A GLOBAL VERSION OF A CLASSICAL RESULT OF JOACHIMSTHAL

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ABSTRACT. A classical result attributed to Joachimsthal in 1846 states that if two surfaces intersect with constant angle along a line of curvature of one surface, then the curve of intersection is also a line of curvature of the other surface. In this note we prove a global analogue of this result, as follows.

Suppose that two closed convex surfaces intersect with constant angle along a curve that is not umbilic in either surface. We prove that the principal foliations of the two surfaces along the curve are either both orientable, or both non-orientable.

We prove this by characterizing the constant angle intersection of two surfaces in Euclidean 3-space as the intersection of a surface and a hypersurface in the space of oriented lines. The surface is Lagrangian, while the hypersurface is null, with respect to the canonical neutral Kaehler structure. We establish a relationship between the principal directions of the two surfaces along the intersection curve in Euclidean space, which yields the result.

This method of proof is motivated by topology and, in particular, the slice problem for curves in the boundary of a 4-manifold.

1. INTRODUCTION

A classical result of differential geometry states that if two surfaces, S_1 and S_2, in $\mathbb{R}^3$ intersect with constant angle along a line of curvature of S_1, then the curve of intersection is also a line of curvature of S_2. In 1846 Joachimsthal proved a special case of this theorem [3], while the general case was presented by Bonnet in 1853 [1].

In this note we prove a global analogue of this result. That is, consider two closed convex surfaces S_1 and S_2 intersecting transversely along a smooth simple closed curve C. Decompose $S_1 = D_1^+ \cup_C D_1^-$, $S_2 = D_2^+ \cup_C D_2^-$, where $D_j^\pm$ are closed discs, with D_1^- defined to be the disc with outward pointing normal given by the normal to S_2 projected onto S_1, and similarly for D_2^-. We prove

Main Theorem. *If S_1 and S_2 intersect with constant angle along a curve that is not umbilic in either S_1 or S_2, then the principal foliations of the two surfaces along the curve are either both orientable, or both non-orientable.*

The Main Theorem is proven by characterizing the constant angle intersection of surfaces in $\mathbb{R}^3$ as the intersection of a surface Σ and a hypersurface $\mathcal{H}_\epsilon$ in the

space $\mathbb{L}(\mathbb{R}^3)$ of oriented lines of $\mathbb{R}^3$. Here $\epsilon = \tan(\alpha/2)$, where α is the angle of intersection. In this characterization, the surface Σ is a Lagrangian section and $\mathcal{H}_\epsilon$ is a null hypersurface with respect to the neutral Kähler structure on the space of oriented lines. From these geometric properties we relate the principal directions of the surfaces along the intersection (equation (2.9)) and the result follows.

Our methods are motivated by topology: the ultimate goal is to investigate whether a knot in the boundary of a 4-manifold is the boundary of a properly embedded disc - the slice problem. As a new geometric model, we propose subsets of $\mathbb{L}(\mathbb{R}^3)$ and invariants derived from the neutral Kähler structure, which we claim can be utilized to detect obstructions to sliceness.

For surfaces in $\mathbb{R}^3$, umbilic points are generically isolated and have a half-integer index associated with them given by the winding number of the principal foliation about the umbilic point. Let D be a disc with non-umbilic boundary and only isolated umbilic points in the interior. The total umbilic index $i(D)$ is defined to be the sum of the umbilic indices on D and when the boundary of D is non-umbilic, it is equal to the winding number of the principal foliation along ∂D.

From our perspective, umbilic points on S correspond to complex points on Σ and the total complex index $I(\Sigma) = 2i(D)$ of the corresponding disc in $\mathbb{L}(\mathbb{R}^3)$ with totally real boundary is a smooth topological invariant [5]. Thus our main Theorem can be restated as:

$$I(\Sigma_1^+) = I(\Sigma_2^+) \bmod 2$$

Alternatively, our result provides a situation where one can bound this invariant by geometric data at the boundary hypersurface. Further results on this topic and its relation to the slice problem will be reported in a forthcoming paper.

2. Proof of the Main Theorem

Let S_1 and S_2 be smooth convex surfaces which intersect along a curve C at a constant angle $0 < \alpha < \pi$.

Split the convex surfaces so that $S_1 = D_1^+ \cup_C D_1^-$, $S_2 = D_2^+ \cup_C D_2^-$, where $D_j^\pm$ are closed discs, with D_1^- defined to be the disc with outward pointing normal given by the normal to S_2 projected onto S_1, and similarly for D_2^-, as in the diagram below.
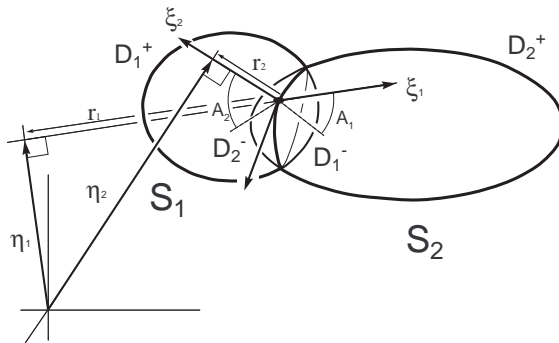
Now transfer this geometric data to the space $\mathbb{L}(\mathbb{R}^3)$ of oriented lines of Euclidean $\mathbb{R}^3$. $\mathbb{L}(\mathbb{R}^3)$ admits a canonical neutral Kähler structure $(\mathbb{J}, \mathbb{G}, \Omega)$ that is invariant

under the induced Euclidean action [4], and, since $\mathbb{L}(\mathbb{R}^3)$ is diffeomorphic to TS^2, write $T\mathbb{S}^2$ for $\mathbb{L}(\mathbb{R}^3)$ endowed with this structure.

The surfaces S_1 and S_2 in $\mathbb{R}^3$ will be treated differently: define $\mathcal{H}_\epsilon \subset T\mathbb{S}^2$ to be the set oriented lines that intersect S_1, forming an angle α with the outward normal, and define $\Sigma \subset T\mathbb{S}^2$ be the set of oriented lines normal to S_2. Clearly these are 3- and 2-dimensional submanifolds of $T\mathbb{S}^2$, the latter being diffeomorphic to a 2-sphere while the former is diffeomorphic to a circle bundle over the 2-sphere .

The submanifolds $\mathcal{H}_\epsilon$ and Σ intersect transversely along the curve $\mathcal{C} \subset T\mathbb{S}^2$ which consists of the oriented normal lines to S_2 along the curve $C = S_1 \cap S_2 \subset \mathbb{R}^3$. The constant angle condition in $\mathbb{R}^3$ has been translated to an intersection condition for certain submanifolds in $T\mathbb{S}^2$. We now describe the geometric properties of these submanifolds.

Starting with Σ, the set of oriented normals to the convex surface S_2. This is a Lagrangian section of the bundle $T\mathbb{S}^2 \to \mathbb{S}^2$ and is locally described by the gradient of the support function $r_2 : \mathbb{S}^2 \to \mathbb{R}$.

We now utilize local holomorphic coordinates (ξ, η) on $T\mathbb{S}^2$ to describe this - see [4] and references therein for further details. The complex coordinate ξ is the standard holomorphic coordinate on $\mathbb{S}^2 - \{\text{south pole}\}$ giving the direction of the oriented line, while the complex coordinate η gives the perpendicular distance of the line to the origin. Together they yield

$$(\xi, \eta) \leftrightarrow \eta \frac{\partial}{\partial \xi} + \bar{\eta} \frac{\partial}{\partial \bar{\xi}} \in T_\xi \mathbb{S}^2.$$

The surface Σ is the section $\xi \mapsto (\xi, \eta = \eta_2(\xi, \bar{\xi}))$ where

$$\eta_2 = \tfrac{1}{2}(1 + \xi\bar{\xi})^2 \frac{\partial r_2}{\partial \xi}, \tag{2.1}$$

and this is Lagrangian as

$$\frac{\partial}{\partial \xi}\left(\frac{\eta_2}{(1 + \xi\bar{\xi})^2}\right) = \frac{\partial}{\partial \bar{\xi}}\left(\frac{\bar{\eta}_2}{(1 + \xi\bar{\xi})^2}\right).$$

Define

$$\sigma_2 = -\frac{\partial \bar{\eta}_2}{\partial \xi} \qquad \psi_2 = r_2 + (1 + \xi\bar{\xi})^2 \frac{\partial}{\partial \xi}\left(\frac{\eta_2}{(1 + \xi\bar{\xi})^2}\right) \qquad \kappa_2 = \psi_2^2 - |\sigma_2|^2.$$

If R_1 and R_2 are the radii of curvature of S_2, then

$$\sigma_2 \bar{\sigma}_2 = \tfrac{1}{4}(R_1 - R_2)^2 \qquad \psi_2 = \tfrac{1}{2}(R_1 + R_2) \qquad \kappa_2 = R_1 R_2.$$

Thus umbilic points on S_2 are precisely the complex points on Σ, while the half of the argument of σ_2 determines the principal directions of the surface S_2.

Let $(\xi, \eta = \eta_1(\xi, \bar{\xi}))$, r_1, σ_1, ψ_1 and κ_1 be the analogous quantities for the surface S_1 and its associated Lagrangian section.

We turn now to $\mathcal{H}_\epsilon$, the oriented lines making a fixed angle α with S_1. To write this in local coordinates, note the following:

Lemma 1. *Given $\xi_1 \in \mathbb{S}^2$, the points on $\mathbb{S}^2$ forming an angle $0 < \alpha < \pi$ with ξ_1 is a circle parameterized by*

$$\xi_2 = \frac{\xi_1 + \epsilon e^{iA_1}}{1 - \bar{\xi}_1 \epsilon e^{iA_1}},$$

for $A_1 \in [0, 2\pi)$, where $\epsilon = \tan(\alpha/2)$.

Proof. If $\xi_1 \leftrightarrow \vec{X}_1$ and $\xi_2 \leftrightarrow \vec{X}_2$, then

$$\vec{X}_1 \cdot \vec{X}_2 = \frac{2\xi_1\bar{\xi}_2 + 2\xi_2\bar{\xi}_1 + (1 - \xi_1\bar{\xi}_1)(1 - \xi_2\bar{\xi}_2)}{(1 + \xi_1\bar{\xi}_1)(1 + \xi_2\bar{\xi}_2)} = \frac{1 - \epsilon^2}{1 + \epsilon^2} = \cos\alpha,$$

as claimed. $\square$

Note that the above relationship can be inverted to

$$\xi_1 = \frac{\xi_2 + \epsilon e^{iA_2}}{1 - \bar{\xi}_2\epsilon e^{iA_2}},$$

where the angles A_1 and A_2 are related by

$$e^{iA_2} = -\frac{e^{iA_1} - \epsilon\xi_1}{e^{-iA_1} - \epsilon\bar{\xi}_1}e^{-iA_1}. \tag{2.2}$$

Now suppose we have an oriented line (ξ_1, η_1) and a point (z, t) on this line. Thus, for some $r_1 \in \mathbb{R}$

$$z = \frac{2(\eta_1 - \bar{\xi}_1^2\bar{\eta}_1)}{(1 + \xi_1\bar{\xi}_1)^2} + \frac{2\xi_1}{1 + \xi_1\bar{\xi}_1}r_1 \qquad\qquad t = -\frac{2(\xi_1\bar{\eta}_1 + \bar{\xi}_1\eta_1)}{(1 + \xi_1\bar{\xi}_1)^2} + \frac{1 - \xi_1\bar{\xi}_1}{1 + \xi_1\bar{\xi}_1}r_1. \tag{2.3}$$

Suppose further that (ξ, η) is another oriented line that passes through (z, t) and forms an angle α with (ξ_1, η_1). The former fact means that:

$$\eta = \tfrac{1}{2}(z - 2t\xi_2 - \bar{z}\xi_2^2).$$

Substituting the expressions for z and t, and for ξ_2 from Lemma 1, we conclude that the set of oriented lines that intersect (ξ_1, η_1) at a parameter $r = r_1$, forming an angle α, is a circle in $T\mathbb{S}^2$ parameterized by $A_1 \in [0, 2\pi)$ as follows

$$\xi = \frac{\xi_1 + \epsilon e^{iA_1}}{1 - \bar{\xi}_1\epsilon e^{iA_1}} \qquad\qquad \eta = \frac{\eta_1 - \epsilon^2 e^{2iA_1}\bar{\eta}_1 - \epsilon(1 + \xi_1\bar{\xi}_1)e^{iA_1}r_1}{(1 - \bar{\xi}_1\epsilon e^{iA_1})^2},$$

where $\epsilon = \tan(\alpha/2)$.

As (ξ_1, η_1) vary over the surface set of normal lines to S_1, the above equations trace out the hypersurface $\mathcal{H}_\epsilon \subset T\mathbb{S}^2$, which is in fact null with respect to the neutral Kähler metric.

By assumption Σ and $\mathcal{H}_\epsilon$ intersect in the curve $\mathcal{C}$, consisting of the oriented normal lines to S_2 along the curve of intersection $C = S_1 \cap S_2$. Parameterize the curve so that C is given by equations (2.3) where everything now depends upon a parameter u.

Note that

$$\frac{d\eta_1}{du} = \frac{\partial\eta_1}{\partial\xi_1}\frac{d\xi_1}{du} + \frac{\partial\eta_1}{\partial\bar{\xi}_1}\frac{d\bar{\xi}_1}{du} = \left(\psi_1 - r_1 + \frac{2\bar{\xi}_1\eta_1}{1 + \xi_1\bar{\xi}_1}\right)\frac{d\xi_1}{du} - \bar{\sigma}_1\frac{d\bar{\xi}_1}{du}, \tag{2.4}$$

and

$$\frac{dr_1}{du} = \frac{\partial r_1}{\partial\xi_1}\frac{d\xi_1}{du} + \frac{\partial r_1}{\partial\bar{\xi}_1}\frac{d\bar{\xi}_1}{du} = \frac{2\bar{\eta}_1}{(1 + \xi_1\bar{\xi}_1)^2}\frac{d\xi_1}{du} + \frac{2\eta_1}{(1 + \xi_1\bar{\xi}_1)^2}\frac{d\bar{\xi}_1}{du}, \tag{2.5}$$

where we have used equation (2.1) and the definitions of σ_1 and ψ_1. Identical expressions hold for the derivatives of η_2 and r_2 with all subscripts changed from 1 to 2.

This yields the following:

Lemma 2. *To fix unit parameterization of the curve C set*

$$\frac{d\xi_1}{du} = \frac{(1 + \xi_1 \bar{\xi}_1)}{2\kappa_1} \left[\psi_1 e^{i(A_1 + \beta)} + \bar{\sigma}_1 e^{-i(A_1 + \beta)} \right], \tag{2.6}$$

for $\beta(u) \in [0, 2\pi)$. The curve $\mathcal{C}$ of oriented lines are tangent to C iff $\beta = 0$ or π, while, the oriented lines are normal to C iff $\beta = \pi/2$ or $3\pi/2$.

In summary, the intersection curve $\mathcal{C}$ in $T\mathbb{S}^2$ is the solution to the two complex equations

$$\xi_2 = \frac{\xi_1 + \epsilon e^{iA_1}}{1 - \bar{\xi}_1 \epsilon e^{iA_1}} \qquad \eta_2 = \frac{\eta_1 - \epsilon^2 e^{2iA_1} \bar{\eta}_1 - (1 + \xi_1 \bar{\xi}_1) \epsilon e^{iA_1} r_1}{(1 - \bar{\xi}_1 \epsilon e^{iA_1})^2}, \tag{2.7}$$

where all quantities depend on u. In the second equation we use the given graph function $\eta_1(u) = \eta_1(\xi_1(u), \bar{\xi}_1(u))$, and similarly for η_2. Thus we have five real unknowns encoded in (ξ_1, ξ_2, A_1) and four real equations, yielding a one dimensional solution set parameterized by u.

Now differentiate the first of these equations with respect to u and use equation (2.6) with $\beta = pi/2$ to get

$$\frac{d\xi_2}{du} = \frac{i(1 + \xi_1 \bar{\xi}_1) e^{iA_1}}{2(1 - \bar{\xi}_1 \epsilon e^{iA_1})^2} \left[2\epsilon \frac{dA_1}{du} + \epsilon(\epsilon e^{2iA_1} + e^{iA_1} \xi_1) \frac{\sigma_1}{\kappa_1} \right.$$
$$\left. + (e^{-iA_1} \epsilon \bar{\xi}_1 - e^{-2iA_1}) \frac{\bar{\sigma}_1}{\kappa_1} - (\epsilon(e^{iA_1} \bar{\xi}_1 + e^{-iA_1} \xi_1) + \epsilon^2 - 1) \frac{\psi_1}{\kappa_1} \right]. \tag{2.8}$$

Differentiate the second equation of (2.7) with respect to u and use equations (2.5), (2.6) and (2.8), along with (2.4) and the analogous equation for η_2. The result, after some computation and rearrangement, is

$$(1 - e^{iA_1} \epsilon \bar{\xi}_1)^2 \bar{\sigma}_2 \left[2\epsilon \kappa_1 \frac{dA_1}{du} + (-e^{2iA_1} + e^{iA_1} \epsilon \xi_1) \sigma_1 \right.$$

$$\left. + \epsilon(e^{-iA_1} \bar{\xi}_1 + \epsilon e^{-2iA_1}) \bar{\sigma}_1 - (\epsilon(e^{iA_1} \bar{\xi}_1 + e^{-iA_1} \xi_1) + \epsilon^2 - 1) \psi_1 \right]$$

$$+ (e^{iA_1} - \epsilon \xi_1)^2 \psi_2 \left[2\epsilon \kappa_1 \frac{dA_1}{du} + \epsilon(\epsilon e^{2iA_1} + e^{iA_1} \xi_1) \sigma_1 \right.$$

$$\left. + (e^{-iA_1} \epsilon \bar{\xi}_1 - e^{-2iA_1}) \bar{\sigma}_1 - (\epsilon(e^{iA_1} \bar{\xi}_1 + e^{-iA_1} \xi_1) + \epsilon^2 - 1) \psi_1 \right]$$

$$- (1 + \epsilon^2) \kappa_1 (e^{iA_1} - \epsilon \xi_1)^2 = 0.$$

Taking the complex conjugate of this equation and eliminating the derivative of A_1 we get

$$\frac{\sigma_1 e^{2A_1 i} - \bar{\sigma}_1 e^{-2A_1 i}}{\kappa_1} - \frac{\sigma_2}{\kappa_2} \frac{(e^{iA_1} - \epsilon \xi_1)^2}{(1 - e^{iA_1} \epsilon \bar{\xi}_1)^2} + \frac{\bar{\sigma}_2}{\kappa_2} \frac{(1 - e^{iA_1} \epsilon \bar{\xi}_1)^2}{(e^{iA_1} - \epsilon \xi_1)^2} = 0.$$

Finally, recalling equation (2.2), we obtain

$$\frac{\sigma_1 e^{2A_1 i} - \bar{\sigma}_1 e^{-2A_1 i}}{\kappa_1} - \frac{\sigma_2 e^{2A_2 i} - \bar{\sigma}_2 e^{-2A_2 i}}{\kappa_2} = 0.$$

To transfer this to back to Euclidean 3-space, denote the principal curvatures of S_1 by (λ_1, μ_1) and those of S_2 by (λ_2, μ_2). Let ϕ_1 be the angle between the principal direction of S_1 and the intersection curve and ϕ_2 the corresponding angle on S_2. The last equation can now be succinctly written

$$(\lambda_1 - \mu_1)\sin\phi_1 - (\lambda_2 - \mu_2)\sin\phi_2 = 0. \tag{2.9}$$

Thus we obtain Joachimsthal's Theorem: C is a line of curvature on S_1 iff it is a line of curvature on S_2. A further result of Joachimsthal's also follows: if S_1 is a sphere (so that $\sigma_1 = 0$) and the surface S_2 intersects it at constant angle, then $\sigma_2 e^{2iA_2} - \bar{\sigma}_2 e^{-2iA_2} = 0$ and so C is a line of curvature on S_2. Note that this equation states that the geodesic torsion of the intersection curve on the two surfaces are equal [2].

Moreover, if C contains no umbilic points on S_1 or S_2, the winding numbers of the maps $\phi_1, \phi_2 : S^1 \to S^1$ count the total umbilic index inside the curve on each surface, where the inside is determined by the outward pointing normals - see Theorem 4 of [4].

For simplicity, write $f_1 = \lambda_1 - \mu_1$ and $f_2 = \lambda_2 - \mu_2$ so that the derivative of equation (2.9) with respect to u is

$$f_1'\sin\phi_1 + f_1\cos\phi_1\ \phi_1' = f_2'\sin\phi_2 + f_2\cos\phi_2\ \phi_2' \tag{2.10}$$

If $\phi_1 \neq \phi_2$ everywhere along the curve then the winding numbers of the principal foliations are equal and the Theorem holds. Otherwise, choose a point $p \in C = S_1 \cap S_2$ such that $\phi_1(p) = \phi_2(p)$. By a deformation of the surfaces preserving constant angle we can also assure that $\phi_1'(p) \neq 0$ (an open condition).

By a choice of parameterization of the intersection curve C we can set p to be $u = 0$ and $u = 2\pi$. Furthermore, by a rotation of the complex coordinates ξ we can set $\phi_1(0) = \phi_2(0) = 0$. Then $\phi_1(2\pi) = n\pi$ and $\phi_2(2\pi) = m\pi$ for some $n, m \in \mathbb{Z}$. Each principal foliation is orientable or non-orientable along C according to whether the winding numbers n or m is even or odd, respectively.

Now evaluating equation (2.10) at $u = 0$ and $u = 2\pi$ we find

$$f_1(p)\phi_1'(p) = f_2(p)\phi_2'(p) \qquad\qquad (-1)^n f_1(p)\phi_1'(p) = (-1)^m f_2(p)\phi_2'(p).$$

Since none of f_1, f_2, ϕ_1' or ϕ_2' vanish at p, both of these equations hold only if $n = m + 2k$ for some $k \in \mathbb{Z}$. Thus the foliations are both orientable or both non-orientable, as claimed.

$\square$

References

[1] O. Bonnet, *Mémoire sur la théorie générale des surfaces*, J. École Polytechnique **35** (1853).

[2] M.P. DoCarmo, Differential geometry of curves and surfaces, Prentice-Hall, New Jersey, 1976.

[3] F. Joachimsthal, *Demonstrationes theorematum ad superficies curvas spectantium*, J. Reine Angew. Math. **30** (1846) 347–350.

[4] B. Guilfoyle and W. Klingenberg, *An indefinite Kähler metric on the space of oriented lines*, J. London Math. Soc. **72**, (2005) 497–509.

[5] H.F. Lai, *Characteristic classes of real manifolds immersed in complex manifolds*, Trans. Amer. Math. Soc. **172** (1972), 1–33.

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