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FINITE DYNAMICAL SYSTEMS, HAT GAMES, AND CODING THEORY*

MAXIMILIEN GADOULEAU†

Abstract. The properties of finite dynamical systems (FDSs) have been investigated in the context of coding theoretic problems, such as network coding and index coding, and in the context of hat games, such as the guessing game and Winkler's hat game. In this paper, we relate the problems mentioned above to properties of FDSs, including the number of fixed points, their stability, and their instability. We first introduce the guessing dimension and the coset dimension of an FDS and their counterparts for directed graphs. Based on the coset dimension, we then refine the existing equivalences between network coding and index coding. We also introduce the concept of the instability of FDSs and we study the stability and the instability of directed graphs. We prove that the instability always reaches the size of a minimum feedback vertex set for large enough alphabets. We also obtain some nonstable bounds independent of the number of vertices of the graph. We then relate the stability and the instability to the guessing number. We also exhibit a class of sparse graphs with large girth that have high stability and high instability; our approach is code-theoretic and uses the guessing dimension. Finally, we prove that the affine instability is always asymptotically greater than or equal to the linear guessing number.

Key words. finite dynamical systems, network coding, index coding, hat games, stability, guessing number

AMS subject classifications. 91A12, 94B05, 05C20

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1. Introduction.

1.1. Finite dynamical systems.

Many entities (such as genes, neurons, persons, computers, etc.) organize themselves as complex networks, where each entity has a finitely valued state and a function which updates the value of the state. Since entities influence each other, this local update function depends on the states of some of the entities. Such a network is modeled by a *finite dynamical system* (FDS).[1] The main problem when studying an FDS is to determine its dynamics, such as the number of its fixed points, or how the trajectory of a state depends on the initial state.

FDSs have been used to represent a network of interacting entities as follows. Each entity v has a local state x_v taking its value in a finite alphabet $[q] = \{0, 1, \ldots, q-1\}$. The state of the whole network is then $x = (x_1, \ldots, x_n) \in [q]^n$. This state then evolves according to a deterministic function $f = (f_1, \ldots, f_n) : [q]^n \to [q]^n$, where $f_v : [q]^n \to [q]$ represents the update of the local state x_v. Although different update schedules have been studied, we are focusing on the parallel update schedule, where all entities update their state at the same time, and x becomes $f(x)$. FDSs have been used to model gene networks [29, 46, 48, 27, 28], neural networks [34, 26, 23], social interactions [38, 25], and more (see [47, 24]).

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†Department of Computer Science, Durham University, Durham, UK (m.r.gadouleau@durham.ac.uk).
[1]FDSs (or limited versions) appear under different names, such as Boolean networks [29, 46], Boolean automata networks [37], multivalued networks [5], etc.

The structure of an FDS $f : [q]^n \to [q]^n$ can be represented via its *interaction graph* $G(f)$, which indicates which update functions depend on which variables. More formally, $G(f)$ has $\{1, \dots, n\}$ as vertex set and there is an arc from u to v if $f_v(x)$ depends on x_u. In different contexts, e.g., in molecular biology, the interaction graph is known (or at least well approximated), while the actual update functions are not [32, 48]. One main problem of research on FDSs is then to predict their dynamics according to their interaction graphs.

1.2. Hat games. *Hat games* are an increasingly popular topic in combinatorics. Typically, a hat game involves n players, each wearing a hat that can take a color from a given set of q colors. No player can see their own hat, but each player can see some subset of the other hats. All players are asked to guess the color of their own hat at the same time.

For an extensive review of different hat games, see [31]. Different variations have been proposed: for instance, the players can be allowed to pass [15], or the players can guess their respective hat's color sequentially [30]. The variation in [15] mentioned above has been investigated further (see [31]) for it is related to coding theory via the concept of covering codes [14]; in particular, some optimal solutions for that variation involve the well-known Hamming codes [16].

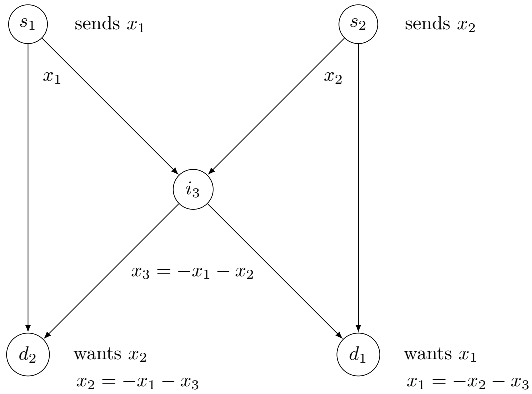
In the variation called the "*guessing game*," players are not allowed to pass, and must guess simultaneously [42]. The team wins if everyone has guessed their color correctly; the aim is to maximize the number of hat assignments which are correctly guessed by all players. This version of the hat game then aims to determine the guessing number of a directed graph (see section 1.3).

In *Winkler's hat game*, the players are not allowed to pass, and must guess simultaneously. The team then scores as many points as there are players guessing correctly. The aim is then to construct a guessing function f which guarantees a score for any possible configuration of hats [49]. The relation between Winkler's hat game and auctions has been revealed in [1] and developed in [8].

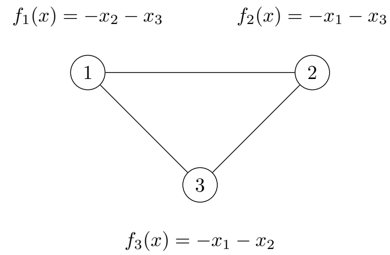
The guessing game and Winkler's hat game can be recast as problems about FDSs as follows. Let D be a directed graph on n vertices where (u, v) is an arc if and only if player v sees the color x_u of player u's hat. Then v's guess is a function $f_v(x_{\mathrm{inn}(v)})$, where $\mathrm{inn}(v)$ is the in-neighborhood of v in D. More generally, a configuration of hats is $x = (x_1, \dots, x_n)$ and the team's guess is $f(x)$, where $f : [q]^n \to [q]^n$ is some FDS whose interaction graph is contained in D. In the guessing game, the team wins whenever $f(x) = x$, i.e., x is a fixed point of f. Thus the goal of the guessing game is to maximize the number of fixed points of f. On the other hand, the score in Winkler's hat game is given by $n - d_{\mathrm{H}}(x, f(x))$, where d_{H} is the Hamming distance. Thus, the goal here is to maximize $s(f) := \min_{x \in [q]^n}(n - d_{\mathrm{H}}(x, f(x)))$, a quantity we shall refer to as the stability of f.

1.3. Network coding and fixed points of FDSs. The aim of this section is to give an overview of network coding, index coding, and their relationships to different problems about FDSs. Since those relationships have been established in the literature, and since network coding and index coding are not the main focus of this paper, we shall only give an informal presentation. The interested reader will find more details in the cited references.

Network coding is a technique to transmit information through networks, which can significantly improve upon routing in theory [2, 51]. At each intermediate node v, the received messages $x_{u_1}, \dots, x_{u_k}$ are combined, and the combined message $f_v(x_{u_1}, \dots, x_{u_k})$ is then forwarded toward its destination. The main problem is to



(a) Butterfly network



(b) Guessing game on K_3

FIG. 1. *Butterfly network and guessing game on K_3.*

determine which functions f_v can transmit the most information. In particular, the *network coding solvability problem* tries to determine whether a certain situation, with a given set of sources, destinations, and messages, is solvable, i.e., whether all messages can be transmitted to their destinations.

The network coding solvability problem can be recast in terms of hat games and fixed points of FDSs as follows [42, 41]. The so-called *guessing number* [42] of a directed graph D is the logarithm in base q of the maximum number of fixed points over all FDSs $f : [q]^n \to [q]^n$ whose interaction graph is a subgraph of D: $G(f) \subseteq D$. The guessing number is always upper bounded by the size of a minimum feedback vertex set of D (a set of vertices whose removal yields an acyclic graph); if equality holds, we say that D is *solvable* and the FDS f reaching this bound is called a solution. Then, a network coding instance N is solvable if and only if the directed graph D_N, obtained by "gluing" each source-receiver pair into a single vertex, is solvable. For instance, the canonical example of (linear) network coding is the butterfly network, depicted in Figure 1(a). A solution to the butterfly network is depicted in the figure. (All operations are done modulo q.) This can be converted into a problem of the guessing number on the triangle K_3, as seen from Figure 1(b): the FDS f has exactly q^2 fixed points, namely, every x such that $x_1 + x_2 + x_3 = 0$. The reader interested in this conversion is referred to [42, 41], where it was proposed, or to the review in [22]. We shall take this conversion for granted and focus on FDSs only, though our results can be interpreted in terms of network coding.

In the network coding literature, coding functions are typically not placed on vertices, as in Figure 1, but on the edges instead. It is easy to verify that those two conventions are equivalent from a solvability point of view, provided the network is modified slightly. For instance, one of the numerous forms of the butterfly network with coding functions on the edges is given in Figure 2. In that case, s_1 still wants to transmit x_1 to d_1 and s_2 wants to transmit x_2 to d_2. This time, the bottleneck is the arc (i_3, i'_3) and the message on that arc could be $x_3 = -x_1 - x_2$ in order to solve the problem.

Linear network coding is the most popular kind of network coding, where the intermediate nodes can only perform linear combinations of the packets they receive [33]. The network coding instance N is then linearly solvable if and only if D_N admits a linear solution. Many interesting classes of linearly solvable directed graphs have

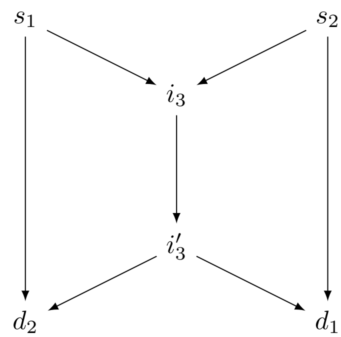


FIG. 2. *The butterfly network with coding functions on the edges.*

been given in the literature (see [40, 22]). On the other hand, graphs which are not linearly solvable have been exhibited in [39, 43, 20].

Index coding is a means to broadcast information to different receivers who have different partial information [9, 7], which we will only briefly describe. For example, suppose that two people wish to know some information $(x_1, x_2) \in [q]^2$, but the first person only has $x_1 \in [q]$ while the second only has $x_2 \in [q]$. We only need to broadcast $x_3 = -x_1 - x_2$ to both of them so that each can recover the pair (x_1, x_2). This example is equivalent to the butterfly network.

The problem of index coding in general is to find the smallest index code, i.e., the minimum amount of information to transmit to all destinations so that they all can gather the same amount of information. Network coding solvability is closely related to index coding [44, 7]. In particular, the length of a minimal index code (for a given directed graph) is the same as the *information defect* [42, 22]. Since a graph is solvable if and only if it is solvable in the sense of information defect [42, 18], there is an equivalence between index coding and network coding. Index coding and network coding were independently proved to be equivalent in [44, 17]; Mazumdar [36] shows that they are also equivalent to determining the storage capacity. In fact, there are two additional equivalences: an asymptotic version and a version in the linear case of the equivalence between network coding and index coding can be found in [22].

1.4. Outline. This paper has three main contributions.

1. We relate the (in)stability of finite dynamical systems to the guessing number, the information defect, and Winkler's hat game.
2. We show that the (in)stability of FDSs can be studied by coding theoretic concepts, such as the covering radius and the remoteness of codes.
3. We investigate the properties of the (in)stability of FDSs based on their interaction graph.

The rest of the paper is organized as follows. We first review the relevant background in section 2. In section 3, we introduce the guessing dimension and the coset dimension of an FDS and their counterparts for directed graphs. Based on the coset dimension, we then refine the known equivalences between guessing number and information defect in Theorem 1. In section 4, we introduce the instability of FDSs and we study the stability and the instability of directed graphs. We prove that the instability always reaches the size of a minimum feedback vertex set in Theorem 2. We also give some nonstable bounds based only on the size of a minimum feedback vertex set (and independent on the number of vertices of the graph) in Theorem 3.

We then relate the stability and the instability to the guessing number in Theorem 4. Finally, we obtain results for linear and affine FDSs in section 5. In Theorem 5, we exhibit a class of sparse graphs with high girth and high affine (in)stability; our approach is code-theoretic and uses the guessing dimension. We also prove that the affine instability is always asymptotically greater than or equal to the linear guessing number in Theorem 6.

2. Background.

2.1. Notation for directed graphs and finite dynamical systems.

Let n be a positive integer, $V = \{1, \dots, n\}$, and D be a (loopless directed) graph on V, i.e., $D = (V, E)$ with $E \subseteq V^2 \setminus \{(v, v) : v \in V\}$. Paths and cycles are always directed. The *girth* of D is the minimum length of a cycle in D. The maximum number of vertex disjoint cycles in D is denoted $\nu(D)$. A *feedback vertex set* is a set of vertices I such that $D - I$ has no cycles. The minimum size of a feedback vertex set is denoted $\tau(D)$. If $J \subseteq V$, then $D[J]$ is the subgraph of D induced by J. If this graph is acyclic, then the vertices of J can be sorted in *acyclic ordering* (also referred to as topological sort): $J = \{j_1, \dots, j_k\}$, where $(j_a, j_b) \in E$ only if $a < b$. The in-neighborhood of a vertex v in D is denoted by $\mathrm{inn}(v)$, and its in-degree is denoted by $\mathrm{ind}(v)$. We denote the maximum in-degree of D by $\Delta_{\mathrm{in}}(D)$; similarly, we denote the maximum out-degree by $\Delta_{\mathrm{out}}(D)$. We say that a graph D is undirected if for any $(u, v) \in E$, we have $(v, u) \in E$ as well. (In other words, we identify bidirectional edges with undirected edges.) If D is undirected, then $\Delta_{\mathrm{in}}(D) = \Delta_{\mathrm{out}}(D) = \Delta(D)$.

Let $q \geq 2$, we denote $[q] = \{0, 1, \dots, q-1\}$. For all $x = (x_1, \dots, x_n) \in [q]^n$, we use the following shorthand notation for all $J = \{j_1, \dots, j_k\} \subseteq V$: $x_J = (x_{j_1}, \dots, x_{j_k})$. For all $x, y \in [q]^n$ we set $\Delta(x, y) := \{i \in [n] : x_i \neq y_i\}$. The Hamming distance between x and y is denoted $d_{\mathrm{H}}(x, y) = |\Delta(x, y)|$. The Hamming weight of $x \in [q]^n$ is $w_{\mathrm{H}}(x) = \{i : x_i \neq 0\} = d_{\mathrm{H}}(x, (0, \dots, 0))$. The volume of a ball of Hamming radius t in $[q]^n$ does not depend on its center, and hence we denote it by

$$V_{\mathrm{H}}(q, n, t) = |\{x \in [q]^n : w_{\mathrm{H}}(x) \leq t\}| = \sum_{d=0}^{t} \binom{n}{d} (q-1)^d.$$

An FDS is any function $f : [q]^n \to [q]^n$. We denote the set of FDSs $f : [q]^n \to [q]^n$ by $F(n, q)$. The image of f is denoted by $\mathrm{Im}(f)$. We write the FDS as $f = (f_1, \dots, f_n)$, where $f_v : [q]^n \to [q]$, $v \in [n]$ is a *local function* of f. We also use the shorthand notation $f_J : [q]^n \to [q]^{|J|}$, $f_J = (f_{j_1}, \dots, f_{j_k})$ for any $J = \{j_1, \dots, j_k\} \subseteq [n]$. We associate with f a graph $G(f)$, referred to as the *interaction graph* of f, defined by the following: the vertex set is V; and for all $u, v \in V$, there exists an arc (u, v) if and only if f_v depends essentially on x_u, i.e., there exist $x, y \in [q]^n$ that only differ by $x_u \neq y_u$ such that $f_v(x) \neq f_v(y)$. For a graph D, we denote by $F(D, q)$ the set of FDSs $f \in F(n, q)$ with $G(f) \subseteq D$.

Example 1. We shall illustrate different properties of an FDS by the following running example. Let $q = 2$, $n = 4$, and $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in F(4, 2)$ be defined as

$$\begin{aligned}
\phi_1(x) &= x_2 + x_3 + 1, \\
\phi_2(x) &= x_3 + x_1 x_4, \\
\phi_3(x) &= x_1 + 1, \\
\phi_4(x) &= x_1 x_2,
\end{aligned}$$

where all operations are done modulo 2. The values of $\phi(x)$ are listed in Table 1.

TABLE 1
The values of $\phi(x)$.

x	$\phi(x)$
0000	1010
0001	1010
0010	0110
0011	0110
0100	0010
0101	0010
0110	1110
0111	1110
1000	1000
1001	1000
1010	0100
1011	0000
1100	0001
1101	0001
1110	1101
1111	1001

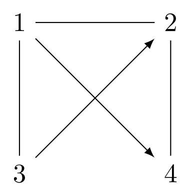


FIG. 3. *The interaction graph $G(\phi)$.*

The interaction graph $D = G(\phi)$ of ϕ is depicted in Figure 3. We then have $\tau(D) = \nu(D) = 2$.

If q is a prime power, we shall endow $[q]$ with the finite field structure $\mathrm{GF}(q)$. In this case, we say that f is *linear* if every local function is a linear combination of the local variables in x: $f_v(x) = \sum_{i=1}^n m_{i,v}x_i$. In other words, f is linear if it is of the form $f(x) = xM$ for some matrix $M \in \mathrm{GF}(q)^{n\times n}$. We say that f is *affine* if $f = xM + y$ for some matrix $M \in \mathrm{GF}(q)^{n\times n}$ and some vector $y \in \mathrm{GF}(q)^n$. We denote the set of linear (affine, respectively) functions $f : \mathrm{GF}(q)^n \to \mathrm{GF}(q)^n$ by $F_{\mathrm{lin}}(n,q)$ ($F_{\mathrm{aff}}(n,q)$, respectively). We shall use the corresponding subscripts throughout this paper.

2.2. Guessing game. The *guessing number* comes from the hat game called "guessing game" [40, 42, 22], where the team wins if and only if all players guess correctly, and the aim is to maximize the number of winning configurations. More formally, for any f, the set of fixed points of f is denoted $\mathrm{fix}(f) = \{x \in [q]^n : f(x) = x\}$. Then the *guessing number* of f is defined by

$$g(f) := \log_q |\mathrm{fix}(f)|.$$

Then the *q-guessing number* of D is $g(D,q) := \max_{f\in F(D,q)} g(f)$.

Example 2 (continued from Example 1). Since ϕ has exactly one fixed point, namely, 1000, its guessing number is equal to 0.

The guessing number has been studied in the context of network coding solvability [40, 42, 22]. Most importantly,

$$\nu(D) \leq g(D,q) \leq \tau(D)$$

for all $q \geq 2$. If $g(D,q) = \tau(D)$, we then say that D is *q-solvable.* The guessing number of the complete graph is $g(K_n, q) = n-1$ for all $q \geq 2$, where the solution is

$$f_v(x) = -\sum_{u \neq v} x_u \mod q,$$

and hence the complete graph is q-solvable for all q. Moreover, the guessing number tends to a limit for large q: the guessing number of D is $g(D) := \sup_q g(D,q) = \lim_{q\to\infty} g(D,q)$ [13]. If $g(D) = \tau(D)$, then we say that D is *asymptotically solvable.* Clearly, if D is q-solvable for some q, then it is asymptotically solvable. Third, we can restrict the choice of FDSs to linear ones, thus yielding the linear guessing number $g_{\mathrm{lin}}(D,q)$. If $g_{\mathrm{lin}}(D,q) = \tau(D)$, we say that D is *q-linearly solvable.* We also denote $g_{\mathrm{lin}}(D) := \max_q g_{\mathrm{lin}}(D,q)$. It is easy to check that $g_{\mathrm{aff}}(D,q) = g_{\mathrm{lin}}(D,q)$.

The guessing game on the triangle K_3 for $q = 3$ was illustrated on Figure 1.

We consider two interesting families of graphs for the guessing number. First, the family of *odd undirected cycles* $\{C_{2k+1} : k \geq 2\}$ satisfies $n = 2k+1$ and [42, 13]

$$\nu(C_{2k+1}) = k < g(C_{2k+1}) = k + 1/2 < \tau(C_{2k+1}) = k+1.$$

These graphs are interesting for they are not asymptotically solvable.

Second, the *strong product* of two graphs D_1 and D_2, denoted by $D_1 \boxtimes D_2$ is defined as follows. Its vertex set is the cartesian product $V(D_1) \times V(D_2)$, and there is an arc from (u_1,u_2) to (v_1,v_2) if and only if either $u_1 = v_1$ and $(u_2,v_2) \in E(D_2)$, or $u_2 = v_2$ and $(u_1,v_1) \in E(D_1)$, or $(u_1,v_1) \in E(D_1)$ and $(u_2,v_2) \in E(D_2)$. Equivalently, the adjacency matrix of the strong product is given by

$$A_{D_1 \boxtimes D_2} = (I_{n_1} + A_{D_1}) \otimes (I_{n_2} + A_{D_2}) - I_{n_1 n_2},$$

where $\otimes$ denotes the Kronecker product of matrices. The graph $\vec{C}_3^2 = \vec{C}_3 \boxtimes \vec{C}_3$ is displayed in Figure 4. Then, for every value of the girth $\gamma \geq 3$, the sequence $\{\vec{C}_\gamma^k : k \geq 1\}$ satisfies $n = \gamma^k$, $\nu(\vec{C}_\gamma^k) = \gamma^{k-1}$, and

$$\tau(\vec{C}_\gamma^k) = g_{\mathrm{lin}}(\vec{C}_\gamma^k, q) = g(\vec{C}_\gamma^k) = \gamma^k - (\gamma-1)^k$$

for any prime power q [22]. Hence these graphs are always linearly solvable and their guessing number is relatively close to the number of vertices.

2.3. Index coding, information defect, and the guessing graph. The *q-information defect* of D is defined as the smallest amount of information that the players in the guessing game need to guess correctly for any configuration of hats. Suppose a helper outside of the team wants to make the team win the guessing game for every possible configuration of hats. The helper can give some information to the whole team, say, it transmits to them the value $a \in \{1, \dots, b\}$. Based on a, the team then agrees to use a guessing strategy f^a, such that $f^a(x) = x$. For instance, for $D = K_3$ and $q = 3$, we see that the players only need one symbol of information to be correct: if they know the value of $x_1 + x_2 + x_3$, then they know the color of their own hats.

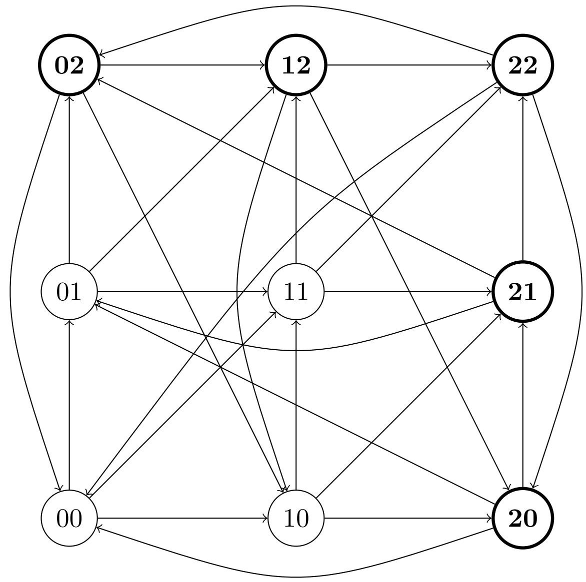


FIG. 4. *The graph $\vec{C}_3^2$ with linear guessing number* 5. *A minimum feedback vertex set is highlighted.*

Formally, the q-information defect is the logarithm in base q of the minimum number of parts in a partition of $[q]^n$ into sets of fixed points [40, 42]:

$$b(D,q) := \min \left\{ \log_q |B| : B \subseteq F(D,q), \bigcup_{f \in B} \mathrm{fix}(f) = [q]^n \right\}.$$

We always have $g(D,q)+b(D,q) \geq n$ [22]. If we restrict the set of functions B to only contain affine functions, then we obtain the q-affine information defect $b_{\mathrm{aff}}(D,q)$. We denote the asymptotic information defect of D by $b(D) := \inf_{q \geq 2} b(D,q)$.

Graph solvability is equivalent to graph "information defect" solvability [22]. In fact, there are three kinds of equivalence between the guessing number and the information defect, given below.

1. *Solvability equivalence.* For any D and q, $g(D,q) = \tau(D)$ if and only if $b(D,q) = n - \tau(D)$.
2. *Asymptotic equivalence.* For any D, $b(D) = n - g(D)$ [22].
3. *Linear-affine equivalence.* For any D and q, $b_{\mathrm{aff}}(D,q) = n - g_{\mathrm{lin}}(D,q)$ (see [22], for instance).

In particular, for the complete graph K_n, we have $b(K_n,q) = 1 = n - \tau(K_n)$ for all q.

The *guessing graph* $\mathrm{G}(D,q)$ is the undirected graph with vertex set $[q]^n$ in which $x, y \in [q]^n$ are adjacent if and only if there is no $f \in F(D,q)$ such that $x, y \in \mathrm{fix}(f)$. More concretely, $\mathrm{G}(D,q)$ has vertex set $[q]^n$ and edge set $E = \bigcup_{v=1}^n \{\{x,y\} : x_{\mathrm{inn}(v)} = y_{\mathrm{inn}(v)}, x_v \neq y_v\}$. The guessing graph was first introduced in [6, 3], where it was referred to as "confusion graph." It was then independently introduced in [22] and extended in two different fashions in [18, 21]. By definition, any set of fixed points of some function $f \in F(D,q)$ is an independent set of $\mathrm{G}(D,q)$. Conversely, any

TABLE 2
The values of $n - d_{\mathrm{H}}(x, \phi(x))$.

x	$\phi(x)$	$n - d_{\mathrm{H}}(x, \phi(x))$
0000	1010	2
0001	1010	1
0010	0110	3
0011	0110	2
0100	0010	2
0101	0010	1
0110	1110	3
0111	1110	2
1000	1000	4
1001	1000	3
1010	0100	1
1011	0000	1
1100	0001	1
1101	0001	2
1110	1101	2
1111	1001	2

independent set of the guessing graph is fixed by some FDS in $F(D, q)$; thus

$$g(D, q) = \log_q \alpha(\mathrm{G}(D, q)),$$
$$b(D, q) = \log_q \chi(\mathrm{G}(D, q)),$$

where α denotes the independence number and χ denotes the chromatic number (see [22] and [3]).

2.4. Winkler's hat game. Winkler's hat game is based on the same setting as the guessing game, but now the team scores a point for every correct guess. The main problem is as follows: How many points can the team be guaranteed to score for any possible configuration of hats? In the language of FDS, let us define the *stability* of an FDS f as the number of points the team is guaranteed to score if they use the guessing strategy f, i.e.,

$$s(f) := \min_{x \in [q]^n} (n - d_{\mathrm{H}}(x, f(x))).$$

In other words, for any x, f always fixes at least $s(f)$ coordinates of x. We also define the q-*stability* of D as $s(D, q) = \max_{f \in F(D,q)} s(f)$.

Example 3 (continued from Examples 1 and 2). Let us determine the stability of ϕ. We compute $n - d_{\mathrm{H}}(x, \phi(x))$ for all x in Table 2.

We then have $s(\phi) = 1$.

Then on a clique of size q, the optimal solution is to cover all possibilities of the sum of all x_i's, i.e.,

$$f_v(x) = v - \sum_{u \neq v} x_u \mod q.$$

This guarantees exactly one correct guess (for $v = \sum_{u=1}^{n} x_u \mod q$) for any value of x. By double counting, this is the best possible. We note the similarity between the solutions for the guessing game and for Winkler's hat game on the clique. In general, packing disjoint copies of K_q in the complete graph K_n yields

$$s(K_n, q) = \left\lfloor \frac{n}{q} \right\rfloor.$$

In particular, $s(K_n, q) = 0$ if $q > n$ and hence for any D $s(D, q) = 0$ if $q > n$.

Some work has been done on $s(D,q)$, or on determining whether $s(D,q) > 0$, in which case D is *q-stable*. If D is undirected, then $s(D,2) = M$, the size of a maximum matching in D; in general, $\nu(D) \leq s(D,2) \leq \tau(D)$ [11]. For any q, there exists a q-stable bipartite undirected graph D (first proved in [11], then refined in [19]). Moreover, there exists a 4-stable oriented graph [19]. In [19], the authors ask whether there exists a q-stable oriented graph for all $q \geq 5$; we shall give an affirmative answer to that question in Theorem 5.

Conversely, some graphs have been proved to be not q-stable. If D is an undirected tree, then D is not 3-stable [11]. The complete bipartite graph $K_{m,s}$ is not $(m+2)$-stable for any $s \geq 1$ [19]. If $\tau(D) = 1$, then D is not 3-stable [19]. The undirected cycle C_n is 3-stable if and only if $n = 4$ or n is divisible by 3; moreover, C_n is not 4-stable for all n [45].

3. The guessing and coset dimensions. We remark that the information defect is defined for a graph, but cannot be defined for a function, since it considers arbitrary subsets of $F(D,q)$. To replace it, define the *coset dimension* of a function $f \in F(D,q)$ as follows:

$$c(f) := \min\left\{\log_q |S| : S \subseteq [q]^n, \bigcup_{a\in S}(\mathrm{fix}(f) - a) = [q]^n\right\}.$$

Thus, the *q-coset dimension* of D is

$$c(D,q) := \min_{f\in F(D,q)} c(f),$$

and we also denote $c(D) := \inf_{q\geq 2} c(D,q)$.

Lemma 1. *For any D and q,*

$$c(D,q) \geq b(D,q) \geq n - g(D,q) \geq n - \tau(D).$$

Proof. For any $f \in F(D,q)$ and any $a \in [q]^n$, define $f^a \in F(D,q)$ by $f^a(x) = f(x+a) - a$; then $\mathrm{fix}(f^a) = \mathrm{fix}(f) - a$. Denoting $B(f,S) = \{f^a : a \in S\}$ for all $S \subseteq [q]^n$, we obtain

$$\begin{aligned} c(D,q) &= \min_{f,S}\left\{\log_q |S| : \bigcup_{a\in S}\mathrm{fix}(f^a) = [q]^n\right\} \\ &= \min_{f,S}\left\{\log_q |B(f,S)| : \bigcup_{g\in B(f,S)}\mathrm{fix}(g) = [q]^n\right\} \\ &\geq b(D,q). \end{aligned}$$ ∎

For any f, the *guessing code* and the *guessing dimension* of f are

$$\begin{aligned} C_f &:= \{f(x) - x : x \in [q]^n\}, \\ l(f) &:= \log_q |C_f|. \end{aligned}$$

Using the guessing game intuition, the guessing code is all the possible ways in which players guess incorrectly if they use the guessing strategy f.

TABLE 3
The values of $\phi(x) - x$.

x	$\phi(x)$	$\phi(x) - x$
0000	1010	1010
0001	1010	1011
0010	0110	0100
0011	0110	0101
0100	0010	0110
0101	0010	0111
0110	1110	1000
0111	1110	1001
1000	1000	0000
1001	1000	0001
1010	0100	1110
1011	0000	1011
1100	0001	1101
1101	0001	1100
1110	1101	0011
1111	1001	0110

Example 4 (continued from Examples 1 through 3). To determine the guessing code and the guessing dimension of ϕ, we determine the values of $\phi(x) - x$ in Table 3. Therefore, $C_\phi = \{0,1\}^4 \setminus \{0010, 1111\}$.

The *q-guessing dimension* of D is then denoted by $l(D,q) := \min_{f \in F(D,q)} l(f)$, and also $l(D) := \inf_{q \geq 2} l(D,q)$. The guessing dimension of a graph is closely related to the information defect $b(D,q)$.

LEMMA 2. *For any D and q,*

$$l(D,q) = \min \left\{ \log_q |S| : S \subseteq [q]^n,\, \exists f \in F(D,q) : \bigcup_{a \in S} \mathrm{fix}(f - a) = [q]^n \right\},$$

whence $l(D,q) \geq b(D,q) \geq n - g(D,q)$.

Proof. We have

$$\begin{aligned}
\bigcup_{a \in S} \mathrm{fix}(f - a) = [q]^n &\Leftrightarrow \forall x \in [q]^n\ \exists a \in S : x \in \mathrm{fix}(f - a) \\
&\Leftrightarrow \forall x \in [q]^n\ \exists a \in S : f(x) - x = a \\
&\Leftrightarrow C_f \subseteq S,
\end{aligned}$$

and the equation follows. ∎

We refine the triple equivalence between guessing number and information defect by replacing the information defect by the coset dimension. This is a refinement because $c(D,q) \geq b(D,q)$ for all D and q, and similarly in the asymptotic ($c(D) \geq b(D)$) and linear ($c_{\mathrm{aff}}(D,q) \geq b_{\mathrm{aff}}(D,q)$) cases.

THEOREM 1. *We have three kinds of equivalence.*

1. Solvability equivalence. *For any D and q, the following are equivalent.*
   (a) $c(D,q) = n - \tau(D)$.
   (b) $b(D,q) = n - \tau(D)$.
   (c) $g(D,q) = \tau(D)$.

2. Asymptotic equivalence. *For any D,*

$$c(D) = \lim_{q\to\infty} c(D,q) = \lim_{q\to\infty} c(D,q),$$
$$b(D) = \lim_{q\to\infty} b(D,q) = \lim_{q\to\infty} b(D,q),$$
$$c(D) = b(D) = n - g(D).$$

3. Linear-affine equivalence. *For any D and q,*

$$l_{\mathrm{lin}}(D,q) = l_{\mathrm{aff}}(D,q),$$
$$c_{\mathrm{lin}}(D,q) = c_{\mathrm{aff}}(D,q),$$
$$g_{\mathrm{lin}}(D,q) = g_{\mathrm{aff}}(D,q),$$
$$l_{\mathrm{aff}}(D,q) = c_{\mathrm{aff}}(D,q) = b_{\mathrm{aff}}(D,q) = n - g_{\mathrm{aff}}(D,q).$$

Proof. 1. Solvability equivalence. By Lemma 1, 1(a) $\Rightarrow$ 1(b) $\Rightarrow$ 1(c). Now, if $g(D,q) = \tau(D)$, let $\mathrm{fix}(f) = \{z^i : i = 1,\dots,q^{\tau(D)}\}$ be an independent set of $\mathrm{G}(D,q)$ of size $q^{\tau(D)}$. Let I be a minimum feedback vertex set of D.

First, $z^i_I \neq z^j_I$ for all $i \neq j$. Indeed, the set $\Delta(z^i,z^j)$ contains a cycle of D, and hence it intersects I. Second, for any $a_J \in [q]^{n-\tau(D)}$, let $a = (0_I, a_J)$. Then for any i,j, $\Delta(z^i - a, z^j - a) = \Delta(z^i,z^j)$ and hence $z^i - a \not\sim z^j - a$. Thus denoting $S := \{a : a_J \in [q]^{n-\tau(D)}\}$, we have $\bigcup_{a\in S}(\mathrm{fix}(f) - a) = [q]^n$ and by Lemma 2, $c(D,q) = n - \tau(D)$.

2. Asymptotic equivalence. By Lemma 1, we only need to prove that $c(D,q)$ tends to $n - g(D)$ as q tends to infinity. We shall adapt the argument (attributed to Szegedy) in the proof of [4, Proposition 3.12] to prove that

$$c(D,q) \le n - g(D,q) + \log_q \ln q^n.$$

Let $Z = \mathrm{fix}(f)$ with $f \in F(D,q)$ be an independent set of the guessing graph of size $q^{g(D,q)}$. Let Q be the family of sequences of $[q]^n$ of length $s = \lceil q^{n-g(D,q)} \ln q^n \rceil$, and let $T \in Q$ maximize the cardinality of the translate $Z + T$ (where we view T as a set). We then have the following double summation:

$$\begin{aligned}
\Sigma &:= \sum_{S\in Q} \sum_{x\in[q]^n} \mathbb{1}\{x \notin Z + S\} \\
&= \sum_{S\in Q} (q^n - |Z+S|) \\
&\ge q^{ns}(q^n - |Z+T|), \\
\Sigma &= \sum_{x\in[q]^n} \sum_{S\in Q} \mathbb{1}\{x \notin Z + S\} \\
&= \sum_{x\in[q]^n} (q^n - |Z|)^s \\
&= q^n (q^n - |Z|)^s,
\end{aligned}$$

where $\mathbb{1}\{P\}$ returns 1 when property P is satisfied and 0 otherwise. Thus,

$$\begin{aligned} q^n - |Z+T| &\le q^n \left(1 - \frac{q^{g(D,q)}}{q^n}\right)^s \\ &< q^n \exp\left(-q^{g(D,q)-n} \lceil q^{n-g(D,q)} \ln q^n \rceil\right) \\ &< 1. \end{aligned}$$

We hence obtain $Z + T = [q]^n$ and $c(D,q) \le \log_q s$.

3. Linear-affine equivalence. The equivalences between affine guessing dimension, coset dimension, and guessing number and their linear counterparts are easy to see (by translation). Moreover, let f be linear, i.e., $f(x) = xM$, then

$$l(f) = c(f) = \operatorname{rank}(M - I) = n - g(f). \qquad \square$$

Problem 1. *Can we obtain the counterparts of the solvability equivalence and of the asymptotic equivalence, if we replace the coset dimension with the guessing dimension?*

4. Stability and instability. For an FDS f, we define the *instability* of f as

$$i(f) := \min_{x \in [q]^n} d_{\mathrm{H}}(x, f(x)).$$

Hence, for any x, f is guaranteed to modify at least $i(f)$ coordinates of x.

Example 5 (continued from Examples 1 to 4). Since ϕ has a fixed point, its instability is equal to zero.

For any graph D, the q-instability of D is $i(D,q) := \max_{f \in F(D,q)} i(f)$, and again the instability of D is $i(D) = \max_{q \ge 2} i(D,q)$.

4.1. General properties.

Proposition 1. *For every graph D we have $i(D,2) = s(D,2)$. Moreover, for every graph D and $q \ge 2$ we have*

$$\begin{aligned} i(D,q) &\le i(D,q+1), \\ s(D,q) &\ge s(D,q+1). \end{aligned}$$

Proof. For every $f \in F(D,2)$, we have $i(f) = s(\neg f)$ and $\neg f \in F(D,2)$, and hence $i(D,2) = s(D,2)$ and $i(D,2) = s(D,2)$.

For every $x \in [q+1]^n$ let $x' \in [q]^n$ be defined by $x'_i = \min(x_i, q-1)$ for all $i \in [n]$. Let $f : [q]^n \to [q]^n$ and let $f' : [q+1]^n \to [q+1]^n$ be defined by $f'(x) = f(x')$ for all $x \in [q+1]^n$. It is easy to see that $G(f') = G(f)$. Furthermore $d_{\mathrm{H}}(x, f'(x)) \ge d_{\mathrm{H}}(x', f(x'))$ for all $x \in [q+1]^n$, and thus $i(f') \ge i(f)$. The proof for the stability number is similar. $\square$

Proposition 2. *For every graph D and $q \ge 2$ we have*

$$\nu(D) \le i(D,q) \le \tau(D).$$

Proof. For a cycle C we have $i(C,2) = 1$, by the following function:

$$f_i(x) = \begin{cases} \neg x_n & \text{if } i = 1, \\ x_{i-1} & \text{if } 2 \le i \le n. \end{cases}$$

Indeed, if x is a fixed point, we have $x_n + 1 = x_1 = x_2 = \cdots = x_n$. By Proposition 1, we obtain $i_{\text{aff}}(C, q) \geq 1$ for all $q \geq 2$ and thus the lower bound is obtained by packing negative cycles. For the upper bound, let $f \in F(D, q)$ and let I be a feedback vertex set of D with of minimum size. Since $D - I$ has no cycle we deduce that for every $x \in [q]^I$ there exists at least one $x' \in [q]^n$ such that $x'_I = x$ and $\Delta(x', f(x')) \subseteq I$. Then $i(D, q) \leq d_{\text{H}}(x', f(x')) \leq |I| = \tau(D)$. ∎

Proposition 3. *For every $n \geq q \geq 2$,*

$$i(K_n, q) = n - \left\lceil \frac{n}{q} \right\rceil.$$

Proof. Let $n = kq + r$, where k and r are integers and $0 \leq r < q$. The classical solution of Winkler's hat game shows that $i(K_p, p) = p - 1$ for every $p \geq 2$. Thus if $p \leq n$, then $i(K_p, n) \geq p - 1$. Therefore, by taking the union of k disjoint copies of K_q and one residual K_r we obtained a spanning subgraph H of K_n such that if $r > 0$,

$$i(K_n, q) \geq i(H, q) \geq k(q - 1) + r - 1 = n - \left\lceil \frac{n}{q} \right\rceil,$$

and the same end result holds for $r = 0$.

Conversely, let $f \in F(K_n, q)$ with $i(f) = i(K_n, q)$. By double counting, again we obtain

$$q^n i(K_n, q) = q^n i(f) \leq n(q - 1)q^{n-1}$$

showing that $i(K_n, q) \leq \lfloor n - \frac{n}{q} \rfloor$. ∎

For any f, we have $s(f) + i(f) \leq n$ by definition; this observation is refined below.

Corollary 1. *For any D and q, $s(D, q) + i(D, q) \leq n$.*

4.2. Suprema of stability and instability. We know that the stability $s(D, q) = 0$ for q large enough; we shall also prove in Theorem 2 below that $i(D, q) = \tau(D)$ for q large enough. This is particularly interesting when compared to the guessing number, which does not always reach the feedback bound, for odd cycles, for instance [40]. Therefore, we also investigate how fast these asymptotic bounds are reached.

For any D, let $E'(D)$ be the set of chordless cycles of D, let $L(D)$ be the undirected graph on $E'(D)$ such that two chordless cycles are adjacent if they meet in at least one vertex of D, and let $\chi'(D)$ be the chromatic number of $L(D)$. In particular, if D is undirected, $E'(D)$ is the edge set of D, $L(D)$ is the line graph of D, and $\chi'(D)$ is the chromatic index of D. According to Vizing's theorem, $\chi'(D) \in \{\Delta(D), \Delta(D)+1\}$ if D is undirected [10].

Theorem 2. *For any graph D,*

$$i(D) = \lim_{q \to \infty} i(D, q) = \tau(D).$$

Moreover, we have $i(D, q) = \tau(D)$ if $q = 2^{\chi'(D)}$ or if $q = 2^{\Delta(D)}$ and D is undirected.

Before detailing the proof, we first explain the basic idea. Let $C^1, \ldots, C^k$ be the set of all cycles in D. Each cycle has instability equal to 1, meaning that for any $i = 1, \ldots, k$, there is a function $f^i \in F(C^i, 2)$ with the following property: for any $x^i \in [2]^{|C^i|}$ there exists $v^i \in C^i$, where $f^i(x^i)$ differs from x^i. By taking the direct product of the alphabets $x = (x^1, \ldots, x^k) \in [2]^{nk}$ (where we extend each x^i to length

n), the resulting function $f(x) = (f^1(x^1), \dots, f^k(x^k))$ has the following property: for any x, there exist $v^1, \dots, v^k$, where $f(x)$ differs from x. Since $\{v^1, \dots, v^k\}$ is a set of vertices which intersects every cycle, they are a feedback vertex set and hence $d_{\mathrm{H}}(f(x), x) \geq |\{v^1, \dots, v^k\}| \geq \tau(D)$.

Proof. If D is acyclic, then $i(D, q) = 0 = \tau(D)$ for all Q. Otherwise, by definition, we can partition the set of chordless cycles into $\chi = \chi'(D)$ parts

$$\{C^{1,1}, \dots, C^{1,p_1}\}, \dots, \{C^{\chi,1}, \dots, C^{\chi,p_\chi}\}$$

such that $C^{\alpha,i}, C^{\alpha,j}$ are disjoint for all $1 \leq \alpha \leq \chi$ and $1 \leq i < j \leq p_\alpha$. Denote each chordless cycle by

$$C^{\alpha,i} = (u_1^{\alpha,i}, \dots, u_{l_{\alpha,i}}^{\alpha,i}).$$

Let $q = 2^\chi$: we then view $x_v \in [q]$ as $x_v = (x[v,1], \dots, x[v,\chi])$. Then consider the function $f \in F(D, q)$, where

$$f_v(x) = (f[v,1](x), \dots, f[v,\chi](x)),$$
$$f[v,\alpha](x) = \begin{cases} \neg x[u_{l_{\alpha,i}}^{\alpha,i}, \alpha] & \text{if } v = u_1^{\alpha,i}, \\ x[u_{k-1}^{\alpha,i}, \alpha] & \text{if } v = u_k^{\alpha,i}, k > 1, \\ 0 & \text{otherwise.} \end{cases}$$

For any x, let J denote the set of coordinates j such that $f_j(x) = x_j$. We claim that J is acyclic. Indeed, if J contains the chordless cycle $C^{\alpha,i}$, then

$$x[u_1^{\alpha,i}, \alpha] = x[u_2^{\alpha,i}, \alpha] = \cdots = x[u_{l_\alpha}^{\alpha,i}, \alpha] = \neg x[u_1^{\alpha,i}, \alpha].$$

Since J is acyclic, its complement is a feedback vertex set, of cardinality at least $\tau(D)$.

If D is undirected, for all $v \in [n]$ let $\mathrm{inn}(v) = \{u_1, \dots, u_{\mathrm{ind}(v)}\}$ sorted in increasing order. Then P be the $n \times \Delta(D)$ matrix such that

$$P(v, d) = \begin{cases} u_d & \text{if } 1 \leq d \leq \mathrm{ind}(v), \\ 0 & \text{if } d > \mathrm{ind}(v). \end{cases}$$

Also, let Q be the $n \times n$ matrix such that for all v and $d \leq \mathrm{ind}(v)$,

$$Q(P(v, d), v) = d,$$

and all other entries are zero. Therefore,

$$P[P(v, d), Q(v, P(v, d))] = v, \quad Q\{P(v, d), P[P(v, d), Q(v, P(v, d))]\} = d.$$

Let $q = 2^{\Delta(D)}$: we then view $x_v \in [q]$ as $x_v = (x[v,1], \dots, x[v, \Delta(D)])$. Consider the function $f \in F(D, q)$, where

$$f_v(x) = (f[v,1](x), \dots, f[v, \Delta(D)](x)),$$
$$f[v,d](x) = \begin{cases} x[P(v,d), Q(v, P(v,d))] & \text{if } 1 \leq P(v,d) < v, \\ \neg x[P(v,d), Q(v, P(v,d))] & \text{if } v < P(v,d) \leq n, \\ 0 & \text{if } P(v,d) = 0. \end{cases}$$

Again, for every x, denote the set of coordinates j such that $f_j(x) = x_j$ by J. We claim that J is an independent set. Indeed, if J contains the edge $\{v, P(v,d)\}$ with $P(v,d) < v$, then

$$\begin{aligned} x[v,d] &= f[v,d](x) \\ &= x[P(v,d), Q(v, P(v,d))] \\ &= f[P(v,d), Q(v,P(v,d))](x) \\ &= \neg x[P[P(v,d), Q(v,P(v,d))], Q\{P(v,d), P[P(v,d), Q(v,P(v,d))]\}] \\ &= \neg x[v,d]. \end{aligned}$$ ∎

Corollary 2. *For any $k \geq 2$,*

$$k = i(C_{2k+1}, 2) < g(C_{2k+1}, 2) < k + 1/2 = g(C_{2k+1}) < k + 1 = i(C_{2k+1}).$$

In particular, there exists a graph D such that for all q large enough, $g(D,q) < i(D,q) = \tau(D)$.

We note that the value of q such that $i(D,q) = \tau(D)$ in Theorem 2 is far from being optimal. For instance, for $D = K_n$, we have $i(K_n, n) = n - 1 = \tau(K_n)$, while Theorem 2 only implies $i(K_n, 2^{n-1}) = n - 1$. In general, the value in Theorem 2 is exponential in some parameter of D, but we believe that this may be strengthened.

Problem 2. *There is an absolute constant $c > 0$ such that for n large enough, $i(D, n^c) = \tau(D)$ for all graphs D on n vertices.*

We now move on to the stability. As noted previously, D is not $(n+1)$-stable. In Theorem 3, we give the first known upper bound on the stability entirely based on the minimum size of a feedback vertex set of D. In particular, this shows that D is not $Q(\tau(D))$-stable for some function of $\tau(D)$ only (and in particular, this is independent of n).

Theorem 3. *For any D and q and $m \geq 1$, let $Q(m) = 2 + \sum_{a=1}^{m} a^a$, then*

$$\begin{aligned} s(D,q) &\leq \frac{\tau(D)}{\lfloor (q-1)^{1/\tau(D)} \rfloor}, \\ s(D, Q(m)) &\leq \tau(D) - m, \end{aligned}$$

and in particular D is not $Q(\tau(D))$-stable.

Proof. Let $\tau = \tau(D)$, I be a minimum feedback vertex set of D, and $J = V \setminus I = \{j_1, \dots, j_{n-\tau}\}$ in acyclic ordering. Let $f \in F(D,q)$ with $s(f) = s(D,q)$.

Claim. For any set $X \subseteq [q]^\tau$ such that $|X| < q$, there exists $y^{(X)} \in [q]^{n-\tau}$ such that

$$d_{\mathrm{H}}(y^{(X)}, f_J(x, y^{(X)})) = n - \tau$$

for all $x \in X$.

Proof of claim. Recursively define $y = y^{(X)} \in [q]^{n-\tau}$ such that

$$\begin{aligned} y_{j_1} &\notin f_{j_1}(X), \\ y_{j_2} &\notin f_{j_2}(X, y_{j_1}) \\ &\vdots \\ y_{j_{n-\tau}} &\notin f_{j_{n-\tau}}(X, y_{j_1}, \dots, y_{j_{n-\tau-1}}). \end{aligned}$$ ∎

First, let $p = \lfloor (q-1)^{1/\tau} \rfloor$, so that $q \geq p^\tau + 1$ and consider the set $X = [p]^\tau \subseteq [q]^\tau$. Then we claim that there exists $x \in X$ such that

$$d_{\mathrm{H}}(x, f_I(x, y^{(X)})) \geq \tau - s(D[I], p).$$

Indeed, otherwise the function $g \in F(D[I], p)$ defined as $g(x) = \max\{p-1, f_I(x, y^{(X)})\}$ has stability greater than $s(D[I], p)$, which is a contradiction. Thus, $s(D, q) \leq s(D[I], p) \leq s(K_\tau, p) \leq \frac{\tau}{p}$.

Second, let $q = Q(m)$ and consider the sets $X_l \subseteq [l+1]^\tau \subseteq [q]^\tau$ defined recursively by $X_1 = \{(0, 0, \dots, 0), (1, 0, \dots, 0)\}$ and for all $2 \leq l \leq m$,

$$\begin{aligned} A_l &= \{x \in [q]^\tau : x_{1,\dots,l} \in [l]^\tau, x_{l+1} = \cdots = x_\tau = 0\}, \\ B_l &= \{x \in [q]^\tau : (x_{1,\dots,l-1}, 0, \dots, 0) \in X_{l-1}, x_l = l, x_{l+1} = \cdots = x_\tau = 0\}, \\ X_l &= A_l \cup B_l. \end{aligned}$$

Then $|X_l| = Q(l) - 1$ and we claim that for all $1 \leq l \leq m$, there exists $x \in X_l$ such that

$$d_{\mathrm{H}}(x_{1,\dots,l}, f_{1,\dots,l}(x, y^{(X_m)})) = l.$$

We prove it by induction on l; the case $l = 1$ is clear. Suppose it holds for $l-1$ but not for l. Note that in X_l, the value of $(x_{l+1}, \dots, x_\tau)$ is fixed, so we write $f_i(x_{1,\dots,l})$ for any $1 \leq i \leq l$. Then in A_l, there is always one player from 1 to l who guesses correctly, and hence for every value of $x_{1,\dots,l-1} \in [l]^\tau$, there exists x_l such that $x_{1,\dots,l}$ is guessed correctly by player l. In other words, $f_l(x_{1,\dots,l-1}) \in [l]$ for any $x_{1,\dots,l-1} \in [l]^{l-1}$. Now in B_l, the players 1 to $l-1$ cannot always guess correctly, by induction hypothesis. Thus, there exists $z \in [l]^{l-1}$ such that $f_l(z) = l$, which is a contradiction. Thus, $s(D, q) \leq \tau - m$. ∎

Corollary 3. *If $\tau(D) = 1$, then $s(D, 3) = 0$; if $\tau(D) = 2$, then $s(D, 7) = 0$. Also, for any D, $s(D, 3) \leq \tau(D) - 1$, i.e., the $\tau(D)$ upper bound on the stability can only be reached for $q = 2$.*

The first statement of Corollary 3 is tight: the cycle C_n satisfies $\tau(C_n) = 1$ and $s(C_n, 2) = 1$. We now prove that the second statement is also tight. For any disjoint sets $L = \{l_1, l_2\}$ and $R = \{r_1, \dots, r_b\}$, the split graph $S_{2,b}$ is the undirected graph with edges

$$E = \{(u, v) : u \neq v \text{ and } (u \in L \text{ or } v \in L)\}.$$

In other words, $S_{2,b}$ is the complete bipartite graph $K_{2,b}$ with an additional unidrected edge on the left part.

Proposition 4. *Let $b = 6^{6^2}$ and $D = S_{2,b}$. Then $\tau(D) = 2$ and $s(D, 6) = 1$.*

Proof. First, we have $\tau(D) = 2$, and hence $s(D, 6) \leq \tau(D) - 1 = 1$. Let $L = \{l_1, l_2\}$ and $R = \{r_1, \dots, r_b\}$ be the set of all functions $r_i : [6]^2 \to [6]$. We construct a function $f \in F(D, 6)$ with stability 1 as follows. First, $f_{r_i}(x_L) = r_i(x_L)$ for all $r_i \in R$.

Then, we claim that for any $y = (y_1, \dots, y_b)$, there are at most five possible choices for x_L such that $f_{r_i}(x_L) \neq y_i$ for all $i = 1, \dots, b$. Indeed, suppose $f_i(x_L^k) \neq y_i$ for all $i = 1, \dots, b$ and all $k \in [6]$. Then consider $r_i \in R$ such that $r_i(x_L^k) = k$ for all k: we have $r_i(x_L^{y_i}) = y_i$, which is a contradiction.

Now, we construct $f_L(x) = (f_1(x_2), f_2(x_1))$ by fixing x_R and assuming that $f_{r_i}(x_L) \neq x_{r_i}$ for all i. We then have $x_L = x_L^k$ for some $k \in \{0, \dots, 4\}$. Depending on the"shape" of the set $X = \{x_L^0, \dots, x_L^4\}$, the function f_L will behave differently. For $i = 1, 2$, let $d_i = |\{x_i^0, \dots, x_i^4\}|$ denote the number of values taken by the ith coordinate in X. Without loss, we assume $d_1 \leq d_2$; moreover, we have $d_1 + d_2 \geq 5$, and hence $d_2 \geq 3$.

1. $d_2 = 5$. All elements of X have distinct second coordinate values, say, $x_2^0 = c^0, \dots, x_2^4 = c^4$, then let $f_1(x_2^a) = c^a$, i.e., l_1 guesses all elements of X correctly.
2. $d_2 = 4$. Similarly as above, l_1 can guess four elements of X correctly. The fifth one can then be guessed correctly by l_2 instead.
3. $d_2 = 3$. Then it can be easily shown that X can be partitioned in two parts: a part of three elements, say, x^0, x^1, x^2, with pairwise distinct first coordinate, and a part with two elements, say, x^3, x^4, with distinct second coordinate. Therefore, l_1 can guess x^0, x^1, x^2 correctly while l_2 can guess x^3, x^4 correctly instead. ∎

4.3. Relation with the guessing number. We can relate the guessing number of a graph with its stability and instability.

THEOREM 4. *For every graph D we have*

$$g(D,q) \geq \tau(D) - \log_q \left[q^{\tau(D)} - V_{\mathrm{H}}(q, \tau(D), i(D,q) - 1) \right],$$
$$g(D,q) \geq \tau(D) - \log_q V_{\mathrm{H}}(q, \tau(D), \tau(D) - s(D,q)).$$

Proof. We begin with an important property of acyclic sets.

Claim. Let I be a feedback vertex set of D and $J = V \setminus I$. Then for any $f \in F(D,q)$, $x \in [q]^I$, $a \in [q]^J$, there exists $x^{(a)} \in [q]^n$ such that $x_I^{(a)} = x$ and $f(x^{(a)}) - x^{(a)} = a$.

Proof of claim. We sort J in acyclic ordering $J = \{j_1, \dots, j_k\}$ and we construct $x_J^{(a)}$ recursively. We have

$$\begin{aligned}
x_{j_1}^{(a)} &= f_{j_1}(x_I) - a_{j_1}, \\
x_{j_2}^{(a)} &= f_{j_2}(x_I, x_{j_1}^{(a)}) - a_{j_2} \\
&\vdots \\
x_{j_k}^{(a)} &= f_{j_1}(x_I, x_{j_1}^{(a)}, \dots, x_{j_{k-1}}^{(a)}) - a_{j_k}.
\end{aligned}$$ ∎

Let I be a feedback vertex set of D of size $\tau = \tau(D)$ and $J = V \setminus I$. Let $f \in F(D,q)$ with maximal instability, and

$$\mathcal{Y} = \{y \in [q]^\tau : w_{\mathrm{H}}(y) \geq i(f)\}.$$

By the claim, for every $x \in [q]^I$ there exists a unique point $x' \in [q]^n$ such that $x'_I = x$ and $y = f(x') - x'$ satisfies $y_J = (0, \dots, 0)$, and hence $y_I \in \mathcal{Y}$. The function $x \mapsto \delta(x) = y_I$ is thus a function from $[q]^I$ to $\mathcal{Y}$, and hence there exists $a \in \mathcal{Y}$ such that

$$|\delta^{-1}(a)| \geq \frac{|[q]^I|}{|\mathcal{Y}|} = \frac{q^\tau}{q^\tau - V_{\mathrm{H}}(q, \tau, i(D,q) - 1)}.$$

Consider then the FDS $f' \in F(D,q)$ defined as

$$f'_v(x) = \begin{cases} f_v(x) - a_v & \text{if } v \in I, \\ f_v(x) & \text{if } v \in J. \end{cases}$$

For every $x \in \delta^{-1}(a)$, x' is a fixed point of f'. Since $x \mapsto x'$ is an injection, $g(f') \geq \log_q |\delta^{-1}(a)|$, which combined with the above, proves the result.

The proof for the stability is similar. Let $f \in F(D,q)$ with maximal instability, I and J as above, and

$$\mathcal{Z} = \{z \in [q]^\tau : w_{\mathrm{H}}(z) \leq \tau - s(f)\}.$$

By the claim, for every $x \in [q]^I$ there exists a unique point $x'' \in [q]^n$ such that $x''_I = x$ and $z = f(x'') - x''$ satisfies $z_J = (1, \dots, 1)$, and hence $z_I \in \mathcal{Z}$. The function $x \mapsto \epsilon(x) = z_I$ is thus a function from $[q]^I$ to $\mathcal{Z}$, and hence there exists $b \in \mathcal{Z}$ such that

$$|\epsilon^{-1}(b)| \geq \frac{|[q]^I|}{|\mathcal{Z}|} = \frac{q^\tau}{V_{\mathrm{H}}(q, \tau, \tau - s(D,q))}.$$

Consider then $f' \in F(D,q)$ defined as

$$f''_v(x) = \begin{cases} f_v(x) - b_v & \text{if } v \in I, \\ f_v(x) - 1 & \text{if } v \in J. \end{cases}$$

For every $x \in \epsilon^{-1}(b)$, x'' is a fixed point of f''. Since $x \mapsto x''$ is an injection, $g(f'') \geq \log_q |\epsilon^{-1}(b)|$, which combined with the above, proves the result. ∎

Corollary 4. *If $i(D,2) = s(D,2) = \tau(D)$, then $g(D,2) = \tau(D)$.*

The implication does not hold for all q. Indeed, for q large enough we have $i(D,q) = \tau(D)$ for all D, while there are graphs D for which $g(D,q) < \tau(D)$ for all q (see Corollary 2, for example). Moreover, we remark that for q large enough, then the bounds in Theorem 4 become trivial.

5. Linear and affine (in)stability.

5.1. Digraphs with high affine stability and instability. The canonical way to obtain a graph which is q-stable for large q is to include a clique of size q, as seen from section 2.4. Previous work on Winkler's hat game has shown that graphs without large cliques may be q-stable for large q as well. For instance, [11] shows that for any q, there exists a q-stable bipartite graph whose size is doubly exponential in q. In [19], that construction is refined to a bipartite graph of size only exponential in q; graphs without very large cliques and with a linear number of vertices are also designed. Moreover, in [19], the authors exhibit a 4-stable oriented graph (i.e., a graph without K_2, or equivalently a graph with girth of 3 or more), and leave it as an open problem whether there exist q-stable oriented graphs for all q.

In this section, we give an affirmative answer to that open problem: for any constant γ, there exist a q-stable graph with girth γ. It is remarkable that the proof technique is based on the guessing dimension and the coset dimension, which were used in section 3 to refine the equivalences between the guessing number and the information defect. The graphs we construct exhibit high (i.e., close to the $n/2$ upper bound) binary (in)stability and high (i.e., close to n) asymptotic instability as well.

Theorem 5. *For any $\gamma \geq 3$, any $\epsilon > 0$, and any $q \geq 2$ there exists D with girth γ such that*

$$s_{\mathrm{aff}}(D,2) = i_{\mathrm{aff}}(D,2) > (1-\epsilon)\frac{n}{2}, \qquad i_{\mathrm{aff}}(D) = \tau(D) > (1-\epsilon)n, \qquad s_{\mathrm{aff}}(D,q) > 0.$$

The strategy is to recast the problem in terms of metric properties of codes, and then to use our results on the guessing code. A q-ary code C of length n is a subset of $[q]^n$. The *remoteness* and the *covering radius* of C are, respectively, defined as [35, 14, 12]

$$\mathrm{rem}(C) := \min_{y \in [q]^n} \max_{c \in C} d_{\mathrm{H}}(c, y),$$
$$\mathrm{cr}(C) := \max_{y \in [q]^n} \min_{c \in C} d_{\mathrm{H}}(c, y).$$

The remoteness of C is the smallest r such that all the codewords can be covered by the same sphere of radius r (in the Hamming metric). Conversely, the covering radius of C is the smallest r such that spheres of radius r centered around the codewords in C cover the whole space.

Example 6 (continued from Examples 1 through 5). For $C_\phi = \{0,1\}^4 \setminus \{0010, 1111\}$, we have $\mathrm{cr}(C_\phi) = 1$ and $\mathrm{rem}(C_\phi) = 3$.

LEMMA 3. *For any D and any q,*

$$i(D, q) = \max_{f \in F(D,q)} \mathrm{cr}(C_f),$$
$$s(D, q) = \max_{f \in F(D,q)} \left(n - \mathrm{rem}(C_f)\right).$$

The same results hold in the affine case.

Proof. We only prove the first equality, the second one being very similarly proved.

First, we can express f as $f = \phi + f(0)$, where $\phi(0) = 0$ and $G(f) = G(\phi)$ (where 0 denotes the all-zero vector of length n). Thus, for any graph D, the set $F(D, q)$ can be partitioned into classes of the form $\{\phi - y : y \in [q]^n\}$. Then

$$\begin{aligned} i(D, q) &= \max_{\phi \in F(D,q), \phi(0)=0} \max_{y \in [q]^n} \min_{x \in [q]^n} d_{\mathrm{H}}(x, \phi(x) - y) \\ &= \max_{\phi \in F(D,q), \phi(0)=0} \max_{y \in [q]^n} \min_{x \in [q]^n} d_{\mathrm{H}}(\phi(x) - x, y) \\ &= \max_{\phi \in F(D,q), \phi(0)=0} \max_{y \in [q]^n} \min_{c \in C_\phi} d_{\mathrm{H}}(c, y) \\ &= \max_{\phi \in F(D,q), \phi(0)=0} \mathrm{cr}(C_\phi). \end{aligned}$$

Finally, since $C_f = C_{\phi + f(0)} = C_\phi + f(0)$, we have $\mathrm{cr}(C_f) = \mathrm{cr}(C_\phi)$, which concludes the proof. ∎

If C_f is small, it has a high covering radius, yielding high instability; it also has low remoteness, thus yielding high stability as well.

LEMMA 4. *For any D and q,*

$$\log_q V_{\mathrm{H}}(q, n, i(D, q)) \geq n - l(D, q),$$
$$\log_q(q^n - V_{\mathrm{H}}(q, n, n - s(D, q) - 1)) \geq n - l(D, q).$$

The same results hold in the affine case.

Proof. The sphere-covering bound [14] states that for any code C of covering radius ρ, $|C| V_{\mathrm{H}}(q, n, \rho) \geq q^n$. Moreover, if C has remoteness r, then $|C|$ $(q^n - V_{\mathrm{H}}(q, n, r - 1)) \geq q^n$ [12]. The results then follow from applying these bounds to $C = C_f$ of cardinality $q^{l(D,q)}$. ∎

Proof of Theorem 5. Let $H_2(p) = -p\log_2 p - (1-p)\log_2(1-p)$ denote the binary entropy for $0 \le p \le 1$. Let $D = \vec{C}_\gamma^k$, where k is chosen such that

$$\tau(D) > n \max\left\{H_2\left(\frac{1-\epsilon}{2}\right), 1-\epsilon, \log_q(q-1)\right\}.$$

By Theorem 1, $l_{\text{aff}}(D,2) = l_{\text{aff}}(D,q) = \tau(D)$.

First, by [35, Chapter 10, Corollary 9], we obtain

$$H_2\left(\frac{i(D,2)}{n}\right) \ge \log_2 V_{\text{H}}(2,n,i(D,2)) \ge \tau(D) > H_2\left(\frac{1-\epsilon}{2}\right),$$

and hence $i(D,2) > (1-\epsilon)\frac{n}{2}$. Second, $i(D) = \tau(D) = n(1-\epsilon)$. Third, suppose $s(D,q) = 0$, then

$$\log_q\left(q^n - V_{\text{H}}(q,n,n-s(D,q)-1)\right) = \log_q(q-1)^n < \tau(D) = n - l(D,q),$$

which violates Lemma 4. ∎

5.2. Additional properties. First, note that $i_{\text{lin}}(D,q) = 0$, since any linear FDS fixes the all-zero vector. Also, from our preliminary results on the stability and instability, we have $i_{\text{aff}}(D,2) = s_{\text{aff}}(D,2)$ and $i_{\text{aff}}(D,q) \ge \nu(D)$.

We now prove that the stability of a linear FDS is severely limited by its interaction graph.

Proposition 5. *For any linear FDS f,*

$$s(f) \le n - \Delta_{\text{out}}(G(f)) - 1.$$

Proof. We have $f(x) - x = xM$, where M has support $I_n + A_{G(f)}$. Therefore,

$$s(f) = n - \max_{x\in[q]^n} d_{\text{H}}(x, f(x)) = n - \max_{c\in C_f} w_{\text{H}}(c).$$

Since the rows of M are codewords of C_f, the maximum weight is at least $\Delta_{\text{out}}(G(f)) + 1$. ∎

This bound is trivially achieved if $\Delta_{\text{out}}(G(f)) = n-1$, in which case $s(f) = 0$. More interestingly, it is also achieved for the graphs constructed from the cyclic simplex codes ($n = 2^r - 1$, $\Delta = 2^{r-1}-1$, $s_{\text{lin}}(D,2) = 2^{r-1}-1$). See [22] for how to construct a graph from a cyclic code. In particular, for $r = 3$ we obtain another example of an oriented graph D with $i(D,2) = \lfloor n/2 \rfloor$, namely, the Paley tournament on seven vertices displayed on Figure 5, the first example being the directed triangle.

Although we do not know whether the affine instability always reaches the feedback upper bound, we can prove that it always exceeds the linear guessing number.

Theorem 6. *For any D, $i_{\text{aff}}(D) := \sup_{q\ge 2} i_{\text{aff}}(D,q) \ge g_{\text{lin}}(D)$.*

Proof. Due to [50, Theorem 4.3], it is easy to check that $g_{\text{lin}}(D,p^m) \ge g_{\text{lin}}(D,p)$ for any prime power p and any integer $m \ge 1$. Therefore, there exists q large enough so that $n\log_q 2 < \epsilon$ and $n - l_{\text{aff}}(D,q) = g_{\text{lin}}(D,q) = g_{\text{lin}}(D)$. Then let $i := i_{\text{aff}}(D,q)$; we have $V_{\text{H}}(q,n,i) \le 2^n q^i$ and hence by Lemma 4 and Theorem 1,

$$i + \epsilon > \log_q V_{\text{H}}(q,n,i) \ge n - l_{\text{aff}}(D,q) = g_{\text{lin}}(D).$$ ∎

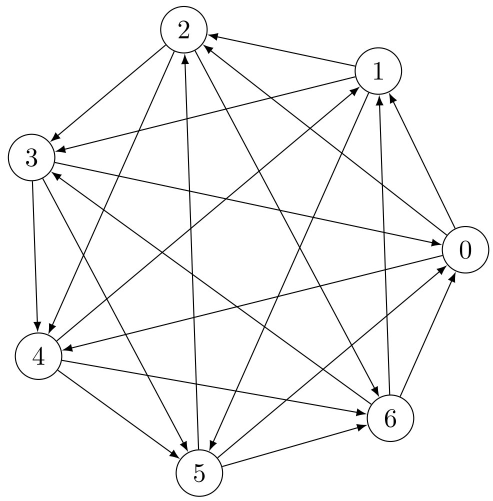


Fig. 5. *The Paley tournament on seven vertices with binary instability* 3.

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