SCHRÖDINGER OPERATOR WITH NON-ZERO ACCUMULATION POINTS OF COMPLEX EIGENVALUES

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Abstract. We study Schrödinger operators $H = -\Delta + V$ in $L^2(\Omega)$ where Ω is $\mathbb{R}^d$ or the half-space $\mathbb{R}^d_+$, subject to (real) Robin boundary conditions in the latter case. For $p > d$ we construct a non-real potential $V \in L^p(\Omega) \cap L^\infty(\Omega)$ that decays at infinity so that H has infinitely many non-real eigenvalues accumulating at every point of the essential spectrum $\sigma_{\mathrm{ess}}(H) = [0, \infty)$. This demonstrates that the Lieb-Thirring inequalities for selfadjoint Schrödinger operators are no longer true in the non-selfadjoint case.

1. Introduction

In three seminal papers [15, 16, 17] from the 1960s, Pavlov studied Schrödinger operators $H = -\Delta + V$ in $L^2(0, \infty)$ with real-valued rapidly decaying potentials V, subject to a non-selfadjoint Robin boundary condition $f'(0) = hf(0)$ for some $h \in \mathbb{C}$. In contrast to the selfadjoint case, for non-real h the discrete eigenvalues are complex and can, in principle, accumulate at a *non-zero* point of the essential spectrum $[0, \infty)$. Using inverse spectral theory, Pavlov proved the existence of a potential V and a boundary condition so that H has infinitely many non-real eigenvalues that accumulate at a prescribed point λ of the essential spectrum $\sigma_{\mathrm{ess}}(H) = [0, \infty)$. He further studied the structure of the set of accumulation points. Since then, it has been an open question whether these results can be modified so that the non-selfadjointness is not coming from the boundary conditions but from a *non-real* potential V.

The aim of the present paper is to fill this gap by proving the following two results. In the first theorem we address non-selfadjoint Schrödinger operators in $L^2(\mathbb{R}^d)$ for any dimension $d \in \mathbb{N}$.

Theorem 1. *Let $p > d$ and $\mathcal{E} > 0$. There exists $V \in L^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $\max\{\|V\|_\infty, \|V\|_p\} \leq \mathcal{E}$ that decays at infinity so that the Schrödinger operator*

$$H := -\Delta + V, \quad \mathcal{D}(H) := W^{2,2}(\mathbb{R}^d),$$

has infinitely many eigenvalues in the open lower complex half-plane that accumulate at every point in $[0, \infty)$.

In the second main result we replace the whole Euclidean space $\mathbb{R}^d$ by the half-space $\mathbb{R}^d_+ := \{x = (x_1, \dots, x_d)^t \in \mathbb{R}^d : x_d > 0\}$ and impose (real) Robin boundary conditions.

Theorem 2. *Let $p > d$ and $\mathcal{E} > 0$, and let $\phi \in [0, \pi)$. There exists $V \in L^\infty(\mathbb{R}^d_+) \cap L^p(\mathbb{R}^d_+)$ with $\max\{\|V\|_\infty, \|V\|_p\} \leq \mathcal{E}$ that decays at infinity so that the Schrödinger operator*

$$H := -\Delta + V, \quad \mathcal{D}(H) := \left\{ f \in W^{2,2}(\mathbb{R}^d_+) : \cos(\phi)\partial_{x_d} f + \sin(\phi) f = 0 \text{ on } \partial\mathbb{R}^d_+ \right\},$$

has infinitely many eigenvalues in the open lower complex half-plane that accumulate at every point in $[0, \infty)$.

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Theorem 1 is also relevant in the context of *Lieb-Thirring inequalities* (after Lieb and Thirring [14], see also [12] for an overview) and their (possible) generalisation to complex potentials [8, 13, 5]. In the selfadjoint case the Lieb-Thirring inequalities state that, if

$$p \geq \frac{d}{2} \quad \text{for } d \geq 3; \quad p > 1 \quad \text{for } d = 2; \quad p \geq 1 \quad \text{for } d = 1, \tag{1}$$

then there exists $C_{d,p} > 0$ so that for every real $V \in L^p(\mathbb{R}^d)$ the negative eigenvalues of the Schrödinger operator $H = -\Delta + V$ satisfy

$$\sum_{\lambda \in \sigma(H) \setminus [0,\infty)} |\lambda|^{p-\frac{d}{2}} \leq C_{d,p} \|V\|_p^p \tag{2}$$

where in the sum each eigenvalue is repeated according to its algebraic multiplicity. In fact, the inequality remains true if V on the right hand side is replaced by the negative part $V_- := \max\{0, -V\}$. Now Theorem 1 demonstrates that, if $p > d$, an inequality like (2) cannot hold in the non-selfadjoint case since, for the constructed V in Theorem 1, the left hand side is infinite whereas the right hand side is finite (and, in fact, arbitrarily small). The sharpness of $p > d$ (in relation to p in (1)) is discussed in Remark 1 below. For possible modifications of Lieb-Thirring inequalities see [6] and the references therein.

Theorem 1 is proved in Section 2, and Theorem 2 in Section 3. In contrast to Pavlov's inverse spectral theory approach using an elaborate analysis of Weyl m-functions, our proofs are constructive. For both $\Omega = \mathbb{R}^d$ and $\Omega = \mathbb{R}^d_+$ the proof relies on the following two main ingredients (see Lemmas 1, 2 and 3, 4 for the precise formulation):

(I) For an arbitrary $\lambda \in (0, \infty)$ we construct $V_0 \in L^\infty(\Omega) \cap L^p(\Omega)$ with arbitrarily small $\|V_0\|_\infty$, $\|V_0\|_p$ and that decays at infinity so that $-\Delta + V_0$ in $L^2(\Omega)$ has an eigenvalue μ close to λ.

(II) For two potentials $V_1 \in L^\infty(\Omega)$, $V_2 \in L^\infty(\mathbb{R}^d)$ decaying at infinity, consider the corresponding Schrödinger operators

$$H_1 := -\Delta + V_1 \quad \text{in} \quad L^2(\Omega), \quad H_2 := -\Delta + V_2 \quad \text{in} \quad L^2(\mathbb{R}^d),$$

and assume that there exists $\mu \in \sigma(H_2) \setminus \sigma(H_1)$. If we shift V_2 in direction of the d-th coordinate vector e_d to $V_2(\cdot - te_d)$ for a sufficiently large $t > 0$, then $H_1 + \chi_\Omega V_2(\cdot - te_d)$ in $L^2(\Omega)$ has an eigenvalue μ_t close to μ.

The potential V in Theorems 1, 2 is then an infinite sum of functions V_j, $j \in \mathbb{N}$, that we construct inductively using (I) and (II) above.

Since we do not know the exact value of the "sufficiently large" shift t in (II), we cannot control the exact decay rate of V at infinity. For $\Omega = \mathbb{R}^3$ or $\Omega = (0, \infty)$, subject to the boundary condition $f(0) = 0$ or $f'(0) = hf(0)$, $h \in \mathbb{C}$, in the half-line case, Pavlov [15] proved that if

$$\exists\, \varepsilon > 0 : \quad \sup_{x \in \Omega} |V(x)| \mathrm{e}^{\varepsilon \sqrt{|x|}} < \infty, \tag{3}$$

then $-\Delta + V$ in $L^2(\Omega)$ has only finitely many eigenvalues. Therefore, the potential V in Theorem 1 (for $d = 3$) and Theorem 2 (for $d = 1$) has to decay so slow to violate (3). The condition (3) for $\Omega = (0, \infty)$ is sharp; Pavlov [16] proved that it cannot be relaxed to $\sup_{x \in (0,\infty)} |V(x)| \mathrm{e}^{\varepsilon x^\beta} < \infty$ for any $\beta \in \left(0, \frac{1}{2}\right)$. For an arbitrary odd dimension d, see [9] and the references therein for conditions guaranteeing a finite number of eigenvalues. In addition, in [18] are conditions, for an arbitrary $d \geq 2$, that prevent a dissipative Schrödinger operator (where $\operatorname{Im} V \leq 0$) to have discrete eigenvalues accumulating at zero.

We employ the following notation and conventions. Let $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The open ball in $\mathbb{R}^d$ with radius $r > 0$ around $v \in \mathbb{R}^d$ is $B(v, r) := \{x \in \mathbb{R}^d : |x - v| < r\}$, and analogously $B(z, r) \subset \mathbb{C}$ denotes the open disk of radius $r > 0$ around $z \in \mathbb{C}$. For a subset $\Lambda \subset \mathbb{C}$ the complex conjugated set is $\Lambda^* := \{\overline{\lambda} : \lambda \in \Lambda\}$, and for $z \in \mathbb{C}$ its distance to Λ is $\operatorname{dist}(z, \Lambda) := \inf_{\lambda \in \Lambda} |z - \lambda|$. Take a domain $\Omega \subset \mathbb{R}^d$ and $p \in [1, \infty]$. A function $f \in L^p(\Omega)$ is viewed as an element of $L^p(\mathbb{R}^d)$ by extending it by zero outside Ω, with L^p norm $\|f\|_p$; conversely, if we multiply a function $g \in L^p(\mathbb{R}^d)$ with the characteristic function χ_Ω of Ω, then $\chi_\Omega g \in L^p(\Omega)$. If not specified by an index, the norm $\|\cdot\|$ always refers to the one of the Hilbert space $L^2(\mathbb{R}^d)$. The operator domain, spectrum and resolvent set of an operator H are denoted by $\mathcal{D}(H)$, $\sigma(H)$ and $\varrho(H)$, and the Hilbert space adjoint operator is H^*. An identity operator is denoted by I, and scalar multiples λI for $\lambda \in \mathbb{C}$ are written as λ. Analogously, in $L^2(\mathbb{R}^d)$ the operator of multiplication with an $L^\infty(\mathbb{R}^d)$ function V is simply V; its adjoint operator is the multiplication operator with the complex conjugated function $\overline{V}$. Weak convergence in $L^2(\mathbb{R}^d)$ is denoted by $f_n \xrightarrow{w} f$, and strong operator convergence is $H_n \xrightarrow{s} H$.

2. Schrödinger operator in $L^2(\mathbb{R}^d)$

Throughout this section, all operator domains are $W^{2,2}(\mathbb{R}^d)$. The functions V_j, $j \in \mathbb{N}$, mentioned in the introduction will be of the form

$$U_{c,t,a}(x) := \begin{cases} c, & x \in B(te_d, a), \\ -\dfrac{(d-3)(d-1)}{4|x - te_d|^2}, & x \in \mathbb{R}^d \setminus B(te_d, a), \end{cases}$$

where $c \in \mathbb{C}$, $t \in \mathbb{R}$ and $a > 0$. Note that in dimension $d = 1$ and $d = 3$ the function $U_{c,t,a}$ vanishes outside the ball $B(te_d, a)$.

Before we study finite or infinite sums, we reduce our attention to a potential of the form $U_{c,t,a}$.

Lemma 1. *Let $\lambda \in (0, \infty)$ and $p > d$. For any $\varepsilon, \delta, r > 0$ there exist $a > 0$, $c \in \mathbb{C}$ and $\mu \in \mathbb{C}$ with $\operatorname{Im} \mu < 0$ such that, for every $t \in \mathbb{R}$,*

$$\|U_{c,t,a}\|_p < \varepsilon, \quad \|U_{c,t,a}\|_\infty < \delta, \quad |\mu - \lambda| < r,$$

and μ is an eigenvalue of $-\Delta + U_{c,t,a}$.

Proof. Define $\nu := \sqrt{\lambda} > 0$ and

$$a_m := \frac{\frac{d\pi}{4} + \pi m}{\nu} > 0, \quad m \in \mathbb{N}_0. \tag{4}$$

For $m \in \mathbb{N}_0$ let $\eta_m > 0$ be the unique solution of

$$\eta_m e^{2\eta_m a_m} = \nu. \tag{5}$$

Note that $a_m \to \infty$ and $\eta_m \to 0$ as $m \to \infty$. We set

$$\tau_m := \nu + i\eta_m, \quad m \in \mathbb{N}_0,$$

and

$$k_m := -i \frac{J_{\frac{d}{2}-2}(\tau_m a_m)}{J_{\frac{d}{2}-1}(\tau_m a_m)} \tau_m + \frac{i(d-3)}{2a_m}, \quad m \in \mathbb{N}_0, \tag{6}$$

where J_n is the Bessel function of the first kind of order n (see [2, Chapter 9]). It satisfies

$$J_n'(z) = J_{n-1}(z) - \frac{nJ_n(z)}{z}, \quad z^2 J_n''(z) + zJ_n'(z) = (n^2 - z^2)J_n(z),$$

see [2, Equation 9.1.27]. For a fixed $m \in \mathbb{N}_0$, define the function

$$g_m(r) := \begin{cases} \dfrac{\mathrm{e}^{\mathrm{i}k_m a_m}}{\sqrt{a_m} J_{\frac{d}{2}-1}(\tau_m a_m)} \dfrac{\tau_m^{\frac{d}{2}-1}}{2^{\frac{d}{2}-1}\Gamma(\frac{d}{2})}, & r = 0, \\ \dfrac{\mathrm{e}^{\mathrm{i}k_m a_m}}{\sqrt{a_m} J_{\frac{d}{2}-1}(\tau_m a_m)} \dfrac{J_{\frac{d}{2}-1}(\tau_m r)}{r^{\frac{d}{2}-1}}, & 0 < r \le a_m, \\ \dfrac{\mathrm{e}^{\mathrm{i}k_m r}}{r^{\frac{d-1}{2}}}, & r > a_m. \end{cases}$$

Using (6) and [2, Equation 9.1.10], one may check that both g_m and g_m' are continuous; for small $r > 0$ we expand $g_m(r) = g_m(0) + \mathcal{O}(r^2)$, hence $\lim_{r\to 0} g_m'(r) = 0$. Let $t \in \mathbb{R}$ be arbitrary. Then $f_m(x) := g_m(|x - te_d|)$, $x \in \mathbb{R}^d$, belongs to $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^d)$ and

$$\begin{aligned} -\Delta f_m(x) &= -g_m''(|x-te_d|) - \frac{d-1}{|x-te_d|} g_m'(|x-te_d|) \\ &= \begin{cases} \tau_m^2 f_m(x), & 0 < |x-te_d| \le a_m, \\ k_m^2 f_m(x) + \frac{(d-3)(d-1)}{4|x-te_d|^2} f_m(x), & |x-te_d| > a_m. \end{cases} \end{aligned}$$

Hence

$$-\Delta f_m + U_{c_m,t,a_m} f_m = \mu_m f_m \quad \text{with} \quad \mu_m := k_m^2, \quad c_m := k_m^2 - \tau_m^2.$$

In order to ensure $f_m \in W^{2,2}(\mathbb{R}^d) = \mathcal{D}(-\Delta + U_{c_m,t,a_m})$, we need $\operatorname{Im} k_m > 0$. We use the asymptotics of the Bessel function for $z \in \mathbb{C}$ with $|\arg z| < \pi$ and large $|z|$ (see [2, Equation 9.2.1]),

$$J_n(z) = \sqrt{\frac{2}{\pi z}} \left(\cos\left(z - \frac{(2n+1)\pi}{4} \right) + \mathrm{e}^{|\operatorname{Im} z|} \mathcal{O}(|z|^{-1}) \right).$$

A straight forward calculation reveals that, if

$$\operatorname{Re} z \in \frac{(n+1)\pi}{2} + \pi\mathbb{Z}, \quad \operatorname{Im} z > 0, \tag{7}$$

then for large $|z|$ we have

$$\begin{aligned} \frac{J_{n-1}(z)}{J_n(z)} &= -\frac{\mathrm{e}^{-\operatorname{Im} z} + \mathrm{i}\mathrm{e}^{\operatorname{Im} z} + \mathrm{e}^{\operatorname{Im} z}\mathcal{O}(|z|^{-1})}{\mathrm{i}\mathrm{e}^{-\operatorname{Im} z} + \mathrm{e}^{\operatorname{Im} z} + \mathrm{e}^{\operatorname{Im} z}\mathcal{O}(|z|^{-1})} \\ &= -2\mathrm{e}^{-2\operatorname{Im} z} + \mathrm{i}(\mathrm{e}^{-4\operatorname{Im} z} - 1) + \mathcal{O}(|z|^{-1}). \end{aligned}$$

The point $z = \tau_m a_m$ satisfies (7) for $n = \frac{d}{2} - 1$, and hence, for large m, (6) yields

$$\begin{aligned} k_m &= -\mathrm{i}\tau_m\big(-2\mathrm{e}^{-2\operatorname{Im}\tau_m a_m} + \mathrm{i}(\mathrm{e}^{-4\operatorname{Im}\tau_m a_m} - 1) + \mathcal{O}(|\tau_m a_m|^{-1})\big) + \mathcal{O}(a_m^{-1}) \\ &= -\nu(1 - \mathrm{e}^{-4\eta_m a_m}) - 2\eta_m \mathrm{e}^{-2\eta_m a_m} \\ &\quad + \mathrm{i}\big(2\nu\mathrm{e}^{-2\eta_m a_m} - \eta_m(1 - \mathrm{e}^{-4\eta_m a_m})\big) + \mathcal{O}(a_m^{-1}). \end{aligned}$$

Using that (5) implies $\mathrm{e}^{-2\eta_m a_m} = \frac{\eta_m}{\nu}$ and $a_m = \frac{\ln(\nu/\eta_m)}{2\eta_m}$, we arrive at

$$k_m = -\nu + \mathrm{i}\eta_m\Big(1 + \mathcal{O}\Big(\Big(\ln\frac{\nu}{\eta_m}\Big)^{-1}\Big)\Big).$$

Since $\eta_m > 0$ and $\ln(\nu/\eta_m)^{-1} \to 0$ as $m \to \infty$, we conclude that $\operatorname{Im} k_m > 0$ for all sufficiently large $m \in \mathbb{N}_0$. In addition, for large $m \in \mathbb{N}_0$ the eigenvalue $\mu_m = k_m^2$ satisfies

$$\mu_m = \lambda - \mathrm{i}2\nu\eta_m\Big(1 + \mathcal{O}\Big(\Big(\ln\frac{\nu}{\eta_m}\Big)^{-1}\Big)\Big),$$

and hence Im $\mu_m < 0$ for all sufficiently large $m \in \mathbb{N}_0$. One may check that

$$\mu_m - \lambda = \mathcal{O}(\eta_m), \quad c_m = k_m^2 - \tau_m^2 = \mathcal{O}(\eta_m)$$

converge to 0 as $m \to \infty$. Further note that

$$\|U_{c_m,t,a_m}\|_p^p = \mathrm{Vol}(B(0,1)) \left(\frac{|c_m|^p a_m^d}{d} + \frac{|d-3|^p |d-1|^p}{4^p (2p-d) a_m^{2p-d}} \right) = \mathcal{O}\Big(\eta_m^{p-d}\Big(\ln \frac{\nu}{\eta_m}\Big)^d\Big),$$
$$\|U_{c_m,t,a_m}\|_\infty = \max\left\{ |c_m|, \frac{|d-3||d-1|}{4a_m^2} \right\} = \mathcal{O}(\eta_m).$$

Since $p > d$ by the assumptions, both norms converge to 0 as $m \to \infty$. Altogether, we see that the claim is satisfied if we set $a := a_m$, $c := c_m$, $\mu := \mu_m$ for a sufficiently large $m \in \mathbb{N}_0$. □

Remark 1. In dimension $d = 1$ the assumption $p > d = 1$ of Lemma 1 is sharp. In fact, due to Abramov et al. [1], for every $V \in L^1(\mathbb{R})$ every eigenvalue $\mu \in \sigma(-\mathrm{d}^2/\mathrm{d}x^2 + V) \backslash [0, \infty)$ satisfies

$$|\mu|^{\frac{1}{2}} \le \frac{1}{2}\|V\|_1; \tag{8}$$

hence $\varepsilon > 0$ cannot be chosen arbitrarily small as in Lemma 1. In addition, in Theorem 1 for $d = 1$ it is impossible to construct $V \in L^1(\mathbb{R}^d)$ since then (8) forces the non-real eigenvalues to lie in the disk $B(0, \mathcal{E}^2/4)$, so they cannot accumulate at every point in $[0, \infty)$.

For dimension $d \ge 2$ the sharpness of the assumption $p > d$ is directly related to the following conjecture of Laptev and Safronov [13]: For $p \in \big(\frac{d}{2}, d\big]$ there exists $C_{d,p} > 0$ such that

$$|\mu|^{p-\frac{d}{2}} \le C_{d,p} \|V\|_p^p \tag{9}$$

for every $V \in L^p(\mathbb{R}^d)$ and every $\mu \in \sigma(-\Delta + V) \backslash [0, \infty)$. In [10] the conjecture was proved for *radial* potentials. Note that the potential in Lemma 1 is radial, so $p > d$ is sharp. In general (for non-radial potentials) the conjecture has been confirmed for $p \in \big(\frac{d}{2}, \frac{d+1}{2}\big]$ (see [7]) and is still open for $p \in \big(\frac{d+1}{2}, d\big]$. If the conjecture is false, then it may also be possible to modify Lemma 1 for a non-radial potential and hence prove Theorems 1, 2 for a $p \le d$.

Lemma 2. *Let $V_1, V_2 \in L^\infty(\mathbb{R}^d)$ be decaying at infinity and such that there exists $\mu \in \sigma(-\Delta + V_2) \backslash \sigma(-\Delta + V_1)$. Then there are*

$$\mu_t \in \sigma\big(-\Delta + V_1 + V_2(\cdot - te_d)\big), \quad t > 0,$$

with $\mu_t \to \mu$ as $t \to \infty$.

Proof. First note that

$$\sigma\big(-\Delta + V_1 + V_2(\cdot - te_d)\big) = \sigma\big(-\Delta + V_1(\cdot + te_d) + V_2\big), \quad t > 0. \tag{10}$$

Next we prove that, for every $z \in \mathbb{C}$ with $\mathrm{dist}(z, [0, \infty)) > \|V_1\|_\infty + \|V_2\|_\infty$, we have strong resolvent convergence

$$\big(-\Delta + V_1(\cdot + te_d) + V_2 - z\big)^{-1} \xrightarrow{s} \big(-\Delta + V_2 - z\big)^{-1}, \quad t \to \infty, \tag{11}$$

and the same holds for the adjoint operators. To this end, first note that a Neumann series argument yields

$$\begin{aligned}
&z \in \bigcap_{t>0} \varrho\big(-\Delta + V_1(\cdot + te_d) + V_2\big) \cap \varrho(-\Delta + V_2),\\
&\sup_{t>0} \big\| \big(-\Delta + V_1(\cdot + te_d) + V_2 - z\big)^{-1} \big\| \\
&\leq \|(-\Delta - z)^{-1}\| \sup_{t>0} \big\| \big(I + (V_1(\cdot + te_d) + V_2)(-\Delta - z)^{-1}\big)^{-1} \big\| \\
&\leq \frac{1}{\operatorname{dist}(z,[0,\infty))} \frac{1}{1 - \frac{\|V_1\|_\infty + \|V_2\|_\infty}{\operatorname{dist}(z,[0,\infty))}} = \frac{1}{\operatorname{dist}(z,[0,\infty)) - (\|V_1\|_\infty + \|V_2\|_\infty)}.
\end{aligned}$$

The space $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{2,2}(\mathbb{R}^d)$ and hence a core of $-\Delta + V_2$. Let $f \in C_0^\infty(\mathbb{R}^d)$. Then $f \in W^{2,2}(\mathbb{R}^d)$, and the assumption $V_1(x) \to 0$ as $|x| \to \infty$ yields

$$\big\| \big(-\Delta + V_1(\cdot + te_d) + V_2\big) f - (-\Delta + V_2) f \big\| \leq \sup_{x \in (\operatorname{supp} f + te_d)} |V_1(x)| \|f\| \longrightarrow 0, \quad t \to \infty.$$

Now the strong resolvent convergence in (11) follows from [3, Theorem 3.1, Proposition 2.16 i)], and the strong resolvent convergence of the adjoint operators

$$\big(-\Delta + V_1(\cdot + te_d) + V_2\big)^* = -\Delta + \overline{V_1(\cdot + te_d)} + \overline{V_2}, \quad t > 0,$$

to $(-\Delta + V_2)^* = -\Delta + \overline{V_2}$ is proved analogously.

By [4, Theorem 2.3 i)], in the limit $t \to \infty$ the isolated eigenvalue $\mu \in \sigma\big(-\Delta + V_2\big) \backslash \sigma(-\Delta + V_1)$ is approximated by points $\mu_t \in \sigma\big(-\Delta + V_1(\cdot + te_d) + V_2\big)$, $t > 0$, provided that the so-called *limiting essential spectrum* satisfies

$$\mu \notin \sigma_{\mathrm{ess}}((-\Delta + V_1(\cdot + te_d) + V_2)_{t>0}) \cup \sigma_{\mathrm{ess}}(((-\Delta + V_1(\cdot + te_d) + V_2)^*)_{t>0})^*. \quad (12)$$

This, together with (10), then proves the claim. So it is left to prove (12).

By definition (see [4]), the point μ belongs to set on the right hand side of (12) only if there exist an infinite subset $I \subset (0,\infty)$ and $f_t \in W^{2,2}(\mathbb{R}^d)$, $t \in I$, with $\|f_t\| = 1$, $f_t \stackrel{w}{\to} 0$ and, in the limit $t \to \infty$,

$$\begin{aligned}
&\big\| \big(-\Delta + V_1(\cdot + te_d) + V_2 - \mu\big) f_t \big\| \longrightarrow 0\\
\text{or}\quad &\big\| \big(-\Delta + \overline{V_1(\cdot + te_d)} + \overline{V_2} - \overline{\mu}\big) f_t \big\| \longrightarrow 0.
\end{aligned} \quad (13)$$

It is easy to see that the latter implies that $\|f_t\|_{W^{1,2}(\mathbb{R}^d)}$, $t \in I$, are uniformly bounded. Since, for any $r > 0$, the space $W^{1,2}(B(0,r))$ is compactly embedded in $L^2(B(0,r))$ by the Rellich-Kondrachov theorem, the weak convergence $f_t \stackrel{w}{\to} 0$ implies $\|\chi_{B(0,r)} f_t\| \to 0$ and hence $\|\chi_{B(0,r)} V_2 f_t\| \to 0$ as $t \to \infty$. Moreover, the assumption $V_2(x) \to 0$ as $|x| \to \infty$ yields

$$\sup_{t>0} \|\chi_{\mathbb{R}^d \backslash B(0,r)} V_2 f_t\| \leq \sup_{|x|>r} |V_2(x)| \longrightarrow 0, \quad r \to \infty.$$

Altogether, in the limit $t \to \infty$ we obtain $\|V_2 f_t\| \to 0$ and hence, by (13),

$$\begin{aligned}
&\big\| \big(-\Delta + V_1 - \mu\big) f_t(\cdot - te_d) \big\| = \big\| \big(-\Delta + V_1(\cdot + te_d) - \mu\big) f_t \big\| \longrightarrow 0\\
\text{or}\quad &\big\| \big(-\Delta + \overline{V_1} + \overline{\mu}\big) f_t(\cdot - te_d) \big\| = \big\| \big(-\Delta + \overline{V_1(\cdot + te_d)} + \overline{\mu}\big) f_t \big\| \longrightarrow 0.
\end{aligned}$$

Therefore, in either case μ needs to belong to $\sigma(-\Delta + V_1) = \sigma(-\Delta + \overline{V_1})^*$, which is excluded by the assumptions. This proves the claim (12). □

Now we are ready to prove the main result.

Proof of Theorem 1. Consider an enumeration of $(\mathbb{Q} \cap (0,\infty)) \times \mathbb{N}$, i.e. a bijective map

$$\mathbb{N} \ni n \mapsto (q_n, m_n)^t \in (\mathbb{Q} \cap (0,\infty)) \times \mathbb{N}.$$

Set $\gamma_0 := \infty$. By induction over $n \in \mathbb{N}$ we construct c_n, t_n, a_n and γ_n such that

$$H_n := -\Delta + \sum_{j=1}^{n} U_{c_j,t_j,a_j}$$

satisfies the following:

i) The norms of the functions are bounded by

$$\begin{aligned} \|U_{c_n,t_n,a_n}\|_p &< \varepsilon_n := \frac{6\mathcal{E}}{\pi^2 n^2}, \\ \|U_{c_n,t_n,a_n}\|_\infty &< \delta_n := \frac{6\min\{\gamma_{n-1}, \mathcal{E}\}}{\pi^2 n^2}, \end{aligned} \tag{14}$$

and

$$\exists\, \mu_n \in \sigma(H_n): \quad \operatorname{Im} \mu_n < 0, \quad |\mu_n - q_n| < \frac{1}{2m_n}. \tag{15}$$

ii) We have $0 < \gamma_n \leq \gamma_{n-1}$ and for any $U_n \in L^\infty(\mathbb{R}^d)$ with $\|U_n\|_\infty < \gamma_n$ there is $\lambda_n \in \sigma(H_n + U_n)$ such that

$$|\lambda_n - \mu_n| < \operatorname{dist}(\mu_n, [0, \infty)).$$

We start with $n = 1$. By Lemma 1 applied to

$$\lambda = q_1, \quad \varepsilon = \varepsilon_1, \quad \delta = \delta_1, \quad r = \frac{1}{2m_1}$$

and an arbitrary $t_1 \in \mathbb{R}$, there exist $c_1 \in \mathbb{C}$, $a_1 > 0$ and an eigenvalue satisfying (15) for $n = 1$. By [11, Theorems IV.2.14, 3.16], there exists γ_1 satisfying claim ii) for $n = 1$.

Now assume that for $j = 1, \ldots, n-1$ the constants c_j, t_j, a_j and γ_j have been constructed. We construct c_n, t_n, a_n and γ_n so that H_n satisfies i) and ii). We apply Lemma 1 to

$$\lambda = q_n, \quad \varepsilon = \varepsilon_n, \quad \delta = \delta_n, \quad r = \min\left\{\operatorname{dist}\bigl(\lambda, \sigma(H_{n-1})\bigr), \frac{1}{4m_n}\right\}.$$

In this way we obtain $c_n \in \mathbb{C}$ and $a_n > 0$ such that, for any $t \in \mathbb{R}$, the Schrödinger operator $-\Delta + U_{c_n,t,a_n}$ has an eigenvalue $\mu \in \sigma(-\Delta + U_{c_n,t,a_n}) \setminus \sigma(H_{n-1})$ with $\operatorname{Im} \mu < 0$ and

$$\|U_{c_n,t,a_n}\|_p < \varepsilon_n, \quad \|U_{c_n,t,a_n}\|_\infty < \delta_n, \quad |\mu - q_n| < \frac{1}{4m_n}.$$

Lemma 2 implies that, for $t_n := t$ sufficiently large, the operator $H_n = H_{n-1} + U_{c_n,t_n,a_n}$ has an eigenvalue μ_n with $\operatorname{Im} \mu_n < 0$, $|\mu_n - \mu| < 1/(4m_n)$ and hence $|\mu_n - q_n| < 1/(2m_n)$. This proves claim i), and claim ii) follows again from [11, Theorems IV.2.14, 3.16].

Finally we prove that the potential

$$V := \sum_{j=1}^{\infty} U_{c_j,t_j,a_j}$$

satisfies the claims of the theorem. By Minkowski's inequality and (14),

$$\max\{\|V\|_p, \|V\|_\infty\} < \sum_{j=1}^{\infty} \max\{\varepsilon_j, \delta_j\} \leq \frac{6\mathcal{E}}{\pi^2} \sum_{j=1}^{\infty} \frac{1}{j^2} = \mathcal{E}.$$

Moreover, for $n \in \mathbb{N}$ the $L^\infty(\mathbb{R}^d)$ norm of $U_n := \sum_{j=n+1}^{\infty} U_{c_j,t_j,a_j}$ is estimated as

$$\|U_n\|_\infty < \sum_{j=n+1}^{\infty} \delta_j \leq \frac{6\gamma_n}{\pi^2} \sum_{j=n+1}^{\infty} \frac{1}{j^2} < \gamma_n.$$

So the above claim ii) implies for $H_n + U_n = H$ that

$$\exists\, \lambda_n \in \sigma(H): \quad \big|\lambda_n - \mu_n\big| < \operatorname{dist}(\mu_n, [0,\infty)).$$

Hence $\operatorname{Im}\lambda_n < 0$ and

$$|\lambda_n - q_n| \le \big|\lambda_n - \mu_n\big| + |\mu_n - q_n| < \operatorname{dist}(\mu_n, [0,\infty)) + |\mu_n - q_n| < \frac{1}{m_n},$$

i.e. $\lambda_n \in B(q_n, \frac{1}{m_n})$, $n \in \mathbb{N}$. Now it is easy to see that every point in $[0,\infty)$, which is the closure of $\mathbb{Q} \cap (0,\infty)$, is an accumulation point of $\{\lambda_n : n \in \mathbb{N}\}$. □

3. Schrödinger operator in $L^2(\mathbb{R}^d_+)$

In this section we study Schrödinger operators on the half-space $\mathbb{R}^d_+$, and for the proof of Lemma 4 below also on the shifted half-space $\mathbb{R}^d_+ + te_d$ for some $t \in \mathbb{R}$. We fix an angle $\phi \in [0,\pi)$ which determines the Robin boundary condition. Throughout this section, every operator in $L^2(\mathbb{R}^d_+ + te_d)$ for some $t \in \mathbb{R}$ is assumed to have the operator domain

$$\big\{f \in W^{2,2}(\mathbb{R}^d_+ + te_d) : \cos(\phi)\partial_{x_d} f + \sin(\phi) f = 0 \text{ on } \partial(\mathbb{R}^d_+ + te_d)\big\},$$

and operators in $L^2(\mathbb{R})$ have domains $W^{2,2}(\mathbb{R})$.

The following result is almost the same as Lemma 1; note that here t is not arbitrary but needs to be sufficiently large, and the eigenvalue μ_t depends on t.

Lemma 3. *Let $\lambda \in (0,\infty)$ and $p > d$. For any $\varepsilon, \delta, r > 0$ there exist $a > 0$ and $c \in \mathbb{C}$ with*

$$\|U_{c,t,a}\|_p < \varepsilon, \quad \|U_{c,t,a}\|_\infty < \delta, \tag{16}$$

and such that, for every sufficiently large $t > 0$, the operator

$$-\Delta + \chi_{\mathbb{R}^d_+} U_{c,t,a} \quad \text{in} \quad L^2(\mathbb{R}^d_+)$$

has an eigenvalue μ_t with $\operatorname{Im}\mu_t < 0$ and $|\mu_t - \lambda| < r$.

For the proof we use the following result, which is the analogue of Lemma 2.

Lemma 4. *Let $V_1 \in L^\infty(\mathbb{R}^d_+)$, $V_2 \in L^\infty(\mathbb{R}^d)$ be decaying at infinity, and define the operators*

$$H_1 := -\Delta + V_1 \quad \text{in} \quad L^2(\mathbb{R}^d_+), \qquad H_2 := -\Delta + V_2 \quad \text{in} \quad L^2(\mathbb{R}^d).$$

Assume that there exists $\mu \in \sigma(H_2) \backslash \sigma(H_1)$. Then, for any $t > 0$, the operator

$$H_1 + \chi_{\mathbb{R}^d_+} V_2(\cdot - te_d) \quad \text{in} \quad L^2(\mathbb{R}^d_+)$$

has an eigenvalue μ_t with $\mu_t \to \mu$ as $t \to \infty$.

Proof. Define operators

$$H_{2,t} := -\Delta + \chi_{(\mathbb{R}^d_+ - te_d)} V_2 \quad \text{in} \quad L^2(\mathbb{R}^d_+ - te_d), \quad t > 0.$$

Note that

$$\sigma\big(H_1 + \chi_{\mathbb{R}^d_+} V_2(\cdot - te_d)\big) = \sigma\big(H_{2,t} + V_1(\cdot + te_d)\big), \quad t > 0. \tag{17}$$

Analogously as in the proof of Lemma 2, one can show that for every $z \in \mathbb{C}$ with $\operatorname{dist}(z, [0,\infty))$ sufficiently large, we have strong resolvent convergence

$$\big(H_{2,t} + V_1(\cdot + te_d) - z\big)^{-1} \xrightarrow{s} \big(H_2 - z\big)^{-1}, \quad t \to \infty,$$

and the same holds for the adjoint operators; note that here we use that every $f \in C_0^\infty(\mathbb{R}^d)$ belongs to $\mathcal{D}(H_{2,t})$ for all $t > 0$ so large that $\operatorname{supp} f \subset (\mathbb{R}^d_+ - te_d)$. Therefore, by [4, Theorem 2.3 i)], in the limit $t \to \infty$ the isolated eigenvalue

$\mu \in \sigma(H_2)\backslash\sigma(H_1)$ is approximated by points $\mu_t \in \sigma\big(H_{2,t} + V_1(\cdot + te_d)\big)$, $t > 0$, provided that

$$\mu \notin \sigma_{\mathrm{ess}}\big(\big(H_{2,t} + V_1(\cdot + te_d)\big)_{t>0}\big) \cup \sigma_{\mathrm{ess}}\Big(\big(\big(H_{2,t} + V_1(\cdot + te_d)\big)^*\big)_{t>0}\Big)^*.$$

Similarly as in the proof of Lemma 2, one may check that the set on the right is contained in $\sigma(H_1) = \sigma(H_1^*)^*$, and $\mu \notin \sigma(H_1)$ by the assumptions. This, together with (17), proves the claim. □

Proof of Lemma 3. First we return to the problem on the whole $\mathbb{R}^d$. By Lemma 1 applied to t, ε, δ and $r/2$, there exist $a > 0$ and $c \in \mathbb{C}$ such that $U_{c,t,a}$ satisfies (16), and so that the operator $-\Delta + U_{c,t,a}$ in $L^2(\mathbb{R}^d)$ has an eigenvalue μ (independent of t) with $\operatorname{Im}\mu < 0$ and $|\mu - \lambda| < r/2$. By Lemma 4 applied to $V_1 \equiv 0$, $V_2 = U_{c,0,a}$, for every $t > 0$ sufficiently large, the operator $-\Delta + \chi_{\mathbb{R}^d_+} U_{c,t,a}$ in $L^2(\mathbb{R}^d_+)$ has an eigenvalue μ_t with $\operatorname{Im}\mu_t < 0$ and $|\mu_t - \mu| < r/2$, hence $|\mu_t - \lambda| < r$. □

Now the proof of the main result is straight forward.

Proof of Theorem 2. We proceed analogously as in the proof of Theorem 1 but use Lemmas 3, 4 instead of Lemmas 1, 2. Note that here t_1 is not arbitrary but given (sufficiently large) by Lemma 3. □

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