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Cosmological perturbations in generalised dark Lagrangians

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Abstract: We describe a new method to parameterise dark energy theories including massive gravity, elastic dark energy and tensor-metric theories. We first examine the existing framework which describes any second order Lagrangian which depends on the variation of the metric and find new constraints on the parameters. We extend the method to Lorentz violating theories which depend on the variation of the time and spatial parts of the metric separately. We show how this can describe massive gravity and elastic dark energy, while ruling out the whole class of theories where the Lagrangian depends only on the variation of the time part of the metric.

We further generalise our method to tensor-metric theories, both with and without splitting the metric into time and spatial parts. Our method extends existing physics by providing a mechanism to easily evaluate large classes of dark energy theories.

Keywords: Classical Theories of Gravity, Space-Time Symmetries, Spacetime Singularities

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Contents

1 Introduction

The discovery of the accelerated expansion of the universe [1–8] prompted many attempts to explain the phenomenon, caused by an unknown "dark energy", using modifications to Einstein's theory of General Relativity [9]. These theories generally started with a specific dark Lagrangian which was then investigated to test its compatibility with both dark

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energy and other cosmological and solar system observations. Here we follow the method of [9–17] and look at a *class* of dark Lagrangians, so that future experimental results can immediately rule out a whole swathe of models without the need for further calculation.

We use the formalism introduced in [10–14] which draws on the "Post Parameterised Friedmann" (PPF) approach described in [9, 15–17]. In these papers the Einstein-Hilbert action is modified with a *dark Lagrangian* such that the modified action is written as

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \mathcal{L}_{\text{matter}} - \mathcal{L}_d\right], \tag{1.1}$$

and equations of state for dark sector perturbations are found for the entropy perturbation and the anisotropic stress. The recent observation of a neutron star merger by LIGO [18] severely constrained the speed of gravitational waves, and seemingly placed strict restrictions on many modified gravity theories. However, it was suggested in [19] that although these theories predict the speed of gravitational waves to be different from that of light at low energies, this may not necessarily be the case at the high energies seen in neutron star mergers, due to the unknown UV completions of these theories.

We find the perturbed fluid variables when $\mathcal{L}_{\{2\}} = \mathcal{L}_{\{2\}}(\delta_L g_{\mu\nu})$, in a similar way to [12]. We then impose invariance under time reparameterisation and find the equations of motion. We also find evolution equations for the equation of state parameter w and the elastic bulk modulus. We find restrictions on w such that we can have a realistic (positive and subluminal) sound speed. We find the perturbed fluid variables when we have imposed spatial invariance, and we find the conditions under which the entropy is gauge invariant.

In this paper our main focus is on generalising all of these calculations for the case where $\mathcal{L}_{\{2\}}$ is a function of the change in the time-like and spatial parts of the metric separately, not necessarily packaged together as the spacetime metric.

Elastic dark energy (EDE) discussed in [12, 20] is a development of Carter-Quintana relativistic elasticity theory [21–25]. These models describe the universe as a solid with certain parameters which can be found by observations. By imposing time translational invariance but not spatial invariance, we fulfill the conditions necessary for EDE.

As we are adding a dark energy term to the Einstein-Hilbert action (1.1), we can decompose the stress energy tensor as $T_{\mu\nu} = T^{\text{matter}}_{\mu\nu} + T^{\text{radiation}}_{\mu\nu} + T^{\text{dark}}_{\mu\nu}$, i.e. we have added a source term to the standard model stress energy tensor.

1.1 Decomposition of the metric

We impose spatial isotropy by foliating the four-dimensional (4D) space-time, as in [12, 26], by three-dimensional (3D) sheets with a time-like unit vector, u_μ which is everywhere orthogonal to the sheets. The 4D spacetime has metric $g_{\mu\nu}$, and the 3D sheets have spatial metric $\gamma_{\mu\nu} = \gamma_{(\mu\nu)}$. The $(3 + 1)$ decomposition of the 4D metric is

$$g_{\mu\nu} = \gamma_{\mu\nu} - u_\mu u_\nu, \tag{1.2}$$

where u_μ and $\gamma_{\mu\nu}$ are subject to the orthogonality and normality conditions:

$$u^\mu\gamma_{\mu\nu} = 0, \qquad u^\mu u_\mu = -1. \tag{1.3}$$

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Hence we can write the dark term in the stress-energy tensor in an isotropic spacetime as

$$T_{\mu\nu} = \rho u_\mu u_\nu + P\gamma_{\mu\nu}, \tag{1.4}$$

where ρ is the density of the universe due to this dark term and P is the pressure. The pressure and density are related by $P = w\rho$, where w is the equation of state parameter. From now on, when we write $T_{\mu\nu}$, this will refer to the dark stress-energy tensor rather than the overall energy momentum tensor.

2 $\mathcal{L}_{\{2\}}(\delta_L g_{\mu\nu})$

We will first look at the case $\mathcal{L}_{\{2\}}(\delta_L g_{\mu\nu})$ before generalising to more generic Lagrangians. For any Lagrangian which is a function only of perturbations of the metric and no derivatives thereof, i.e. $\mathcal{L}_{\{2\}} = \mathcal{L}_{\{2\}}(\delta_L g_{\mu\nu})$, then from [12], the most general quadratic Lagrangian for dark sector perturbations is

$$\mathcal{L}_{\{2\}} = \frac{1}{8} W^{\mu\nu\alpha\beta} \delta_L g_{\mu\nu} \delta_L g_{\alpha\beta}, \tag{2.1}$$

where $\delta_L g_{\mu\nu}$ is the metric fluctuation under a perturbation and $W^{\mu\nu\alpha\beta}$ can be thought of as a mass term for the perturbation, not to be confused with the Weyl tensor. From [13], we can use the symmetries of the W tensor:

$$W_{\mu\nu\alpha\beta} = W_{(\mu\nu)(\alpha\beta)} = W_{\alpha\beta\mu\nu}, \tag{2.2}$$

to obtain the most general possible $W_{\alpha\beta\mu\nu}$:

$$\begin{aligned} W_{\mu\nu\alpha\beta} = {} & A_W u_\mu u_\nu u_\alpha u_\beta + B_W \left(u_\mu u_\nu \gamma_{\alpha\beta} + \gamma_{\mu\nu} u_\alpha u_\beta \right) \\ & + C_W u_{(\mu} \gamma_{\nu)(\alpha} u_{\beta)} + D_W \gamma_{\mu\nu} \gamma_{\alpha\beta} + E_W \gamma_{\mu(\alpha} \gamma_{\beta)\nu}. \end{aligned} \tag{2.3}$$

2.1 Contractions of the Eulerian change in the energy-momentum tensor

The deformation vector, which represents possible coordinate changes [14],[1] has both time-like parts and space-like parts

$$\xi_\mu = -\chi u_\mu + m_\mu. \tag{2.4}$$

We can find the various contractions of $\delta_E T^\mu{}_\nu$ by inserting the derivatives of the deformation vector. We use two different variational operators: δ_L and δ_E, which are linked via the Lie derivative along the diffeomorphism-generating vector ξ^μ via

$$\delta_L = \delta_E + L_\xi, \tag{2.5}$$

where δ_L is the Lagrangian perturbation (in co-moving coordinates) and δ_E is the Eulerian perturbation (points are fixed in spacetime). L_ξ is the Lie derivative in the direction of ξ^μ, a vector field representing possible coordinate transformations, so that the Lie derivative acting on a given symmetric tensor field is $L_\xi X_{\mu\nu} = \xi^\alpha \nabla_\alpha X_{\mu\nu} + 2X_{\alpha(\mu} \nabla_\alpha \xi^{\nu)}$. Note that

[1] The equations of motion of General Relativity are independent of ξ_μ, but this is not necessarily true for more general actions. We can decompose $\delta_L g_{\mu\nu}$ as $\delta_L g_{\mu\nu} = h_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)}$ [14].

$L_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}$. We are interested in the perturbed fluid equations, which are derived from the Lagrangian for perturbations, denoted as $\mathcal{L}_{(2)}$. In order to find the perturbed fluid variables, we first need to find the Eulerian change in the stress-energy tensor, $\delta_E T^{\mu\nu}$. We will work in the synchronous gauge, where perturbations to the metric have spatial components only

$$\delta_E g_{\mu\nu} = \gamma^\alpha{}_\mu \gamma^\beta{}_\nu h_{\alpha\beta}, \tag{2.6}$$

Working in the synchronous gauge (2.6), we vary (2.1) to obtain the contractions of the stress energy tensor [12, 14]

$$\begin{aligned}
u_\mu u^\nu \delta_E T^\mu{}_\nu &= (\dot\rho + K(B_W+\rho))\chi - (B_W+\rho)\left(\frac{1}{2}\hat h + \nabla_\alpha m^\alpha\right) - (\rho + A_W)\dot\chi, && (2.7a)\\
u^\nu \gamma^\sigma{}_\mu \delta_E T^\mu{}_\nu &= \left(\frac{1}{4}C_W + P\right)\bar\nabla^\sigma\chi - \frac{1}{4}C_W K^{\sigma\beta} m_\beta + \left(\frac{1}{4}C_W - \rho\right)\dot m^\sigma + \rho K^\sigma{}_\alpha m^\alpha, && (2.7b)\\
\gamma^\nu{}_\mu \delta_E T^\mu{}_\nu &= -\frac{1}{2}\left(\gamma^{\alpha\beta}(3D_W + E_W + P) + 3u^\alpha u^\beta (B_W - P)\right)\delta_E g_{\alpha\beta} && (2.7c)\\
&\quad - \left[3u^\alpha u^\beta (B_W - P) + \gamma^{\alpha\beta}(3D_W + E_W + P)\right]\nabla_\alpha \xi_\beta + 3\chi\dot P && (2.7d)\\
\gamma^\sigma{}_\mu \gamma^\nu{}_\rho \delta_E T^\mu{}_\nu &= \chi\gamma^\sigma{}_\rho \dot P - \frac{1}{2}\left((D_W+P)\gamma^\sigma{}_\rho \hat h + (E_W - 2P)\hat h^\sigma_\rho\right) && (2.7e)\\
&\quad + \gamma^\sigma{}_\rho\left[(P - A_W)\dot\chi - (D_W+P)\left(\bar\nabla_\alpha m^\alpha - \chi K\right)\right]\\
&\quad + (2P - F_W)\left[\bar\nabla^{(\sigma} m_{\rho)} - m_\beta K^{\beta(\sigma} u_{\rho)} - \chi K^\sigma{}_\rho\right],
\end{aligned}$$

where we have defined "time" and "space" differentiation as the derivative operator projected along the time and space directions

$$\dot\psi \equiv u^\mu \nabla_\mu \psi, \qquad \bar\nabla_\mu \psi \equiv \gamma^\nu{}_\mu \nabla_\nu \psi, \tag{2.8}$$

and where $K = 3H$, (where H is the Hubble parameter) is the trace of the extrinsic curvature tensor $K_{\mu\nu} \equiv \nabla_\mu u_\nu$, which satisfies $K_{\mu\nu} = K_{(\mu\nu)}$ and $u^\mu K_{\mu\nu} = 0$.

2.2 Perturbed fluid variables

The components of the perturbed energy-momentum tensor $T^\mu{}_\nu$ are written as

$$\delta_E T^\mu{}_\nu = \delta\rho u^\mu u_\nu + 2(\rho+P)v^{(\mu}u_{\nu)} + \delta P \gamma^\mu{}_\nu + P\Pi^\mu{}_\nu, \tag{2.9}$$

where v^μ is the perturbed dark sector velocity and $P\Pi^\mu{}_\nu$ is the anisotropic part of the stress tensor, which is orthogonal and symmetric. We have dropped the subscript E on the variation of the density and pressure. We can now compare (2.9) and the contractions of the perturbed stress energy tensor (2.7), to obtain the perturbed fluid variables in terms of the deformation vector (2.4), as described in [12, 14]

$$\delta\rho = (\dot\rho + K(B_W+\rho))\,\chi - (\rho + A_W)\dot\chi - (B_W+\rho)\left(\frac{1}{2}\hat h + \bar\nabla_\alpha m^\alpha\right), \tag{2.10a}$$

$$\begin{aligned}\delta P &= (P - B_W)\dot\chi - \frac{1}{3}(3D_W + E_W + P)\left(\frac{1}{2}\hat h + \bar\nabla_\alpha m^\alpha\right) \\ &\quad + \left[\frac{1}{3}(3D_W + E_W + P)K + \dot P\right]\chi,\end{aligned} \tag{2.10b}$$

$$(\rho + P)v^\sigma = \left(\rho - \frac{C_W}{4}\right)\dot m^\sigma - \left(P + \frac{C_W}{4}\right)\bar\nabla^\sigma\chi, \tag{2.10c}$$

$$P\Pi^\sigma{}_\rho = (2P - E_W)\left[\frac{1}{2}\hat h^\sigma_\rho + \bar\nabla^{(\sigma}m_{\rho)} - \frac{1}{3}\gamma^\sigma{}_\rho\left(\frac{1}{2}\hat h + \bar\nabla^\alpha m_\alpha\right)\right], \tag{2.10d}$$

where we have defined $P\Pi^\mu{}_\nu = \left(\gamma^\mu{}_\beta\gamma^\alpha{}_\nu - \frac{1}{3}\gamma^\alpha{}_\beta\gamma^\mu{}_\nu\right)\delta_E T^\beta{}_\alpha$.

2.3 Invariance under time reparameterisation

We now want to discover what constraints invariance under changes in time imposes. This is helpful for [12, 20] because applying time translational reparameterisation invariance but not spatial reparameterisation invariance leads to elastic dark energy. Hence we set the coefficients of χ, $\dot\chi$ and $\bar\nabla^\sigma\chi$ in (2.10) to zero and therefore obtain

$$\begin{aligned}\dot\rho + 3H(P+\rho) &= 0, \\ \dot P + (P + 3D_W + E_W)\,H &= 0.\end{aligned} \tag{2.11}$$

This gives the conservation equation and an equation for the evolution of pressure with time. We can rewrite the perturbed fluid variables (2.10) as

$$\delta\rho = -(P+\rho)\left(\frac{1}{2}\hat h + \bar\nabla_\alpha\xi^\alpha\right), \tag{2.12a}$$

$$\delta P = -\beta\left(\frac{1}{2}\hat h + \bar\nabla_\alpha\xi^\alpha\right), \tag{2.12b}$$

$$v^\sigma = \dot\xi^\sigma, \tag{2.12c}$$

$$P\Pi^\sigma{}_\rho = 2\mu\left[\frac{1}{2}\hat h^\sigma_\rho + \bar\nabla^{(\sigma}\xi_{\rho)} - \frac{1}{3}\gamma^\sigma{}_\rho\left(\frac{1}{2}\hat h + \bar\nabla^\alpha\xi_\alpha\right)\right], \tag{2.12d}$$

and the pressure evolution as

$$\dot P + 3\beta H = 0, \tag{2.13}$$

where we have defined

$$\beta = \frac{1}{3}P + D_W + \frac{1}{3}E_W, \tag{2.14a}$$

$$\mu = P - \frac{1}{2}E_W, \tag{2.14b}$$

as parameters that can be determined by experiment. β and μ correspond to the elastic bulk modulus and the elastic shear modulus, respectively [12, 20].

So far, we have summarised previous work. However, as an aside we note that it is possible to place constraints on the parameters using observational data. In [27], it was noted that it might be possible to place constraints on the parameters of the elastic dark energy model, referred to as Time Diffeomorphism Invariant (TDI) models in their paper. The authors used observational data on cosmic shear and CMB lensing which would give constraints for any given value of w, unless $w \approx -1$, as a wide. Unfortunately, recent Planck data has shown that $w \approx -1$, so these constraint do not apply.

In appendix A, using 2018 Planck data [28], we find a constraint of $-0.004 < \beta < 0.106$ if we assume that the equation of state parameter w of the Lagrangian (2.1), where $P = w\rho$, is constant. If we assume that $w \neq -1$ exactly, then we also find constraints of $-0.0477 \leq \hat{\mu} \leq 0.0599$, where we have defined $\hat{\mu} = \mu/\rho$. However,the assumption that we do not have a phantom equation of state, i.e. we require $w \geq -1$, gives $0 \leq \hat{\mu} \leq 0.0599$.

3 $\mathcal{L}_{\{2\}}(\delta_L u_\mu, \delta_L \gamma_{\mu\nu})$

In the previous section, we summarised the derivation of the perturbed fluid variables for theories where the second variation of the Lagrangian depends only on the change in the metric [12] and found new constraints on the values of the elasticity and rigidity parameters. We now move on to extend this work to more general Lagrangians.

If we take the second variation of the Lagrangian as a function only of the change in the time and spatial parts of the metric separately

$$\mathcal{L}_{\{2\}} = \mathcal{L}_{\{2\}}\left(\delta_L u_\mu, \delta_L \gamma_{\mu\nu}\right), \tag{3.1}$$

then the most general possible quadratic Lagrangian which is a function of $\delta_L u_\mu$ and $\delta_L \gamma_{\mu\nu}$ is

$$\mathcal{L}_{\{2\}} = \frac{1}{8} X^{\mu\nu\alpha\beta} \delta_L \gamma_{\mu\nu} \delta_L \gamma_{\alpha\beta} + \frac{1}{8} Y^{\mu\nu} \delta_L u_\mu \delta_L u_\nu + \frac{1}{4} Q^{\mu\nu\alpha} \delta_L u_\mu \delta_L \gamma_{\nu\alpha}, \tag{3.2}$$

where

$$X^{\mu\nu\alpha\beta} = X^{(\mu\nu)(\alpha\beta)} = X^{\alpha\beta\mu\nu}, \qquad Y^{\mu\nu} = Y^{(\mu\nu)}, \qquad Q^{\mu\nu\alpha} = Q^{\mu(\nu\alpha)}. \tag{3.3}$$

Using the identities [20]

$$\begin{aligned} \delta_L u^\mu &= \frac{1}{2} u^\mu u^\alpha u^\beta \delta_L g_{\alpha\beta}, \\ \delta_L \gamma_{\mu\nu} &= \delta_L g_{\mu\nu} + 2u_{(\mu} \left(\gamma^\alpha{}_{\nu)} - \frac{1}{2} u_{\nu)} u^\alpha \right) u^\beta \delta_L g_{\alpha\beta}, \end{aligned} \tag{3.4}$$

it is possible to write (3.2) as

$$\mathcal{L}_{\{2\}} = \left(W_X^{\mu\nu\alpha\beta} + W_Y^{\mu\nu\alpha\beta} + W_Z^{\mu\nu\alpha\beta} \right) \delta_L g_{\alpha\beta} \delta_L g_{\mu\nu}, \tag{3.5}$$

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where upon comparing (3.2) with (2.1), we can see that the components of $W^{\sigma\rho\phi\lambda} = W_X^{\sigma\rho\phi\lambda} + W_Y^{\sigma\rho\phi\lambda} + W_Q^{\sigma\rho\phi\lambda}$ are

$$\begin{aligned} W_X^{\sigma\rho\phi\lambda} &= \frac{1}{2}X^{\mu\nu\alpha\beta}\Big[2\left(\delta^{(\sigma}{}_\mu\delta^{\rho)}{}_\nu\delta^{(\phi}{}_\alpha\delta^{\lambda)}{}_\beta + \delta^{(\phi}{}_\mu\delta^{\lambda)}{}_\nu\delta^{(\sigma}{}_\alpha\delta^{\rho)}{}_\beta\right) + 4u_{(\mu}H^{\sigma\rho}{}_{\nu)} && \text{(3.6a)} \\ &\qquad \times \delta^{(\phi}{}_\alpha\delta^{\lambda)}{}_\beta + 4u_{(\alpha}H^{\phi\lambda}{}_{\beta)}\delta^{(\sigma}{}_\mu\delta^{\rho)}{}_\nu + 4u_{(\mu}H^{\sigma\rho}{}_{\nu)}u_{(\alpha}H^{\phi\lambda}{}_{\beta)}\Big], && \text{(3.6b)} \\ W_Y^{\sigma\rho\phi\lambda} &= \frac{1}{4}Y^{\mu\nu}\Big[H^{\phi\lambda}{}_\mu H^{\sigma\rho}{}_\nu + H^{\phi\lambda}{}_\nu H^{\sigma\rho}{}_\mu\Big], && \text{(3.6c)} \\ W_Q^{\sigma\rho\phi\lambda} &= 2Q^{\mu\nu\alpha}\left(\delta^{(\sigma}{}_\mu\delta^{\rho)}{}_\nu + u_{(\mu}H^{\sigma\rho}{}_{\nu)}\right)H^{\phi\lambda}{}_\alpha, && \text{(3.6d)} \end{aligned}$$

where we have defined

$$H^{\alpha\beta}{}_\mu \equiv \left(\gamma_\mu^{(\alpha} - \frac{1}{2}u_\mu u^{(\alpha}\right)u^{\beta)}. \tag{3.7}$$

Eq. (3.5) is a very important result — we can rewrite any Lagrangian dependent on the variation of the spatial part of the metric and the time-like part of the metric separately into one dependent on only the variation of the full metric.

In the rest of this section, we will explore how we can use the method of section 2 to find the perturbations of a Lagrangian of the form (2) without any new calculations.

3.1 Decomposition of the X,Y and Q tensors

The most general tensors that satisfy the necessary symmetries (3.3) are:

$$\begin{aligned} X^{\mu\nu\alpha\beta} &= A_X u^\mu u^\nu u^\alpha u^\beta + B_X\left(u^\mu u^\nu\gamma^{\alpha\beta} + \gamma^{\mu\nu}u^\alpha u^\beta\right) + 4C_X u^{(\mu}\gamma^{\nu)(\alpha}u^{\beta)}, && \text{(3.8a)} \\ &\quad + D_X\gamma^{\mu\nu}\gamma^{\alpha\beta} + \frac{1}{2}E_X\gamma^{\mu(\alpha}\gamma^{\beta)\nu}, \\ Y^{\mu\nu} &= A_Y u^\mu u^\nu + B_Y\gamma^{\mu\nu}, && \text{(3.8b)} \\ Q^{\mu\nu\alpha} &= A_Q u^\mu u^\nu u^\alpha + B_Q u^\mu\gamma^{\nu\alpha} + 2C_Q u^{(\nu}\gamma^{\alpha)\mu}. && \text{(3.8c)} \end{aligned}$$

3.2 The specific W tensor for $\mathcal{L}(\delta_L u_\mu, \delta_L\gamma_{\mu\nu})$

Collecting like terms, we obtain

$$\begin{aligned} W^{\sigma\rho\phi\lambda} &= W_X^{\sigma\rho\phi\lambda} + W_Y^{\sigma\rho\phi\lambda} + W_Q^{\sigma\rho\phi\lambda} \\ &= \left(\frac{1}{2}A_X + \frac{1}{8}A_Y + \frac{1}{2}A_Q\right)u^\sigma u^\rho u^\phi u^\lambda + B_X\left(u^\sigma u^\rho\gamma^{\phi\lambda} + \gamma^{\sigma\rho}u^\phi u^\lambda\right) \qquad \text{(3.9)} \\ &\quad + \left(8C_X + \frac{1}{2}B_Y + B_Q + 2C_Q\right)u^{(\sigma}\gamma^{\rho)(\phi}u^{\lambda)} + 2D_X\gamma^{\sigma\rho}\gamma^{\phi\lambda} + E_X\gamma^{\phi(\sigma}\gamma^{\rho)\lambda}. \end{aligned}$$

Comparing to (2.3), we find

$$\begin{aligned} A_W &= \frac{1}{2}A_X + \frac{1}{8}A_Y + \frac{1}{2}A_Q, \\ B_W &= B_X, \\ C_W &= 8C_X + \frac{1}{2}B_Y + B_Q + 2C_Q, \\ D_W &= D_X, \\ E_W &= E_X. \end{aligned} \tag{3.10}$$

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3.3 Invariance under time reparameterisation

We repeat our calculations from earlier, where we fixed our equations to be the same under a change in time and we obtained (2.13).

Plugging in the values from (3.10), we find an equation for the evolution of the pressure

$$\dot{P} = -\left(P + 3D_X + E_X\right) H = -3\beta_g H, \tag{3.11}$$

where $\beta_g \equiv \frac{1}{3}\left(P + 3D_X + E_X\right)$. We also find that the coefficients of the W tensor become

$$\begin{aligned} A_W &= \frac{1}{2}A_X + \frac{1}{8}A_Y + \frac{1}{2}A_Q = \rho, \\ B_W &= B_X = P, \\ C_W &= 8C_X + \frac{1}{2}B_Y + B_Q + 2C_Q = -P, \\ D_W &= D_X, \\ E_W &= E_X. \end{aligned} \tag{3.12}$$

Next we will examine the specific cases where the Lagrangian depends only on the variation of either the time *or* the spatial part of the metric.

3.3.1 $\mathcal{L}_{\{2\}}(\delta_L u_\mu)$

First, if we make $\mathcal{L}_{\{2\}}$ a function of $\delta_L u_\mu$ only, i.e. $\mathcal{L}_{\{2\}}$ is dependent only on the change in the time part of the metric, we obtain $W_X = W_Q = 0$, in which case $\beta = P = 0$, and using (2.14b) we can then rewrite the perturbed fluid variables from (2.12)

$$\delta\rho = -\rho\left(\frac{1}{2}\hat{h} + \bar{\nabla}_\alpha \xi^\alpha\right), \tag{3.13a}$$

$$\delta P = 0, \tag{3.13b}$$

$$(1+w)v^\sigma = \dot{\xi}^\sigma, \tag{3.13c}$$

$$P\Pi^\sigma{}_\rho = 0. \tag{3.13d}$$

We find that the equation of state parameter is

$$w = \dot{w} = 0. \tag{3.14}$$

Using constraints from Planck [28] which show $w \approx -1$, we can therefore rule out this case. While this case might intuitively seem unlikely,[2] we have completely ruled it out without having to examine any specific model. This shows the power of our parameterisation method.

[2]The lack of spatial dependence means that the dark energy Lagrangian is described by a pressureless fluid.

3.3.2 $\mathcal{L}_{\{2\}}(\delta_L \gamma_{\mu\nu})$

If we make $\mathcal{L}$ a function of $\delta_L \gamma_{\mu\nu}$ only, i.e. $\mathcal{L}$ is dependent only on the change in the spatial part of the metric, we get $W_Y = W_Q = 0$, which means

$$\begin{aligned}
A_W &= \frac{1}{2} A_X = \rho, \\
B_W &= B_X = P, \\
C_W &= 8 C_X = -P, \\
D_W &= D_X, \\
E_W &= E_X,
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
\dot{P} &= -\left(P + 3D_X + E_X\right) H \\
&\equiv -3\beta_\gamma H.
\end{aligned} \tag{3.16}$$

where we have defined $\beta_\gamma = \frac{1}{3}\left(P + 3D_X + E_X\right)$. For $\mathcal{L}_{\{2\}}(\delta_L \gamma_{\mu\nu})$ the evolution equation for w (A.2) remains the same but with β now depending on D_X and E_X rather than D and F, which lead to the same result if we use (3.15).

3.3.3 Summary of $\mathcal{L}_{\{2\}}(\delta_L u_\mu, \delta_L \gamma_{\mu\nu})$

The analysis of section 3 shows that the effect of the metric split is simply to change the coefficients as shown in (3.10).

If we have mandated time reparameterisation invariance by decoupling χ, then the only relevant contribution to β and μ come from the spatial part of the metric $\gamma_{\mu\nu}$, i.e. only the first term in (3.2) has any effect on the system.

3.4 Comparison with elastic dark energy theories

We can compare to [20] to find the properties of the dark energy material, as this Lagrangian can be described using elastic dark energy.

We perform a scalar-vector-tensor (SVT) decomposition, where we decompose the perturbation to the metric h_{ij} as [29]

$$\frac{1}{2} h_{ij} = H_L^S Q_{ij}^S + H_T^S Q_{ij}^S + H^V Q_{ij}^V + H^T Q_{ij}^T, \tag{3.17}$$

where one can decompose a spatial tensor field as

$$\eta^{ij} \nabla_i \nabla_j Q^{S,V,T} = -k^2 Q^{S,V,T}, \tag{3.18}$$

scalars can be constructed from vectors and tensors as [30]

$$\nabla_i Q^S = -k Q_i^S, \qquad \nabla_i \nabla_j Q^S + \frac{1}{3} k^2 \eta_{ij} Q^S = Q_{ij}^S, \tag{3.19}$$

and vectors can be constructed from tensors as

$$\nabla_{(i} Q_{j)}^V = -k Q_{ij}^V, \tag{3.20}$$

with the requirement that $Q_i^{V|i} = Q_{ij}^{T|i} = Q_i^{Ti} = 0$. We now find the entropy perturbation, which is defined by

$$w\Gamma = \left(\frac{\delta P}{\delta\rho} - \frac{dP}{d\rho}\right)\delta. \tag{3.21}$$

The entropy perturbation and scalar anisotropic stress are given by

$$\begin{aligned} w\Gamma &= 0, \\ w\Pi^S &= -2\frac{\mu}{\rho+P}\left[\delta - 3(1+w)\eta\right], \end{aligned} \tag{3.22}$$

where we have defined $\delta = \frac{\delta\rho}{\rho}$, and $\eta = -\left(H_L^S + \frac{1}{3}H_T^S\right)$. This is the same result as [20] finds when looking at elastic mediums.

4 Tensor-metric theories

So far, we have looked at theories where $\mathcal{L}_{\{2\}} = \mathcal{L}_{\{2\}}(\delta g_{\mu\nu}, \delta f_{\mu\nu})$. In this section, we now apply our analysis to theories where the dark Lagrangian is a function of both the variation of the metric $g_{\mu\nu}$ and of an unspecified non-dynamical symmetric rank-2 tensor $f_{\mu\nu}$ [31, 32].[3]

If $\mathcal{L}_{\{2\}} = \mathcal{L}_{\{2\}}(\delta g_{\mu\nu}, \delta f_{\mu\nu})$, then the most general quadratic Lagrangian we can have is

$$\mathcal{L}_{\{2\}} = \frac{1}{8}A^{\mu\nu\alpha\beta}\delta_L g_{\mu\nu}\delta_L g_{\alpha\beta} + \frac{1}{4}B^{\mu\nu\alpha\beta}\delta_L g_{\mu\nu}\delta_L f_{\alpha\beta} + \frac{1}{8}C^{\mu\nu\alpha\beta}\delta_L f_{\mu\nu}\delta_L f_{\alpha\beta}. \tag{4.1}$$

The tensors obey the following symmetries

$$A^{\mu\nu\alpha\beta} = A^{(\mu\nu)(\alpha\beta)} = A^{\alpha\beta\mu\nu}, \qquad B^{\mu\nu\alpha\beta} = B^{(\mu\nu)(\alpha\beta)}, \qquad C^{\mu\nu\alpha\beta} = C^{(\alpha\beta)(\mu\nu)}. \tag{4.2}$$

The decomposition of these tensors, called "coupling tensors" because they prescribe how the fields combine in the Lagrangian, is

$$\begin{aligned} A^{\mu\nu\alpha\beta} &= A_X u^\mu u^\nu u^\alpha u^\beta + B_X\left(u^\mu u^\nu \gamma^{\alpha\beta} + \gamma^{\mu\nu}u^\alpha u^\beta\right) + 4C_X u^{(\mu}\gamma^{\nu)(\alpha}u^{\beta)} \\ &\quad + D_X\gamma^{\mu\nu}\gamma^{\alpha\beta} + 2E_X\gamma^{\mu(\alpha}\gamma^{\beta)\nu}, \end{aligned} \tag{4.3a}$$

$$\begin{aligned} B^{\mu\nu\alpha\beta} &= A_Y u^\mu u^\nu u^\alpha u^\beta + B_Y u^\mu u^\nu \gamma^{\alpha\beta} + 4C_Y u^{(\mu}\gamma^{\nu)(\alpha}u^{\beta)} \\ &\quad + D_Y\gamma^{\mu\nu}\gamma^{\alpha\beta} + 2E_Y\gamma^{\mu(\alpha}\gamma^{\beta)\nu} + F_Y\gamma^{\mu\nu}u^\alpha u^\beta, \end{aligned} \tag{4.3b}$$

$$\begin{aligned} C^{\mu\nu\alpha\beta} &= A_Z u^\mu u^\nu u^\alpha u^\beta + B_Z\left(u^\mu u^\nu \gamma^{\alpha\beta} + \gamma^{\mu\nu}u^\alpha u^\beta\right) + 4C_Z u^{(\mu}\gamma^{\nu)(\alpha}u^{\beta)} \\ &\quad + D_Z\gamma^{\mu\nu}\gamma^{\alpha\beta} + 2E_Z\gamma^{\mu(\alpha}\gamma^{\beta)\nu}. \end{aligned} \tag{4.3c}$$

[3]It should be noted that at this point there are no derivatives of $f_{\mu\nu}$ in the action and therefore no kinetic term for $f_{\mu\nu}$. We could choose a form of $f_{\mu\nu}$ that includes derivatives of a vector field or a scalar, and therefore generates a kinetic term.

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4.1 Equations of motion

We use (4.3) to find

$$\begin{aligned}\delta_E T^\mu{}_\nu = &-\frac{1}{2}\left(B_X u^\mu u_\nu + D_X\gamma^\mu{}_\nu + T^\mu{}_\nu\right)h - E_X h^\mu{}_\nu - \frac{1}{2}B^\mu{}_\nu{}^{\alpha\beta}k_{\alpha\beta} \\ &-\left(\nabla_\alpha T^\mu{}_\nu - \frac{1}{2}B^\mu{}_\nu{}^{\sigma\beta}\nabla_\alpha f_{\sigma\beta}\right)\xi^\alpha \\ &+\left(2T^{\alpha(\mu}g^\beta{}_{\nu)} - B^\mu{}_\nu{}^{\sigma\alpha}f^\beta{}_\sigma - A^\mu{}_\nu{}^{\alpha\beta} - T^\mu{}_\nu g^{\alpha\beta}\right)\nabla_\alpha\xi_\beta\end{aligned} \tag{4.4}$$

where we have defined $k_{\mu\nu} \equiv \delta_E f_{\mu\nu}$.

We find the perturbed fluid variables by assuming the unperturbed tensor $f_{\mu\nu}$ is homogenous, i.e. $\bar\nabla_\alpha f_{\mu\nu} = 0$, as chosen in [33], recalling the definition of "space" differentiation $\bar\nabla_\mu\psi \equiv \gamma^\nu{}_\mu\nabla_\nu\psi$ from (2.8)

$$\begin{aligned}\delta\rho =& -(B_X+\rho)\left(\frac{1}{2}h+\bar\nabla^\alpha m_\alpha\right)-\frac{1}{2}\left(B_Y\gamma^{\alpha\beta}+A_Y u^\alpha u^\beta\right)k_{\alpha\beta} \\ &+\left[\dot\rho+\frac{1}{2}\dot f^{\alpha\beta}\left(B_Y\gamma_{\alpha\beta}+A_Y u_\alpha u_\beta\right)+B_Y f^{\alpha\beta}K_{\alpha\beta}+(\rho+B_X)K\right]\chi \\ &-(A_X+A_Y f^\beta{}_\sigma u^\sigma u_\beta+\rho)\dot\chi - f^{\alpha\beta}B_Y\bar\nabla_\alpha m_\beta,\end{aligned} \tag{4.5a}$$

$$\begin{aligned}(\rho+P)v^\lambda =& (\rho-C_X)\dot m^\lambda-(P+C_X-C_Y u^\sigma u_\beta f^\beta{}_\sigma)\bar\nabla^\lambda\chi \\ &-C_Y\left\{\gamma^{\lambda(\alpha}u^{\beta)}k_{\alpha\beta}+\gamma^{\lambda\sigma}f^\beta{}_\sigma\dot m_\beta - m_\beta f^{\beta\sigma}K^\lambda{}_\sigma\right\},\end{aligned} \tag{4.5b}$$

$$\begin{aligned}\delta P =& -\frac{1}{3}(3D_X+P+2E_X)\left(\frac{1}{2}h+\bar\nabla^\beta m_\beta\right)-\left(\frac{1}{2}B_Y u^\alpha u^\beta+\frac{1}{6}(3D_Y+2E_Y)\gamma^{\alpha\beta}\right)k_{\alpha\beta} \\ &-f^{\alpha\beta}\frac{1}{3}(3D_Y+2E_Y)\bar\nabla_\alpha m_\beta+(P-B_X+B_Y f^{\alpha\beta}u_\alpha u_\beta)\dot\chi \\ &+\left[\dot P+\frac{1}{3}(P+3D_X+2E_X)K+\frac{1}{3}(3D_Y+2E_Y)f^{\alpha\beta}K_{\alpha\beta}\right. \\ &\left.-\frac{1}{2}\dot f^{\alpha\beta}\left(\frac{1}{3}(3D_Y+2E_Y)\gamma_{\alpha\beta}+A_Y u_\alpha u_\beta\right)\right]\chi,\end{aligned} \tag{4.5c}$$

$$\begin{aligned}P\Pi^\rho{}_\lambda =& 2(P-2E_X)\left[\frac{1}{2}h^\rho{}_\lambda+\bar\nabla^{(\rho}m_{\lambda)}-\frac{1}{3}\gamma^\rho{}_\lambda\left(\frac{1}{2}h+\bar\nabla_\alpha m^\alpha\right)\right] \\ &-E_Y\left[\left(\gamma^{\rho(\alpha}\gamma^{\beta)}{}_\lambda-\frac{1}{3}\gamma^\rho{}_\lambda\gamma^{\alpha\beta}\right)k_{\alpha\beta}+2f^{\alpha\beta}\left(\gamma_\alpha{}^{(\rho}\bar\nabla_{\lambda)}m_\beta - -\frac{1}{3}\gamma^\rho{}_\lambda\bar\nabla_\alpha m_\beta\right)\right].\end{aligned} \tag{4.5d}$$

Assuming time reparameterisation invariance gives

$$\delta\rho = -(A_X+\rho)\left(\frac{1}{2}h+\bar\nabla^\alpha\xi_\alpha\right)-\frac{1}{2}\left(B_Y\gamma^{\alpha\beta}+A_Y u^\alpha u^\beta\right)k_{\alpha\beta}-B_Y f^{\alpha\beta}\bar\nabla_\alpha\xi_\beta, \tag{4.6a}$$

$$(\rho+P)v^\lambda = (\rho-C_X)\dot\xi^\lambda - C_Y\left[\gamma^{\lambda(\alpha}u^{\beta)}k_{\alpha\beta}+f^{\alpha\beta}\left(\gamma^\lambda{}_\alpha\dot\xi_\beta-K^\lambda{}_\alpha\xi_\beta\right)\right], \tag{4.6b}$$

$$\begin{aligned}\delta P &= -\frac{1}{3}(3D_X+P+2E_X)\left(\frac{1}{2}h+\bar\nabla^\beta\xi_\beta\right)-\left(\frac{1}{2}B_Yu^\alpha u^\beta+\frac{1}{6}(3D_Y+2E_Y)\gamma^{\alpha\beta}\right)k_{\alpha\beta}\\ &\quad -\frac{1}{3}f^{\alpha\beta}(3D_Y+2E_Y)\bar\nabla_\alpha\xi_\beta, \qquad (4.6c)\\ P\Pi^\rho{}_\lambda &= 2(P-2E_X)\left[\frac{1}{2}h^\rho{}_\lambda+\bar\nabla^{(\rho}\xi_{\lambda)}-\frac{1}{3}\gamma^\rho{}_\lambda\left(\frac{1}{2}h+\bar\nabla_\alpha\xi^\alpha\right)\right]\\ &\quad -E_Y\left\{\left(\gamma^{\rho(\alpha}\gamma^{\beta)}{}_\lambda-\frac{1}{3}\gamma^\rho{}_\lambda\gamma^{\alpha\beta}\right)k_{\alpha\beta}+2f^{\alpha\beta}\left(\gamma_\alpha{}^{(\rho}\bar\nabla_{\lambda)}\xi_\beta-\frac{1}{3}\gamma^\rho{}_\lambda\bar\nabla_\alpha\xi_\beta\right)\right\}, \qquad (4.6d)\end{aligned}$$

and we obtain evolution equations for P and ρ

$$\begin{aligned}\dot\rho &= -\frac{1}{2}\dot f^{\alpha\beta}\left(B_Y\gamma_{\alpha\beta}+A_Yu_\alpha u_\beta\right)-B_Yf^{\alpha\beta}K_{\alpha\beta}-(\rho+B_X)K,\\ \dot P &= \frac{1}{3}(3D_Y+2E_Y)f^{\alpha\beta}K_{\alpha\beta}+\frac{1}{2}(D_Y\gamma_{\alpha\beta}-B_Yu_\alpha u_\beta)\dot f^{\alpha\beta}-\frac{1}{3}(P+3D_X+2E_X)K.\end{aligned} \qquad (4.7)$$

5 No preferred direction in the coupling tensors

5.1 Preferred direction

Using u_μ and $\gamma_{\mu\nu}$ in the decomposition of the coupling tensors means that we have chosen a preferred direction for the Lagrangian. If we assume there is no preferred direction and only the full metric is seen in the coupling tensors, then (4.3) becomes

$$A^{\mu\nu\alpha\beta} = A_Ag^{\mu\nu}g^{\alpha\beta}+2B_Ag^{\alpha(\mu}g^{\nu)\beta}, \qquad (5.1a)$$
$$B^{\mu\nu\alpha\beta} = A_Bg^{\mu\nu}g^{\alpha\beta}+2B_Bg^{\alpha(\mu}g^{\nu)\beta}, \qquad (5.1b)$$
$$C^{\mu\nu\alpha\beta} = A_Cg^{\mu\nu}g^{\alpha\beta}+2B_Cg^{\alpha(\mu}g^{\nu)\beta}. \qquad (5.1c)$$

In order to obtain (5.1a) we must set the coefficients in (4.3) as follows

$$\begin{array}{lll} A_X=A_A+2B_A & A_Y=A_B+2B_B & A_Z=A_C+2B_C\\ B_X=-A_A & B_Y=-A_B & B_Z=-A_C\\ C_X=-B_A & C_Y=-B_B & C_Z=-B_C\\ D_X=A_A & D_Y=A_B & D_Z=A_C\\ E_X=B_A & E_Y=B_B & E_Z=B_C\\ & F_Y=-A_B. & \end{array} \qquad (5.2)$$

5.2 The perturbed fluid variables

When we use the coupling tensors (5.2), then we find that the perturbed fluid variables when χ is decoupled are

$$\delta\rho=(A_A-\rho)\left(\frac{1}{2}h+\bar\nabla_\alpha\xi^\alpha\right)+A_B\left(f^{\alpha\beta}\nabla_\alpha\xi_\beta+\frac{1}{2}k\right)-\frac{1}{3}B_Bk^{\alpha\beta}u_\alpha u_\beta, \qquad (5.3a)$$

$$(\rho+P)v^\sigma = B_B\left[u^\nu\gamma^\sigma{}_\mu k^\mu{}_\nu + B_B\left(\gamma^\sigma{}_\mu f^{\mu\beta}\dot{\xi}_\beta - f^{\alpha\beta}K^\sigma{}_\alpha\xi_\beta\right)\right] + (B_A+\rho)\dot{\xi}^\sigma, \tag{5.3b}$$

$$\delta P = -\left(\frac{1}{3}(2B_A+P+3A_A)\left(\frac{1}{2}h+\bar{\nabla}_\alpha\xi^\alpha\right)+\frac{1}{2}A_Bk+\frac{1}{3}B_B\gamma^\nu{}_\mu k^\mu{}_\nu\right) \tag{5.3c}$$
$$-\frac{1}{3}(3A_B+2B_B)f^{\alpha\beta}\bar{\nabla}_\alpha\xi_\beta,$$

$$P\Pi^\rho{}_\lambda = 2(P-2B_A)\left(\frac{1}{2}h^\rho{}_\lambda+\bar{\nabla}^{(\rho}\xi_{\lambda)}-\frac{1}{3}\gamma^\rho{}_\lambda\left(\frac{1}{2}h+\bar{\nabla}_\alpha\xi^\alpha\right)\right) \tag{5.3d}$$
$$-2B_B\left[\gamma^\rho{}_\mu\gamma^\nu{}_\lambda k^\mu{}_\nu-\frac{1}{3}\gamma^\nu{}_\mu k^\mu{}_\nu\gamma^\rho{}_\lambda+2f^{\alpha\beta}\left(\gamma_{\alpha(\lambda}\bar{\nabla}^{\rho)}\xi_\beta-\frac{1}{3}\gamma^\rho{}_\lambda\bar{\nabla}_\alpha\xi_\beta\right)\right],$$

and we can write the evolution equations (4.7) as

$$\dot{\rho} = A_B(f^{\alpha\beta}K_{\alpha\beta}+\frac{1}{2}\dot{f})-B_B\dot{f}^{\alpha\beta}u_\alpha u_\beta+(A_A-\rho)K \tag{5.4a}$$

$$\dot{P} = -\frac{1}{6}\left(2(P+3A_A+2B_A)K+3A_B\left(\dot{f}+2f^{\alpha\beta}K_{\alpha\beta}\right)+2B_B\left(\dot{f}^{\alpha\beta}\gamma_{\alpha\beta}+2f^{\alpha\beta}K_{\alpha\beta}\right)\right). \tag{5.4b}$$

6 Choosing a form for $f_{\mu\nu}$

6.1 Conformal and disformal choices

An obvious choice for $f_{\mu\nu}$ is

$$f_{\mu\nu} = A_f\phi g_{\mu\nu} + B_f\nabla_\mu\phi\nabla_\nu\phi \tag{6.1}$$

where A_f and B_f are constants and ϕ is a scalar field. However, because this means that the variation of $f_{\mu\nu}$ contains the variation of the metric, we must modify (4.1). In fact, this choice means that (4.1) can be rewritten as the second variation of a Lagrangian of the form $\mathcal{L}_{(2)}(\delta_O Lg_{\mu\nu}, \delta_L\phi, \nabla_\mu\delta_L\phi)$, i.e.

$$\mathcal{L}_{(2)} = \mathcal{A}(\delta_L\phi)^2+\mathcal{B}^\mu\delta_L\phi\nabla_\mu\delta_L\phi+\frac{1}{2}\mathcal{C}^{\mu\nu}\nabla_\mu\delta_L\phi\nabla_\nu\delta_L\phi$$
$$+\frac{1}{4}\left[\mathcal{Y}^{\alpha\mu\nu}\nabla_\alpha\delta_L\phi\delta_Lg_{\mu\nu}+\mathcal{V}^{\mu\nu}\delta_L\phi\delta_Lg_{\mu\nu}+\frac{1}{2}\mathcal{W}^{\mu\nu\alpha\beta}\delta_Lg_{\mu\nu}\delta_Lg_{\alpha\beta}\right], \tag{6.2}$$

which was studied in [13]. The terms on the second line of (6.2) could be possibly be generated by "Beyond Horndeski" theories [34, 35]. The Beyond Horndeski theories contain terms in a specific combination to avoid the Ostrogradski instability, which would place constraints on the couplings A, B^μ and $C^{\mu\nu}$.

6.2 Flat reference metric

Another choice is a flat reference metric, i.e. $f_{\mu\nu} = \eta_{\mu\nu}$, although this choice does not simplify our perturbed fluid variables greatly while we are still using the generalised lagrangian (4.1).

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7 Summary of results

In this paper, we have

- summarised the calculations given in [14] for the perturbed fluid variables for a general dark Lagrangian of the form $\mathcal{L}_{\{2\}} = \mathcal{L}_{\{2\}}(\delta_L g_{\mu\nu})$ by working in the synchronous gauge and in the perfectly elastic case, both with and without the imposition of time reparameterisation invariance

- obtained new constraints on the values of w and μ for a realistic sound speed under various conditions, using data from the Planck satellite

- rewritten the perturbed fluid variables for general dark Lagrangians of the form $\mathcal{L}_{\{2\}} = \mathcal{L}_{\{2\}}(\delta_L \gamma_{\mu\nu}, \delta_L u_\mu)$, $\mathcal{L}_{\{2\}} = \mathcal{L}_{\{2\}}(\delta_L u_\mu)$ and $\mathcal{L}_{\{2\}} = \mathcal{L}_{\{2\}}(\delta_L \gamma_{\mu\nu})$ both in general and when time reparameterisation invariance is imposed

- obtained new evolution equations for $\dot{P}$ and w for these new Lagrangians and repeated these calculations for tensor-metric theories where $\mathcal{L}_{\{2\}} = \mathcal{L}_{\{2\}}(\delta_L g_{\mu\nu}, \delta_L f_{\mu\nu})$, using either a time-spatial metric split in the coupling tensors or using only the full metric

- found new evolution equations for ρ and P and rewritten the perturbed fluid variables in these theories

There are many different modified gravity theories which attempt to explain the accelerated expansion of the universe. By parameterising theories based on the second variation of the Lagrangian, we were able to develop a framework which can very quickly rule out various theories, and indeed we can rule out any theory which is purely a function of the variation of the time part of the metric. Our method could be used both in future models of massive gravity or to rule out large classes of theories when new observational results are found.

We have examined the connections between these theories and elastic dark energy. For our first case, the elastic dark energy framework can be straightforwardly used. We also placed constraints on the parameters of elastic dark energy using Planck results. Future work could examine whether other dark energy models can use this framework.

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A Evolution of w

We want to find an evolution equation for w, the equation of state parameter where $P = w\rho$. Using the conservation equation, (2.11) and (2.13), and defining

$$\hat{\beta} \equiv \frac{\beta}{\rho}, \tag{A.1}$$

we obtain

$$\dot{w} = 3\left[w(1+w) - \hat{\beta}\right] H, \tag{A.2}$$

which notably does not depend on μ. Hence w is constant if

$$\hat{\beta} = w(1+w), \tag{A.3}$$

and so

$$w = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1+4\hat{\beta}}, \tag{A.4}$$

gives a stable universe. Using 2018 Planck data [28], the 68% constraint on w is $w = -1.028 \pm 0.032$ which in turn gives a constraint of $-0.004 < \beta < 0.106$.

A.1 Sound speed

The sound speed for elastic dark energy is given by [20]

$${c_s}^2 \equiv \frac{\hat{\beta} + \frac{4}{3}\hat{\mu}}{1+w}, \tag{A.5}$$

where $\hat{\mu} = \frac{\mu}{\rho}$. Using (A.3), and in order that the sound speed fulfills $0 \leq {c_s}^2 \leq 1$, i.e. is real and sub-luminal, then for constant w we must have

$$-\frac{3}{4}w(1+w) \leq \hat{\mu} \leq \frac{3}{4}(1-w^2). \tag{A.6}$$

If we set $\hat{\mu} = 0 = \dot{w}$, then (A.6) gives that either $w = -1$ exactly, or

$$0 \leq w \leq 1, \tag{A.7}$$

which leads to a contradiction with the acceleration of the universe, as acceleration requires $w < -\frac{1}{3}$. This means that either $w = -1$ or *a stable universe with zero shear modulus cannot support acceleration of the universe* and we therefore require a non-zero μ. Using the 2018 Planck data [28] together with (A.6) gives us constraints of $-0.0477 \leq \hat{\mu} \leq 0.0599$. However,the assumption that we do not have a phantom equation of state, i.e. we require $w \geq -1$, gives $0 \leq \hat{\mu} \leq 0.0599$.

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