $i \in \{1, \dots, r\}$  do
  if  $G_i$  does not contain an induced  $C_5$  then
    | apply Theorem 8 on  $G_i$  to find  $\chi(G_i)$ 
  else
    | determine the set  $I$  for  $G_i$ 
    | apply Theorem 9 on  $G_i - I$  to find  $\chi(G_i - I)$ 
    |  $\chi(G_i) \leftarrow \chi(G_i - I)$ 
  end
end
return  $\max\{\chi(G_1), \dots, \chi(G_r)\}$ 

```

Claim 9. Suppose $S, T \subseteq \{1, 2, 3, 4, 5\}$ with $|S| = 2$ and $|T| = 3$. Then at least one of V_S and V_T is small.

Suppose for contradiction that there are large sets V_S and V_T for $S, T \subseteq \{1, 2, 3, 4, 5\}$ with $|S| = 2$ and $|T| = 3$. By Claim 2 and symmetry, we may assume that $S = \{1, 3\}$. By Claim 5 and symmetry, we may assume that $T = \{i, i+1, i+2\}$ for some $i \in \{1, 2, 3\}$. We consider each of these cases in turn.

Suppose $x \in V_{1,3}$ is non-adjacent to $y \in V_{1,2,3}$. Then $G[v_1, v_3, v_2, x, y]$ is a $\overline{P_2 + P_3}$, a contradiction. Therefore $V_{1,3}$ is complete to $V_{1,2,3}$. Suppose $x, x' \in V_{1,3}$ and $y \in V_{1,2,3}$. Then y must be adjacent to both x and x' . By Claim 3, x is non-adjacent to x' . Therefore $G[v_4, v_5, x, y, x']$ is a $P_2 + P_3$, a contradiction. It follows that if $V_{1,3}$ is large then $V_{1,2,3}$ is empty.

Suppose $x \in V_{1,3}$ is adjacent to $y \in V_{2,3,4}$. Then $G[x, v_2, v_3, v_1, y]$ is a $\overline{P_2 + P_3}$, a contradiction. Therefore $V_{1,3}$ is anti-complete to $V_{2,3,4}$. Suppose $x, x' \in V_{1,3}$ and $y \in V_{2,3,4}$. Then y is non-adjacent to both x and x' . By Claim 3, x is non-adjacent to x' . Therefore $G[v_4, y, x, v_1, x']$ is a $P_2 + P_3$, a contradiction. It follows that if $V_{1,3}$ is large then $V_{2,3,4}$ is empty.

Suppose $x \in V_{1,3}$ is non-adjacent to $y \in V_{3,4,5}$. Then $G[v_4, y, v_2, v_1, x]$ is a $P_2 + P_3$, a contradiction. Therefore $V_{1,3}$ is complete to $V_{3,4,5}$. Suppose $x, x' \in V_{1,3}$ and $y \in V_{3,4,5}$. Then y must be adjacent to both x and x' . By Claim 3, x is non-adjacent to x' . Therefore $G[x, x', y, v_1, v_3]$ is a $\overline{P_2 + P_3}$, a contradiction. It follows that if $V_{1,3}$ is large then $V_{3,4,5}$ is empty. This completes the proof of Claim 9.

Claim 9 implies that we can have large sets V_S with $|S| = 2$ or large sets V_T with $|T| = 3$ (or neither), but not both. Again, since $\overline{C_5} = C_5$, the graph \overline{G} contains a C_5 on vertices v_1, v_3, v_5, v_2, v_4 , in that order. A vertex in G with exactly two (resp. three) neighbours in C will have exactly three (resp. two) neighbours in this new cycle in \overline{G} . Therefore, if there is a large set V_T with $|T| = 3$ then V_S is small whenever $|S| = 2$. In this case we replace G by \overline{G} and replace C by this new cycle (we may do this by Fact 2). We may therefore assume that V_T is small for all T with $|T| = 3$. Note that after doing this we no longer have the symmetry between the situation for G and that for \overline{G} . Indeed, in Claim 7 we showed that because G has no clique separator, we may assume that V_\emptyset is complete to $V_{1,2,3,4,5}$ (Claim 8). However, we cannot guarantee that \overline{G} has no clique separator. Therefore, if we do complement G above, then the

sets V_\emptyset and $V_{1,2,3,4,5}$ will be swapped and will become anti-complete to each other, instead of complete to each other.

Suppose V_S is small for some set $S \subseteq \{1, 2, 3, 4, 5\}$. Then, by Fact 1, we may delete the vertices of V_S . We may therefore assume that for each $S \subseteq \{1, 2, 3, 4, 5\}$, V_S is either large or empty. By Claims 2 and 5 and our assumption that V_T is small for all T with $|T| = 3$ it follows that the only sets V_S that can be large are V_\emptyset , $V_{1,2,3,4,5}$ and $V_{i,i+2}$ for $i \in \{1, 2, 3, 4, 5\}$.

Claim 10. $V_{1,2,3,4,5}$ is complete to $V_{i,i+2}$ for all $i \in \{1, 2, 3, 4, 5\}$.

Suppose for contradiction that $x \in V_{1,3}$ is non-adjacent to $y \in V_{1,2,3,4,5}$. Then $G[v_1, v_3, v_2, x, y]$ is a $P_2 + P_3$, a contradiction. Claim 10 follows by symmetry.

Claim 11. We may assume that $V_{1,2,3,4,5}$ is empty.

By Claim 10, $V_{1,2,3,4,5}$ is complete to each set $V_{i,i+2}$. By Claim 8, V_\emptyset is complete to $V_{1,2,3,4,5}$ or, if we complemented G then it is anti-complete to $V_{1,2,3,4,5}$. By Fact 3, we may apply a bipartite complementation between $V_{1,2,3,4,5}$ and $V(C) \cup \bigcup_{i \in \{1,2,3,4,5\}} V_{i,i+2}$. If V_\emptyset is complete to $V_{1,2,3,4,5}$ then we apply a bipartite complementation between $V_{1,2,3,4,5}$ and V_\emptyset . These operations disconnect $G[V_{1,2,3,4,5}]$ from the rest of the graph. By Claim 4, $G[V_{1,2,3,4,5}]$ is a complete graph, so it has clique-width at most 2. We may therefore assume that $V_{1,2,3,4,5}$ is empty. This completes the proof of Claim 11.

Claim 12. For $i \in \{1, 2, 3, 4, 5\}$, the edges between $V_{i,i+2}$ and V_\emptyset form a matching.

By symmetry, it suffices to prove the claim for $i = 1$. Note that V_\emptyset and $V_{1,3}$ are independent sets by Claims 1 and 3, respectively. If the claim is false then there must be a vertex y in one of these sets that has two neighbours x and x' in the other set. Then x and x' are non-adjacent, so $G[v_4, v_5, x, y, x']$ is a $P_2 + P_3$. This contradiction completes the proof of Claim 12.

Claim 13. For distinct $i, j \in \{1, 2, 3, 4, 5\}$, if $x \in V_{i,i+2}$ is adjacent to $y \in V_\emptyset$ and $z \in V_{j,j+2}$ is non-adjacent to y then x is adjacent to z .

Suppose, for contradiction, that the claim is false. By symmetry, we may assume that $i = 1$, $j \in \{2, 3\}$. Indeed, suppose $x \in V_{1,3}$ is adjacent to $y \in V_\emptyset$ and $z \in V_{j,j+2}$ is non-adjacent to x and y . Then $G[x, y, v_5, v_4, z]$ or $G[x, y, v_4, v_5, z]$ is a $P_2 + P_3$ if $j = 2$ or $j = 3$, respectively. This contradiction completes the proof of Claim 13.

Claim 14. We may assume that V_\emptyset is empty.

Recall that $V_{i,i+2}$ is an independent set for $i \in \{1, 2, 3, 4, 5\}$, by Claim 3. For $i \in \{1, 2, 3, 4, 5\}$, let $V'_{i,i+2}$ be the set of vertices in $V_{i,i+2}$ that have a neighbour in V_\emptyset and let $G' = G[V_\emptyset \cup \bigcup_{i \in \{1,2,3,4,5\}} V'_{i,i+2}]$. By Claim 13, for distinct $i, j \in \{1, 2, 3, 4, 5\}$, we find that $V'_{i,i+2}$ is complete to $V_{j,j+2} \setminus V'_{j,j+2}$. By Fact 3 we may apply a bipartite complementation between every such pair of sets and also between $\{v_i, v_{i+2}\}$ and $V'_{i,i+2}$ for $i \in \{1, 2, 3, 4, 5\}$. This disconnects G' from the rest of the graph. We now prove that G' has bounded clique-width.

Claim 12 implies that for $i \in \{1, 2, 3, 4, 5\}$, every vertex in V_\emptyset has at most one neighbour in $V'_{i,i+2}$ and every vertex in $V'_{i,i+2}$ has exactly one neighbour in V_\emptyset . Furthermore, for distinct $i, j \in \{1, 2, 3, 4, 5\}$, if $x \in V'_{i,i+2}$ and $y \in V'_{j,j+2}$ have different neighbours in V_\emptyset then x is adjacent to y by Claim 13. Now let G'' be the graph obtained from G' by applying a bipartite complementation between $V'_{i,i+2}$ and $V'_{j,j+2}$ for each pair of distinct $i, j \in \{1, 2, 3, 4, 5\}$. Then, for distinct $i, j \in \{1, 2, 3, 4, 5\}$, if $x \in V'_{i,i+2}$ and $y \in V'_{j,j+2}$ are adjacent in G'' then they must have the same unique neighbour in V_\emptyset in the graph G'' . Therefore every component of G'' contains exactly one vertex in V_\emptyset and at most one vertex in each set $V'_{i,i+2}$, so each component contains at most six vertices. It follows that G'' has clique-width at most 6. By Fact 3, G' has bounded clique-width. We may therefore assume that V_\emptyset is empty. This completes the proof of Claim 14.

Note that in the proof of Claim 14 we may remove vertices from $V_{i,i+2}$ for some $i \in \{1, 2, 3, 4, 5\}$. If this causes a set $V_{i,i+2}$ to become small, then again by Fact 1, we may delete all remaining vertices of $V_{i,i+2}$. Therefore, we may again assume that each set $V_{i,i+2}$ is either large or empty. We now analyse the edges between these sets.

Claim 15. *For $i \in \{1, 2, 3, 4, 5\}$, the edges between $V_{i,i+2}$ and $V_{i+1,i+3}$ form a co-matching.*

Suppose for contradiction that the claim is false. Without loss of generality, assume there is a vertex $x \in V_{1,3}$ with two non-neighbours $y, y' \in V_{2,4}$. Then y must be non-adjacent to y' by Claim 3, so $G[x, v_1, y, v_4, y']$ is a $P_2 + P_3$, a contradiction. This completes the proof of Claim 15.

Claim 16. *For $i \in \{1, 2, 3, 4, 5\}$, suppose $x \in V_{i,i+2}$ is adjacent to $y \in V_{i+1,i+3}$. If $z \in V_{i+3,i}$ then z is not complete to $\{x, y\}$.*

Suppose, for contradiction that $x \in V_{1,3}$ is adjacent to $y \in V_{2,4}$ and $z \in V_{4,1}$ is complete to $\{x, y\}$. Then $G[v_1, y, x, v_2, z]$ is a $\overline{P_2 + P_3}$, a contradiction. This completes the proof of Claim 16.

Claim 17. *For $i \in \{1, 2, 3, 4, 5\}$, if $V_{i,i+2}$ is large then $V_{i-1,i+1}$ is anti-complete to $V_{i+1,i+3}$.*

Suppose the claim is false. Without loss of generality, assume that $V_{2,4}$ is large and $x \in V_{1,3}$ is adjacent to $z \in V_{3,5}$. By Claim 15, the vertices x and z each have at most one non-neighbour in $V_{2,4}$. Since $V_{2,4}$ is large, there must therefore be a vertex $y \in V_{2,4}$ that is adjacent to both x and z . Then $G[v_3, y, x, v_2, z]$ is a $\overline{P_2 + P_3}$, a contradiction. This completes the proof of Claim 17.

For distinct $i, j \in \{1, 2, 3, 4, 5\}$, we say that $V_{i,i+2}$ and $V_{j,j+2}$ are *consecutive* sets if v_i and v_j are adjacent vertices of the cycle C and *opposite* sets if they are not. We consider cases depending on which sets $V_{i,i+2}$ are large.

Case 1. $V_{i,i+2}$ is large for all $i \in \{1, 2, 3, 4, 5\}$.

By Claim 3, every set $V_{i,i+2}$ is independent. Then, by Claim 17, if $V_{i,i+2}$ and $V_{j,j+2}$ are opposite then they must be anti-complete to each other. By Claim 15, if $V_{i,i+2}$ and $V_{j,j+2}$ are consecutive, then the edges between these sets form a co-matching. By Fact 1, we may delete the vertices of the cycle C . Applying a bipartite complementation between each pair of consecutive sets $V_{i,i+2}$ and $V_{j,j+2}$, we therefore obtain a graph of maximum degree 2. By Lemma 2 and Fact 3, G has bounded clique-width. This completes Case 1.

Case 2. $V_{i,i+2}$ is large for $i \in \{1, 2, 3\}$ and empty for $i = 5$. ($V_{i,i+2}$ may be large or empty for $i = 4$.)

By Claim 3, every set $V_{i,i+2}$ is independent. Then, by Claim 17, $V_{1,3}$ is anti-complete to $V_{3,5}$ and $V_{2,4}$ is anti-complete to $V_{4,1}$ (note that this also holds in the case where $V_{4,1} = \emptyset$). Therefore $V_{1,3} \cup V_{3,5}$ and $V_{2,4} \cup V_{4,1}$ are independent sets, so $G[V_{1,3} \cup V_{3,5} \cup V_{2,4} \cup V_{4,1}]$ is a bipartite $(P_2 + P_3)$ -free graph, and thus has bounded clique-width by Lemma 4. By Fact 1, we may delete the vertices of the cycle C . All remaining vertices are in $V_{1,3} \cup V_{3,5} \cup V_{2,4} \cup V_{4,1}$. Therefore G has bounded clique-width. This completes Case 2.

Case 3. $V_{i,i+2}$ is large for at most two values $i \in \{1, 2, 3, 4, 5\}$.

By Claim 3, every set $V_{i,i+2}$ is independent. By Fact 1, we may delete the vertices of the cycle C . Therefore the remainder of the graph consists of the at most two sets $V_{i,i+2}$, and is therefore a bipartite $(P_2 + P_3)$ -free graph, so it has bounded clique-width by Lemma 4. Therefore G has bounded clique-width. This completes Case 3.

By symmetry, only one case remains:

Case 4. $V_{i,i+2}$ is large for $i \in \{1, 2, 4\}$ and empty for $i \in \{3, 5\}$.

We consider two subcases:

Case 4a. No vertex of $V_{4,1}$ has a neighbour in both $V_{1,3}$ and $V_{2,4}$.

By Claim 3, $V_{1,3}$, $V_{2,4}$ and $V_{4,1}$ are independent. By Fact 1, we may delete the vertices of the cycle C . All remaining vertices belong to $V_{1,3} \cup V_{2,4} \cup V_{4,1}$. Let $V_{4,1}^*$ be the set of vertices in $V_{4,1}$ that have a neighbour in $V_{1,3}$ and let $V_{4,1}^{**} = V_{4,1} \setminus V_{4,1}^*$. Note that if a vertex $x \in V_{4,1}$ has a neighbour in $V_{2,4}$ then $x \in V_{4,1}^{**}$. Now $V_{2,4} \cup V_{4,1}^*$ and $V_{1,3} \cup V_{4,1}^{**}$ are independent sets, so $G[V_{2,4} \cup V_{4,1}^* \cup V_{1,3} \cup V_{4,1}^{**}]$ is a bipartite $(P_2 + P_3)$ -free graph and thus has bounded clique-width by Lemma 4. Therefore G has bounded clique-width. This completes Case 4a.

Case 4b. There is a vertex $x \in V_{4,1}$ with neighbours $y \in V_{1,3}$ and $z \in V_{2,4}$.

By Claim 3, $V_{1,3}$, $V_{2,4}$ and $V_{4,1}$ are independent. Then, by Claim 16, y must be non-adjacent to z . By Claim 3, $V_{1,3}$, $V_{2,4}$ and $V_{4,1}$ are independent. By Claim 15, the edges between $V_{1,3}$ and $V_{2,4}$ form a co-matching, so y is complete to $V_{2,4} \setminus \{z\}$ and z is complete to $V_{1,3} \setminus \{y\}$. By Claim 16, it follows that x has no neighbours in $(V_{1,3} \setminus \{y\}) \cup (V_{2,4} \setminus \{z\})$. Then, since $V_{1,3}$ and $V_{2,4}$ are large, there must be adjacent vertices $y' \in V_{1,3} \setminus \{y\}$ and $z' \in V_{2,4} \setminus \{z\}$. Now $G[x, y, z', y', z]$ is a C_5 , with vertices in that order; we denote this cycle by C' . By Fact 1, we may delete the vertices of the original cycle C . Repeating the arguments of Claim 2, but applied with respect to the cycle C' instead of C , we find that at most five vertices outside C' have exactly one neighbour in C' . By Fact 1, we may delete any such vertices.

Let $y'' \in V_{1,3} \setminus \{y, y'\}$. By Claim 3, y'' is non-adjacent to y and y' . Since z is complete to $V_{1,3} \setminus \{y\}$, it follows that y'' is adjacent to z . By Claim 16 and the fact that z is adjacent to x and y'' , it follows that x cannot be adjacent to y'' . Since y'' cannot have exactly one neighbour on C' , we conclude that y'' must be adjacent to z' . Therefore every vertex of $V_{1,3} \setminus \{y, y'\}$ is adjacent to z and z' , but no other vertices of C' . Similarly, every vertex of $V_{2,4} \setminus \{z, z'\}$ is adjacent to y and y' , but no other vertices of C' .

Let $x' \in V_{4,1} \setminus \{x\}$. By Claim 3, the set $V_{4,1}$ is independent, so x' is non-adjacent to x . Therefore x' must have at least two neighbours in $\{y, y', z, z'\}$, but by Claim 16, it cannot be complete to $\{y, z'\}$, $\{y', z'\}$ or $\{y', z\}$. Therefore the neighbourhood of x' in C' must be $\{y, y'\}$, $\{z, z'\}$ or $\{y, z\}$. If x' is adjacent to y then by Claim 16 and the fact that y is complete to $V_{2,4} \setminus \{z\}$, it follows that x' is anti-complete to $V_{2,4} \setminus \{z\}$. Similarly, if x' is adjacent to z , then it is anti-complete to $V_{1,3} \setminus \{y\}$. This means that if x' is adjacent to y and z then it has no other neighbours, so it is a false twin of x and by Lemma 3 we may delete x' in this case. Therefore x' is either adjacent to y and y' , in which case x' is anti-complete to $V_{2,4}$, or x' is adjacent to z and z' , in which case x' is anti-complete to $V_{1,3}$.

Let $V_{4,1}^*$ be the set of vertices in $V_{4,1}$ that are adjacent to y and y' and let $V_{4,1}^{**}$ be the remaining vertices of $V_{4,1}$. Now $V_{2,4} \cup V_{4,1}^*$ and $V_{1,3} \cup V_{4,1}^{**}$ are independent sets. Deleting the vertex x , we obtain the graph $G[V_{2,4} \cup V_{4,1}^* \cup V_{1,3} \cup V_{4,1}^{**}]$, which is a bipartite $(P_2 + P_3)$ -free graph and thus has bounded clique-width by Lemma 4. By Fact 1, it follows that G has bounded clique-width. This concludes Case 4b and therefore Case 4.

We have proved that G has bounded clique-width. By Theorem 9, this completes the proof of the theorem. \square

4.3 The Final Proof

We are now ready to prove Theorem 3.

Theorem 3 (restated). *Let H be a graph with $H, \overline{H} \notin \{(s+1)P_1 + P_3, sP_1 + P_4 \mid s \geq 2\}$. Then COLOURING is polynomial-time solvable for (H, \overline{H}) -free graphs if H or \overline{H} is an induced subgraph of $K_{1,3}$, $P_1 + P_4$, $2P_1 + P_3$, $P_2 + P_3$, P_5 , or $sP_1 + P_2$ for some $s \geq 0$ and it is NP-complete otherwise.*

Proof. We first consider the polynomial-time cases. The COLOURING problem is solvable in polynomial time for $(K_{1,3}, \overline{K_{1,3}})$ -free graphs [34], $(P_1 + P_4, \overline{P_1 + P_4})$ -free graphs (by Theorem 9 combined with the fact that they have bounded clique-width [3]), $(2P_1 + P_3, \overline{2P_1 + P_3})$ -free graphs [1], $(P_2 + P_3, \overline{P_2 + P_3})$ -free graphs (by Theorem 11), $(P_5, \overline{P_5})$ -free graphs [22] and $(sP_1 + P_2, \overline{sP_1 + P_2})$ -free graphs for $s \geq 0$ [14].

Let H be a graph and suppose that COLOURING is not NP-complete for (H, \overline{H}) -free graphs and that neither H nor \overline{H} is isomorphic to $(s+1)P_1 + P_3$ or $sP_1 + P_4$ for $s \geq 2$. We will show that one of the polynomial-time cases above holds. For any $p \geq 3$, COLOURING is NP-complete for (C_3, \dots, C_p) -free graphs due to Lemma 1. Therefore, we may assume without loss of generality that H is a forest.

First suppose that H contains a vertex of degree at least 3. Then $K_{1,3} \subseteq_i H$. We may assume H contains a vertex x not in this $K_{1,3}$, otherwise we are done. Since H is a forest, x can have at most one neighbour on the $K_{1,3}$. Then $\overline{2P_1 + P_2} \subseteq_i \overline{H}$ if x is adjacent to a leaf vertex of the $K_{1,3}$ and $K_4 \subseteq_i \overline{H}$ if it is not. This means that the class of (H, \overline{H}) -free graphs contains the class of $(K_{1,3}, K_4, \overline{2P_1 + P_2})$ -free graphs for which COLOURING is NP-complete [27]. We may therefore assume that H does not contain a vertex of degree 3, so H must be a linear forest.

Now H must be $(P_1 + 2P_2, 2P_3, P_6)$ -free, otherwise the problem is NP-complete on the class of (H, \overline{H}) -free graphs by Theorem 6. Let H_1, \dots, H_r be the components of H with $|V(H_1)| \geq \dots \geq |V(H_r)|$ for some $r \geq 1$ and note that each component H_j is isomorphic to a path.

If $r \geq 2$ then $|V(H_2)| \leq 2$ since H is $2P_3$ -free. Suppose $|V(H_2)| = 2$. Then $r \leq 2$ and $|V(H_1)| \leq 3$, since H is $(P_1 + 2P_2)$ -free. This means that $H \subseteq_i P_2 + P_3$, so we are done. We may therefore assume that all components apart from H_1 are trivial, that is, $H = sP_1 + P_t$ for some $s, t \geq 0$. Now $t \leq 5$, since H is P_6 -free. If $t = 5$ then $s = 0$ since H is $(P_1 + 2P_2)$ -free, so $H = P_5$ and we are done. If $t = 4$ then $s \leq 1$ by assumption, so $H \subseteq_i P_1 + P_4$ and we are done. If $t = 3$ then $s \leq 2$ by assumption, so $H \subseteq_i 2P_1 + P_3$ and we are done. If $t \leq 2$ then $H \subseteq_i sP_1 + P_2$ for some $s \geq 0$ and we are done. This completes the proof. \square

5 The Proofs of Theorems 4 and 5

To prove Theorems 4 and 5, we use a construction introduced by Huang [23, Section 2]. Recall that the chromatic number and clique number of a graph G are denoted by $\chi(G)$ and $\omega(G)$, respectively. A graph G is k -critical if and only if $\chi(G) = k$ and $\chi(G - v) < k$ for every vertex v in G . A k -critical graph G is *nice* if and only if it contains an independent set on three vertices c_1, c_2, c_3 , such that $\omega(G) = \omega(G - \{c_1, c_2, c_3\}) = k - 1$.

We describe the construction of Huang. Let I be an instance of 3-SAT with variables $X = \{x_1, \dots, x_n\}$ and clauses $\mathcal{C} = \{C^1, \dots, C^m\}$. We may assume that the variables in each clause are pairwise distinct. Let H be a nice k -critical graph. By definition, H contains three pairwise non-adjacent vertices c_1, c_2, c_3 such that $\omega(H) = \omega(H - \{c_1, c_2, c_3\}) = k - 1$. We construct the graph $G_{H,I}$ as follows:

- For each variable x_i , we create a pair of literal vertices x_i and $\overline{x_i}$ joined by an edge. We call these vertices X -type.
- For each variable x_i , we create a variable vertex d_i . We call these vertices D -type.

- For each clause C^j , we create a subgraph H_j isomorphic to H . The three vertices of H_j corresponding to c_1, c_2, c_3 are denoted c_1^j, c_2^j, c_3^j , respectively, and are called C -type; they are understood to each represent a distinct literal of C^j . The remaining vertices of H_j are called U -type.

We have the following additional adjacencies:

- Every U -type vertex is adjacent to every D -type and every X -type vertex.
- Every C -type vertex is adjacent to the X -type vertex and D -type vertex that represent the corresponding literal and variable.

The following are [23, Lemma 1] and [23, Lemma 2].

Lemma 5 ([23]). *Let H be a nice k -critical graph. An instance I of 3-SAT is satisfiable if and only if $G_{H,I}$ is $(k+1)$ -colourable.*

Lemma 6 ([23]). *Let H be a nice k -critical graph. If H is P_t -free for any integer $t \geq 6$, then $G_{H,I}$ is also P_t -free.*

Theorem 4 (restated). *4-COLOURING is NP-complete for $(P_7, \overline{P_8})$ -free graphs, and thus for $(P_8, \overline{P_8})$ -free graphs.*

Proof. It follows from Lemma 5 that we need only exhibit a nice 3-critical graph H such that $G_{H,I}$ is $(P_7, \overline{P_8})$ -free for any 3-SAT instance I . We claim that C_7 will suffice. This is the graph used by Huang [23, Theorem 6] to show that 4-COLOURING is NP-complete for P_7 -free graphs. He noted (and it is trivial to check) that C_7 is a P_7 -free nice 3-critical graph and so, by Lemma 6, $G_{C_7,I}$ is also P_7 -free. It only remains to show that $G_{C_7,I}$ is $\overline{P_8}$ -free. Suppose, for contradiction, that $G_{C_7,I}$ contains a $\overline{P_8}$, whose vertex set we denote P , as an induced subgraph.

We observe that any four vertices of P induce at least three edges and that any three vertices induce at least one edge. So P cannot contain four vertices that are each either X -type or D -type as they would induce at most two edges.

So P contains at least five vertices that belong to copies of C_7 . We recall that vertices from distinct copies of C_7 are not adjacent in $G_{C_7,I}$. So if three of these five vertices belong to three distinct copies of C_7 , they induce no edge, a contradiction. Hence there must be a copy of C_7 that contains at least three of the vertices. As two of them must be non-adjacent, no other vertex of the five can belong to another copy of C_7 , otherwise we again have three vertices that induce no edge. Therefore the five vertices must all belong to the same copy. Considering the subgraphs of C_7 on five vertices, we see that these five vertices induce one of $P_1 + P_4$, $P_2 + P_3$ and P_5 , each of which contains an independent set on three vertices. Therefore P contains an independent set on three vertices. This contradiction completes the proof. \square

Theorem 5 (restated). *5-COLOURING is NP-complete for $(P_2 + P_5, P_6, \overline{P_1 + P_6})$ -free graphs, and thus for $(P_1 + P_6, \overline{P_1 + P_6})$ -free graphs.*

Proof. It follows from Lemma 5 that we need only to exhibit a nice 4-critical graph H such that $G_{H,I}$ is $(P_2 + P_5, P_6, \overline{P_1 + P_6})$ -free for any 3-SAT instance I . We claim that the graph H of Fig. 5 will suffice. This graph was used by Chudnovsky, Huang, Spirkl and Zhong [8, Theorem 1.8] and by Huang [23, Theorem 5] to show that 5-COLOURING is NP-complete for $(P_2 + P_5)$ -free and P_6 -free graphs, respectively. Huang [23] noted, and it is easy to verify, that H is a P_6 -free nice 4-critical graph and so, by Lemma 6, $G_{H,I}$ is also P_6 -free. Chudnovsky, Huang, Spirkl and Zhong [8] showed that $G_{H,I}$ is also $(P_2 + P_5)$ -free. It only remains to show that $G_{H,I}$ is $\overline{P_1 + P_6}$ -free. Suppose, for contradiction, that $G_{H,I}$ contains $\overline{P_1 + P_6}$ as an induced subgraph. Thus $G_{H,I}$ contains

a set, denoted P , of six vertices that induce a $\overline{P_6}$ and a further vertex q adjacent to every vertex of P .

We observe that any four vertices of P induce at least three edges and that any three vertices induce at least one edge.

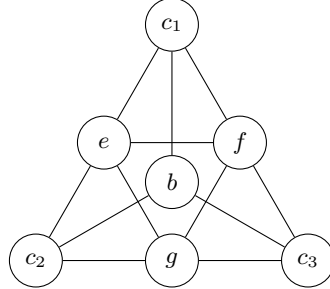


Fig. 5. The graph H used in the proof of Theorem 5.

Suppose that q is either D -type or X -type. Then q has at most one neighbour that is also D -type or X -type, and so P contains at least five vertices that belong to copies of H . Vertices from distinct copies of H are not adjacent in $G_{H,I}$. If three of these five vertices belong to three distinct copies of H , they induce no edge, a contradiction. Hence the five vertices belong to at most two distinct copies of H . If there are two copies of H that each contain at least two of the five vertices, then there are four vertices that induce at most two edges, a contradiction. Therefore at least four of the five vertices belong to the same copy of H and so, in fact, they must all belong to the same copy (otherwise there is a single vertex in a distinct copy that is adjacent to none of the other four, a contradiction). Label the vertices in that copy as in Fig. 5. We notice that P contains at most one C -type vertex as q is adjacent to only one C -type vertex in each copy of H (recall that we assumed that the clauses of the 3-SAT instance contain three distinct variables). So without loss of generality P contains c_1, b, e, f and g , but these vertices do not induce a subgraph of $\overline{P_6}$ (notice, for example, that there are three vertices not adjacent to b), a contradiction.

Let us suppose instead that q is a C -type vertex. Then q has degree 5 in $G_{H,I}$ so it cannot be adjacent to every vertex of P , a contradiction.

Finally we assume that q is a U -type vertex and that it belongs to a copy of H with the labelling of Fig. 5. By the symmetry of H , without loss of generality, only two cases remain: namely $q = b$ or $q = e$. If $q = b$, then the possible vertices of P are c_1, c_2, c_3 and D -type and X -type vertices. Hence P contains zero, one or two C -type vertices (it cannot contain all three since P does not contain an independent set of size 3). Then P contains at least four D -type and X -type vertices, but these four vertices can induce at most two edges, a contradiction.

So we must have $q = e$ and P contains vertices from c_1, c_2, f, g and the D -type and X -type vertices. Then P cannot contain both c_1 and c_2 as every pair of non-adjacent vertices in a $\overline{P_6}$ has a common neighbour (since every pair of adjacent vertices in a P_6 has a common non-neighbour), but c_1 and c_2 have no possible common neighbour as they represent distinct variables and so are joined to different D -type and X -type vertices. Observe that for each vertex v of P , there is at least one other vertex of P that is not adjacent to v . So, looking for possible non-neighbours, we have that

- if f is in P , then c_2 is in P , and

– if g is in P , then c_1 is in P ,

and so P cannot contain both f and g . Thus P contains at least four D -type and X -type vertices, but these four vertices can induce at most two edges. This contradiction completes the proof. \square

We finish this section and our paper with two remarks on the question marks in Table 1.

Remark 1. There is no nice k -critical graph H with the property that $G_{H,I}$ is $\overline{P_6}$ -free for every 3-SAT instance I . This can be seen as follows. Let H_j and H_ℓ be two copies of H in $G_{H,I}$ corresponding to distinct clauses C^j and C^ℓ . Let c and u be adjacent vertices of H_j that are of C -type and U -type, respectively. Note that these two vertices must exist, as H is connected and the three C -type vertices are independent. Let u' be a U -type vertex in H_ℓ . Assume without loss of generality that c is adjacent to an X -type vertex x_h , and thus also adjacent to d_h , but not to $\overline{x_h}$. Then the vertices $x_h, d_h, \overline{x_h}, c, u', u$, in that order, form an induced P_6 in $\overline{G_{H,I}}$ and thus an induced $\overline{P_6}$ in $G_{H,I}$. Hence, Lemma 5 cannot be used to find an integer $k \geq 5$ (if one exists) such that k -COLOURING is NP-complete for $(P_6, \overline{P_6})$ -free graphs (recall that 4-COLOURING is polynomial-time solvable for P_6 -free graphs [10,11] and that 3-COLOURING is polynomial-time solvable even for P_7 -free graphs [2]).

Remark 2. The P_7 -free graph $G_{C_7,I}$ used in the proof of Theorem 4 is not $\overline{P_7}$ -free. In order to see this, we pick a copy H_j of $H = C_7$ with vertices a_1, \dots, a_7 in that order. Without loss of generality, we may assume a_1, a_3 and a_5 are of C -type, and a_2, a_4, a_6 and a_7 are of U -type. Assume without loss of generality that a_1 is adjacent to an X -type vertex x_h , and thus also adjacent to d_h , but not to $\overline{x_h}$. Then the vertices $x_h, d_h, \overline{x_h}, a_1, a_6, a_2, a_7$, in that order, form an induced P_7 in $\overline{G_{C_7,I}}$ and thus an induced $\overline{P_7}$ in $G_{C_7,I}$. Here, the relevant property is that H contains an induced path on four vertices that are of U -type, U -type, C -type and U -type, respectively; we call this property the *UUCU-property*. We leave the existence of a P_7 -free graph $G_{H,I}$ that is also $\overline{P_7}$ -free (in which case H does not have the *UUCU-property*) as an open question. Chudnovsky, Goedgebeur, Schaudt and Zhong [7] gave an infinite family of 4-critical P_7 -free graphs (they credited the graph construction to Pokrovskiy via private communication). Using a computer they also found all 4-critical P_7 -free graphs on less than 16 vertices. It can be verified that all of these graphs H either have clique number not equal to 3 or have the *UUCU-property* for every valid choice of c_1, c_2, c_3 . As such, they do not lead to a P_7 -free graph $G_{H,I}$ that is also $\overline{P_7}$ -free.

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