Properties of Bounded Stochastic Processes Employed in Biophysics

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Abstract

Realistic stochastic modeling is increasingly requiring the use of bounded noises. In this work, properties and relationships of commonly employed bounded stochastic processes are investigated within a solid mathematical ground. Four families are object of investigation: the Sine-Wiener (SW), the Doering-Cai-Lin (DCL), the Tsallis-Stariolo-Borland (TSB), and the Kessler-Sørensen (KS) families. We address mathematical questions on existence and uniqueness of the processes defined through SDEs, which often conceal non-obvious behavior, and we explore the behavior of the solutions near the boundaries of the state space. The expression of the time-dependent probability density of the Sine-Wiener noise is provided in closed form, and a close connection with the Doering-Cai-Lin noise is shown. Further relationships among the different families are explored, pathwise and in distribution. Finally, we illustrate an analogy between the Kessler-Sørensen family and Bessel processes, which allows to relate the respective local times at the boundaries.

Keywords: Bounded noises; Stochastic Differential Equations; Strong Uniqueness; Local times; Transformations.

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1 Introduction

The dynamics of a number of phenomena of the physical world, especially in biology, are affected by intrinsic or extrinsic randomness, and in some cases by both. In biophysics and mathematical biology, the influence of extrinsic sources of stochasticity in otherwise deterministic biological systems is frequently taken into account by elementarily perturbing a deterministic system. Namely, the deterministic dynamical system that is adopted in the absence of the above-mentioned sources is often perturbed by adding stochastic fluctuations modeled with a Gaussian white noise or a colored Gaussian perturbation.

This approach frequently allows to make analytical or semi-analytical inferences. However, it can lead to artifacts, sometimes hidden. To give an example, as stressed in [1, 2, 3], modeling the extrinsic perturbations affecting an anti-tumor cytotoxic therapy by means of a white noise can allow the possibility that the therapy adds tumor cells instead of killing them, as a consequence of the unbounded stochastic fluctuations. The unboundedness of the perturbation implies a second more subtle but equally relevant artifact in the above model: the possibility of an excessive instantaneous killing of tumor cells.

Another important limitation is the fact that white noise perturbations only apply to parameters on which a system depends linearly, which severely limits their applicability. The Ornstein-Uhlenbeck noise is an alternative to white noise which does not require linear dependence; however, in many cases it is not a correct choice, *e.g.* see the models described in [2]. Such examples suggest that, in many applications, Gaussian noises should not be employed to model the real world randomness, due to their unboundedness. An alternative strategy which is becoming increasingly important [3] consists in modeling parametric perturbations by bounded noises: these allow to preserve the positiveness and boundedness of the perturbed parameters and can also be employed to model the fluctuations on which a system depends nonlinearly.

In the last two decades, a large literature has been devoted to the application of bounded stochastic processes to many scientific areas. For example: noise-induced transitions [4], stochastic resonance [5], Kramers problem [6], bifurcation theory [7], parametric resonance [8], fractional mechanics [9], nonlinear mechanics [10], chaotic systems [11], tumor biophysics [2, 12], cell biology [13], ecology [14], environmental sciences [15], interacting cellular populations [16], delayed systems [17], neurosciences [18], chemistry [19], and population genetics [20]. However, the best known and oldest example of bounded stochastic process is probably the dichotomous Markov noise, also known as telegraph noise [15]. This process is not continuous, thus it is optimally suited to model stochastic transitions of a system between two or more discrete states, as in the important case of gene activation/deactivation [21].

In order to realistically model continuous stochastic fluctuations of a parameter, continuous stochastic processes are needed. The simplest recipe to get a continuous bounded noise is to apply a continuous bounded function to any continuous stochastic process. This is the approach used to generate one of most widely employed bounded noises, the so called Sine-Wiener (SW) noise [22, 23]. Other popular families of bounded stochastic processes are the Doering-Cai-Lin (DCL) [24, 25], the Tsallis-Stariolo-Borland (TSB) [4] and the Kessler-Sørensen (KS) [26] families, which are generated by means of appropriate stochastic differential equations. Given the above-summarized increasing relevance of bounded stochastic processes, and since the vast majority of works on these classes of models are of heuristic nature, in the following we apply rigorous methods of stochastic analysis to investigate their properties and to make new analytical inferences of practical interest.

After introducing the above-mentioned families and their main properties (Section 2), in Sections 3 and 4 we investigate the well-posedness of the SDEs defining the TSB and DCL noises and the boundedness of their solutions, for different values of the relevant parameters. In Section 5 we obtain the analytical expression of the time-dependent density of the SW noise and hence assess the characteristic autocorrelation time, needed for the process to be considered stationary in practical applications. Sections 6 and 7 explore similarities and differences among the first three families, both in the strong (pathwise) and in the weak (in distribution) sense. In the last section, we show that the KS family can be obtained as a transformation of the DCL family, but that the relationship is not one-to-one. Uniqueness and boundedness of the SDE can be lost after the transformation, a fact that shares similarities with the theory of Bessel processes, as we shall show.

2 Different Families of Bounded Noises

Let us first set the basic notation used throughout. Given a filtered probability space $\big(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}\big)$, in this paper with the term Bounded Noise we denote a real stochastic process $X : \Omega \times [0\,, \infty) \to \mathbb{R}$ of nonzero finite quadratic variation which takes values on a bounded interval $I \subset \mathbb{R}$ with probability one:

$$\exists\, B > 0 \;:\; \mathbb{P}\big[\,|X_t| < B \;\; \forall\, t \geq 0\big] = 1\,. \tag{1}$$

For the sake of simplicity, we shall always rescale X so that $B = 1$. Throughout this work, we will denote by I the closed interval $[-1, 1]$ and by $\mathring{I}$ its interior, $\mathring{I} = (-1, 1)$.

Remark. We have included the condition on the quadratic variation to identify stochastic processes with a certain, quite canonical, level of roughness, so that it is reasonable to call them "noise". However, this definition does not aim to be comprehensive. It is equally meaningful to call noise other irregular processes with different levels of roughness, for instance bounded processes based on fractional Brownian motion in place of Brownian motion; in such a case the quadratic variation may be zero or infinite or could not exist, so the definition requires to be enlarged.

We now introduce different families of bounded noises generated via different methods, and concisely list their key properties. Proofs are provided in later sections.

2.1 *The Sine-Wiener Noise*

A first simple method to generate a bounded noise X is to apply a bounded deterministic function f to a stochastic process Y. In the recent literature of bounded noises [3, 23], the case where $f(y) = \sin(y)$ and the process Y is a rescaled Wiener process has mainly been considered:

$$X_t = \sin\left(\sqrt{\frac{2}{\tau}}\, W_t\right), \quad \tau > 0\,. \tag{2}$$

This bounded stochastic process has first been introduced by Dimentberg [22] and will be hereafter referred to as Sine-Wiener (SW) noise. The autocovariance function of the process (also termed un-normalized autocorrelation function – see Appendix A, also for the definition of characteristic autocorrelation time) can be computed from first principles. Its expression is as follows [23]:

$$R_{XX}(s,t) = \mathbb{E}[X_s X_t] = \frac{1}{2}\left(1 - \exp\left(-\frac{4s}{\tau}\right)\right) \cdot \exp\left(-\frac{t-s}{\tau}\right), \quad s < t\,. \tag{3}$$

Hence, the parameter τ is the characteristic autocorrelation time of the process. The stationary density of the Sine-Wiener noise is instead the following, shown in Figure 1:

$$p^{st}_{SW}(x) = \frac{1}{\pi\sqrt{1-x^2}}\,. \tag{4}$$

2.2 *The Doering-Cai-Lin Family*

Another way of generating bounded noises is by means of Stochastic Differential Equations (SDEs):

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t\,. \tag{5}$$

A summary on existence and uniqueness of the solutions of SDEs is provided in Appendix B. In this section, we concentrate on the case where the drift is linear and decreasing: $\mu(x) = -\alpha x$, $\alpha > 0$ [25]. In order to get a solution bounded in $I = [-1, 1]$,

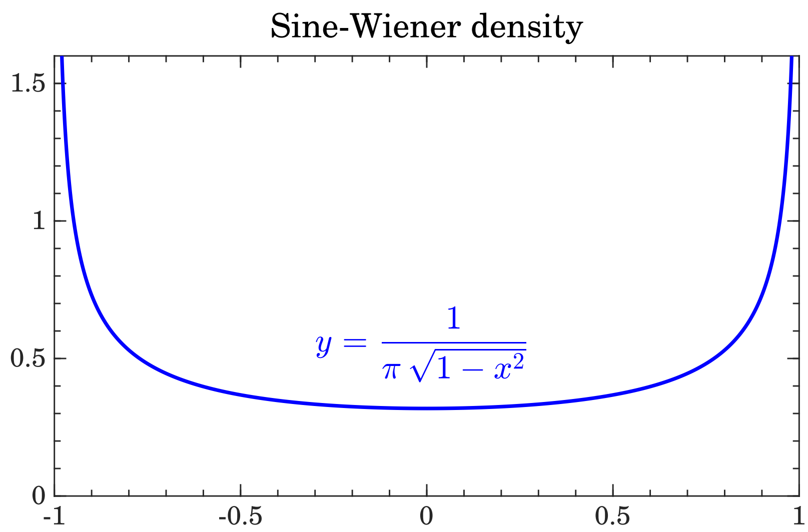

Figure 1: Stationary probability density (4) of the Sine-Wiener noise.

the diffusion σ must vanish at the boundaries. The most popular choice - introduced in [24, 27] and adopted in [25] - is the following:

$$\sigma(x) \propto \sqrt{1-x^2}\ . \tag{6}$$

Together with the introduction of appropriate parameters θ and δ, this yields the following family of SDEs which will be hereafter referred to as the Doering-Cai-Lin (DCL) family:

$$dX_t = -\frac{1}{\theta} X_t \, dt \; + \; \sqrt{\frac{1-{X_t}^2}{\theta(\delta+1)}} \, dW_t\,, \quad \theta > 0,\ \delta > -1\,. \tag{7}$$

For any value of δ and θ as above, and any initial condition $X_0 \in I = [-1, 1]$, this equation admits a unique strong solution bounded in I. The stationary probability density of the DCL family depends on the parameter δ only and reads as follows [24, 25]:

$$p^{st}_{\text{DCL}}(x) = Z^{-1}\left(1-x^2\right)^{\delta}, \quad Z = \frac{\sqrt{\pi}\,\Gamma(1+\delta)}{\Gamma(1.5+\delta)}\,. \tag{8}$$

Notice the transition from unimodality to bimodality when going from positive to negative values of δ (Figure 2). A closed analytical form of the autocovariance function of the DCL family is probably not available. However, as we shall prove in Section 4, the characteristic autocorrelation time is equal to the positive parameter θ.

2.3 *The Tsallis-Stariolo-Borland Family*

A different family of bounded noises, the Tsallis-Stariolo-Borland (TSB) family, can again be obtained as solution to a parametric SDE. In the case considered here, the diffusion

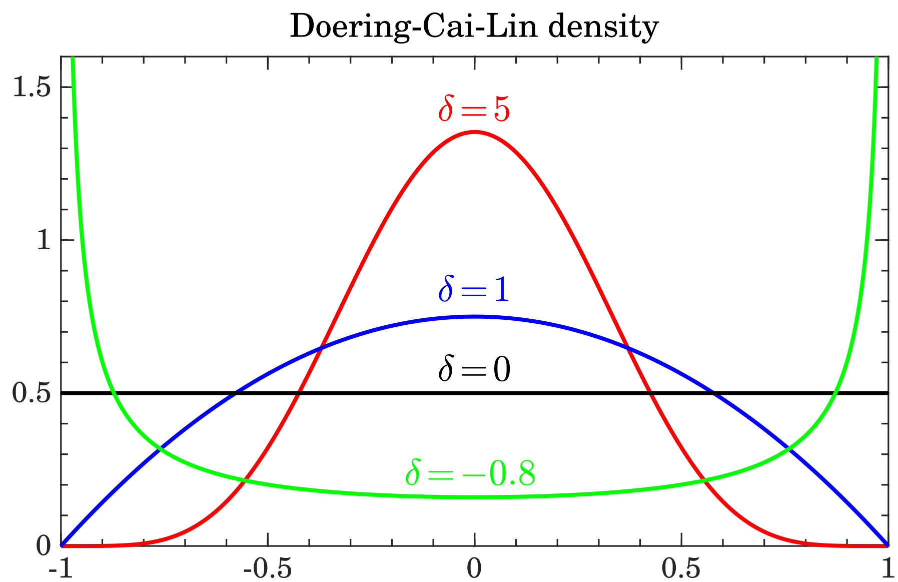

Figure 2: Stationary probability density (8) of the Doering-Cai-Lin family for different values of $\delta > -1$.

of the process is constant, while the drift tends to infinity (thus repelling the solution) as long as the boundaries of $\mathring{I} = (-1, 1)$ are approached [4]:

$$dX_t = -\frac{1}{\theta}\,\frac{X_t}{1 - {X_t}^2}\,dt \;+\; \sqrt{\frac{1-q}{\theta}}\;dW_t\,, \quad \theta > 0,\; q < 1\,. \tag{9}$$

Questions of existence, uniqueness and boundedness of the solution to the above SDE are strongly related to the particular value of q. In [28] it is shown that uniqueness and boundedness are lost for $q < 0$, and a physical interpretation of the phenomenon is provided. For $q \in [0, 1)$, instead, equation (9) admits a unique strong solution bounded in $\mathring{I}$ (as shown in Section 3), whose stationary density is given by

$$p^{st}_{\mathrm{TSB}}(x) = Z^{-1}\,(1 - x^2)^{\frac{1}{1-q}}\,, \qquad q \in [0, 1). \tag{10}$$

Notice that this has the same functional form of the DCL stationary density (8): here, however, only unimodal densities are allowed by the condition $q \in [0, 1)$, as Figure 3 shows. Moreover, an approximate formula is available for the characteristic autocorrelation time of the process [4]:

$$\tau_{\mathrm{TSB}} \simeq \frac{2}{5 - 3q}\,\theta\,. \tag{11}$$

We conclude by only noticing that, by choosing the drift of equation (9) proportional to $\tan(x)$, the following SDE is obtained, introduced in [26]:

$$dY_t = -\alpha\,\tan\left(\frac{\pi}{2}Y_t\right)\,dt \;+\; C\,dW_t\,, \quad \alpha, C > 0\,. \tag{12}$$

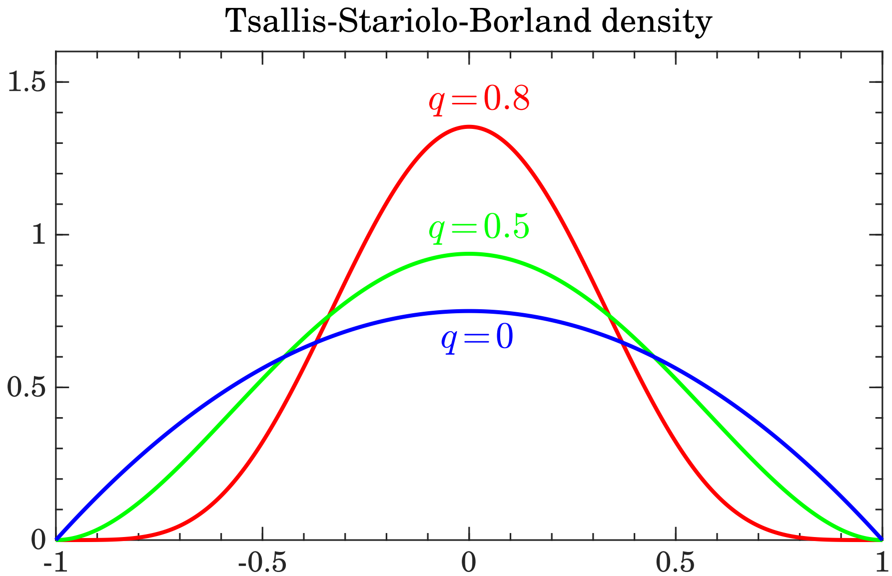

Figure 3: Stationary probability density (10) of the Tsallis-Stariolo-Borland family, for different values of $q \in [0, 1)$.

In Section 8 we recover this equation as a transformation of the DCL equation (7). The properties of the SDE and its solution will be explored for the different values of α.

3 Well-posedness and Boundedness of TSB Equation ($q \in [0, 1)$)

The SDEs (7) and (9) defining the DCL and the TSB families share an important feature: their coefficients do not satisfy Lipschitz conditions. Hence, questions such as their existence and uniqueness, as well as the boundedness of their solutions, need to be investigated. We consider here the case of TSB equation, and investigate in detail the DCL equation in Section 4. For the sake of self-containment, most of the stochastic tools employed in the the following analyses are summarized in Appendix C.

Theorem 3.1. *Suppose $q \in [0, 1)$ and set $\mathring{I} = (-1, 1)$. Then, for any $x_0 \in \mathring{I}$, there exists a unique, strong solution X_t of the TSB equation (9) with initial condition x_0. Moreover, the solution does never leave the interval $\mathring{I}$ with probability one.*

Proof. While the diffusion of (9) is extremely regular, the drift presents two asymptotes in ± 1 and hence is not Lipschitz on $\mathring{I}$. However, it is locally Lipschitz: together with the non-reachability of the boundaries, this will yield strong existence and uniqueness. Let us formalize the reasoning.

For small $\varepsilon > 0$, define μ_ε as the following continuous extension of μ outside the interval $(-1+\varepsilon, 1-\varepsilon)$:

$$\mu_\varepsilon(x) = \begin{cases} \mu(x) & \text{if } |x| \leq 1-\varepsilon \\ \mu(1-\varepsilon) & \text{if } x \geq 1-\varepsilon \\ \mu(-1+\varepsilon) & \text{if } x \leq -1+\varepsilon \end{cases} \tag{13}$$

Since μ_ε is Lipschitz on $\mathbb{R}$, the stochastic differential equation

$$dX_t = \mu_\varepsilon(X_t)dt + \sqrt{\frac{1-q}{\theta}}\, dW_t \tag{14}$$

has a unique (global in time) strong solution with initial condition x_0. Now take ε such that $|x_0| < 1-\varepsilon$ and denote by $X^{(\varepsilon)}$ the solution of (14) corresponding to such ε and with initial condition x_0. Moreover, set

$$T_\varepsilon = \inf\left\{ t \geq 0 \;\middle|\; |X_t^{(\varepsilon)}| \geq 1-\varepsilon \right\}.$$

Till the random time T_ε, the process $X^{(\varepsilon)}$ is also a solution of the original TSB equation (9). Hence, at least till T_ε, a solution to (9) exists and is unique (uniqueness is intrinsically a local problem). However, by means of the tools and results summarized in Appendix C, we now show that with probability one a solution of the TSB equation does never reach the endpoints of I, which will yield a unique global strong solution of (9).

The scale function of the TSB equation (9), as computed in [28], reads as follows:

$$s(x) = \int_0^x \left(1-z^2\right)^{-\frac{1}{1-q}} dz. \tag{15}$$

Set $\alpha = (1-q)^{-1}$: hence $\alpha \geq 1$ since $q \in [0,1)$. We have:

$$s(1) = \int_0^1 \frac{1}{(1-z^2)^\alpha} dz = \int_0^1 \frac{1}{(1+z)^\alpha (1-z)^\alpha} dz \geq 2^{-\alpha} \int_0^1 \frac{1}{(1-z)^\alpha} = \infty,$$

since $\alpha \geq 1$. Similarly, $s(-1) = -\infty$. Now, denote by T the first exit time from $\mathring{I}$ of the - locally well defined and unique - solution of the TSB equation: theorem C.3(i) assures that T is almost surely infinite. Hence, the locally unique strong solution of (9) starting from x_0 does never reach the endpoints of $\mathring{I}$ with probability one, which translates into the fact that the local strong solution is global in time, and it does never leave $\mathring{I}$. This completes the proof. □

The case $q < 0$ has been investigated in detail in [28], where it is shown that the solution X_t would in this case reach the endpoints ± 1 in finite time, almost surely. Notice that the result of Theorem 3.1 can be extended to any random initial condition X_0 with state space in $\mathring{I}$.

4 Properties of the DCL Noise

In this section we investigate questions concerning the strong existence, uniqueness and boundedness of the DCL equation (7). Some technical results provided in [29] will be exploited, but we shall discover in the end that the equation does not present irregular and unexpected behavior as in the case of the TSB equation. Although the sign of the parameter δ affects the behavior of the DCL trajectories near the boundaries ± 1, it has no consequences on the uniqueness of the SDE and on the boundedness of its solution.

4.1 *Strong Existence, Uniqueness and Boundedness*

Let us first extend the coefficients of the DCL SDE (7) to the whole real line in a bounded and Hölder-continuous way, where results of Appendix B can be applied. Once proven that the auxiliary equation has solutions which never leave the interval $I = [-1, 1]$, it will become clear that also the original SDE (7) has the desired properties, and that the particular extension of drift and diffusion outside I plays no role. Hence, let us consider

$$dX_t = \frac{1}{\theta}\,\mu^{(1)}(X_t)\,dt \;+\; \beta\,\sigma^{(1)}(X_t)\,dW_t\,, \tag{16}$$

where

$$\mu^{(1)}(x) = \begin{cases} 1 & \text{for } x \leq -1 \\ -x & \text{for } |x| \leq 1 \\ -1 & \text{for } x \geq 1 \end{cases} \tag{17}$$

and

$$\sigma^{(1)}(x) = \begin{cases} \sqrt{1-x^2} & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}\;. \tag{18}$$

The constant β is $\beta = [\theta\,(\delta+1)]^{-\frac{1}{2}}$, as in (7). This equation has the desired properties of existence and uniqueness.

Theorem 4.1. *For any given random variable X_0 with state space $I = [-1, 1]$, strong existence and uniqueness hold for the auxiliary equation (16) with initial condition X_0.*

Proof. The diffusion (18) is, on I, the product of two bounded 1/2-Hölder continuous functions ($\sqrt{1-x}$ and $\sqrt{1+x}$), and it is constant outside I. It is therefore 1/2-Hölder continuous itself, that is

$$|\sigma(x) - \sigma(y)|^2 \leq C\,|x-y| \quad \forall\, x, y \in \mathbb{R}\,, \tag{19}$$

for some constant $C > 0$ ($C = 2$ works). The same condition can be rewritten as

$$|\sigma(x) - \sigma(y)|^2 \leq \rho(|x - y|) \quad \forall\, x, y \in \mathbb{R}\,, \tag{20}$$

where the function $\rho(z) = Cz$ is such that $1/\rho$ in not integrable in any neighborhood of zero. Hence Proposition B.5 in Appendix applies (the drift (17) is trivially Lipschitz-continuous), which yields strong uniqueness for equation (16).

In order to prove the existence of a strong solution for the same equation, thanks to the Yamada-Watanabe theorem, it is enough to prove the existence of a weak solution. Now, since X_0 has support in $I = [-1, 1]$, for any choice of $m > 1$ we have

$$\mathbb{E}[\,|X_0|^{2m}] \leq 1 < \infty\,.$$

Moreover, both $\mu^{(1)}$ and $\sigma^{(1)}$ are bounded and continuous, as per their definition in (17),(18). Hence, Proposition B.6 ensures that a weak solution to (16) exists. Together with the already proved pathwise uniqueness, Yamada-Watanabe yields strong existence as well. This completes the proof.

□

In order to prove strong existence and uniqueness of the original DCL equation (7), we want to show that the unique strong solution of the slightly different equation (16) never leaves the interval I, where the coefficients of both equations coincide. The boundedness of the solution for positive values of δ might be deduced from the results of Section 4.2, where we show that in this case the trajectories do not even attain the boundaries ± 1. However, for the remaining values of δ the boundaries are reached, and proving directly that the solution is reflected to the interior of I is not straightforward.

Therefore, in order to provide a more unified treatment, in Theorem 4.2 we prove the boundedness of the DCL trajectories for all values of δ, positive and negative, by means of the so-called Comparison Theorem proposed in [29] (see Appendix, Proposition B.8). Then, in Section 4.2, we analyze the behavior of the DCL trajectories near the boundaries of its state space, for the different values of δ.

Theorem 4.2. *Let X_0 be a random variable taking values in $I = [-1, 1]$. Then a unique strong solution of the Doering-Cai-Lin SDE (7) with initial condition X_0 exists and satisfies:*

$$\mathbb{P}[X_t \in I \ \textit{for all} \ \ t \geq 0] = 1\,.$$

Proof. Let us first prove the boundedness of the auxiliary equation (16), by means of the comparison theorem in Appendix, Proposition B.8. To this end, let us denote by $X^{(1)}$ the unique strong solution (Theorem 4.1) of equation (16), with starting random variable $X_0^{(1)} = X_0$. Let us also denote by $\mu^{(1)}(x)$ the drift coefficient of such equation, as in (17).

Now consider $\mu^{(2)}(x)$ as follows

$$\mu^{(2)}(x) = \begin{cases} 1 & \text{for } x \leq -1 \\ -x & \text{for } -1 \leq x \leq 0 \\ 0 & \text{for } x \geq 0 \end{cases}, \qquad \mu^{(1)}(x) \leq \mu^{(2)}(x) \quad \forall\, x \in \mathbb{R}\,,$$

and $X_0^{(2)}$ the constant random variable $X_0^{(2)} \equiv 1$. With a proof similar to the one of Theorem 4.1, one shows that there exists a unique strong solution of the following system:

$$\begin{cases} dY_t &= \dfrac{1}{\theta} b^{(2)}(Y_t)\, dt \;+\; \beta\, \sigma(Y_t)\, dW_t \\ Y_0 &= X_0^{(2)} \end{cases}. \tag{21}$$

Indeed, notice that the only difference in the new system is the drift, which is however still Lipschitz-continuous and bounded as it is needed in the proof.

The process $X_t^{(2)} \equiv 1$ is a solution of (21), since $\mu^{(2)}(1) = \sigma(1) = 0$ and clearly $dX_t^{(2)} = 0$. Thus, $X^{(2)}$ is the unique solution of that system. As already done earlier in the proof, denote by $X^{(1)}$ the solution of (16). All the hypotheses of the Comparison Theorem B.8 hold ($X_0^{(1)} \leq X_0^{(2)}$ clearly holds since $|X_0^{(1)}| \leq 1$ by assumption) and we therefore get

$$\mathbb{P}\big[X_t^{(1)} \leq X_t^{(2)} \;\; \forall\, t \geq 0\big] = 1\,, \tag{22}$$

namely

$$\mathbb{P}\big[X_t^{(1)} \leq 1 \;\; \forall\, t \geq 0\big] = 1\,. \tag{23}$$

By means of a symmetrical reasoning, one shows that $X_t^{(1)} \geq -1$ holds with probability one as well. Thus, the unique strong solution $X_t^{(1)}$ of the auxiliary equation (16) with starting condition $X_0^{(1)} = X_0$ satisfies

$$\mathbb{P}\big[X_t^{(1)} \in I \;\; \forall\, t \geq 0\big] = 1\,.$$

Since the dynamics in I of the original DCL equation (7) coincides with that of equation (16), and since we have just proven that the unique solution $X_t^{(1)}$ to (16) starting in I never leaves the interval, it follows that $X_t \equiv X_t^{(1)}$ is also the unique strong solution of the DCL equation (7), and that X_t is confined in I. This completes the proof. □

4.2 *Behavior near the Boundaries*

It has just been shown that the unique solution of SDE (7) starting in $I = [-1, 1]$ remains confined in I, independently of the value of δ (provided of course $\delta > -1$). Whether the solution reaches the boundaries ± 1 depends however on the value of the parameter δ, as we are going to show.

First, however, a comparison with the TSB case is crucial. In that case, as shown in [28], attaining the boundaries ($q < 0$) has major consequences on the uniqueness and boundedness of the solution. In the case of the DCL SDE, instead, both the drift and the diffusion of the SDE are well defined at ± 1, and the unique solution of the SDE remains bounded for any value of δ and θ (Theorem 4.2), even if the boundaries are attained. However, knowing whether ± 1 are attainable is of interest in itself as well as in applied contexts, and will also be used in later sections of this work (*e.g.* Section 8).

In order to study the behavior of the noise near the boundaries of the state space, results and tools summarized in Appendix C are employed. The scale function of the DCL noise can easily be obtained by substituting the drift and diffusion of equation (7) into expression (C.2). One gets:

$$s(x) = \int_0^x \frac{1}{\left(1 - z^2\right)^{\delta+1}}\, dz\,. \tag{24}$$

It can be noticed that this has the same functional form of the scale function of the TSB noise for $q \in [0, 1)$, as in (15).

Theorem 4.3. *Let* $\mathring{I} = (-1, 1)$ *and* X_t *be the unique solution of the DCL SDE (7) with initial condition* $X_0 \in \mathring{I}$*. Then:*

(i) If $\delta \geq 0$*, the solution* X_t *will a.s. never reach the endpoints* ± 1

(ii) If $-1 < \delta < 0$*, the solution* X_t *will a.s. reach one of the endpoints* ± 1 *in finite time.*

Proof. To prove the statement in the case $\delta \geq 0$, it is enough to observe that, in this case,

$$s(\pm 1) = \pm \infty\,, \tag{25}$$

where $s(x)$ is as in (24). The claim then follows from Theorem C.3(i).

As far as the case $-1 < \delta < 0$ is concerned, we have

$$|s(\pm 1)| < \infty\,. \tag{26}$$

This is not enough to conclude that the endpoints of I are reached in finite time, and one needs to resort to Theorem C.6, involving the speed measure of the noise and the function $v(x)$ as introduced in Definitions C.4 and C.5. A simple calculation reveals that the speed measure associated with (7) is as follows:

$$m(dy) = \frac{2dy}{\sigma^2(y)\, s'(y)} = C \left(1 - y^2\right)^{\delta} dy\,. \tag{27}$$

Hence, neglecting for simplicity the constant C, the function $v(x)$ in (118) satisfies:

$$
\begin{aligned}
v(1) &= \int_0^1 \big(s(1) - s(y)\big)\, m(dy) \\
&= \int_0^1 \left(\int_y^1 \left(1 - z^2\right)^{-(\delta+1)} dz \right) \left(1 - y^2\right)^{\delta} dy \\
&\leq \int_0^1 \left. \frac{(1-z)^{-\delta}}{-\delta} \right|_1^y \left(1 - y^2\right)^{\delta} dy && (28) \\
&= -\frac{1}{\delta} \int_0^1 (1-y)^{-\delta} \left(1 - y^2\right)^{\delta} dy && (29) \\
&= -\frac{1}{\delta} \int_0^1 (1+y)^{\delta}\, dy < \infty
\end{aligned}
$$

The hypotheses $\delta + 1 > 0$ and $\delta < 0$ have been exploited, respectively, in (28) and (29). Being $v(x)$ an even function, we also have

$$
v(-1) = v(1) < \infty \,. \tag{30}
$$

By Theorem C.6(a), the time to reach one of the boundaries ± 1 is almost surely finite. This proves the second point of the statement, and completes the proof. □

5 Time-dependent Density of the SW Noise

Differently from the case of the TSB and the DCL noises, both defined as solutions to an SDE, the SW noise (2) has an explicit analytic expression. We recall it below:

$$
X_t = \sin\left(\sqrt{\frac{2}{\tau}}\, W_t \right), \quad \tau > 0 \,. \tag{31}
$$

The form of the SW stationary density is provided in Section 2, equation (4). Here, we derive the time-dependent density of the noise, from which the stationary density can be recovered as limit as t tends to infinity. For convenience, we separately recall the following elementary result of probability theory.

Lemma 5.1 (Transformation of densities)**.** *Let I, $J \subseteq \mathbb{R}$ be two real intervals, $Z \colon \Omega \to I$ a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $F \colon I \to J$ a diffeomorphism of real intervals. If $\rho_Z \colon I \to [0, +\infty)$ is the density of Z, then the random variable $X = F(Z)$ has density $\rho_X \colon J \to [0, +\infty)$ given by*

$$
\rho_X(x) = \frac{\rho_Z\left(F^{-1}(x)\right)}{\left|F'\left(F^{-1}(x)\right)\right|} \,. \tag{32}
$$

Proposition 5.2. *Let $W\colon \Omega \to \mathbb{R}$ be a real random variable with density $\rho_{_W}$, and set $X = \sin(W)$. Then, the density of X is given by:*

$$\rho_{_X}(x) = \frac{\rho_{_Z}(\,\arcsin(x)\,)}{\sqrt{1-x^2}}\,, \tag{33}$$

where

$$\rho_{_Z}(z) = \sum_{k=-\infty}^{\infty} \rho_{_W}\Big((-1)^k z + k\pi\Big)\,. \tag{34}$$

Proof. Let us first define a random variable

$$Z\colon \Omega \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \tag{35}$$

as follows. For each $\omega \in \Omega$, consider the only $z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin(z) = \sin(W(\omega))$; hence define $Z(\omega) = z$. Thus we have:

$$X(\omega) = \sin(W(\omega)) = \sin(Z(\omega)) \quad \forall\, \omega \in \Omega\,. \tag{36}$$

The density of Z can be written as

$$\rho_{_Z}(z) = \sum_{w\in G(z)} \rho_{_W}(w)\,, \tag{37}$$

where $G(z)$ is the set of all $w \in \mathbb{R}$ such that $\sin(w) = \sin(z)$, $z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. These are:

- $w = z + 2k\pi$, $k \in \mathbb{Z}$ (in the first or fourth quadrant);
- $w = (\pi - z) + 2k\pi = -z + (2k+1)\pi$, $k \in \mathbb{Z}$ (in the second or third quadrant).

With a unifying expression,

$$G(z) = \Big\{(-1)^k z + k\pi \;\big|\; k \in \mathbb{Z}\Big\}\,. \tag{38}$$

Thus, the density of Z takes the form (34). Since the function $F(z) = \sin(z)$ is a diffeomorphism between the intervals $[-\pi/2, \pi/2]$ and $[-1, 1]$, Lemma 5.1 applied to $X = F(Z)$ yields

$$\rho_{_X}(x) = \frac{\rho_{_Z}(\,\arcsin(x)\,)}{|\cos(\arcsin(x))|} = \frac{\rho_{_Z}(\,\arcsin(x)\,)}{\sqrt{1-x^2}}\,,$$

as it was to be proved. □

Remark. Notice that equation (34) can be written as

$$\rho_{_Z}(z) = \psi(z) + \psi(\pi - z)\,, \tag{39}$$

where

$$\psi(z) = \sum_{k=-\infty}^{\infty} \rho_{_W}\,(z + 2k\pi)\;. \tag{40}$$

We will use this later in this section.

Proposition 5.2 can easily be applied to the case where W is a Gaussian random variable, in order to obtain an explicit expression of the time-dependent density of the SW noise. The following Jacobi theta function allows us to write the SW density in a compact and elegant form.

Definition 5.3. The function

$$\vartheta_3(z,q) = \sum_{k=-\infty}^{\infty} q^{k^2} \exp\left(2\, ikz\right) \tag{41}$$

is the third version of the Jacobi theta function, where $z \in \mathbb{C}$, $q \in \mathbb{R}$ and $|q| < 1$ [30].

Theorem 5.4. *The time-dependent density of the Sine-Wiener noise (31) has the following form:*

$$p_{SW}(x,t) = \frac{\vartheta_3\left(\frac{z}{2}, e^{-t/\tau}\right) \,+\, \vartheta_3\left(\frac{\pi - z}{2}, e^{-t/\tau}\right)}{2\pi\sqrt{1-x^2}}\,, \tag{42}$$

where $z = \arcsin(x)$.

Proof. At time t, the SW noise (31) is the random variable

$$X = \sin\left(W\right)\,, \quad W \sim N\left(0, 2t/\tau\right)\,. \tag{43}$$

For convenience of notation, let us set $\sigma^2 = 2t/\tau$. Equation (33) of Proposition 5.2 and equation (39) then yield

$$p_{\mathrm{sw}}(x,t) = \frac{1}{\sqrt{1-x^2}}\left[\psi_{\sigma^2}(z) + \psi_{\sigma^2}(\pi - z)\right]\,, \tag{44}$$

where $z = \arcsin(x)$ and, according to (40),

$$\psi_{\sigma^2}(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \sum_{k=-\infty}^{\infty} \exp\left(-\frac{\left(z+2k\pi\right)^2}{2\sigma^2}\right). \tag{45}$$

The expression for ψ_{σ^2} can be linked to the Jacobi ϑ_3 function by means of the Poisson transform, as shown in formula (4.4) of [31]. This reads:

$$\sqrt{\pi a} \sum_{k=-\infty}^{\infty} \exp\left(-a(u+k\pi)^2\right) = \sum_{k=-\infty}^{\infty} \exp\left(-\frac{k^2}{a}\right) \exp(2\, iku)\,, \quad a > 0\,. \tag{46}$$

This last equation for $u = z/2$ reads as follows:

$$2\pi\sqrt{\frac{a/4}{\pi}} \sum_{k=-\infty}^{\infty} \exp\left(-\frac{a}{4}(z+2k\pi)^2\right) = \vartheta_3\left(\frac{z}{2}, e^{-1/a}\right)\,. \tag{47}$$

Setting $4/a = 2\sigma^2$ and recalling equation (45) yields

$$2\pi\,\psi_{\sigma^2}(z) = \vartheta_3\left(\frac{z}{2},\, e^{-\sigma^2/2}\right) . \tag{48}$$

Substituting this expression for ψ_{σ^2} back into equation (44), and recalling that $\sigma^2 = 2t/\tau$, leads to equation (42) and completes the proof.

□

The analytic expression of $p_{\text{SW}}(t,x)$ in (42) can also be used to write down the stationary density of the noise and to assess its characteristic autocorrelation time. If time t tends to infinity, both addends of the numerator of (42) tend to the constant one: indeed,

$$\lim_{q\to 0}\vartheta_3(z,q) = 1 \quad \forall\, z \in \mathbb{C}\,,$$

as it can be seen by definition (41). Thus, the stationary density of the SW noise is

$$p^{st}_{\text{SW}}(x) = \lim_{t\to\infty} p_{\text{SW}}(t,x) = \frac{1+1}{2\pi\sqrt{1-x^2}} = \frac{1}{\pi\sqrt{1-x^2}}\,, \tag{49}$$

as stated in (4). Moreover, the convergence takes place at the same speed at which the exponential function $e^{-t/\tau}$ attains zero (again by definition in (41), one sees that the ϑ_3 function tends to 1 linearly in q, as q tends to 0). Hence, by definition, τ is the characteristic autocorrelation time of the process: the full expression of its autocovariance has been presented in (3).

6 Relationship between SW and DCL

The stationary density (49) of the SW noise is the same as the one of the DCL noise – equation (8) – in the case $\delta = -1/2$. This raises the question of whether the two noises, SW and DCL with $\delta = -1/2$, share further properties. To investigate this, it appears convenient to apply Itô's formula in order to find the SDE satisfied by the SW noise, and compare this to the one of the DCL case with $\delta = -1/2$.

Lemma 6.1. *The SW process X_t in (31) satisfies the following SDE:*

$$dX_t = -\frac{1}{\tau}\,X_t\,dt + \sqrt{\frac{2}{\tau}}\,\sqrt{1-{X_t}^2}\;\text{sgn}\left(\cos\left(\sqrt{2/\tau}\,W_t\right)\right)\,dW_t\,. \tag{50}$$

Proof. Let us apply Itô's lemma to the process $X_t = F(Z_t)$, where $F(z) = \sin z$ and $Z_t = \sqrt{2/\tau}\,W_t$. We have:

$$\begin{cases} F'(Z_t) \;=\; \cos(Z_t) = \text{sgn}\left(\cos(Z_t)\right)\sqrt{1-F(Z_t)^2} \\ F''(Z_t) \;=\; -\sin(Z_t) = -F(Z_t) \end{cases} \tag{51}$$

and therefore

$$
\begin{aligned}
d\,X_t &= F'(Z_t)\,dZ_t \;+\; \frac{1}{2}\,F''(Z_t)\,\sigma_z^2\,dt \\
&= \operatorname{sgn}\left(\cos\left(\sqrt{2/\tau}\,W_t\right)\right)\,\sqrt{1-{X_t}^2}\,\sqrt{\frac{2}{\tau}}\,dW_t \;+\; \frac{1}{2}\,(-X_t)\;\frac{2}{\tau}\,dt\,. \qquad (52)
\end{aligned}
$$

This coincides with expression (50). □

The SDE (50) is very similar to the one of the DCL noise in (7) for $\delta=-1/2$, which we rewrite for convenience:

$$
dX_t = -\frac{1}{\tau}\,X_t\,dt \;+\; \sqrt{\frac{2}{\tau}}\,\sqrt{1-{X_t}^2}\,dW_t\,, \quad \tau>0\,. \qquad (53)
$$

The drift is the same in both cases, while the two diffusions differ by the factor $\operatorname{sgn}(\cos(\sqrt{2/\tau}\,W_t))$. However, being the square of this factor constantly equal to one, the two processes share the same Fokker-Planck equation: this yields the same time-dependent density for the two processes, if the initial distribution is the same. In fact, this yields the same overall law for the two processes, as we show in Theorem 6.2.

Theorem 6.2. *The SW noise (31) and the DCL noise with $\delta=\frac{1}{2}$ in (53) are the same process in distribution.*

Proof. Let us define the process

$$
\widetilde{W}_t = \int_0^t \operatorname{sgn}\left(\cos\left(\sqrt{2/\tau}\,W_s\right)\right)\,dW_s\,, \qquad (54)
$$

so that the SW SDE (50) can be rewritten as

$$
dX_t = -\frac{1}{\tau}\,X_t\,dt \;+\; \sqrt{\frac{2}{\tau}}\,\sqrt{1-{X_t}^2}\,d\widetilde{W}_t\,. \qquad (55)
$$

We can exploit the classical martingale characterization of Brownian Motion due to Paul Lévy [32], [33, Theorem 3.16], to show that $\widetilde{W}$ is still a Brownian Motion. Indeed, $\widetilde{W}_t$ is a (continuous) martingale, since it is the stochastic integral of a bounded function [29, 33] and its quadratic variation can be computed as:

$$
[\,\widetilde{W},\,\widetilde{W}\,]_t = \int_0^t \left[\operatorname{sgn}\left(\cos\left(\sqrt{2/\tau}\,W_s\right)\right)\right]^2 ds = \int_0^t 1\,ds = t\,. \qquad (56)
$$

$\widetilde{W}_t$ is therefore a continuous martingale with quadratic variation equal to t: thanks to Lévy's characterization, it is a Brownian Motion.

Equations (53) and (55) therefore represent the same SDE, only driven by different Brownian Motions. Hence, their unique strong solutions (Theorem 4.2) have the same law, as it was to be proved.

□

As a consequence of Theorem 6.2, any property related to the law of the SW process also holds for the DCL process X_t with $\delta = -1/2$ and $X_0 = 0$. For example, it follows that the autocovariance function of X_t is given by expression (3) and its characteristic autocorrelation time by τ (called θ in the original SDE (7)). This last property holds in more generality, for all values of δ.

Theorem 6.3. *The autocovariance function R_{XX} of the Doering-Cai-Lin noise (7), starting from any symmetric $X_0 \in [-1, 1]$, satisfies the following inequality for all $\delta > -1$:*

$$| \, R_{XX}(s,t) \, | \leq \exp\left(-\frac{t-s}{\theta}\right) \qquad \forall \, s \leq t \in [0, \infty) \, . \tag{57}$$

Hence, the characteristic autocorrelation time of the process is equal to the positive parameter θ.

Proof. Let us denote the diffusion coefficient of the DCL equation (7) by $\sigma(x)$, $\sigma(x) \propto \sqrt{1-x^2}$. For fixed $s \leq t$, we can write equation (7) in its integral form starting from time s, and then multiply both sides by X_s. We obtain:

$$X_t X_s = {X_s}^2 \, + \int_s^t -\frac{1}{\theta} X_r X_s \, dr \, + \int_s^t \sigma(X_r) X_s \, dW_r \, . \tag{58}$$

Since both the diffusion $\sigma(\cdot)$ and the DCL process X_t are bounded, the stochastic integral on the right-hand side of (58) has zero mean. Thus, by taking the expectation of both sides of (58), we get

$$R_{XX}(s,t) := \mathbb{E}\left[X_s X_t\right] = R_{XX}(s,s) \, + \int_s^t -\frac{1}{\theta} \, R_{XX}(s,r) \, dr \, . \tag{59}$$

Let us now set $F(t) = R_{XX}(s,t)$ as a function of t only, for fixed s. Equation (59) represents the integral form of the ODE

$$F'(t) = -\frac{1}{\theta} \, F(t) \, , \tag{60}$$

which yields

$$F(t) = F(s) \, \exp\left(-\frac{t-s}{\theta}\right) \, . \tag{61}$$

Now simply observe that

$$|F(s)| = \left| \, \mathbb{E}\left[{X_s}^2\right] \, \right| \leq 1 \tag{62}$$

since the process X_s is itself bounded by 1. The claim immediately follows by taking the absolute value of expression (61). □

7 Relationship between DCL and TSB

Driven by the result of Theorem 6.2, it is natural to investigate whether similar relationships can be found between the DCL noise X and the TSB noise Y. We do not simply investigate here whether the two processes have the same law; we address the more general question of whether a C^2 bijection F exists, such that the transformed process $F(X)$ and the process Y have the same law. To this aim, we preliminary relate the trajectories of $F(X)$ and Y. Hence, in general, let

$$dX_t = a(X_t)\,dt \,+\, c(X_t)\,dW_t \tag{63}$$

and

$$dY_t = \alpha(Y_t)\,dt \,+\, \gamma(Y_t)\,dW_t \tag{64}$$

be two stochastic differential equations, each admitting a unique strong solution bounded in $I = [-1, 1]$. Let also $F \in C^2(I, I)$ be a bijection of I. Thanks to Itô's formula, if $F(X_t) = Y_t$ holds, then F represents a solution to the following system of ordinary differential equations:

$$\frac{1}{2}\,F''(x)\,c^2(x) + F'(x)\,a(x) = \alpha\left(F(x)\right) \tag{65}$$

$$F'(x)\,c(x) = \gamma\left(F(x)\right)\,. \tag{66}$$

Proposition 7.1 then follows.

Proposition 7.1. *Let X_t and Y_t denote the DCL and the TSB processes with initial conditions $X_0, Y_0 \in \mathring{I}$. There exists no bijection $F \in C^2(\mathring{I}, \mathring{I})$ such that, almost surely, $Y_t = F(X_t)$ for all $t \geq 0$.*

Proof. Suppose by contradiction such an F exists. Then, the two ODEs (65) and (66) hold, where

$$a(x) = -\frac{1}{\theta}\,x\,, \qquad c(x) = \beta\sqrt{1-x^2}\,, \tag{67}$$

$$\alpha(y) = -\frac{1}{\theta}\,\frac{y}{1-y^2}\,, \quad \gamma(y) = \eta, \tag{68}$$

and

$$\eta^2 = \frac{1-q}{\theta}\,, \qquad \beta^2 = \frac{1}{\theta\,(\delta+1)}\,. \tag{69}$$

Compare indeed with the coefficients of the SDEs (7) and (9). Equation (66) reads

$$F'(x)\,\beta\sqrt{1-x^2} = \eta\,, \tag{70}$$

which yields

$$F(x) = F(0) + \frac{\eta}{\beta} \arcsin(x)\,. \tag{71}$$

Since F is a bijection of the interval $\mathring{I} = (-1, 1)$, we immediately deduce that $F(0) = 0$ and $\eta/\beta = 2/\pi$:

$$F(x) = \frac{2}{\pi} \arcsin(x)\,. \tag{72}$$

Such a function F does not however satisfy the second-order linear ODE (65): after simplifying, this indeed reads

$$\frac{C}{\pi} \frac{x}{\sqrt{1-x^2}} = \frac{\pi/2\ \arcsin(x)}{\pi^2/4\ -\ \arcsin^2(x)}\,, \tag{73}$$

where $C = (2\delta + 1)/(\delta + 1)$: it can be checked that equation (73) cannot hold for all $x \in \mathring{I}$ (for example by checking that the ratio between the two sides of (73) has non-zero derivative). This completes the proof. □

Proposition 7.1 excludes coincidence of the two processes $F(X)$ and Y. This, however, still allows the possibility that the laws of the two processes coincide, although the two processes have different paths. The following result excludes this case, hence showing that both the dynamics and the law of the DCL and TSB processes are intrinsically different, even after a transformation that is smooth in the interior of I.

Theorem 7.2. *Let X and Y denote the DCL and the TSB process, respectively. There is no bijection $F \in C^2(\mathring{I}, \mathring{I})$ such that $F(X)$ and Y share the same law.*

Proof. Let us denote by $a(x)$ and $c(x)$ the drift and the diffusion of the DCL noise X, and by $\alpha(y)$ and $\gamma(y)$ the drift and the diffusion of the TSB noise Y. Explicit expressions are provided in (67), (68), (69). Define $\widetilde{Y}_t := F(X_t)$. The process $\widetilde{Y}$ satisfies an SDE with drift

$$\widetilde{\alpha}(y) = \frac{1}{2} F''(x)\, c^2(x) + F'(x)\, a(x)\Big|_{x=F^{-1}(y)} \tag{74}$$

and diffusion

$$\widetilde{\gamma}(y) = F'(x)\, c(x)|_{x=F^{-1}(y)}\,. \tag{75}$$

The solutions to this SDE and to the SDE defining Y_t have the same law if and only if

$$\widetilde{\alpha}(y) = \alpha\,(y) \tag{76}$$

$$\widetilde{\gamma}^{\,2}(y) = \gamma^2(y) \tag{77}$$

(see [34]). Equation (77) reads

$$\left(F'(x)\right)^2 \beta^2 (1 - x^2) = \eta^2\,. \tag{78}$$

This means that at every $x \in \mathring{I}$ we must have

$$F'(x) = -\frac{\eta}{\beta\sqrt{1-x^2}} \quad \text{or} \quad F'(x) = +\frac{\eta}{\beta\sqrt{1-x^2}}\,, \tag{79}$$

the sign possibly depending on the point x. However, the function F' is continuous on $\mathring{I}$ and it never vanishes, hence it is either always positive or always negative, according to whether the bijection F is increasing or decreasing. In each of the two cases, the final part of the proof of Proposition 7.1 applies. Either choice of sign in (79) leads to a solution F which does not satisfy the second order ODE (76), which indeed in both cases reads as in (73). By contradiction, this completes the proof. □

8 The Kessler-Sørensen SDE

8.1 *KS as Transformation of DCL* ($\delta \geq 0$)

In Section 7 we have shown that any deterministic transformation of the DCL noise cannot have the same law as the TSB noise. However, imposing that the transformed noise satisfies an SDE with constant diffusion (as the TSB noise does) yields to the SDE (12), whose properties are investigated in this Section. The hypothesis $\delta \geq 0$ plays a crucial role in the following result.

Proposition 8.1. *Let X_t be the DCL noise where $\delta \geq 0$, and define $Y_t = F(X_t)$, where F is a C^2 bijection of the open interval $\mathring{I} = (-1, 1)$. If Y_t satisfies an SDE with constant diffusion, then*

$$F(x) = \frac{2}{\pi}\arcsin(x) \tag{80}$$

and the SDE is

$$dY_t = -\frac{f}{\pi\theta}\,\tan\left(\frac{\pi}{2}\,Y_t\right)dt + \frac{2}{\pi\,\sqrt{\theta\,(\delta+1)}}\,dW_t, \tag{81}$$

where θ is positive and the quantity $f = f(\delta)$ reads as follows:

$$f = \frac{2\delta+1}{\delta+1}. \tag{82}$$

We will refer to equation (81) as to the Kessler-Sørensen SDE [26].

Proof. Let us consider the DCL noise X_t with associated drift $a(x)$ and diffusion $c(x)$ as follows:

$$a(x) = -\frac{1}{\theta}\,x\,, \quad c(x) = \beta\,\sqrt{1-x^2}\,, \tag{83}$$

where

$$\beta^2 = \frac{1}{\theta\,(\delta+1)}\,. \tag{84}$$

The drift $\alpha(y)$ and diffusion $\gamma(y)$ of the process Y_t satisfy equations (65) and (66): Itô's formula indeed applies since Theorem 4.3(i) guarantees that in the case $\delta \geq 0$ the process takes values in the open interval $\mathring{I}$, where the function F is C^2 by hypothesis. By imposing that the diffusion $\gamma(y)$ be a constant C, equation (66) reads

$$F'(x)\sqrt{1-x^2} = \frac{C}{\beta}\,. \tag{85}$$

By imposing $F(0) = 0$ and $C/\beta = 2/\pi$ in order for the range of F to be $\mathring{I} = (-1, 1)$, this gives

$$F(x) = \frac{2}{\pi}\arcsin(x) \quad \text{ and } \quad \gamma(y) \equiv C = \frac{2\,\beta}{\pi}\,. \tag{86}$$

These expressions coincide with the function F in (80) and with the diffusion of SDE (81). It only remains to show that the drift $\alpha(y)$ of the SDE satisfied by Y_t is the one of equation (81): for this, we can use the second order ODE (65). A simple substitution of a, c, F, F' and F'' from (83) and (86) leads to:

$$\alpha\left(\frac{2}{\pi}\arcsin(x)\right) = -\frac{f}{\pi\theta}\,\frac{x}{\sqrt{1-x^2}}\,, \qquad \text{where } \ f = \frac{2\delta+1}{\delta+1}\,. \tag{87}$$

Hence,

$$\alpha(y) = -\frac{f}{\pi\theta}\,\frac{\sin(\pi y/2)}{|\cos(\pi y/2)|} = -\frac{f}{\pi\theta}\,\tan\left(\frac{\pi}{2}\,y\right)\,, \tag{88}$$

since $\cos(\phi) > 0$ if $\phi \in (-\pi/2, \pi/2)$. This completes the proof. □

In Proposition 8.1 we assumed $\delta \geq 0$, hence the process $Y_t = F(X_t)$ does not attain ± 1. In this case, equation (81) has not only strong existence, but also pathwise uniqueness, since the coefficients of the SDE are locally Lipschitz in the open interval $(-1, 1)$: see Section 3 for the proof of the same result applied to the TSB equation. The scenario for the $\delta < 0$ case is instead different and is investigated in the next section.

8.2 *Non-uniqueness after Transformation* ($\delta < 0$)

If $\delta < 0$, the DCL process X attains the boundaries ± 1 almost surely; hence, Itô's formula cannot be applied to the function F in (80) since this is not C^2 on the closed interval $I = [-1, 1]$. Nonetheless, one may still explore the behavior of the SDE (81) for negative values of δ, acknowledging that this may not be the same SDE obtained after transforming the SDE (7) through F. A computation of the scale function $s(y)$ and the speed measure $m(dy)$ associated to the SDE (81) reveals that

$$s(y) = \int_0^y \left(\cos\left(\frac{\pi}{2}z\right)\right)^{-(2\delta+1)} dz \tag{89}$$

and

$$m(dy) \propto \left(\cos\left(\frac{\pi}{2}y\right)\right)^{2\delta+1} dy\,. \tag{90}$$

Hence, with the tools summarized in Appendix C, one sees that for

$$-\frac{1}{2} < \delta < 0\,, \tag{91}$$

i.e. $0 < f(\delta) < 1$, the solution to (81) attains the boundaries with probability one. Following the lines of [28], uniqueness and boundedness are lost. Notice that the same conclusion holds true even more in the case

$$-1 < \delta \leq -\frac{1}{2}\,, \tag{92}$$

where the coefficient $f(\delta)$ appearing in the drift of (81) is negative (thus even the deterministic force alone actively drives the noise towards the boundaries).

On the other hand, recall that, under either of conditions (91) or (92), strong uniqueness holds for the DCL SDE (7), and the solution X_t does attain the boundaries ± 1. It is then possible to investigate which SDE the transformed process $Y_t = F(X_t)$ satisfies in each of the two cases, where F is as in equation (80). In analogy with the theory of Bessel processes, we think that under condition (91) the process Y_t still satisfies the same SDE (81) as in the case $\delta > 0$, while under condition (92) the process satisfies a more difficult equation in which a local time appears. We do not enter here all the details of this more difficult case, which involves the computation of the local time of X_t, before Itô-Tanaka's formula for non-smooth functions can be applied. In Section 8.3, we explain the connection with Bessel processes, and clarify the heuristic reasons behind our considerations when $\delta < 0$.

Notice, however, that the above provides an explicit example of an interesting stochastic phenomenon. Indeed, if

$$-\frac{1}{2} < \delta < 0\,, \tag{93}$$

we saw that equation (7) has strong existence and uniqueness, while uniqueness is lost for the transformed SDE (81). Boundedness is lost as well after the transformation, as pointed out immediately after equation (91). However, by calling X_t the unique strong solution of (7), the process

$$Y_t = \frac{2}{\pi}\arcsin(X_t) \tag{94}$$

does represent a bounded strong solution to the SDE (81). Other solutions, not bounded in $[-1, 1]$, are also present in accordance with the non-uniqueness.

This result establishes a further analogy with the case of Bessel processes. The squared Bessel process $\mathcal{B}_t^2$ satisfies an SDE for which strong uniqueness is well known to hold.

However, the SDE obtained by transforming the latter through the square root function is satisfied not only by the process $\mathcal{B}_t$, but also by other processes which are not constrained in $[0, +\infty)$. This is shown by Cherny in [35]. More on the analogy between the KS family and the Bessel family is provided in the following section. In the following proposition, we summarize the results of this section about the Kessler-Sørensen SDE.

Proposition 8.2. *Consider the Kessler-Sørensen SDE (81) and define* $\mathring{I} = (-1, 1)$.

1. *If* $\delta \geq 0$, *then the equation has strong existence and uniqueness, and for any initial condition* $Y_0 \in \mathring{I}$ *the solution* Y_t *remains in* $\mathring{I}$ *for all times* $t > 0$ *with probability one.*

2. *If* $\delta < 0$, *the SDE has neither weak nor strong uniqueness and any solution* Y_t *starting in* $\mathring{I}$ *attains of the boundaries of* $\mathring{I}$ *in finite time with probability one. However, if* $-1/2 < \delta < 0$, *one strong solution is given by*

$$Y_t = \frac{2}{\pi} \arcsin(X_t) \tag{95}$$

where X_t *is the unique strong solution to the DCL equation (7) with parameter* δ.

8.3 *Local Time: Analogy with the Bessel Process*

In this last section, we provide a justification of the claims which have not been proven in Section 8.2 about the process Y_t in (95) and the SDE (81). Consider the DCL process X_t in a neighborhood of $x = 1$, where the function $F = 2/\pi \arcsin(x)$ is irregular (the case near $x = -1$ is identical). To emphasize this viewpoint, consider the auxiliary process

$$Z_t = 1 - X_t \in [0, 2]\,, \tag{96}$$

solution of

$$dZ_t = \frac{1}{\theta}(1 - Z_t)\,dt + \left(-\sqrt{\frac{2 - Z_t}{\theta(\delta + 1)}}\right)\sqrt{Z_t}\,dW_t\,. \tag{97}$$

There is a paradigmatic equation in the literature, similar to (97), for which several facts have been understood. This is the equation for the squared Bessel process $\mathcal{B}_t^2$ [29], which we denote here by $\widetilde{Z}_t$ for convenience of analogy with Z_t:

$$d\widetilde{Z}_t = \widetilde{\delta} dt + 2\sqrt{\widetilde{Z}_t}\,dW_t. \tag{98}$$

For any initial condition in $(0, \infty)$, this equation admits a unique strong solution $\mathcal{B}_t^2$ taking values in $[0, \infty)$. For integer values of $\widetilde{\delta}$, the process $\mathcal{B}_t$, square root of such solution, can be realized as the norm of a $\widetilde{\delta}$-dimensional Brownian Motion (see [29], Chapter XI §1).

As a preliminary step, therefore, let us apply a time-change to reduce equation (97) to a form as close as possible to (98).

Lemma 8.3. *For $\lambda = 2\theta(\delta+1)$, the process $Z_t^{(\lambda)} := Z_{\lambda t}$ satisfies the equation*

$$dZ_t^{(\lambda)} = \widetilde{\delta}\left(1 - Z_t^{(\lambda)}\right) dt + 2\sqrt{1 - \frac{Z_t^{(\lambda)}}{2}}\sqrt{Z_t^{(\lambda)}}\, dW_t^{(\lambda)} \tag{99}$$

where

$$W_t^{(\lambda)} := -\frac{1}{\sqrt{\lambda}} W_{\lambda t} \tag{100}$$

is a standard Brownian motion and

$$\widetilde{\delta} = 2(\delta + 1). \tag{101}$$

Let us now translate known results for the squared Bessel process $\widetilde{Z}_t$ into results for our process $Z_t^{(\lambda)}$, being sure that, close to $z = 0$, the drift and the diffusion of $Z_t^{(\lambda)}$ behave as the ones of $\widetilde{Z}_t$:

$$\widetilde{\delta}(1-z) \sim \widetilde{\delta} \quad \text{and} \quad 2\sqrt{1 - \frac{z}{2}}\sqrt{z} \sim 2\sqrt{z} \qquad \text{if } z \sim 0. \tag{102}$$

The results claimed here on the squared Bessel process are taken from [29] Chapter XI §1, in particular Proposition 1.5. We assume to start with an initial condition greater than zero.

- For $\widetilde{\delta} < 2$, namely $\delta < 0$, the point $z = 0$ is reached almost surely, but it is instantaneously reflected and the local time at zero, L_t^0, is zero.
- For $\widetilde{\delta} \geq 2$, namely $\delta \geq 0$, $z = 0$ is never reached; hence, trivially, L_t^0 is zero.

Let us continue with a translation of results, for the process Y_t in (95). Given the identity (96), we have

$$Y_t = \phi(Z_t)$$

where

$$\phi(z) = \frac{2}{\pi}\arcsin(1 - z). \tag{103}$$

The function $\phi(z)$ near $z = 0^+$ behaves like $1 - c\sqrt{z}$ for a suitable constant c. Indeed, by solving $\phi(z) = r$ around $r = 1^-$, we find

$$\begin{aligned} z &= 1 - \sin\left(\frac{\pi}{2} r\right) \\ z &\sim \frac{1}{8}\pi^2(1-r)^2 \\ r &\sim 1 - \frac{2\sqrt{2}}{\pi}\sqrt{z}. \end{aligned} \tag{104}$$

Hence, at the second order,

$$\phi(z) \sim 1 - \frac{2\sqrt{2}}{\pi}\sqrt{z}\,. \tag{105}$$

Therefore, investigating $Y_t = \phi(Z_t)$ near $z = 0^+$ (that is, $y = 1^-$) is qualitatively analogous to investigating the process $\sqrt{Z_t}$ near 0. In turn, the latter has the same behavior as the Bessel process $\mathcal{B}_t = \sqrt{\widetilde{Z}_t}$ due to the analogy illustrated in (102).

The following facts are translation of known results about the process $\mathcal{B}_t$ (see [29], Chapter XI, Exercise 1.26) for our process Y_t. They recover the claims of Section 8.2 about the SDE satisfied by the process Y_t in the case $\delta < 0$.

- For $\widetilde{\delta} > 1$, namely $\delta > -\frac{1}{2}$, the process $Y_t = \phi(Z_t)$ satisfies the equation (81) expected on the ground of Itô's formula.
- For $\widetilde{\delta} \leq 1$, namely $\delta \leq -\frac{1}{2}$, the process Y_t satisfies an identity which involves a local time.

A short review of similar results about Bessel processes can also be found in [36].

9 Concluding Remarks

Realistic stochastic modeling of natural phenomena is increasingly requiring the use of bounded stochastic processes. Indeed, these are important to avoid to obtain results that are mathematically correct but contain artifacts from the application point of view. An heuristic approach is often adopted in the literature of modeling with bounded noises, which can however lead to misleading results. An example has been stressed in [28], where it is shown that an apparently bounded process can in reality be unbounded for some important range of its parameters. Thus, it is important to put the applications of bounded noises on firmer mathematical ground. In this work, we pursue this aim in the case of commonly employed bounded noises: the Sine-Wiener noise, the Doering-Cai-Lin family, and the Tsallis-Stariolo-Borland family. In the last section, we also investigate mathematical properties of an additional family, the Kessler-Sørensen family.

Specifically, we have characterized the range of parameters of the DCL and TSB families which give rise to strongly unique and bounded solutions. In the case of the DCL family, we have also shown that positive values of the parameter δ generate trajectories which never even attain the boundary of the state space. Moreover, we have analytically inferred the time evolution of the SW density for the first time and have shown that the noise can be recovered as a particular case of the DCL noise. We have investigated the relationships between the SDEs defining the DCL and the TSB families, showing that the two noises are intrinsically different from both a strong (pathwise) and a weak (in law) point of view. In addition, this investigation has lead to the SDE already introduced in an example of an earlier work of Kessler and Sørensen [26]. We have shown that the SDE can be obtained as a deterministic transformation of DCL SDE, only, however, for positive values of the parameter δ. Boundedness and uniqueness of the transformed SDE are instead lost in the case $\delta < 0$, in analogy with the case of Bessel processes.

As future lines of research, we mention two interesting points we are investigating. The first concerns a rigorous mathematical approach to the numerical simulations of SDEs which generate bounded noises. Indeed, the tendency of the simulated trajectories to go outside their theoretical bounds, due to the necessary numerical discretization, makes the simulation stiff. The second one refers to the fact that all processes here investigated are endowed with symmetric stationary densities, whereas in the real world the stochastic fluctuations can be asymmetric.

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A Autocorrelation Time

Definition A.1. Let X be a zero-mean stochastic process endowed with second order moments. The autocovariance function of X is the two variable function

$$R_{XX}(s,t) = \mathbb{E}[X_s X_t] = \mathrm{Cov}(X_s, X_t)\,, \quad s,t \geq 0\,. \tag{106}$$

Notice that in statistical physics this is also termed (un-normalized) autocorrelation function [37], which can cause some ambiguities. A stationary autocovariance function is one where $R_{XX}(s,t)$ only depends on the quantity $|\,t-s\,|$. If

$$\exists\, \tau > 0 \;:\; |\, R_{XX}(s,t)\,| \leq |\, R_{XX}(s,s)\,| \, \exp\left(-\frac{|\,t-s\,|}{\tau}\right) \tag{107}$$

at least for $s,t \geq K$ (for some $K>0$), then the smallest of such τ will be referred to as the characteristic **autocorrelation time** of the process X.

B Existence and Uniqueness of SDE Solutions

Definition B.1. We write a Stochastic Differential Equation (SDE) in the form

$$dX_t = \mu(X_t)\, dt \,+\, \sigma(X_t)\, dW_t\,, \tag{108}$$

where W_t is a standard Wiener Process on a filtered probability space $(\Omega, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$. The coefficients $\mu, \sigma \colon I \to \mathbb{R}$, where $I \subset \mathbb{R}$, are called drift and diffusion, respectively.

Definition B.2. A solution of (108) consists of a filtered probability space and a pair (X, W) such that the integral form (109) of the SDE holds with probability one, uniformly in t:

$$X_t = X_0 \,+\, \int_0^t \mu(X_s)\, ds \,+\, \int_0^t \sigma(X_s)\, dW_s\,. \tag{109}$$

A solution of (108) is called a **strong solution** if the process X is adapted to the filtration generated by the Brownian motion W. Otherwise, it is simply called a **weak solution**.

Definition B.3. There is **weak uniqueness** or *uniqueness in law* for a SDE if, whenever two solutions (X, W) and $(\widetilde{X}, \widetilde{W})$ possibly defined on different spaces are such that X_0 and $\widetilde{X}_0$ have the same distribution, then the whole laws of X and $\widetilde{X}$ coincide.

Definition B.4. There is **strong uniqueness** or *pathwise uniqueness* for a SDE if for every two strong solutions (X, W) and $(\widetilde{X}, \widetilde{W})$ defined on the same probability space, and with $X_0 = \widetilde{X}_0$ $\mathbb{P}$-a.s., the processes X and $\widetilde{X}$ are indistinguishable (that is, $\mathbb{P}[\,X_t = \widetilde{X}_t \;\forall\, t \geq 0\,] = 1$).

Although Lipschitz growth of the coefficients is a standard sufficient condition for strong existence and uniqueness of a SDE, these can also be obtained under weaker assumptions. We state here some technical results on the case of bounded or Hölder functions, which are exploited in Section 4. Proofs can be found in [29, 33].

Proposition B.5 ([29] Theorem 3.5, Chap. IX)**.** *Let μ and σ be the drift and diffusion coefficients of an SDE. Suppose that μ is Lipschitz continuous and that σ satisfies*

$$|\sigma(x) - \sigma(y)|^2 \leq \rho(|x - y|)\,, \tag{110}$$

where $\rho : (0, \infty) \to (0, \infty)$ is such that

$$\int_0^{\varepsilon} \frac{1}{\rho(z)}\, dz = \infty \quad \forall\, \varepsilon > 0\,. \tag{111}$$

Then, strong uniqueness holds for the SDE.

Proposition B.6. *Let $\mu, \sigma : \mathbb{R} \to \mathbb{R}$ be bounded and continuous functions, and X_0 a random variable with*

$$\mathbb{E}[\,|X_0|^{2m}] < \infty \tag{112}$$

for some $m > 1$. Then, there exists a weak solution of the SDE

$$dX_t = \mu(X_t)\, dt + \sigma(X_t) dW_t$$

with initial condition X_0.

Theorem B.7 (Yamada-Watanabe, 1971)**.** *Suppose pathwise uniqueness holds for a SDE and that a weak solution also exists. Then, a strong solution of the SDE also exists (such solution being unique in the strong sense thanks to the first assumption).*

Proposition B.8 (Comparison theorem, [29] Chap. IX §3)**.** *Let*

$$dX_t = \mu^{(i)}(X_t)\, dt + \sigma(X_t)\, dW_t \qquad \text{for } i = 1, 2 \tag{113}$$

be two stochastic differential equations, whose coefficients satisfy:

i) $|\sigma(x) - \sigma(y)|^2 \leq \rho(|x - y|)$, *$\rho(z)$ as in (111),*

ii) *at least one between $\mu^{(1)}$ and $\mu^{(2)}$ satisfies a Lipschitz condition, and the inequality $\mu^{(1)}(x) \leq \mu^{(2)}(x)$ holds everywhere.*

Further, let $X_0^{(i)}$ be two random variables, with $X_0^{(1)} \leq X_0^{(2)}$ $\mathbb{P}$-a.s., and let $X^{(i)}$ be solutions of (113) with starting conditions $X_0^{(i)}$, for $i = 1, 2$. Then,

$$\mathbb{P}\left[X_t^{(1)} \leq X_t^{(2)} \;\; \text{for all} \;\; t \geq 0\right] = 1\,.$$

C Attaining the Boundaries of the State Space

In the following we provide necessary and sufficient conditions to establish whether a stochastic process with bounded state space I does or does not attain the boundaries of I. We limit ourselves to the definitions and results of interest for this work, and refer the reader, for example, to [33] Chapter 5.5 and [29] Chapter VII §3 for proofs.

In the following, $I \subset \mathbb{R}$ denotes a bounded open, semi-open or closed real interval, $\mathring{I} = (l, r)$ its interior, and $x_0 \in \mathring{I}$ the deterministic initial condition of a stochastic process X_t with state space I. For $x \in I$, the first time the process X hits x will be denoted by

$$T_x = \inf\{\, t \geq 0 \mid X_t = x \,\} .$$

Definition C.1. A function $s : I \to \mathbb{R}$ is called a scale function for the process X (with $X_0 \equiv x_0$) if it is strictly increasing and, for any $a < x_0 < b \in I$, it holds

$$\mathbb{P}\left[T_a < T_b\right] = \frac{s(b) - s(x_0)}{s(b) - s(a)} .$$

Remark. If $s(x)$ is a scale function, also $\tilde{s}(x) = \alpha s(x) + \beta$ is a scale function for any $\alpha > 0$ and $\beta \in \mathbb{R}$. Hence, we can arbitrarily set $s(x_0) = 0$. Monotonicity then implies $s(a) < 0$, $s(b) > 0$.

Roughly speaking, the modulus of the scale function at a point x quantifies the "inaccessibility" of that point when starting from x_0. The bigger $|s(x)|$ is with respect to $|s(y)|$, the less likely x is reached before y. The scale function of a process can be easily calculated if the infinitesimal generator of the process is known, or if the process is solution of an SDE. Here, we consider the second case, and suppose that X_t is a (weak) solution of

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \tag{114}$$

where

$$\sigma^2(x) > 0 \quad \text{for } x \in \mathring{I} \tag{115}$$

$$\forall\, x \in \mathring{I}\,, \;\; \exists\, \varepsilon > 0 \;:\; \int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + |\mu(y)|}{\sigma^2(y)}\, dy < \infty . \tag{116}$$

Notice that, if the coefficients are continuous in I and (115) holds true, then the local integrability condition (116) is trivially fulfilled.

Proposition C.2. *Let X be a weak solution of (114) where μ and σ satisfy (115) and (116). Then, the scale function $s(x)$ can be computed as*

$$s(x) = \int_c^x \exp\left(-\int_c^y 2\,\frac{\mu(z)}{\sigma^2(z)}\, dz\right) dy\,, \quad c \in \mathring{I} . \tag{117}$$

Theorem C.3, proved in [33], Chap. 5, Prop. 5.22, relates the finiteness of the scale function to the attainability of the boundaries. Even if the endpoints l or r are not in I, we shall use the following notation:

$$s(l) = \lim_{x \to l^+} s(x)\,, \quad s(r) = \lim_{x \to r^-} s(x)\,.$$

Theorem C.3. *Let X be a weak solution of SDE (114), under conditions (115), (116), with initial condition $x_0 \in \mathring{I}$. Moreover, call T the random time*

$$T := \inf\Big\{ t \geq 0 \;\Big|\; X_t \notin \mathring{I} = (l, r) \Big\} = T_l \wedge T_r\,.$$

Then:

(i) *if both* $|s(l)|, |s(r)| = \infty$,

$$\mathbb{P}\big[T = \infty\big] = 1 = \mathbb{P}\big[\inf_{t \geq 0} X_t = l\big] = \mathbb{P}\big[\sup_{t \geq 0} X_t = r\big]\,;$$

(ii) *if both* $|s(l)|, |s(r)| < \infty$,

$$\mathbb{P}\big[\lim_{t \to T} X_t = l\big] = \frac{s(r) - s(x_0)}{s(r) - s(l)} = 1 - \mathbb{P}\big[\lim_{t \to T} X_t = r\big]\,;$$

(iii) *if* $|s(l)| < \infty$, $|s(r)| = \infty$,

$$\mathbb{P}\big[\lim_{t \to T} X_t = l\big] = 1 = \mathbb{P}\big[\sup_{t \geq 0} X_t < r\big]\,;$$

(iv) *if* $|s(l)| = \infty$, $|s(r)| < \infty$,

$$\mathbb{P}\big[\lim_{t \to T} X_t = r\big] = 1 = \mathbb{P}\big[\inf_{t \geq 0} X_t > l\big]\,.$$

Case (i) of previous theorem guarantees that, if s explodes at both endpoints, then the process never reaches the endpoints (i.e, T is infinite). However, no conclusion about the finiteness of T can be drawn in any of the three other cases. To present equivalent conditions to the almost surely finiteness of T, other two definitions are needed.

Definition C.4. Suppose the process X solves (114), under conditions (115), (116), and let $s(x)$ be the scale function associated with X. The speed measure associated with the process X and the scale function s is the measure on $(I, \mathcal{B}(\mathbb{R}))$ given by

$$m(dx) = \frac{2dx}{s'(x)\,\sigma^2(x)}\,, \quad x \in I\,.$$

Definition C.5. By denoting with $s(x)$ the scale function and $m(dx)$ the speed measure of a process X, we set

$$v(x) := \int_c^x \big(s(x) - s(y)\big)\, m(dy)\,, \qquad x \in I\,. \tag{118}$$

Notice that $v(x)$ is always positive and its finiteness does not depend on the choice of c.

The following theorem provides a characterization of the almost sure finiteness of T ([33], Chap. 5, Prop. 5.32).

Theorem C.6. *Under the usual assumptions (115), (116), T is almost surely finite if and only if one of the following mutually exclusive conditions holds:*

(a) *both $v(l) < \infty$ and $v(r) < \infty$,*

(b) *$v(l) < \infty$ and $v(r) = \infty$, but also $s(r) = \infty$,*

(c) *$v(r) < \infty$ and $v(l) = \infty$, but also $s(l) = -\infty$.*

Moreover, in case (a), the stronger condition $\mathbb{E}\big[T\big] < \infty$ holds true.