NON-ARITHMETIC MONODROMY OF HIGHER HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. We show that all the currently known non-arithmetic lattices in $\mathrm{PU}(2,1)$ are monodromy groups of higher hypergeometric functions.

1. Introduction

A classical result of Schwarz [16] says that any (orientation preserving) triangle group is the monodromy a hypergeometric differential equation. That is, it is the group of linear maps that record how pairs of hypergeometric functions solving this equation vary when analytically continued around a singularity of the equation. This generalises the well known fact that the modular group $\mathrm{PSL}(2,\mathbb{Z})$ is the monodromy group of an elliptic function. The latter connection raises questions about the arithmetic nature of such monodromy groups. Vinberg [23] and Takeuchi [19] gave criteria for the arithmeticity of reflection groups and Fuchsian groups, respectively. Subsequently, Takeuchi [20] showed that all but finitely many triangle groups in $\mathrm{PSL}(2,\mathbb{R})$ are non-arithmetic. Combining the results of Schwarz and Takeuchi gives infinitely many non-arithmetic hypergeometric monodromy groups.

We now discuss two generalisations of Schwarz's result about hypergeometric monodromy. First, in [3] and [11] Deligne and Mostow considered the monodromy of hypergeometric functions in n variables, originally constructed by Picard [14]. These monodromy groups live in $\mathrm{PU}(n,1)$ (the case $n=1$ gives the classical case since $\mathrm{PU}(1,1)$ is conjugate to $\mathrm{PSL}(2,\mathbb{R})$). Mostow [10] generalised Vinberg's arithmeticity criterion to $\mathrm{PU}(n,1)$ and gave the first examples of non-arithmetic lattices in $\mathrm{PU}(2,1)$. Deligne and Mostow showed that Mostow's lattices are monodromy groups of second order hypergeometric equations in two variables and they produced other examples. In particular, for many years all the known examples of non-arithmetic lattices in $\mathrm{PU}(2,1)$ and $\mathrm{PU}(3,1)$ were contained in the Deligne-Mostow list, and hence were monodromy groups for hypergeometric functions. (It is an open question whether $\mathrm{PU}(n,1)$ contains non-arithmetic lattices for $n \geq 4$.)

In [6] and [7] Deraux, Paupert and I gave some new examples of non-arithmetic lattices in $\mathrm{PU}(2,1)$, which are not commensurable to groups on the Deligne-Mostow list. (Recall, that two groups are said to be commensurable if, after conjugating one of them if necessary, their intersection has finite index in each of them. It is natural to only study lattices up to commensurability.) When discussing these examples, a question we have frequently been asked is whether the new non-arithmetic lattices constructed in [6] and [7] arise as

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monodromy groups for any functions or differential equations. In this paper we give an answer to this question. For completeness, we briefly mention that Deraux [4], [5] has shown that some of the new lattices we construct in [6] and [7] had already been found by Couwenberg, Heckman and Looijenga [2], who do not discuss their arithmetic properties.

A second generalisation of hypergeometric monodromy was studied in detail by Beukers and Heckman [1]. They consider higher order hypergeometric equations in one variable, first constructed by Thomae [21]. In particular, they give a characterisation of hypergeometric groups due to Levelt [9] and also a method of calculating the signature of the Hermitian form preserved by such a group. We discuss this in Section 2 below. Arithmetic monodromy groups of these higher hypergeometric equations have been studied by Singh and Venkataramana [18] and by Fuchs, Mieri and Sarnak [8].

Our main result is:

Theorem 1.1. *With the exception of* $\mathcal{T}(p, \mathbf{E}_2)$ *for* $p = 3$, 6, 12, *all the lattices in* $\mathrm{PU}(2,1)$ *constructed by Deraux, Parker and Paupert in* [7] *are commensurable to monodromy groups of third order hypergeometric equations.*

Using the tables in Section 3.2, we see that an immediate consequence is:

Corollary 1.2. *Each currently known commensurability class of non-arithmetic lattices in* $\mathrm{PU}(2,1)$ *contains a monodromy group for a third order hypergeometric equation.*

We remark that a consequence is that the Deligne-Mostow lattices in $\mathrm{PU}(2,1)$ are commensurable both to monodromy groups of second order hypergeometric equations in two variables and to monodromy groups of third order hypergeometric equations in one variable. It is not clear whether, for each group, there is any relationship between these equations.

It would be interesting to know whether there are any additional higher hypergeometric equations with non-arithmetic monodromy, and perhaps this could give a place to start looking for more non-arithmetic lattices.

In Section 2 we review the necessary background on higher order hypergeometric equations and their solutions, following Beukers and Heckman [1]. In Section 3 we review groups generated by three complex reflections. Our main reference is Deraux, Parker and Paupert [7]. In Section 4 we combine the previous two sections in order to prove Theorem 1.1. The proof is split into different cases according to the families considered in [7]. In each case we exhibit a set of generators for the monodromy group satisfying Levelt's criterion and we give the possible values of the angle parameters α_j, β_j of the associated higher hypergeometric equation. These results are Propositions 4.1, 4.6, 4.10, 4.13 and 4.15 respectively.

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2. Hypergeometric functions

We review hypergeometric equations and functions together with the associated monodromy groups. We do not discuss the case of hypergeometric functions in several variables. This material is discussed at length by Deligne and Mostow [3], [11].

2.1. **The classical case.** We begin with a brief review the classical hypergeometric equation and hypergeometric functions; see Chapter XIV of Whittaker and Watson [24].

We write the Pochhammer symbol

$$(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}. \tag{1}$$

The classical hypergeometric function ${}_2F_1$ is defined by the series

$${}_2F_1(\alpha,\beta,\gamma;z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\cdot\frac{z^k}{k!}.$$

It is a solution to the hypergeometric equation

$$z(1-z)\frac{d^2w}{dz^2} + \big(\gamma-(\alpha+\beta+1)z\big)\frac{dw}{dz} - \alpha\beta w = 0.$$

This equation has three singular points at 0, 1 and ∞. Analytically continuing a pair of independent solutions to this equation along a closed path around these singular points yields two new solutions which are linear combinations of the two initial solutions. The resulting 2×2 matrix is the monodromy of the equation associated to this path. The monodromy group was investigated by Schwarz [16]. In particular, if $1-\gamma$, $\alpha-\beta$ and $\gamma-\alpha-\beta$ are rational numbers with denominators p, q, r, all at least 2, then the monodromy group is the (orientation preserving) (p,q,r) triangle group, which is discrete. This group preserves a Hermitian form which is positive definite when $1/p+1/q+1/r>1$, degenerate when $1/p+1/q+1/r=0$ and indefinite when $1/p+1/q+1/r<1$.

2.2. **Higher hypergeometric equations and functions.** This section is a review of higher order hypergeometric equations and functions and it closely follows the paper [1] by Beukers and Heckman. Our aim is to include the necessary background for later sections. For a fuller account readers should look at [1].

The higher hypergeometric function ${}_nF_{n-1}$ (see equation (1.3) of Beukers and Heckman [1]) is defined to be:

$${}_nF_{n-1}(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_{n-1};z) = \sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_n)_k}{(\beta_1)_k\cdots(\beta_{n-1})_k}\cdot\frac{z^k}{k!}$$

where once again $(\alpha)_k$ denotes the Pochhammer symbol (1). We use the differential operator $\theta = z\frac{d}{dz}$ and define

$$\begin{aligned} D(\alpha;\beta) &= D(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n)\\ &= (\theta+\beta_1-1)\cdots(\theta+\beta_n-1) - z(\theta+\alpha_1)\cdots(\theta+\alpha_n). \end{aligned}$$

Following equation (2.5) of [1], we write the higher hypergeometric equation as

$$D(\alpha;\beta)w = D(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_n)w = 0. \tag{2}$$

If no pair of $\beta_1,\dots,\beta_n$ differ by an integer, then n independent solutions of the equation (2) are given, for $i=1,\dots,n$, by

$$z^{1-\beta_i}\,{}_nF_{n-1}(1+\alpha_1-\beta_i,\dots,1+\alpha_n-\beta_i;1+\beta_1-\beta_i,\overset{\vee}{\cdots},1+\beta-\beta_n-\beta_i;z)$$

where $\vee$ denotes the variable $1+\beta_i-\beta_i$ has been omitted; equation (2.9) of [1].

Beukers and Heckman give the following definition of a hypergeometric group.

Definition 2.1. *(Definition 3.1 of Beukers and Heckman* [1]*) Suppose that $a_1,\dots,a_n$; $b_1,\dots,b_n\in\mathbb{C}-\{0\}$ with $a_j\neq b_k$ for all $j,k\in\{1,\dots,n\}$. A* **hypergeometric group** *$H(a;b)=H(a_1,\dots,a_n;b_1,\dots,b_n)$ with numerator parameters $a_1,\dots,a_n$ and denominator parameters $b_1,\dots,b_n$ is a subgroup of* $\mathrm{GL}(n,\mathbb{C})$ *generated by A and B which have characteristic polynomials*

$$\chi_A(t)=\det(tI-A)=\prod_{j=1}^n(t-a_j),\quad \chi_B(t)=\det(tI-B)=\prod_{j=1}^n(t-b_j)$$

and so that BA^{-1} is a complex reflection, that is $(BA^{-1}-I)$ has rank one.

A **scalar shift** *of the hypergeometric group $H(a;b)$ is a hypergeometric group $H(da;db)=H(da_1,\dots,da_n,db_1,\dots,db_n)$ for some $d\in\mathbb{C}-\{0\}$.*

Hypergeometric groups are monodromy groups of higher hypergeometric equations:

Proposition 2.2 (Proposition 3.2 of Beukers and Heckman [1])**.** *Suppose $a_1,\dots,a_n$; $b_1,\dots,b_n\in\mathbb{C}-\{0\}$ with $a_j\neq b_k$ for all $j,k\in\{1,\dots,n\}$. Let $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n$ be complex numbers satisfying $a_j=e^{2\pi i\alpha_j}$ and $b_j=e^{2\pi i\beta_j}$ for $j=1,\dots,n$. The monodromy group of the hypergeometric equation $D(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_n)w=0$ is a hypergeometric group with parameters $a_1,\dots,a_n$, $b_1,\dots,b_n$.*

Hypergeometric groups were characterised by Levelt:

Theorem 2.3 (Theorem 1.1 of Levelt [9]; Theorem 3.5 of Beukers and Heckman [1])**. .** *Suppose $a_1,\dots,a_n$; $b_1,\dots,b_n\in\mathbb{C}-\{0\}$ with $a_j\neq b_k$ for all $j,k\in\{1,\dots,n\}$. Let $A_1,\dots,A_n$, $B_1,\dots,B_n$ be defined by*

$$\prod_{j=1}^n(t-a_j)=t^n+A_1t^{n-1}+\cdots+A_n,\quad \prod_{j=1}^n(t-b_j)=t^n+B_1t^{n-1}+\cdots+B_n.$$

Let A and B in $\mathrm{GL}(n,\mathbb{C})$ *be defined by:*

$$A=\begin{pmatrix}0&0&\cdots&0&-A_n\\1&0&\cdots&0&-A_{n-1}\\0&1&\cdots&0&-A_{n-2}\\\vdots&\vdots&&\vdots&\vdots\\0&0&\cdots&1&-A_1\end{pmatrix},\quad B=\begin{pmatrix}0&0&\cdots&0&-B_n\\1&0&\cdots&0&-B_{n-1}\\0&1&\cdots&0&-B_{n-2}\\\vdots&\vdots&&\vdots&\vdots\\0&0&\cdots&1&-B_1\end{pmatrix}.$$

Then the matrices A and B generate a hypergeometric group $H(a_1,\dots,a_n,b_1,\dots,b_n)$. Moreover, any hypergeometric group with the same parameters is conjugate inside $\mathrm{GL}(n,\mathbb{C})$ *to this one. Also any scalar shift $H(da_1,\dots,da_n,db_1,\dots,db_n)$ for $d\in\mathbb{C}-\{0\}$ of this group can be obtained by multiplying A and B by d with $|d|=1$.*

In fact, it will be more convenient for us to use a normalisation where the $-A_j$ (respectively $-B_j$) occur in the first row of A (respectively B) rather than the last column. To be precise, we require the following normal forms:

$$A=\begin{pmatrix}-A_1 & -A_2 & \cdots & -A_{n-1} & -A_n\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{pmatrix},\quad B=\begin{pmatrix}-B_1 & -B_2 & \cdots & -B_{n-1} & -B_n\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{pmatrix}$$

It is clear that we can go from one form to the other by taking the transpose then performing the same permutation of rows and columns in both matrices.

2.3. **Monodromy when $n=3$.** For the rest of the paper we restrict our attention to the case of 3×3 matrices. We begin with matrices A and B in the previous section with $n=3$.

$$(3)\quad A=\begin{pmatrix}-A_1 & -A_2 & -A_3\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}=\begin{pmatrix}a_1+a_2+a_3 & -a_1a_2-a_2a_3-a_3a_1 & a_1a_2a_3\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix},$$

$$(4)\quad B=\begin{pmatrix}-B_1 & -B_2 & -B_3\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}=\begin{pmatrix}b_1+b_2+b_3 & -b_1b_2-b_2b_3-b_3b_1 & b_1b_2b_3\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}.$$

Also,

$$A^{-1}=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1/A_3 & -A_1/A_3 & -A_2/A_3\end{pmatrix},\quad B^{-1}=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1/B_3 & -B_1/B_3 & -B_2/B_3\end{pmatrix}.$$

Thus

$$BA^{-1}=\begin{pmatrix}B_3/A_3 & (B_3A_1-B_1A_3)/A_3 & (B_3A_2-B_2A_3)/A_3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

It is clear that $(BA^{-1}-I)$ has rank one, and so it is a complex reflection.

For $j=1,\,2,\,3$ it is easy to see that the vector $\mathbf{a}_j$ below spans the a_j-eigenspace of A. We also write down the image of $\mathbf{a}_j$ under B.

$$(5)\qquad \mathbf{a}_j=\begin{pmatrix}a_j^2\\ a_j\\ 1\end{pmatrix},\quad B\mathbf{a}_j=\begin{pmatrix}-B_1a_j^2-B_2a_j-B_3\\ a_j^2\\ a_j\end{pmatrix}=\begin{pmatrix}a_j^3-\chi_B(a_j)\\ a_j^2\\ a_j\end{pmatrix}.$$

Let U be the following matrix whose columns are eigenvectors for A:

$$U = \begin{pmatrix} a_1^2/(a_1-a_2)(a_1-a_3) & a_2^2/(a_2-a_1)(a_2-a_3) & a_3^2/(a_3-a_1)(a_3-a_2) \\ a_1/(a_1-a_2)(a_1-a_3) & a_2/(a_2-a_1)(a_2-a_3) & a_3/(a_3-a_1)(a_3-a_2) \\ 1/(a_1-a_2)(a_1-a_3) & 1/(a_2-a_1)(a_2-a_3) & 1/(a_3-a_1)(a_3-a_2) \end{pmatrix}.$$

Then

$$U^{-1} = \begin{pmatrix} 1 & -a_2-a_3 & a_2a_3 \\ 1 & -a_3-a_1 & a_1a_3 \\ 1 & -a_1-a_2 & a_1a_2 \end{pmatrix}.$$

Define $A' = U^{-1}AU$ and $B' = U^{-1}BU$. Then, using equation (5) we have:

$$A' = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad B' = \begin{pmatrix} a_1(1+c_1) & a_2c_2 & a_3c_3 \\ a_1c_1 & a_2(1+c_2) & a_3c_3 \\ a_1c_1 & a_2c_2 & a_3(1+c_3) \end{pmatrix} \tag{6}$$

where for $j,\ k \in \{1,\ 2,\ 3\}$:

$$c_j = \frac{-\chi_B(a_j)}{a_j\prod_{k\neq j}(a_k-a_j)} = \frac{(b_j-a_j)}{a_j}\prod_{k\neq j}\frac{(b_k-a_j)}{(a_k-a_j)}. \tag{7}$$

Note that $\mathrm{tr}(BA^{-1}) = 2+(b_1b_2b_3)/(a_1a_2a_3)$ and $\mathrm{tr}(B'A'^{-1}) = 3+c_1+c_2+c_3$. Since these matrices are conjugate we have the identity:

$$b_1b_2b_3 = a_1a_2a_3(c_1+c_2+c_3+1). \tag{8}$$

Beukers and Heckman show that when the eigenvalues of A and B satisfy $|a_j| = |b_j| = 1$, for $j = 1,\ 2,\ 3$, then A and B preserve a Hermitian form and they give a recipe for finding the signature of of this form. We give a direct proof of their result in the 3×3 case.

Theorem 2.4 (Beukers and Heckman, Theorem 4.5 of [1])**.** *Suppose that the eigenvalues $a_1,\ a_2,\ a_3;\ b_1,\ b_2,\ b_3$ of A and B are complex numbers with $|a_j| = |b_j| = 1$ for $j = 1,\ 2,\ 3$. Suppose also that the a_j are distinct. Let $A' = U^{-1}AU$ and $B' = U^{-1}BU$ be given by (6). For $j,\ k \in \{1,2,3\}$ write*

$$d_j \quad = \quad -i\left(\frac{b_1b_2b_3}{a_1a_2a_3}\right)^{-1/2}\frac{(b_j-a_j)}{a_j}\prod_{k\neq j}\frac{(b_k-a_j)}{(a_k-a_j)} = -i\big(\det(BA^{-1})\big)^{-1/2}c_j.$$

Then A' and B' preserve a diagonal Hermitian form $D = \mathrm{diag}(d_1,d_2,d_3)$. Furthermore, any Hermitian form preserved by A' and B' is a real multiple of D.

Proof. Beukers and Heckman give a proof valid for all $n\times n$ matrices of the given form. We give a short direct proof for 3×3 matrices. It is possible to extend this proof to the $n\times n$ case, but we will not need that below.

Let c_j be given by (7). Observe that, since $|a_i| = |b_i| = 1$ for all i, we have

$$\frac{c_j}{\bar{c}_j} = \frac{(b_j-a_j)\bar{a}_j}{a_j(\bar{b}_j-\bar{a}_j)}\prod_{k\neq j}\frac{(b_k-a_j)(\bar{a}_k-\bar{a}_j)}{(a_k-a_j)(\bar{b}_k-\bar{a}_j)} = -\frac{b_1b_2b_3}{a_1a_2a_3}.$$

Define $\psi \in [0, \pi)$ by

$$e^{i\psi} = \left(\frac{b_1 b_2 b_3}{a_1 a_2 a_3}\right)^{1/2} = \left(\det(BA^{-1})\right)^{1/2}. \tag{9}$$

Thus, for $j = 1, 2, 3$, we have $c_j/\bar{c}_j = -e^{2i\psi}$ and so $c_j = ie^{i\psi} d_j$ for some real number d_j. Also note that, using (8), we have

$$d_1 + d_2 + d_3 = -ie^{-i\psi}(c_1 + c_2 + c_3) = -ie^{-i\psi}\left(\frac{b_1 b_2 b_3}{a_1 a_2 a_3} - 1\right) = 2\sin(\psi). \tag{10}$$

Let D be the diagonal matrix $D = \mathrm{diag}(d_1, d_2, d_3)$. We claim that the matrices $A' = U^{-1}AU$ and $B' = U^{-1}BU$ given by (6) satisfy $A'^*DA' = D$ and $B'^*DB' = D$. The first of these is obvious since $A' = U^{-1}AU$ is diagonal with entries a_j where $|a_j| = 1$. A short calculation shows that the ijth entry of $B'^*DB' - D$ is

$$\bar{a}_i a_j \big(\bar{c}_i c_j (d_1 + d_2 + d_3) + \bar{c}_i d_j + c_j d_i\big).$$

Using the definition of d_j and equation (10) we have

$$\bar{c}_i c_j (d_1 + d_2 + d_3) + \bar{c}_i d_j + c_j d_i = d_i d_j 2\sin(\psi) + ie^{i\psi} d_i d_j - ie^{-i\psi} d_i d_j = 0.$$

Hence $B'^*DB' = D$ as claimed.

Finally, provided the a_i are distinct and the c_i are non-zero, we see that $\langle A', B'\rangle$ is irreducible. Hence, any Hermitian form preserved by A' and B' is a real multiple of D. The above construction allows us to give a short direct proof.

First, observe that since A' is diagonal with distinct entries then any Hermitian form it preserves must be diagonal, say $D' = \mathrm{diag}(d_1', d_2', d_3')$. Now consider $B'^*D'B' - D'$. Arguing as above, we see that for $i, j \in \{1, 2, 3\}$ we must have

$$\bar{c}_i c_j (d_1' + d_2' + d_3') + \bar{c}_i d_j' + c_j d_i' = 0.$$

Therefore for all i, j, k we have

$$\begin{aligned} 0 &= c_k\big(\bar{c}_i c_j (d_1' + d_2' + d_3') + \bar{c}_i d_j' + c_j d_i'\big) - c_j\big(\bar{c}_i c_k (d_1' + d_2' + d_3') + \bar{c}_i d_k' + c_k d_i'\big) \\ &= \bar{c}_i (c_k d_j' - c_j d_k'). \end{aligned}$$

Thus there is $\lambda \in \mathbb{C}$ with $\lambda d_j' = c_j$ for $j = 1, 2, 3$. Since d_j' must be real, we see that $\lambda = ie^{i\psi}$, giving the result. □

We can give the d_j in terms of angle parameters as follows. The formula below appears on page 335 of Beukers and Heckman [1] where it is written $F(u_j, u_j)$.

Corollary 2.5. *Let A and B be as in Theorem 2.4. If $a_j = e^{2\pi i\alpha_j}$ and $b_j = e^{2\pi i\beta_j}$ for $j = 1, 2, 3$ then the Hermitian form $D = \mathrm{diag}(d_1, d_2, d_3)$ preserved by A' and B' has entries:*

$$d_j = 2\sin(\pi\beta_j - \pi\alpha_j)\prod_{j\neq k}\frac{\sin(\pi\beta_k - \pi\alpha_j)}{\sin(\pi\alpha_k - \pi\alpha_j)}.$$

Proof. In the expression for d_i in Theorem 2.4 we distribute the square root terms so that each bracket becomes purely imaginary. This yields

$$d_j = -i\big(b_j^{1/2}\bar{a}_j^{1/2} - \bar{b}_j^{1/2}a_j^{1/2}\big)\prod_{k\neq j}\frac{\big(b_k^{1/2}\bar{a}_j^{1/2} - \bar{b}_k^{1/2}a_j^{1/2}\big)}{\big(a_k^{1/2}\bar{a}_j^{1/2} - \bar{a}_k^{1/2}a_j^{1/2}\big)}.$$

Then substituting for $a_j^{1/2} = e^{\pi i\alpha_j}$ and $b_j^{1/2} = e^{\pi i\beta_j}$ gives the result. □

We now briefly connect our proof of Theorem 2.4 with the one given by Beukers and Heckman on page 335 of [1]. They write ζ for a solution of $-1 = \zeta^2(b_1b_2b_3)/(a_1a_2a_3)$. In Proposition 4.4 of [1], they define a vector $\mathbf{u}$ satisfying

$$\zeta\big(B'A'^{-1} - I\big)\mathbf{z} = \pm\langle\mathbf{z},\mathbf{u}\rangle\mathbf{u}.$$

where $A' = U^{-1}AU$ and $B' = U^{-1}BU$ as in (6). We have

$$-ie^{-i\psi}\big(B'A'^{-1} - I\big) = -ie^{-i\psi}\begin{pmatrix} c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}(1,\ 1,\ 1)\begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}.$$

Therefore we may take $\zeta = \mp ie^{i\psi}$ and

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Following [1], we decompose this vector as $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$ where $\mathbf{u}_j$ is an a_j-eigenvector of A'. Since A' is diagonal, it is clear that $\mathbf{u}_j$ is the jth vector in the standard basis for $\mathbb{C}^3$. Since the Hermitian form is $\mathrm{diag}(d_1, d_2, d_3)$ this yields $\langle\mathbf{u}_j,\mathbf{u}_j\rangle = d_j$, which agrees with the statement given on page 335 of [1].

Beukers and Heckman give a list of angle parameters α_j and β_j corresponding to finite primitive hypergeometric groups, that is groups where the Hermitian form is positive definite; Table 8.3 of [1]. For convenience we reproduce this table below. We also add the order of BA^{-1}, which we denote by p. This can be recovered from the α_j and β_j since $e^{2\pi i/p} = (b_1b_2b_3)/(a_1a_2a_3)$ where $a_j = e^{2\pi i\alpha_j}$ and $b_j = e^{2\pi i\beta_j}$. All except one of these groups is a Shephard-Todd group [17] and Beukers and Heckman give the Shephard-Todd number (ST in the table below). (See also [5] for further connections with Shephard-Todd groups.) They also give the field generated by the coefficients of the characteristic polynomials of A

and B.

BH	α_1	α_2	α_3	β_1	β_2	β_3	p	Field	ST
2	3/14	5/14	13/14	0	1/3	2/3	2	$\mathbb{Q}(i\sqrt{7})$	24
3	3/14	5/14	13/14	0	1/4	3/4	2	$\mathbb{Q}(i\sqrt{7})$	24
4	3/14	5/14	13/14	1/7	2/7	4/7	2	$\mathbb{Q}(i\sqrt{7})$	24
5	0	1/5	4/5	1/6	1/2	5/6	2	$\mathbb{Q}(\sqrt{5})$	23
6	0	1/5	4/5	1/10	1/2	9/10	2	$\mathbb{Q}(\sqrt{5})$	23
7	1/6	11/30	29/30	0	1/5	4/5	2	$\mathbb{Q}(\sqrt{5}, i\sqrt{3})$	27
8	1/6	11/30	29/30	0	1/4	3/4	2	$\mathbb{Q}(\sqrt{5}, i\sqrt{3})$	27
9	1/6	2/3	5/6	0	1/4	3/4	3	$\mathbb{Q}(i\sqrt{3})$	25
10	1/9	4/9	7/9	0	1/6	1/2	3	$\mathbb{Q}(i\sqrt{3})$	25
11	1/9	4/9	7/9	0	1/4	3/4	−3	$\mathbb{Q}(i\sqrt{3})$	25
12	1/12	7/12	5/6	0	1/4	3/4	2	$\mathbb{Q}(i\sqrt{3})$	−−

For the group on the last line, the reflection group $\langle A^k(BA^{-1})A^{-k}\rangle$ is imprimitive (see Section 5 of Beukers and Heckman), meaning that there is a direct sum decomposition of $\mathbb{R}^3$ that is permuted by $\langle A^k(BA^{-1})A^{-k}\rangle$. Specifically, writing $\omega = e^{2\pi i/3}$, we have:

$$A = \begin{pmatrix} -\omega & -\bar{\omega} & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The complex reflections $A^k(BA^{-1})A^{-k}$ preserve the decomposition $\mathbb{R}^3 = V_1 \oplus V_2 \oplus V_3$ where

$$V_1 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ -\omega \end{pmatrix}\right\}, \quad V_2 = \text{span}\left\{\begin{pmatrix} 1 \\ \omega \\ -1 \end{pmatrix}\right\}, \quad V_3 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ -\bar{\omega} \end{pmatrix}\right\}.$$

In later sections we relate the groups in this table to groups generated by complex reflections. For ease of reference, we refer to them as BH2 to BH12.

3. Groups generated by three complex reflections

In this section we consider subgroups of $\mathrm{PGL}(3, \mathbb{C})$ generated by three complex reflections. These groups preserve a Hermitian form. Depending on the signature of this form the group acts on $\mathbf{P}^2_{\mathbb{C}}$, $\mathbf{E}^2_{\mathbb{C}}$ or $\mathbf{H}^2_{\mathbb{C}}$. We are interested when the group is a lattice, meaning that it is discrete and the quotient of the above space by this group has finite volume. Our main reference is Deraux, Parker and Paupert [7], which builds on several earlier papers including Mostow [10], Parker and Paupert [12] and Thompson [22].

3.1. **Parameters and angles.** Recall that an element R of $\mathrm{PGL}(3, \mathbb{C})$ is a complex reflection with angle ψ if $(R - I)$ has rank one and R has determinant $e^{i\psi}$. We consider subgroups of $\mathrm{PGL}(3, \mathbb{C})$ generated by three complex reflections, each with angle $\psi = 2\pi/p$. The space of conjugacy classes of such groups has four dimensions; see Pratoussevitch [15] for example.

Following Mostow [10] we normalise the three complex reflections R_1, R_2 and R_3 so that the $e^{2\pi i/p}$-eigenspace of R_j is spanned by the jth standard basis vectors. Note that, rather than normalising the determinants to be 1, we normalise that $(R_j - I)$ has rank one. (See also Section 2.5 of Parker, Paupert [12] for a similar normalisation in the case of 3-fold symmetry.) Specifically:

$$\begin{aligned} R_1 &= \begin{pmatrix} e^{2\pi i/p} & \rho & -\bar{\tau} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ R_2 &= \begin{pmatrix} 1 & 0 & 0 \\ -e^{2\pi i/p}\bar{\rho} & e^{2\pi i/p} & \sigma \\ 0 & 0 & 1 \end{pmatrix}, \\ R_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{2\pi i/p}\tau & -e^{2\pi i/p}\bar{\sigma} & e^{2\pi i/p} \end{pmatrix}. \end{aligned}$$

These matrices preserve the Hermitian form (which is $1/2\sin(\pi/p)$ times the form in [7]):

$$H = \begin{pmatrix} 2\sin(\pi/p) & -ie^{-i\pi/p}\rho & ie^{-i\pi/p}\bar{\tau} \\ ie^{i\pi/p}\bar{\rho} & 2\sin(\pi/p) & -ie^{-i\pi/p}\sigma \\ -ie^{-i\pi/p}\tau & ie^{i\pi/p}\bar{\sigma} & 2\sin(\pi/p) \end{pmatrix}. \tag{11}$$

We claimed above that the space of conjugacy classes of triples of complex reflections all with angle $2\pi/p$ has dimension four. Hence, there is some redundancy in the above parametrisation by three complex numbers. Here is a precise statement which combines results found in Proposition 3.3 of [7], Section 10 of Pratoussevitch [15] or Section 2.3 of Thompson [22].

Proposition 3.1. *Let R_j, R'_j for $j = 1, 2, 3$ be complex reflections with angle $2\pi/p$. Let ρ, σ, τ and ρ', σ', τ' be as defined above. The triples $R_1\, R_2$, R_3 and R'_1, R'_2, R'_3 are conjugate in* $\mathrm{PGL}(3, \mathbb{C})$ *if and only if one of the following is true:*

(1) *If $\rho\sigma\tau \neq 0$ and $p \geq 3$ then*
$$|\rho'| = |\rho|, \quad |\sigma'| = |\sigma|, \quad |\tau'| = |\tau|, \quad \arg(\rho'\sigma'\tau') = \arg(\rho'\sigma'\tau).$$
(2) *If $\rho\sigma\tau \neq 0$ and $p = 2$ then*
$$|\rho'| = |\rho|, \quad |\sigma'| = |\sigma|, \quad |\tau'| = |\tau|, \quad \arg(\rho'\sigma'\tau') = \pm\arg(\rho'\sigma'\tau).$$
(3) *If $\rho\sigma\tau = 0$ then*
$$|\rho'| = |\rho|, \quad |\sigma'| = |\sigma|, \quad |\tau'| = |\tau|.$$

Note that this result means we can freely choose the arguments of two of ρ, σ and τ. Furthermore, if $\rho\sigma\tau = 0$ we may assume, without loss of generality, that $\sigma = 0$ and that ρ, τ are non-negative real numbers.

Let n be a natural number. We say that A and B satisfy a braid relation of length n, written $\mathrm{br}_n(A, B)$, if
$$(AB)^{n/2} = (BA)^{n/2}.$$

This notation means the alternating products of A and B with n terms starting with A and B respectively. So $\mathrm{br}_2(A,B)$ is $AB=BA$, that is A and B commute, and $\mathrm{br}_3(A,B)$ is $ABA=BAB$, which is the classical braid relation. We write $\mathrm{br}(A,B)=n$ for the smallest positive integer for which the braid relation $\mathrm{br}_n(A,B)$ holds.

Using Pratoussevitch's formulae [15], or by direct calculation we find:

$$\mathrm{tr}(R_1R_2)=e^{2\pi i/p}(2-|\rho|^2)+1, \qquad \mathrm{tr}(R_2R_3)=e^{2\pi i/p}(2-|\sigma|^2)+1,$$
$$\mathrm{tr}(R_3R_1)=e^{2\pi i/p}(2-|\tau|^2)+1, \qquad \mathrm{tr}(R_1R_3^{-1}R_2R_3)=e^{2\pi i/p}(2-|\sigma\tau-\bar{\rho}|^2)+1.$$

Using this, following Proposition 2.3 of [7], we observe that

- if $|\rho|=2\cos(\pi/c)$ with $c\in\mathbb{N}$ then $\mathrm{br}(R_1,R_2)=c$;
- if $|\sigma|=2\cos(\pi/a)$ with $a\in\mathbb{N}$ then $\mathrm{br}(R_2,R_3)=a$;
- if $|\tau|=2\cos(\pi/b)$ with $b\in\mathbb{N}$ then $\mathrm{br}(R_3,R_1)=b$;
- if $|\sigma\tau-\bar{\rho}|=2\cos(\pi/d)$ with $d\in\mathbb{N}$ then $\mathrm{br}(R_1,R_3^{-1}R_2R_3)=d$.

In the case above we say the group has **braiding parameters** $(a,b,c;d)$. Hence the braiding parameters $(a,b,c;d)$ completely determine $|\rho|$, $|\sigma|$, $|\tau|$ and $\Re(\rho\sigma\tau)$. Thus, together with p, they determine two possible groups $\langle R_1,R_2,R_3\rangle$ up to conjugacy, the ambiguity coming from the sign of $\Im(\rho\sigma\tau)$; see Remark 3.1 of [7]. However, conjugating by an antiholomorphic map (for example complex conjugation) has the effect of changing the sign of both p and $\arg(\rho\sigma\tau)$. Hence we may assume $\arg(\rho\sigma\tau)\in[0,\pi]$ if we allow p to be negative. Thus, it follows from Proposition 3.1 that, if we allow p to be negative, the braiding parameters $(a,b,c;d)$ determine the group $\langle R_1,R_2,R_3\rangle$ up to conjugacy (possibly by an antiholomorphic map).

There is a special case when $\sigma=0$ and so $a=2$. This means that $\mathrm{br}(R_2,R_3)=2$, that is R_2 and R_3 commute. Also, $\arg(\rho\sigma\tau)$ is not defined, since $\sigma=0$. Moreover, we have $2\cos(\pi/d)=|\bar{\rho}|=2\cos(\pi/c)$ and so $d=c$. As we observed above, $|\rho|$ and $|\tau|$ completely determine the group and so, together with p, the parameters b and c determine the group, which has braiding parameters $(2,b,c;c)$.

3.2. **Lattices and arithmeticity.** In the tables below we write down the four braiding parameters $(a,b,c;d)$ (as described in the previous section) for each of the lattices constructed in Deraux-Parker-Paupert [7]. For each of these sets of parameters we give the values of p (which may be negative) where the corresponding group is discrete. Depending on the signature of the Hermitian form H these act on one of $\mathbf{P}^2_{\mathbb{C}}$, $\mathbf{E}^2_{\mathbb{C}}$ or $\mathbf{H}^2_{\mathbb{C}}$. We indicate this by giving three columns with the heading p corresponding, respectively, to these three cases. In the last of these cases, we write the value of p in bold face when $\langle R_1,R_2,R_3\rangle$ is non-arithmetic. The arithmeticity criterion we use is due to Mostow [10]. We will not go into details of how to apply this criterion here, since it has been discussed at length for these groups in the paper [7]. In the next section (Propositions 4.1, 4.6, 4.10, 4.13 and 4.15 respectively) we will show that all of these groups are monodromy groups of higher hypergeometric functions except those for $(3,4,4;4)$ with p a multiple of 3. We indicate these values of p in parenthesis.

In the last column we give notes indicating more information about these groups. If a group is written in square brackets then it indicates that the two groups are commensurable but that the standard generators do not correspond to the generators we give. When the braiding parameters are $(3, 3, 3; m)$ then $\langle R_1, R_2, R_3\rangle$ is the Mostow group $\Gamma(p, t)$ where $t = 1/p + 2/m - 1/2$ is Mostow's parameter; see Section A.9 of [7]. The sporadic groups $\mathcal{S}(p, -)$ and Thompson groups $\mathcal{T}(p, -)$ are described in Sections A.1 to A.8 of [7]. Commensurability relations between them are given in Section 7 of [7]. The relationship between these groups and the Couwenberg-Heckman-Looijenga lattices is given in Theorem 1 of Deraux [5]. The groups with rotations of order p and braiding parameters $(n, n, n; n)$ for $n = 4$, 5 are subgroups of $\Gamma(n, 1/n + 2/p - 1/2)$, that is lattices with rotations of order n and braiding parameters $(3, 3, 3; p)$; see Proposition 5.1 of Parker and Paupert [12]. Finally, we indicate which of the Beukers-Heckman groups (acting on $\mathbf{P}^2_{\mathbb{C}}$) lie in this family and we indicate the value of p, so BH$N(p)$ refers to the Nth Beukers-Heckman group and for this group the complex reflection has order p; see Propositions 4.2, 4.3, 4.4, 4.7, 4.8 and 4.11 below.

(1) Three-fold symmetry $(n, n, n; m)$. The three columns headed p corresponding to groups acting on $\mathbf{P}^2_{\mathbb{C}}$, $\mathbf{E}^2_{\mathbb{C}}$ or $\mathbf{H}^2_{\mathbb{C}}$ respectively.

n	m	p	p	p	Notes
3	2	2, 3	4	5, 6, 7, 8, **9**, 10, 12, 18	$\Gamma(p, 1/p + 1/2)$, $\mathcal{C}(p, G_{25})$, BH11(3)
3	3	2		4, 5, 6, **7**, **8**, 9, **10**, 12, 18	$\Gamma(p, 1/p + 1/6)$, [BH12(2)]
3	4	2		3, 4, **5**, **6**, 8, 12	$\Gamma(p, 1/p)$
3	5	2		3, **4**, 5, 10	$\Gamma(p, 1/p - 1/10)$
3	6	2		3, **4**, **6**	$\Gamma(p, 1/p - 1/6)$
3	7	2		**3**, 7	$\Gamma(p, 1/p - 3/14)$
3	8	2		3, 4	$\Gamma(p, 1/p - 1/4)$
3	9	2		3	$\Gamma(p, 1/p - 5/18)$
3	10	2		3	$\Gamma(p, 1/p - 3/10)$
3	12	2		3	$\Gamma(p, 1/p - 1/3)$
4	3	2		3, **4**, **5**, **6**, **8**, **12**	$\mathcal{S}(p, \bar{\sigma}_4)$, $\mathcal{C}(p, G_{24})$, BH2(2)
4	4		2	3, 4, **5**, **6**, 8, 12	$[\Gamma(4, 2/p - 1/4)]$
4	5			2, **3**, **4**	$\mathcal{S}(p, \sigma_5)$
5	3	2		3, 4, 5, 10	$\mathcal{S}(p, \sigma_{10})$, $\mathcal{C}(p, G_{23})$, BH5(2),
5	5			2, 3, **4**, 5, 10	$[\Gamma(5, 2/p - 3/10)]$
6	4		2	**3**, **4**, **6**	$\mathcal{S}(p, \sigma_1)$

(2) Two-fold symmetry $(n, n, m; m)$

n	m	p	p	p	Notes
3	3	2		4, 5, 6, **7**, **8**, 9, **10**, 12, 18	BH12(2)
3	4	2		3, **4**, **5**, **6**, **8**, **12**	$\mathcal{T}(p, \mathbf{S}_1)$
3	5			2, **3**, 5, 10, −5	$\mathcal{T}(p, \mathbf{H}_2)$
4	3	2	3	4, 5, 6, 8, 12	$[\mathcal{T}(p, \mathbf{S}_4)]$
4	4		2	3, 4, **5**, **6**, 8, 12	$[\Gamma(4, 2/p - 1/4)]$
5	4	2		3, **4**, **5**	$[\mathcal{T}(p, \mathbf{S}_2)]$
5	5			2, 3, **4**, 5, 10	$[\Gamma(5, 2/p - 3/10)]$

(3) $(3, 3, 4; n)$

n	p	p	p	Notes
3	2		3, 4, **5**, **6**, 8, 12	$[\Gamma(p, 1/p)]$
4	2		3, **4**, **5**, **6**, **8**, **12**	$\mathcal{T}(p, \mathbf{S}_1)$, $[\mathcal{S}(p, \bar{\sigma}_4)]$
5	2		3, **4**, **5**	$\mathcal{T}(p, \mathbf{S}_2)$, $\mathcal{C}(p, G_{27})$, BH7(2)
6		2	**3**, **4**, **5**	$\mathcal{T}(p, \mathbf{E}_1)$, $[\mathcal{S}(p, \sigma_1)]$
7			2, 7	$\mathcal{T}(p, \bar{\mathbf{H}}_1)$

(4) $(2, 3, n; n)$

n	p	p	p	Notes
3	2, 3	4	5, 6, 7, 8, **9**, 10, 12, 18	$\mathcal{T}(p, \mathbf{S}_3)$, $[\Gamma(p, 1/p + 1/2)]$
4	2	3	4, 5, 6, 8, 12	$\mathcal{T}(p, \mathbf{S}_4)$, $\mathcal{C}(p, G_{26})$, $[\Gamma(p, 3/p - 1/2)]$
5	2		3, 4, 5, 10	$\mathcal{T}(p, \mathbf{S}_5)$, $[\mathcal{S}(p, \sigma_{10})]$
6		2	3, 4, 6	$\mathcal{T}(p, \mathbf{E}_3)$

(5) $(3, 4, 4; 4)$

p	p	p	Notes
	2	(3), **4**, (6), (12)	$\mathcal{T}(p, \mathbf{E}_2)$

4. Groups generated by three reflections as higher monodromy groups

In this section we prove the main result of the paper. To do so, we compare the hypergeometric groups described in Section 2 and the groups generated by three complex reflections described in Section 3, in particular non-arithmetic lattices in $\mathrm{PU}(2,1)$ with this property. The main issue is that the hypergeometric groups have two generators whereas the complex reflection groups have three generators. We get around this in several ways. First, for some groups there is an extra symmetry, which means that the group generated by complex reflections has finite index in a two-generator group. Secondly, for other groups we use known relations between the generators to find a generating set with two elements. In the final subsection, we discuss how this approach goes wrong for the groups with braiding parameters $(3, 4, 4; 4)$ and angle $2\pi/p$ where p is divisible by 3.

We consider the groups from [7] in families corresponding to the tables is Section 3, treating each family in a separate section. For each family of groups (possibly by adjoining symmetries) we give generators satisfying Levelt's criterion from Theorem 2.3, showing that the reflection group is (commensurable to) a hypergeometric group. We explicitly write down the parameters α_j and β_j of the associated hypergeometric equation (2) for each group. We also give connections of these groups to Beukers and Heckman's groups BH2 to BH12 [1]. Finally, for the first two families we show how to pass between the Hermitian form D given in [1] and the form H given in [7]. A similar construction applies in the other cases, but the precise expressions are more complicated.

4.1. **Lattices with three-fold symmetry.** We suppose we have braid relations

$$\mathrm{br}(R_1,R_2)=\mathrm{br}(R_2,R_3)=\mathrm{br}(R_3,R_1)=n,\quad \mathrm{br}(R_1,R_3^{-1}R_2R_3)=m.$$

(Such groups with $n=3$ were studied by Mostow [10].) In this case $\rho=\sigma=\tau$ and there is a symmetry J which (projectively) has order 3 and satisfies $R_2=JR_1J^{-1}$ and $R_3=JR_2J^{-1}=J^{-1}R_1J$. The parameter τ is determined up to complex conjugation by

$$|\tau|=2\cos(\pi/n),\quad |\tau^2-\bar{\tau}|=2\cos(\pi/m).$$

The particular values of τ giving rise to lattices are:

n	m	τ	n	m	τ
3	k	$-e^{-2\pi i/3k}$	k	k	$e^{4\pi i/3k}+e^{-2\pi i/3k}$
4	3	$(-1-i\sqrt{7})/2$	4	5	$e^{-i\pi/9}(\sqrt{5}+i\sqrt{3})/2$
5	3	$(1+\sqrt{5})/2$	6	4	$-1+i\sqrt{2}$

In the $(3,k)$ case we get $k=2$, 3, 4, 5, 6, 7, 8, 9, 10 or 12; in the (k,k) case we get a lattice when $k=3$, 4 or 5.

The group $\langle R_1,J\rangle$ contains $\langle R_1,R_2,R_3\rangle$ as a subgroup of index 1 or 3. It has generators:

$$A=J=\begin{pmatrix}0&0&e^{-2\pi i/p}\\1&0&0\\0&1&0\end{pmatrix},\quad B=R_1J=\begin{pmatrix}\tau&-\bar{\tau}&1\\1&0&0\\0&1&0\end{pmatrix}. \tag{12}$$

Observe that these matrices are in the form (3) and (4). Moreover, $BA^{-1}=R_1$ is a complex reflection. Therefore, we have:

Proposition 4.1. *The matrices A and B given by* (12) *satisfy Levelt's criterion and so $\langle R_1,J\rangle$ is a hypergeometric group.*

Note that we have $\det(A)=1$, $\mathrm{tr}(A)=\tau$, $\det(B)=e^{-2\pi i/p}$ and $\mathrm{tr}(B)=0$. From this we can find the eigenvalues $a_j=e^{2\pi i\alpha_j}$ and $b_j=e^{2\pi i\beta_j}$ of A and B. The parameters α_j and β_j are given, up to a scalar shift, by:

n	m	α_1	α_2	α_3	β_1	β_2	β_3
3	k	$1/3-1/3p$	$2/3-1/3p$	$1-1/3p$	$1/6k$	$1/2-1/3k$	$1/2+1/6k$
4	3	$1/3-1/3p$	$2/3-1/3p$	$1-1/3p$	$3/7$	$5/7$	$6/7$
4	4	$1/3-1/3p$	$2/3-1/3p$	$1-1/3p$	$1/12$	$1/6$	$3/4$
4	5	$1/3-1/3p$	$2/3-1/3p$	$1-1/3p$	$1/9$	$13/90$	$67/90$
5	3	$1/3-1/3p$	$2/3-1/3p$	$1-1/3p$	0	$1/5$	$4/5$
5	5	$1/3-1/3p$	$2/3-1/3p$	$1-1/3p$	$1/10$	$2/15$	$23/30$
6	4	$1/3-1/3p$	$2/3-1/3p$	$1-1/3p$	$1/8$	$3/8$	$1/2$

Comparing these parameters with those from Beukers and Heckman, we can identify several of the Beukers-Heckman groups, possibly after a scalar shift and a change of generators.

Proposition 4.2. *The groups BH2 and BH4 are isomorphic to the group* $\langle R_1, J\rangle$ *in the triangle group with braiding parameters* $(4,4,4;3)$ *and* $p=2$*; that is* $\mathcal{S}(2,\bar{\sigma}_4)$*. Specifically:*

1. *The group BH2 is a scalar shift of* $\mathcal{S}(2,\bar{\sigma}_4)$ *with generators* $A = R_1J$ *and* $B = J$.
2. *The group BH4 is* $\mathcal{S}(2,\bar{\sigma}_4)$ *with generators* $A = J^{-1}R_1J^{-1}$ *and* $B = (R_1J^{-1})^2$.

Proof. The group $\mathcal{S}(2,\bar{\sigma}_4)$ has $p=2$ and $\tau = (-1-i\sqrt{7})/2 = e^{6\pi i/7} + e^{10\pi i/7} + e^{12\pi i/7}$.

1. The parameters of $\mathcal{S}(2,\bar{\sigma}_4)$ are 3/7, 5/7, 6/7; 1/6, 1/2, 5/6. Adding 1/2 to each of these (mod 1) and reordering (including swapping the roles of A and B) gives 0, 1/3, 2/3; 3/14, 5/14, 13/14, which are the parameters of BH2. This proves (1).
2. In the group $\mathcal{S}(2,\bar{\sigma}_4)$ write $A = J^{-1}R_1J^{-1}$ and $B = (R_1J^{-1})^2$. Note that $BA^{-1} = R_1$. We calculate
$$\det(A) = \det(J^{-1}R_1J^{-1}) = -1, \quad \operatorname{tr}(A) = \operatorname{tr}(J^{-1}R_1J^{-1}) = -\tau.$$
Thus the eigenvalues of $J^{-1}R_1J^{-1}$ are $-e^{10\pi i/7}$, $-e^{12\pi i/7}$, $-e^{6\pi i/7}$, so $\alpha_1 = 3/14$, $\alpha_2 = 5/14$, $\alpha_3 = 13/14$. Similarly,
$$\det(R_1J^{-1}) = 1, \quad \operatorname{tr}(R_1J^{-1}) = \bar{\tau}.$$
Therefore the eigenvalues of R_1J^{-1} are $e^{2\pi i/7}$, $e^{4\pi i/7}$ and $e^{8\pi i/7}$. Hence the eigenvalues of $B = (R_1J^{-1})^2$ are also $e^{2\pi i/7}$, $e^{4\pi i/7}$ and $e^{8\pi i/7}$. Therefore $\beta_1 = 1/7$, $\beta_2 = 2/7$ and $\beta_3 = 4/7$. This means that the parameters of $\langle A, B\rangle$ are the same as the parameters of BH4.

   It remains to show that A and B generate $\langle R_1, J\rangle$. Since R_1J^{-1} has order 7 so $B^3 = (R_1J^{-1})^6 = (R_1J^{-1})^{-1}$. Therefore $R_1 = BA^{-1}$ and $J^{-1} = AB^3$ so $\langle A, B\rangle = \langle R_1, J\rangle$.

□

Proposition 4.3. *The groups BH5 and BH6 are isomorphic to the group* $\langle R_1, J\rangle$ *in the triangle group with braiding parameters* $(5,5,5;3)$ *and* $p=2$*; that is* $\mathcal{S}(2,\sigma_{10})$*. Specifically:*

1. *The group BH5 is* $\mathcal{S}(2,\sigma_{10})$ *with generators* $A = R_1J$ *and* $B = J$.
2. *The group BH6 is* $\mathcal{S}(2,\sigma_{10})$ *with generators* $A = J^{-1}R_1J^{-1}$ *and* $B = (R_1J^{-1})^2$.

Proof. The group $\mathcal{S}(2,\sigma_{10})$ has $p=2$ and $\tau = (1+\sqrt{5})/2 = 1 + e^{2\pi i/5} + e^{8\pi i/5}$.

1. The parameters of BH5 are 0, 1/5, 4/5; 1/6, 1/2, 5/6 which are the same as $\mathcal{S}(2,\sigma_{10})$ after swapping the roles of A and B.
2. This proof is very similar to the proof of Proposition 4.2 (2). The eigenvalues of $A = J^{-1}R_1J^{-1}$ are $-e^{8\pi i/5}$, -1 and $-e^{2\pi i/5}$. Hence we have $\alpha_1 = 3/10$, $\alpha_2 = -1$ and $\alpha_3 = 7/10$. The eigenvalues of R_1J^{-1} are 1, $e^{2\pi i/5}$ and $e^{8\pi i/5}$ and so the eigenvalues of $B = (R_1J^{-1})^2$ are 1, $e^{4\pi i/5}$ and $e^{6\pi i/5}$. So $\beta_1 = 0$, $\beta_2 = 2/5$ and $\beta_3 = 3/5$. Performing a scalar shift by 1/2 gives the parameters of BH6. Finally, $R_1 = BA^{-1}$ and $J^{-1} = AB^2$.

□

Proposition 4.4. *The groups BH9, BH10 and BH11 are isomorphic to the group* $\langle R_1, J\rangle$ *in the triangle group with braiding parameters* $(3,3,3;2)$ *and* $p=3$*; that is the Mostow group* $\Gamma(3,5/6)$*. Specifically:*

1. *The group BH9 is* $\Gamma(3,5/6)$ *with generators by* $A=R_1J^{-1}$ *and* $B=R_1^{-1}J^{-1}$*.*
2. *The group BH10 is* $\Gamma(3,5/6)$ *with generators by* $A=J^{-1}$ *and* $B=R_1J^{-1}$*.*
3. *The group BH11 is* $\Gamma(3,5/6)$ *with generators* $A=J^{-1}$ *and* $B=R_1^{-1}J^{-1}$*.*

Proof. The group $\Gamma(3,5/6)$ has $p=3$ and $\tau=-e^{-\pi i/3}=e^{2\pi i/3}=e^{2\pi i/3}+e^{i\pi/6}+e^{7\pi i/6}$.

1. We take $A=R_1J^{-1}$ and $B=R_1^{-1}J^{-1}$. Note that $AB^{-1}=(R_1J^{-1})(JR_1)=R_1^2=R_1^{-1}$ since R_1 has order 3. We have
$$\det(A)=e^{4\pi i/3},\qquad \mathrm{tr}(A)=-e^{2\pi i/3}\bar\tau=-1,$$
$$\det(B)=1,\qquad \mathrm{tr}(R_1^{-1}J^{-1})=\bar\tau=-e^{i\pi/3}.$$
Therefore the parameters for this group are 1/3, 1/2, 5/6; 5/12, 2/3, 11/12. These differ from the parameters for BH9 by a scalar shift of 1/3.
2. We take $A=J^{-1}$ and $B=R_1J^{-1}$. Note that $AB^{-1}=R_1^{-1}$. We have
$$\det(J^{-1})=e^{2\pi i/3},\qquad \mathrm{tr}(J^{-1})=0,$$
$$\det(R_1J^{-1})=e^{4\pi i/3},\qquad \mathrm{tr}(R_1J^{-1})=-e^{2\pi i/3}\bar\tau=-1.$$
Therefore, the parameters of this group are 1/9, 4/9, 7/9; 1/3, 1/2, 5/6. These differ from the parameters for the group BH10 by a scalar shift by 2/3.
3. We take $A=J^{-1}$ and $B=R_1^{-1}J^{-1}$. Arguing as above, we see the parameters of this group are 1/9, 4/9, 7/9; 5/12, 2/3, 11/12. These differ from the parameters of BH11 by a scalar shift of 1/3.

□

Finally, we compare the Hermitian forms D from Theorem 2.4 and H from (11). From general theory (for example the last part of Theorem 2.4) we know they have the same signature, but it is instructive to make this explicit. We have:
$$H=\begin{pmatrix}2\sin(\pi/p) & -ie^{-i\pi/p}\tau & ie^{-i\pi/p}\bar\tau\\ ie^{i\pi/p}\bar\tau & 2\sin(\pi/p) & -ie^{-i\pi/p}\tau\\ -ie^{i\pi/p}\tau & ie^{i\pi/p}\bar\tau & 2\sin(\pi/p)\end{pmatrix}.$$
Write $\omega=e^{2\pi i/3}$. The eigenvalues of J are $a_1=\omega e^{-2\pi i/3p}$, $a_2=\bar\omega e^{-2\pi i/3p}$ and $a_3=e^{-2\pi i/3p}$. A matrix V whose columns are eigenvectors for J is:
$$V=\begin{pmatrix}\omega e^{-2\pi i/3p}/3 & \bar\omega e^{-2\pi i/3p}/3 & e^{-2\pi i/3p}/3\\ 1/3 & 1/3 & 1/3\\ \bar\omega e^{2\pi i/3p}/3 & \omega e^{2\pi i/3p}/3 & e^{-2\pi i/3p}/3\end{pmatrix}.$$

Then a short calculation shows that $V^*HV = D = \mathrm{diag}(d_1,\, d_2,\, d_3)$ where

$$\begin{aligned} d_1 &= \big(2\sin(\pi/p) - i\bar{\omega}e^{-i\pi/3p}\tau + i\omega e^{i\pi/3p}\bar{\tau}\big)/3, \\ d_2 &= \big(2\sin(\pi/p) - i\omega e^{-i\pi/3p}\tau + i\bar{\omega} e^{i\pi/3p}\bar{\tau}\big)/3, \\ d_3 &= \big(2\sin(\pi/p) - ie^{-i\pi/3p}\tau + i\bar{\tau}\big)/3. \end{aligned}$$

This agrees with the value of d_j found in Theorem 2.4, namely for $\{j,k,\ell\} = \{1,2,3\}$

$$d_j = \frac{ie^{-i\pi/p}(a_j^3 - \tau a_j^2 + \bar{\tau}a_j - 1)}{a_j(a_k - a_j)(a_\ell - a_j)}.$$

4.2. **Lattices with two-fold symmetry.** We suppose we have braid relations

$$\mathrm{br}(R_2,R_3) = \mathrm{br}(R_3,R_1) = n, \quad \mathrm{br}(R_1,R_2) = \mathrm{br}(R_1, R_3^{-1}R_2R_3) = m.$$

These groups were studied by Thompson [22] and by Parker and Sun [13]. Following [13], we choose the normalisation $\sigma = \tau = \sqrt{\rho+\overline{\rho}} = 2\cos(\pi/n)$ and $|\rho| = 2\cos(\pi/m)$. There is a symmetry Q with

$$\begin{aligned} QR_1Q^{-1} &= R_1R_2R_1^{-1}, & QR_2Q^{-1} &= R_1R_3R_1R_3^{-1}R_1^{-1}, \\ QR_3Q^{-1} &= R_1R_3R_1^{-1}, & Q^2 &= R_1R_2R_3. \end{aligned}$$

The values of ρ giving lattices are:

n	m	ρ	n	m	ρ
3	3	$e^{i\pi/3}$	3	4	$(1+i\sqrt{7})/2$
3	5	$-e^{2\pi i/5} - e^{4i\pi/5}$	4	3	1
4	4	$1+i$	5	4	$e^{i\pi/3}(\sqrt{5}-i\sqrt{3})/2$
5	5	$e^{2\pi i/5}+1$			

Note that when $n = m$ we recover groups that also fall into the $(n,n,n;m)$ category, but we obtain a different set of generators.

Lemma 4.5. *The group $\langle R_1,R_2,R_3,Q\rangle$ is generated by R_1 and Q.*

Proof. We have

$$R_2 = R_1^{-1}QR_1Q^{-1}R_1, \quad R_3 = R_2^{-1}R_1^{-1}Q^2 = (R_1^{-1}Q)^2.$$

□

In this case generators are given by:

$$R_1^{-1}Q = \begin{pmatrix} 0 & e^{-2\pi i/p} & 0 \\ 1 & 0 & 0 \\ 0 & \sqrt{\rho+\bar{\rho}} & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} \rho & 1-\rho-\bar{\rho} & \sqrt{\rho+\bar{\rho}} \\ 1 & 0 & 0 \\ 0 & \sqrt{\rho+\bar{\rho}} & -1 \end{pmatrix}.$$

Conjugating by

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{\rho+\overline{\rho}} \end{pmatrix}$$

gives $A = C^{-1}(R_1^{-1}Q)C$ and $B = C^{-1}QC$ in the forms (3) and (4):

$$A = \begin{pmatrix} -1 & e^{-2\pi i/p} & e^{-2\pi i/p} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \rho - 1 & 1 - \overline{\rho} & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{13}$$

Note that $BA^{-1} = C^{-1}R_1C$ is a complex reflection. Therefore, we have:

Proposition 4.6. *The matrices A and B given* (13) *satisfy Levelt's criterion and so $\langle R_1, R_2, R_3, Q\rangle$ is a hypergeometric group.*

Note that $\det(B) = e^{-2\pi i/p}$, $\mathrm{tr}(B) = -1$, $\det(A) = 1$ and $\mathrm{tr}(A) = \rho - 1$. From this we can find the eigenvalues $a_j = e^{2\pi i \alpha_j}$ and $b_j = e^{2\pi i \beta_j}$ of A and B. Up to a scalar shift, the parameters α_j and β_j are given by:

n	m	α_1	α_2	α_3	β_1	β_2	β_3
3	3	$1/2-1/2p$	$1/2$	$1-1/2p$	$1/12$	$1/3$	$7/12$
3	4	$1/2-1/2p$	$1/2$	$1-1/2p$	$1/7$	$2/7$	$4/7$
3	5	$1/2-1/2p$	$1/2$	$1-1/2p$	$1/15$	$1/5$	$11/15$
4	3	$1/2-1/2p$	$1/2$	$1-1/2p$	0	$1/3$	$2/3$
4	4	$1/2-1/2p$	$1/2$	$1-1/2p$	$1/8$	$1/4$	$5/8$
5	4	$1/2-1/2p$	$1/2$	$1-1/2p$	$1/15$	$4/15$	$2/3$
5	5	$1/2-1/2p$	$1/2$	$1-1/2p$	$3/20$	$1/5$	$13/20$

Comparing these parameters with those from Beukers and Heckman, we obtain:

Proposition 4.7. *The group BH3 is isomorphic to the group $\langle R_1, Q\rangle$ in the triangle group with braiding parameters $(3,3,4;4)$ and $p = 2$, that is the group $\mathcal{T}(2, \mathbf{S}_1)$, and generators $A = Q^{-1}$ and $B = R_1Q^{-1}$.*

Proof. We have $p = 2$ and $\rho - 1 = (-1 + i\sqrt{7})/2 = e^{2\pi i/7} + e^{4\pi i/7} + e^{8\pi i/7}$. Therefore

$$\begin{aligned} \det(Q^{-1}) &= 1, & \mathrm{tr}(Q^{-1}) &= \bar{\rho} - 1 = (-1 - i\sqrt{7})/2, \\ \det(R_1Q^{-1}) &= -1, & \mathrm{tr}(R_1Q^{-1}) &= -1. \end{aligned}$$

Therefore the parameters of this group are $\alpha_1 = 3/7$, $\alpha_2 = 5/7$, $\alpha_3 = 6/7$, $\beta_1 = 1/4$, $\beta_2 = 1/2$ and $\beta_3 = 3/4$. These differ from the parameters for BH3 by a scalar shift by $1/2$. □

We remark that Proposition 7.1 (1) of [7] says that $\mathcal{T}(p, \mathbf{S}_1)$ is isomorphic to $\mathcal{S}(p, \bar{\sigma}_4)$. That is, for the same value of p the groups with braiding parameters $(3,3,4;4)$ and $(4,4,4;3)$ are isomorphic. So this result indicates BH3 is isomorphic to BH2 and BH4; see [1].

Proposition 4.8. *The group BH12 is isomorphic to the group $\langle R_1, Q\rangle$ in the triangle group with braiding parameters $(3,3,3;3)$ and $p = 2$, that is the Mostow group $\Gamma(2, 2/3)$, and generators $A = Q$ and $B = R_1^{-1}Q$.*

Proof. We see that the parameters of $\Gamma(2,2/3)$ with these generators are $\alpha_1 = 1/12$, $\alpha_2 = 1/3$, $\alpha_3 = 7/12$, $\beta_1 = 1/4$, $\beta_2 = 1/2$ and $\beta_3 = 3/4$. These differ from the parameters of BH12 by a scalar shift of 1/2. □

Finally, we compare the Hermitian forms H and D using the same method as in the previous section. We have

$$H = \begin{pmatrix} 2\sin(\pi/p) & -ie^{-i\pi/p}\rho & ie^{-i\pi/p}\sqrt{\rho+\bar\rho} \\ ie^{i\pi/p}\bar\rho & 2\sin(\pi/p) & -ie^{-i\pi/p}\sqrt{\rho+\bar\rho} \\ -ie^{i\pi/p}\sqrt{\rho+\bar\rho} & ie^{i\pi/p}\sqrt{\rho+\bar\rho} & 2\sin(\pi/p) \end{pmatrix}.$$

The eigenvalues of $R_1^{-1}Q$ are $a_1 = -e^{-i\pi/p}$, $a_2 = -1$ and $a_3 = e^{-i\pi/p}$ and a matrix V whose columns are eigenvectors for $R_1^{-1}Q$ are

$$V = \begin{pmatrix} -e^{-i\pi/p}/2 & 0 & e^{-i\pi/p}/2 \\ 1/2 & 0 & 1/2 \\ \sqrt{\rho+\bar\rho}/2(1-e^{-i\pi/p}) & \sqrt{\rho+\bar\rho}/2\sin(\pi/p) & \sqrt{\rho+\bar\rho}/2(1+e^{-i\pi/p}) \end{pmatrix}$$

Then we calculate $V^*HV = D = \mathrm{diag}(d_1,\, d_2,\, d_3)$ where

$$\begin{aligned} d_1 &= \frac{4\sin^2(\pi/p) - (1+e^{-i\pi/p})\rho - (1+e^{i\pi/p})\bar\rho}{4\sin(\pi/p)}, \\ d_2 &= \frac{\rho+\bar\rho}{2\sin(\pi/p)}, \\ d_3 &= \frac{4\sin^2(\pi/p) - (1-e^{-i\pi/p})\rho - (1-e^{i\pi/p})\bar\rho}{4\sin(\pi/p)}. \end{aligned}$$

A simple calculation shows that d_j is the value given in Theorem 2.4, namely for $\{j,k,\ell\} = \{1,2,3\}$

$$d_j = \frac{ie^{-i\pi/p}\big(a_j^3 - (\rho-1)a_j^2 + (\bar\rho-1)a_j - 1\big)}{a_j(a_k-a_j)(a_\ell-a_j)}.$$

4.3. $(3,3,4;n)$ **triangle groups.** In this case we suppose that

$$\mathrm{br}(R_1,R_2) = 4, \quad \mathrm{br}(R_2,R_3) = \mathrm{br}(R_1,R_3) = 3, \quad \mathrm{br}(R_1, R_3^{-1}R_2R_3) = n.$$

This means $|\sigma| = |\tau| = 1$ and $|\rho| = \sqrt{2}$. Since we are free to choose the arguments of two of these parameters, we take $\sigma = \tau = 1$; see Thompson [22]. Therefore, $|\rho\sigma - \bar\tau| = |\rho - 1| = 2\cos(\pi/n)$. The values of ρ corresponding to lattices are

n	ρ	n	ρ
3	$1+i$	4	$(1+i\sqrt{7})/2$
5	$e^{2\pi i/3}(\sqrt{5}-i\sqrt{3})/2$	6	$i\sqrt{2}$
7	$e^{6\pi i/7}(-1+i\sqrt{7})/2$		

Proposition 4.9. *Suppose that* $\mathrm{br}(R_1,R_2) = 4$ *and* $\mathrm{br}(R_2,R_3) = \mathrm{br}(R_1,R_3) = 3$ *then* $\langle R_1,R_2,R_3\rangle$ *is generated by* R_2 *and* $R_1R_2R_3$.

Proof. Using $\mathrm{br}(R_2,R_3)=3$, then $\mathrm{br}(R_1,R_3)=3$ and then $\mathrm{br}(R_1,R_2)=4$, we have:

$$\begin{aligned}
&R_2(R_1R_2R_3)^2R_2(R_1R_2R_3)^{-2}R_2^{-1}\\
&= R_2R_1R_2R_3R_1(R_2R_3R_2R_3^{-1}R_2^{-1})R_1^{-1}R_3^{-1}R_2^{-1}R_1^{-1}R_2^{-1}\\
&= R_2R_1R_2(R_3R_1R_3R_1^{-1}R_3^{-1})R_2^{-1}R_1^{-1}R_2^{-1}\\
&= R_2R_1R_2R_1R_2^{-1}R_1^{-1}R_2^{-1}\\
&= R_1.
\end{aligned}$$

Then $R_3=R_2^{-1}R_1^{-1}(R_1R_2R_3)$. That is,

$$(R_1R_2R_3)^2R_2^{-1}(R_1R_2R_3)^{-2}R_2^{-1}(R_1R_2R_3)=R_3.$$

Therefore R_1, R_2 and R_3 are all contained in $\langle R_2,\,(R_1R_2R_3)\rangle$. □

This means that $\langle R_1,R_2,R_3\rangle$ is generated by $(R_1R_2R_3)$ and $(R_2^{-1}R_1R_2R_3)$. Moreover, $(R_1R_2R_3)(R_2^{-1}R_1R_2R_3)^{-1}=R_2$ which is a complex reflection. We multiply both of these matrices by $e^{2\pi i/p}$, which is a scalar shift.

$$\begin{aligned}
e^{-2\pi i/p}R_2^{-1}R_1R_2R_3 &= \begin{pmatrix} \rho-2 & 1 & \rho-1\\ 2-2\bar\rho-e^{-2\pi i/p}\bar\rho & \bar\rho+e^{-2\pi i/p} & 2-\bar\rho\\ 1 & -1 & 1\end{pmatrix},\\
e^{-2\pi i/p}R_1R_2R_3 &= \begin{pmatrix} \rho-2 & 1 & \rho-1\\ 1-\bar\rho & 0 & 1\\ 1 & -1 & 1\end{pmatrix}.
\end{aligned}$$

Conjugating by

$$C=\begin{pmatrix} 0 & 1 & -\rho\\ 1 & 1-\rho & -1\\ 0 & -1 & \rho-1\end{pmatrix}$$

gives matrices in the form (3) and (4):

$$\begin{aligned}
A &= C^{-1}(e^{-2\pi i/p}R_2^{-1}R_1R_2R_3)C\\
&= \begin{pmatrix} \rho+\bar\rho-1+e^{-2\pi i/p} & -1+e^{-2\pi i/p}(1-\rho-\bar\rho) & e^{-2\pi i/3p}\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}, \qquad (14)\\
B &= C^{-1}(e^{-2\pi i/p}R_1R_2R_3)C\\
&= \begin{pmatrix} \rho-1 & 1-\bar\rho & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}, \qquad (15)
\end{aligned}$$

(16)

Observe that $BA^{-1}=C^{-1}R_2C$ is a complex reflection. Therefore, we have:

Proposition 4.10. *The matrices A and B given by* (14) *and* (15) *satisfy Levelt's criterion and so $\langle R_1,R_2,R_3\rangle$ is a hypergeometric group.*

Therefore, we have $\det(B) = 1$, $\mathrm{tr}(B) = \rho - 1$, $\det(A) = e^{-2\pi i/p}$ and

$$\mathrm{tr}(A) = \rho + \overline{\rho} - 1 + e^{-2\pi i/p} = 2 - |\rho - 1|^2 + e^{-2\pi i/p} = -2\cos(2\pi/n) + e^{-2\pi i/p}.$$

From this we can calculate $a_j = e^{2\pi i\alpha_j}$ and $b_j = e^{2\pi i\beta_j}$ the eigenvalues of A and B. We have:

n	α_1	α_2	α_3	β_1	β_2	β_3
3	$1/2 - 1/n$	$1/2 + 1/n$	$1 - 1/p$	$1/8$	$1/4$	$5/8$
4	$1/2 - 1/n$	$1/2 + 1/n$	$1 - 1/p$	$1/7$	$2/7$	$4/7$
5	$1/2 - 1/n$	$1/2 + 1/n$	$1 - 1/p$	$2/15$	$1/3$	$8/15$
6	$1/2 - 1/n$	$1/2 + 1/n$	$1 - 1/p$	$1/8$	$3/8$	$1/2$
7	$1/2 - 1/n$	$1/2 + 1/n$	$1 - 1/p$	$23/42$	$4/7$	$37/42$

Proposition 4.11. *The groups BH7 and BH8 are isomorphic to the group with braiding parameters* $(3, 3, 4; 5)$ *and* $p = 2$*; that is* $\mathcal{T}(2, \mathbf{S}_2)$*. Specifically:*

(1) *The group BH7 is* $\mathcal{T}(2, \mathbf{S}_2)$ *with generators* $A = R_3R_2R_1$ *and* $B = R_2R_3R_2R_1$*.*
(2) *The group BH8 is* $\mathcal{T}(2, \mathbf{S}_2)$ *with generators* $A = R_3R_2R_1$ *and* $B = R_2R_1$*.*

Proof. (1) Let $A = R_3R_2R_1 = (R_1R_2R_3)^{-1}$ and $B = R_2R_3R_2R_1 = (R_1R_2R_3R_2^{-1})^{-1}$. We find that

$$\det(A) = -1, \qquad \mathrm{tr}(A) = 1 - \bar{\rho} = e^{i\pi/3}(1 + \sqrt{5})/2 = e^{i\pi/3}(1 + e^{2\pi i/5} + e^{8\pi i/5}),$$
$$\det(B) = 1, \qquad \mathrm{tr}(B) = 2 - \rho - \bar{\rho} = (1 + \sqrt{5})/2 = 1 + e^{2\pi i/5} + e^{8\pi i/5}.$$

Therefore the parameters of this group are $\alpha_1 = 1/6$, $\alpha_2 = 11/30$, $\alpha_3 = 29/30$, $\beta_1 = 0$, $\beta_2 = 1/5$ and $\beta_3 = 4/5$. These are the parameters of BH7.

(2) We take $A = R_3R_2R_1$ and $B = R_2R_1$. The parameters α_j are the same as in part (1). We have

$$\det(B) = 1, \quad \mathrm{tr}(B) = 1 = 1 + i - i.$$

Therefore the parameters of B are $\beta_1 = 0$, $\beta_2 = 1/4$ and $\beta_3 = 3/4$. These are the parameters of BH8.

□

We could compare the Hermitian forms D and H as in earlier sections. The same method works, but the matrix V is slightly harder to write down. Therefore we leave the details to the reader.

4.4. $(2, 3, n; n)$ **triangle groups.** In this case we suppose

$$\mathrm{br}(R_2, R_3) = 2, \quad \mathrm{br}(R_1, R_3) = 3, \quad \mathrm{br}(R_1, R_2) = n.$$

This means that $\sigma = 0$, $|\tau| = 1$ and $|\rho| = 2\cos(\pi/n)$. This group is rigid and we may take $\tau = 1$ and $\rho = 2\cos(\pi/n)$ for $n = 3$, 4, 5 or 6; see [7].

Proposition 4.12. *Suppose that* $\mathrm{br}(R_2, R_3) = 2$ *and* $\mathrm{br}(R_1, R_3) = 3$*. Then* $\langle R_1, R_2, R_3\rangle$ *is generated by* R_3 *and* R_1R_2*.*

Proof. Using $\mathrm{br}(R_2, R_3) = 2$ and $\mathrm{br}(R_1, R_3) = 3$ we have

$$\begin{aligned} R_1 &= R_3R_1R_3R_1^{-1}R_3^{-1} = R_3(R_1R_2)R_3(R_1R_2)^{-1}R_3^{-1}, \\ R_2 &= R_1^{-1}(R_1R_2) = R_3(R_1R_2)R_3^{-1}(R_1R_2)^{-1}R_3^{-1}(R_1R_2). \end{aligned}$$

□

We take R_1R_2 and $R_3R_1R_2$ as generators. We do a scalar shift by multiplying both generators by $e^{-2\pi i/p}$. We have

$$\begin{aligned} e^{-2\pi i/p}R_1R_2 &= \begin{pmatrix} 1-|\rho|^2 & \rho & -e^{-2\pi i/p} \\ -\bar\rho & 1 & 0 \\ 0 & 0 & e^{-2\pi i/p} \end{pmatrix}, \\ e^{-2\pi i/p}R_3R_1R_2 &= \begin{pmatrix} 1-|\rho|^2 & \rho & -e^{-2\pi i/p} \\ -\bar\rho & 1 & 0 \\ e^{2\pi i/p} - e^{2\pi i/p}|\rho|^2 & e^{2\pi i/p}\rho & 0 \end{pmatrix}. \end{aligned}$$

Conjugating by

$$C = \begin{pmatrix} 0 & -e^{-2\pi i/p} & e^{-2\pi i/p} \\ 0 & 0 & e^{-2\pi i/p}\bar\rho \\ 1 & |\rho|^2 - 2 & 1 \end{pmatrix}$$

gives

$$\begin{aligned} A &= C^{-1}(e^{-2\pi i/p}R_1R_2)C \\ &= \begin{pmatrix} 2-|\rho|^2+e^{-2\pi i/p} & e^{-2\pi i/p}|\rho|^2 - 2e^{-2\pi i/p} - 1 & e^{-2\pi i/p} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad (17) \\ B &= C^{-1}(e^{-2\pi i/p}R_3R_1R_2)C \\ &= \begin{pmatrix} 2-|\rho|^2 & |\rho|^2-2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, . \qquad (18) \end{aligned}$$

Observe that $BA^{-1} = C^{-1}R_3C$ is a complex reflection. Therefore, we have:

Proposition 4.13. *The matrices A and B given by* (17) *and* (18) *satisfy Levelt's criterion and so $\langle R_1, R_2, R_3\rangle$ is a hypergeometric group.*

Since $|\rho| = 2\cos(\pi/n)$ we see that

$$\mathrm{tr}(A) = e^{-2\pi i/p} - 2\cos(2\pi/n), \quad \mathrm{tr}(B) = 2 - 4\cos^2(\pi/n) = -2\cos(2\pi/n).$$

Thus

n	α_1	α_2	α_3	β_1	β_2	β_3
3	$1/2-1/n$	$1/2+1/n$	$1-1/p$	0	$1/4$	$3/4$
4	$1/2-1/n$	$1/2+1/n$	$1-1/p$	0	$1/3$	$2/3$
5	$1/2-1/n$	$1/2+1/n$	$1-1/p$	0	$2/5$	$3/5$
6	$1/2-1/n$	$1/2+1/n$	$1-1/p$	0	$1/2$	$1/2$

4.5. $(3,4,4;4)$ **triangle groups.** In this case we suppose we have braid relations

$$\mathrm{br}(R_2,R_3)=3,\quad \mathrm{br}(R_1,R_3)=\mathrm{br}(R_1,R_2)=\mathrm{br}(R_1,R_3^{-1}R_2R_3)=4.$$

Following Thompson [22], we choose $\rho=\tau=\sqrt{2}$ and $\sigma=-\overline{\omega}=e^{i\pi/3}$. The group generated by the R_j is called $\mathcal{T}(p,\mathbf{E}_2)$. Among the values of p for which these groups are discrete there is one non-arithmetic lattice, $\mathcal{T}(4,\mathbf{E}_2)$. We show below that this group is a subgroup of a hypergeometric monodromy group. Unfortunately, it seems that this is not true for some other values of p.

There is a symmetry S of order 3 satisfying:

$$SR_1S^{-1}=R_1,\quad SR_2S^{-1}=R_3,\quad SR_3S^{-1}=R_3^{-1}R_2R_3. \tag{19}$$

In particular, $SR_3^{-1}R_2R_3S^{-1}=R_2$ and $SR_2R_3S^{-1}=R_2R_3$.

Proposition 4.14. *If p is not divisible by 3 then $\langle R_1,R_2,R_3,S\rangle$ is generated by R_3 and SR_1.*

Proof. When p, which is the order of R_1, is not a multiple of 3, say $p=3m\pm1$, since S and R_1 commute and S^3 is the identity, we have

$$(SR_1)^{3m}=(S^3)^mR_1^{3m}=R_1^{p\mp1}=R_1^{\mp1}.$$

Thus R_1 (and hence S) lies in the group $\langle SR_1,\,R_3\rangle$. Furthermore $R_2=S^{-1}R_3S$ and so R_1, R_2, R_3 and S all lie in $\langle SR_1,\,R_3\rangle$. □

As a matrix S is given by

$$S=\begin{pmatrix}1&0&0\\0&0&\omega\\0&-1&-\bar{\omega}\end{pmatrix}.$$

We take SR_1 and R_3SR_1 as generators and we perform a scalar shift by multiplying by $e^{-2\pi i/p}$. We have

$$\begin{aligned}
e^{-2\pi i/p}SR_1 &= \begin{pmatrix}1&\sqrt{2}e^{-2\pi i/p}&-\sqrt{2}e^{-2\pi i/p}\\0&0&\omega e^{-2\pi i/p}\\0&-e^{-2\pi i/p}&-\bar{\omega}e^{-2\pi i/p}\end{pmatrix},\\
e^{-2\pi i/p}R_3SR_1 &= \begin{pmatrix}1&\sqrt{2}e^{-2\pi i/p}&-\sqrt{2}e^{-2\pi i/p}\\0&0&\omega e^{-2\pi i/p}\\\sqrt{2}e^{2\pi i/p}&1&-2\end{pmatrix}.
\end{aligned}$$

Conjugating by

$$C=\begin{pmatrix}0&-\sqrt{2}&\sqrt{2}\omega e^{-2\pi i/p}\\0&\omega&-\omega\\e^{2\pi i/p}&-e^{2\pi i/p}&0\end{pmatrix}$$

gives

$$(20)\quad A \;=\; C^{-1}(e^{-2\pi i/p}SR_1)C = \begin{pmatrix} 1-\bar\omega e^{-2\pi i/p} & \bar\omega e^{-2\pi i/p}-\omega e^{-4\pi i/p} & \omega e^{-4\pi i/p} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$(21)\quad B \;=\; C^{-1}(e^{-2\pi i/p}R_3SR_1)C = \begin{pmatrix} -1 & \omega e^{-2\pi i/p} & \omega e^{-2\pi i/p} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note that $BA^{-1} = C^{-1}R_3C$ is a complex reflection. Therefore we have:

Proposition 4.15. *The matrices A and B given by* (20) *and* (21) *satisfy Levelt's criterion. Therefore, when p is not divisible by 3 the group $\langle R_1, R_2, R_3, S\rangle$ is a hypergeometric group.*

Since we have

$$\begin{array}{ll} \det(A) = \omega e^{-4\pi i/p}, & \mathrm{tr}(A) = 1-\bar\omega e^{-2\pi i/p} = 1+e^{-2\pi i/p}+\omega e^{-2\pi i/p}, \\ \det(B) = \omega e^{-2\pi i/p}, & \mathrm{tr}(B) = -1, \end{array}$$

we can calculate the eigenvalues of A as $a_1 = 1$, $a_2 = \omega e^{-2\pi i/p}$, $a_3 = e^{-2\pi i/p}$ and the eigenvalues of B as $b_1 = -\bar\omega e^{-i\pi/p}$, $b_2 = -1$, $b_3 = \bar\omega e^{-i\pi/p}$. Therefore, the group has the following angle parameters:

α_1	α_2	α_3	β_1	β_2	β_3
0	$1/3-1/p$	$1-1/p$	$1/6-1/2p$	$1/2$	$2/3-1/2p$

In contrast, we now show this method will not work for $\langle R_1, R_2, R_3, S\rangle$ when $p = 3$ or 6. Of course, this does not rule out the possibility that these groups may be commensurable to a two generator group.

Proposition 4.16. *Suppose that $p = 3$ or $p = 6$. Then the group $\Gamma = \langle R_1, R_2, R_3, S\rangle$ does not have a presentation with two generators. In particular, it is not a hypergeometric group.*

Proof. When $p = 3, 4, 6$ a presentation for $\langle R_1, R_2, R_3\rangle$ is given in Section A.6 of [7]. It is:

$$\left\langle R_1,\, R_2,\, R_3 \;\middle|\; \begin{array}{c} R_1^p,\, R_2^p,\, R_3^p,\, (R_1R_2R_3)^6,\, \mathrm{br}_3(R_2,R_3), \\ \mathrm{br}_4(R_3,R_1),\, \mathrm{br}_4(R_1,R_2),\, \mathrm{br}_4(R_1,R_2R_3R_2^{-1}),\, \mathrm{br}_6(R_3,R_1R_2R_1^{-1}), \\ (R_1R_2)^{\frac{4p}{p-4}},\, (R_1R_3)^{\frac{4p}{p-4}},\, (R_1R_2R_3R_2^{-1})^{\frac{4p}{p-4}},\, (R_3R_1R_2R_1^{-1})^{\frac{3p}{p-3}} \end{array} \right\rangle.$$

(A relation should be omitted when the denominator of its exponent is zero or negative.)

We now adjoin S and use the relations (19) to get a presentation for $\Gamma = \langle R_1, R_2, R_3, S\rangle$. (Note that since S normalises $\langle R_1, R_2, R_3\rangle$ this is easy to do.) We eliminate the generator R_3 by substituting $R_3 = SR_2S^{-1}$. Note that

$$R_2 = S(R_3^{-1}R_2R_3)S^{-1} = S^{-1}R_2^{-1}S^{-1}R_2SR_2S$$

is equivalent to $\mathrm{br}_4(R_2,S)$. In turn, this implies

$$\begin{aligned} R_3^{-1}R_2^{-1}R_3^{-1}R_2R_3R_2 &= (SR_2^{-1}S^{-1})R_2^{-1}(SR_2^{-1}S^{-1})R_2(SR_2S^{-1})R_2 \\ &= S(R_2^{-1}S^{-1}R_2^{-1}S^{-1})S^{-1}R_2^{-1}S^{-1}(R_2SR_2S)SR_2 \\ &= S(S^{-1}R_2^{-1}S^{-1}R_2^{-1})S^{-1}R_2^{-1}S^{-1}(SR_2SR_2)SR_2 \\ &= 1. \end{aligned}$$

Thus $\mathrm{br}_4(R_2,S)$ implies $\mathrm{br}_3(R_2,R_3)=\mathrm{br}_3(R_2,SR_2S^{-1})$. Hence Γ has a presentation

$$\left\langle R_1,\,R_2,\,S \;\middle|\; \begin{array}{c} R_1^p,\,R_2^p,\,S^3,\,(R_1R_2SR_2S^{-1})^6, \\ \mathrm{br}_2(R_1,S),\,\mathrm{br}_4(R_2,S),\,\mathrm{br}_4(R_1,R_2),\,\mathrm{br}_6(SR_2S^{-1},R_1R_2R_1^{-1}), \\ (R_1R_2)^{\frac{4p}{p-4}},\,(SR_2S^{-1}R_1R_2R_1^{-1})^{\frac{3p}{p-3}} \end{array} \right\rangle.$$

Now consider the abelianisation Γ' of Γ. We claim that Γ' is a direct product of two cyclic groups of order p and a group of order 3. Since $p=3$ or 6 is divisible by 3 we see that this group requires at least three generators. If Γ were to have a two generator presentation then this would lead to a presentation for Γ' with at most two generators, which is a contradiction. Therefore the result follows from this claim.

We now prove the claim. We investigate the effect of abelianisation on each of the relations. Recall we are only considering $p=3$ or $p=6$:

- Since p divides 6 the abelianisation of $(R_1R_2SR_2S^{-1})^6$ follows from R_1^p and R_2^p.
- The abelianisation of any braid relation of even length is the trivial relation.
- Since p divides $4p/(p-4)$ the abelianisation of $(R_1R_2)^{\frac{4p}{p-4}}$ follows from R_1^p and R_2^p.
- When $p=6$, we see that p divides $3p/(p-3)$ and so the abelianisation of the relation $(SR_2S^{-1}R_1R_2R_1^{-1})^{\frac{3p}{p-3}}$ follows from R_2^p.

Hence, the only relations in Γ' are R_1^p, R_2^p, S^3 and that the generators commute. Therefore $\Gamma'=\langle R_1\rangle\times\langle R_2\rangle\times\langle S\rangle$ as claimed. □

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