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CLIQUE-WIDTH FOR GRAPH CLASSES CLOSED UNDER COMPLEMENTATION*

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Abstract. Clique-width is an important graph parameter due to its algorithmic and structural properties. A graph class is hereditary if it can be characterized by a (not necessarily finite) set $\mathcal{H}$ of forbidden induced subgraphs. We study the boundedness of clique-width of hereditary graph classes closed under complementation. First, we extend the known classification for the $|\mathcal{H}| = 1$ case by classifying the boundedness of clique-width for every set $\mathcal{H}$ of self-complementary graphs. We then completely settle the $|\mathcal{H}| = 2$ case. In particular, we determine one new class of $(H, \overline{H})$-free graphs of bounded clique-width (as a side effect, this leaves only five classes of (H_1, H_2)-free graphs, for which it is not known whether their clique-width is bounded). Once we have obtained the classification of the $|\mathcal{H}| = 2$ case, we research the effect of forbidding self-complementary graphs on the boundedness of clique-width. Surprisingly, we show that for *every* set $\mathcal{F}$ of self-complementary graphs on at least five vertices, the classification of the boundedness of clique-width for $(\{H, \overline{H}\} \cup \mathcal{F})$-free graphs coincides with the one for the $|\mathcal{H}| = 2$ case if and only if $\mathcal{F}$ does not include the bull.

Key words. clique-width, hereditary graph class, dichotomy

AMS subject classifications. 05C78, 05C75, 05C69

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1. Introduction.

Many graph-theoretic problems that are computationally hard for general graphs may still be solvable in polynomial time if the input graph can be decomposed into large parts of "similarly behaving" vertices. Such decompositions may lead to an algorithmic speedup and are often defined via some type of graph construction. One particular type is to use vertex labels and to allow certain graph operations, which ensure that vertices labeled alike will always keep the same label and thus behave identically. The clique-width $\mathrm{cw}(G)$ of a graph G is the minimum number of different labels needed to construct G using four such operations (see section 2 for details). Clique-width has been studied extensively both in algorithmic and structural graph theory. The main reason for its popularity is that, indeed, many well-known NP-hard problems [16, 27, 38, 43], such as COLORING and HAMILTON CYCLE, become polynomial-time solvable on any graph class $\mathcal{G}$ of *bounded* clique-width, that is, for which there exists a constant c, such that every graph in $\mathcal{G}$ has clique-width at most c. GRAPH ISOMORPHISM is also polynomial-time solvable on such graph classes [32].

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Having bounded clique-width is equivalent to having bounded rank-width [42] and having bounded NLC-width [36], two other well-known width parameters. However, despite these close relationships, clique-width is a notoriously difficult graph parameter, and our understanding of it is still very limited. For instance, no polynomial-time algorithms are known for computing the clique-width of very restricted graph classes, such as unit interval graphs, or for deciding whether a graph has clique-width at most 4.[1] In order to get a better understanding of clique-width and to identify new "islands of tractability" for central NP-hard problems, many graph classes of bounded and unbounded clique-width have been identified; see, for instance, the Information System on Graph Classes and their Inclusions [26], which keeps a record of such graph classes. In this paper we study the following research question:

What kinds of properties of a graph class ensure that its clique-width is bounded?

We refer to the surveys [33, 37] for examples of such properties. Here, we consider graph complements. The complement $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ and edge set $\{uv \mid uv \notin E(G)\}$ and has clique-width $\mathrm{cw}(\overline{G}) \leq 2\,\mathrm{cw}(G)$ [17]. This result implies that a graph class $\mathcal{G}$ has bounded clique-width if and only if the class consisting of all complements of graphs in $\mathcal{G}$ has bounded clique-width. Due to this, we initiate a *systematic* study of the boundedness of clique-width for graph classes $\mathcal{G}$ *closed under complementation*, that is, for every graph $G \in \mathcal{G}$, its complement $\overline{G}$ also belongs to $\mathcal{G}$.

To get a handle on graph classes closed under complementation, we restrict ourselves to graph classes that are not only closed under complementation but also under vertex deletion. This is a natural assumption, as deleting a vertex does not increase the clique-width of a graph. A graph class closed under vertex deletion is said to be *hereditary* and can be characterized by a (not necessarily finite) set $\mathcal{H}$ of forbidden induced subgraphs. Over the years many results on the (un)boundedness of clique-width of hereditary graph classes have appeared. We briefly survey some of these results below.

A hereditary graph class of graphs is monogenic or H-free if it can be characterized by one forbidden induced subgraph H, and bigenic or (H_1, H_2)-free if it can be characterized by two forbidden induced subgraphs H_1 and H_2. It is well known (see [25]) that a class of H-free graphs has bounded clique-width if and only if H is an induced subgraph of P_4.[2] By combining known results [3, 7, 9, 10, 11, 12, 13, 19, 20, 23, 41] with new results for bigenic graph classes, Dabrowski and Paulusma [25] classified the (un)boundedness of clique-width of (H_1, H_2)-free graphs for all but 13 pairs (H_1, H_2) (up to an equivalence relation). Afterwards, five new classes of (H_1, H_2)-free graphs were identified by Dabrowski, Dross, and Paulusma, [18] and two others were identified by Dabrowski, Lozin, and Paulusma [22] and Bonamy et al. [4], respectively. Other systematic studies were performed for H-free weakly chordal graphs [7], H-free chordal graphs [7] (two open cases), H-free triangle-free graphs [22] (two open cases), H-free bipartite graphs [24], H-free split graphs [6] (two open cases), and $\mathcal{H}$-free graphs, where $\mathcal{H}$ is any set of 1-vertex extensions of the P_4 [8] or any set of graphs on at most four vertices [9]. Clique-width results or techniques for these graph classes impacted upon each other and could also be used for obtaining new results for bigenic graph classes.

[1] It is known that computing clique-width is NP-hard in general [29] and that deciding whether a graph has clique-width at most 3 is polynomial-time solvable [15].

[2] We refer to section 2 for all the notation used in this section.

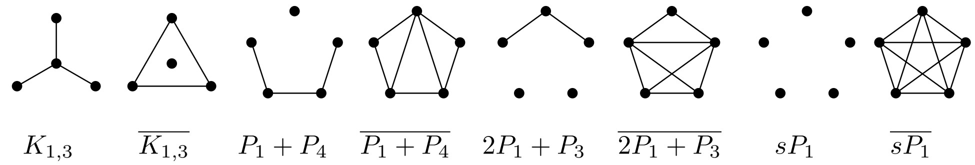

FIG. 1. *Graphs H for which the clique-width of $(H, \overline{H})$-free graphs is bounded. For sP_1 and $\overline{sP_1}$ the $s = 5$ case is shown.*

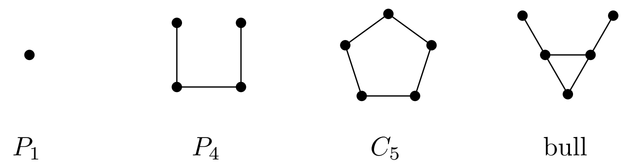

FIG. 2. *The four nonempty self-complementary graphs on fewer than eight vertices* [44].

Our contribution. Recall that we investigate the clique-width of hereditary graph classes closed under complementation. A graph that contains no induced subgraph isomorphic to a graph in a set $\mathcal{H}$ is said to be *$\mathcal{H}$-free*. We first consider the $|\mathcal{H}| = 1$ case. The class of H-free graphs is closed under complementation if and only if H is a self-complementary graph, that is, $H = \overline{H}$. Self-complementary graphs have been extensively studied; see [28] for a survey. From the aforementioned result for P_4-free graphs, we find that the only self-complementary graphs H for which the class of H-free graphs has bounded clique-width are $H = P_1$ and $H = P_4$. In section 3 we prove the following generalization of this result.

THEOREM 1. *Let $\mathcal{H}$ be a set of nonempty self-complementary graphs. Then the class of $\mathcal{H}$-free graphs has bounded clique-width if and only if either $P_1 \in \mathcal{H}$ or $P_4 \in \mathcal{H}$.*

We now consider the $|\mathcal{H}| = 2$ case. Let $\mathcal{H} = \{H_1, H_2\}$. Due to Theorem 1 we may assume $H_2 = \overline{H_1}$ and H_1 is not self-complementary. The class of $(2P_1+P_3, \overline{2P_1 + P_3})$-free graphs was one of the remaining bigenic graph classes, and the only bigenic graph class closed under complementation, for which boundedness of clique-width was open. In section 4 we settle this case by proving that the clique-width of this class is bounded. In the same section we combine this new result with known results to prove the following theorem, which, together with Theorem 1, shows to what extent the property of being closed under complementation helps with bounding the clique-width for bigenic graph classes (see also Figure 1).

THEOREM 2. *For a graph H, the class of $(H, \overline{H})$-free graphs has bounded clique-width if and only if H or $\overline{H}$ is an induced subgraph of $K_{1,3}$, $P_1 + P_4$, $2P_1 + P_3$, or sP_1 for some $s \geq 1$.*

For the $|\mathcal{H}| = 3$ case, where $\{H_1, H_2, H_3\} = \mathcal{H}$, we observe that a class of (H_1, H_2, H_3)-free graphs is closed under complementation if and only if either every H_i is self-complementary, or one H_i is self-complementary and the other two graphs H_j and H_k are complements of each other. By Theorem 1, we only need to consider $(H_1, \overline{H_1}, H_2)$-free graphs, where H_1 is not self-complementary, H_2 is self-complementary, and neither H_1 nor H_2 is an induced subgraph of P_4. The next two smallest self-complementary graphs H_2 are the C_5 and the bull (see also Figure 2). Observe that any self-complementary graph on n vertices must contain $\frac{1}{2}\binom{n}{2}$ edges and this number must be an integer, so $n = 4q$ or $n = 4q + 1$ for some integer $q \geq 0$.

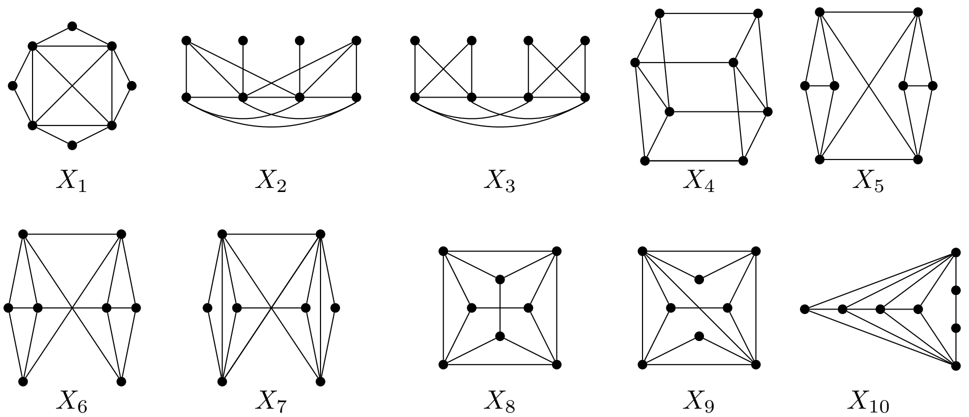

FIG. 3. *The ten self-complementary graphs on eight vertices* [44].

There are exactly ten nonisomorphic self-complementary graphs on eight vertices [44] and we depict these in Figure 3.

It is known that split graphs or, equivalently, $(2P_2, \overline{2P_2}, C_5)$-free graphs have unbounded clique-width [41]. In section 5 we determine three new hereditary graph classes of unbounded clique-width, which imply that the class of $(H, \overline{H}, C_5)$-free graphs has unbounded clique-width if $H \in \{K_{1,3}+P_1, 2P_2, 3P_1+P_2, S_{1,1,2}\}$. By combining this with known results, we discovered that the classification of boundedness of clique-width for $(H, \overline{H}, C_5)$-free graphs coincides with the one of Theorem 2. This raised the question of whether the same is true for other sets of self-complementary graphs $\mathcal{F} \neq \{C_5\}$. If $\mathcal{F}$ contains the bull, then the answer is negative: by Theorem 2, both the class of $(S_{1,1,2}, \overline{S_{1,1,2}})$-free graphs and the class of $(2P_2, C_4)$-free graphs have unbounded clique-width, but both the class of $(S_{1,1,2}, \overline{S_{1,1,2}}, \text{bull})$-free graphs and even the class of $(P_5, \overline{P_5}, \text{bull})$-free graphs have bounded clique-width [8]. However, also in section 5, we prove that the bull is the *only* exception (apart from the trivial cases when $H' \in \{P_1, P_4\}$ which yield bounded clique-width of $(H, \overline{H}, H')$-free graphs for any graph H).

THEOREM 3. *Let $\mathcal{F}$ be a set of self-complementary graphs on at least five vertices not equal to the bull. For a graph H, the class of $(\{H, \overline{H}\} \cup \mathcal{F})$-free graphs has bounded clique-width if and only if H or $\overline{H}$ is an induced subgraph of $K_{1,3}$, P_1+P_4, $2P_1+P_3$, or sP_1 for some $s \geq 1$.*

In section 6 we discuss a number of consequences of our results, in particular for the COLORING problem, and discuss directions for future work.

2. Preliminaries. Throughout our paper we consider only finite, undirected graphs without multiple edges or self-loops. Below we define further graph terminology.

Given two graphs G and H, an *isomorphism* from G to H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. If such an isomorphism exists, we say that G and H are *isomorphic*.

The *disjoint union* $(V(G) \cup V(H), E(G) \cup E(H))$ of two vertex-disjoint graphs G and H is denoted by $G+H$ and the disjoint union of r copies of a graph G is denoted by rG. The *complement* of a graph G, denoted by $\overline{G}$, has vertex set $V(\overline{G}) = V(G)$ and an edge between two distinct vertices if and only if these two vertices are not adjacent in G. A graph G is *self-complementary* if G is isomorphic to $\overline{G}$. For a subset $S \subseteq V(G)$, we let $G[S]$ denote the subgraph of G *induced* by S, which has vertex set S and edge set $\{uv \mid u, v \in S, uv \in E(G)\}$. If $S = \{s_1, \dots, s_r\}$, then, to simplify notation, we may also write $G[s_1, \dots, s_r]$ instead of $G[\{s_1, \dots, s_r\}]$. We use $G \setminus S$ to denote the graph obtained from G by deleting every vertex in S, that is, $G \setminus S = G[V(G) \setminus S]$; if $S = \{v\}$, we may write $G \setminus v$ instead. We write $G' \subseteq_i G$ to indicate that G' is an induced subgraph of G.

A graph $G = (V, E)$ is *empty* if $V = E = \emptyset$, otherwise it is nonempty. The graphs C_r, K_r, $K_{1,r-1}$, and P_r denote the cycle, complete graph, star, and path on r vertices, respectively. The graphs K_3 and $K_{1,3}$ are also called the *triangle* and *claw*, respectively. A graph G is a *linear forest* if every component of G is a path (on at least one vertex). The graph $S_{h,i,j}$, for $1 \leq h \leq i \leq j$, denotes the *subdivided claw*, that is, the tree that has only one vertex x of degree 3 and exactly three leaves, which are of distance h, i, and j from x, respectively. Observe that $S_{1,1,1} = K_{1,3}$. We let $\mathcal{S}$ be the class of graphs each component of which is either a subdivided claw or a path on at least one vertex.

For a set of graphs $\mathcal{H}$, a graph G is $\mathcal{H}$-free (or $(\mathcal{H})$-free) if it has no induced subgraph isomorphic to a graph in $\mathcal{H}$. If $\mathcal{H} = \{H_1, \dots, H_p\}$ for some integer p, then we may write $(H_1, \dots, H_p)$-free instead of $(\{H_1, \dots, H_p\})$-free, or, if $p = 1$, we may simply write H_1-free.

For a graph $G = (V, E)$, the set $N(u) = \{v \in V \mid uv \in E\}$ denotes the *neighborhood* of $u \in V$. A component in G is *trivial* if it contains exactly one vertex, otherwise, it is nontrivial. A graph is *bipartite* if its vertex set can be partitioned into two (possibly empty) independent sets. A graph is *split* if its vertex set can be partitioned into a (possibly empty) independent set and a (possibly empty) clique. Split graphs have been characterized as follows.

Lemma 1 ([30]). *A graph G is split if and only if it is $(2P_2, C_4, C_5)$-free.*

Let X be a set of vertices in a graph $G = (V, E)$. A vertex $y \in V \setminus X$ is *complete* to X if it is adjacent to every vertex of X and *anticomplete* to X if it is nonadjacent to every vertex of X. Similarly, a set of vertices $Y \subseteq V \setminus X$ is complete (resp., anticomplete) to X if every vertex in Y is complete (resp., anticomplete) to X. We say that the edges between two disjoint sets of vertices X and Y form a *matching* (resp., *comatching*) if each vertex in X has at most one neighbor (resp., nonneighbor) in Y and vice versa. A vertex $y \in V \setminus X$ *distinguishes* X if y has both a neighbor and a nonneighbor in X. The set X is a *module* of G if no vertex in $V \setminus X$ distinguishes X. A module X is *nontrivial* if $1 < |X| < |V|$, otherwise it is *trivial*. A graph is *prime* if it has only trivial modules.

To help reduce the amount of case analysis needed to prove Theorems 2 and 3, we prove the following lemma.

Lemma 2. *Let $H \in \mathcal{S}$. Then H is $(K_{1,3} + P_1, 2P_2, 3P_1 + P_2, S_{1,1,2})$-free if and only if H is an induced subgraph of $K_{1,3}$, $P_1 + P_4$, $2P_1 + P_3$, or sP_1 for some $s \geq 1$.*

Proof. Let $H \in \mathcal{S}$. First suppose H is an induced subgraph of $K_{1,3}$, $P_1 + P_4$, $2P_1 + P_3$, or sP_1 for some $s \geq 1$. It is readily seen that H is $(K_{1,3} + P_1, 2P_2, 3P_1 + P_2, S_{1,1,2})$-free.

Now suppose that H is $(K_{1,3} + P_1, 2P_2, 3P_1 + P_2, S_{1,1,2})$-free. If H is not a linear forest then since $H \in \mathcal{S}$, it contains an induced subgraph isomorphic to $K_{1,3}$. We may assume that H is not an induced subgraph of $K_{1,3}$, otherwise we are done. In this case H contains an induced subgraph that is a one-vertex extension of $K_{1,3}$.

Since $H \in \mathcal{S}$, this means that H contains $K_{1,3} + P_1$ or $S_{1,1,2}$ as an induced subgraph, a contradiction. We may therefore assume that H is a linear forest.

Since H is a linear forest, it is isomorphic to $P_{i_1} + P_{i_2} + \cdots + P_{i_k}$ for some positive integers $i_1 \geq i_2 \geq \cdots \geq i_k$. We may assume that $i_1 \geq 2$, otherwise $H = sP_1$ for some $s \geq 1$. Since H is $2P_2$-free, it follows that $i_1 \leq 4$ and, if $k \geq 2$, then $i_2 \leq 1$, so H has exactly one nontrivial component and that component is isomorphic to P_2, P_3, or P_4. So $H = P_s + tP_1$ for some $s \in \{2, 3, 4\}$ and $t \geq 0$. If $s = 4$ then $t \leq 1$, since H is $(3P_1 + P_2)$-free, in which case H is an induced subgraph of $P_1 + P_4$ and we are done. If $s \in \{2, 3\}$ then $t \leq 2$, since H is $(3P_1 + P_2)$-free, in which case H is an induced subgraph of $2P_1 + P_3$ and we are done. This completes the proof. ∎

2.1. Clique-width. The *clique-width* of a graph G, denoted by $\mathrm{cw}(G)$, is the minimum number of labels needed to construct G by using the following four operations:

1. creating a new graph consisting of a single vertex v with label i;
2. taking the disjoint union of two labeled graphs G_1 and G_2;
3. joining each vertex with label i to each vertex with label j ($i \neq j$);
4. renaming label i to j.

Note that the clique-width of a graph is the maximum of the clique-width of its components: we can construct each component separately, then take the disjoint union of the resulting labeled graphs.

A class of graphs $\mathcal{G}$ has bounded clique-width if there is a constant c such that the clique-width of every graph in $\mathcal{G}$ is at most c; otherwise the clique-width of $\mathcal{G}$ is *unbounded*.

Let G be a graph. We define the following operations. For an induced subgraph $G' \subseteq_i G$ (or a vertex set $X \subseteq V(G)$), the *subgraph complementation* operation, acting on G with respect to G' (resp., X), replaces every edge present in G' (resp., $G[X]$) by a nonedge, and vice versa. Similarly, for two disjoint vertex subsets S and T in G, the *bipartite complementation* operation with respect to S and T acts on G by replacing every edge with one end-vertex in S and the other one in T by a nonedge and vice versa.

We now state some useful facts about how the above operations (and some other ones) influence the clique-width of a graph. We will use these facts throughout the paper. Let $k \geq 0$ be a constant and let γ be some graph operation. We say that a graph class $\mathcal{G}'$ is (k,γ)-*obtained* from a graph class $\mathcal{G}$ if the following two conditions hold:

1. every graph in $\mathcal{G}'$ is obtained from a graph in $\mathcal{G}$ by performing γ at most k times, and
2. for every $G \in \mathcal{G}$ there exists at least one graph in $\mathcal{G}'$ obtained from G by performing γ at most k times.

We say that γ *preserves* boundedness of clique-width if for every finite constant k and every graph class $\mathcal{G}$, every graph class $\mathcal{G}'$ that is (k,γ)-obtained from $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}$ has bounded clique-width. Note that condition 1 is necessary for this definition to be meaningful, as without it the class of all graphs (which has unbounded clique-width) would be (k,γ)-obtained from every other graph class. Similarly, condition 2 is necessary, otherwise every graph class would be (k,γ)-obtained from the class of all graphs.

Fact 1. Vertex deletion preserves boundedness of clique-width [39].

Fact 2. Subgraph complementation preserves boundedness of clique-width [37].

Fact 3. Bipartite complementation preserves boundedness of clique-width [37].

As hereditary graph classes are closed under vertex deletion by definition, applying a vertex deletion to a graph from a hereditary graph class $\mathcal{G}$ results in another graph from $\mathcal{G}$. This means that using Fact 1, it is "safe" to apply a bounded number of vertex deletions, as this will not only avoid the clique-width changing by "too much," but also ensure that the graph under consideration does not leave $\mathcal{G}$. However, this does not necessarily hold if we instead apply a subgraph complementation or a bipartite complementation, so we must take more care when applying these operations and using Facts 2 and 3. To ensure this, throughout all our proofs, we only use subgraph complementations or bipartite complementations for four purposes.

1. We can apply a subgraph complementation to a whole graph G; this results in the complement $\overline{G}$. Fact 2 implies that a class of graphs $\mathcal{G}$ has bounded clique-width if and only if the class of graphs whose complements lie in $\mathcal{G}$ has bounded clique-width.
2. When proving that a class of graphs $\mathcal{G}$ has unbounded clique-width, we take a known class $\mathcal{G}'$ of unbounded clique-width and for every graph $G' \in \mathcal{G}'$ we show how to use subgraph complementations and bipartite complementations a bounded number of times to change G' into a graph $G \in \mathcal{G}$. Then Facts 2 and 3 imply that $\mathcal{G}$ also has unbounded clique-width.
3. Similarly to purpose 2, when proving that a class of graphs $\mathcal{G}$ has bounded clique-width, we use the properties of graphs in $\mathcal{G}$ to show that we can apply subgraph complementations and bipartite complementations (along with vertex deletions) to modify an arbitrary graph $G \in \mathcal{G}$ into a graph G' that belongs to some graph class $\mathcal{G}'$ known to have bounded clique-width. As long as we only use these operations a bounded number of times, Facts 1–3 imply that $\mathcal{G}$ must also have bounded clique-width. Note that the obtained graph G' is not necessarily in $\mathcal{G}$ in this case.
4. Again when proving that a class of graphs $\mathcal{G}$ has bounded clique-width, we use subgraph complementations and bipartite complementations (along with vertex deletions) to modify an arbitrary graph $G \in \mathcal{G}$ into the disjoint union of some induced subgraphs of G that have a simpler structure than G itself. We can then deal with these simpler induced subgraphs separately. Since $\mathcal{G}$ is closed under taking induced subgraphs, we can make use of properties we have proved for the class $\mathcal{G}$.

We need the following lemmas on clique-width, the first one of which is easy to show.

LEMMA 3. *The clique-width of a graph of maximum degree at most* 2 *is at most* 4.

LEMMA 4 ([25]). *Let H be a graph. The class of H-free graphs has bounded clique-width if and only if $H \subseteq_i P_4$.*

LEMMA 5 ([40]). *Let $\{H_1, \ldots, H_p\}$ be a finite set of graphs. If $H_i \notin \mathcal{S}$ for all $i \in \{1, \ldots, p\}$ then the class of $(H_1, \ldots, H_p)$-free graphs has unbounded clique-width.*

LEMMA 6 ([17]). *Let G be a graph and let $\mathcal{P}$ be the set of all induced subgraphs of G that are prime. Then* $\mathrm{cw}(G) = \max_{H \in \mathcal{P}} \mathrm{cw}(H)$.

3. The proof of Theorem 1.

We first prove the following lemma, which we will also use in the proof of Theorem 3.

LEMMA 7. *If G is a (C_4, C_5, K_4)-free self-complementary graph then G is an induced subgraph of the bull.*

Proof. Suppose, for contradiction, that G is a (C_4, C_5, K_4)-free self-complementary graph on n vertices that is not an induced subgraph of the bull. Since G is C_5-free and is not an induced subgraph of the bull, it is not equal to P_1, P_4, C_5, or the bull. As these are the only nonempty self-complementary graphs on fewer than eight vertices (see Figure 2), G must have at least eight vertices. Since G is C_4-free and self-complementary, it is also $2P_2$-free, so it is $(C_4, C_5, 2P_2)$-free. Then, by Lemma 1, G must be a split graph, so its vertex set can be partitioned into a clique C and an independent set I. Since G is K_4-free and self-complementary, it is also $4P_1$-free. Therefore $|C|, |I| \leq 3$, so G has at most six vertices, a contradiction. This completes the proof. ∎

We are now ready to prove Theorem 1. Note that this theorem holds even if $\mathcal{H}$ is infinite.

THEOREM 1 (restated). *Let $\mathcal{H}$ be a set of nonempty self-complementary graphs. Then the class of $\mathcal{H}$-free graphs has bounded clique-width if and only if either $P_1 \in \mathcal{H}$ or $P_4 \in \mathcal{H}$.*

Proof. Suppose there is a graph $H \in \mathcal{H} \cap \{P_1, P_4\}$. Then the class of $\mathcal{H}$-free graphs is a subclass of the class of P_4-free graphs, which have bounded clique-width by Lemma 4. Now suppose that $\mathcal{H} \cap \{P_1, P_4\} = \emptyset$. The only nonempty self-complementary graphs on at most five vertices that are not equal to P_1 and P_4 are the bull and the C_5 (see Figure 2). By Lemma 7, it follows that every graph in $\mathcal{H}$ contains an induced subgraph isomorphic to the bull, C_4, C_5, or K_4. Therefore the class of $\mathcal{H}$-free graphs contains the class of $(\text{bull}, C_4, C_5, K_4)$-free graphs, which has unbounded clique-width by Lemma 5. ∎

4. The proof of Theorem 2. In this section we prove Theorem 2 by combining known results with the new result that $(2P_1 + P_3, \overline{2P_1 + P_3})$-free graphs have bounded clique-width. We prove this result in the following way. We first prove three useful structural lemmas, namely, Lemmas 8–10; we will use these lemmas repeatedly throughout the proof. Next, we prove Lemmas 11 and 12, which state that if a $(2P_1 + P_3, \overline{2P_1 + P_3})$-free graph G contains an induced C_5 or C_6, respectively, then G has bounded clique-width. We do this by partitioning the vertices outside this cycle into sets, depending on their neighborhood in the cycle. We then analyze the edges within these sets and between pairs of such sets. After a lengthy case analysis, we find that G has bounded clique-width in both these cases. By Fact 2 it only remains to analyze $(2P_1 + P_3, \overline{2P_1 + P_3})$-free graphs that are also $(C_5, C_6, \overline{C_6})$-free. Next, in Lemma 13, we show that if such graphs are prime, then they are either K_7-free or $\overline{K_7}$-free. In Lemma 15 we use the fact that $(2P_1 + P_3, \overline{2P_1 + P_3})$-free graphs are χ-bounded (that is, their chromatic number is bounded by a function of their clique number) to deal with the case where a graph in the class is K_7-free. Finally, we combine all these results together to obtain the new result (Theorem 4).

We start by proving the aforementioned structural lemmas. Recall that if X and Y are disjoint sets of vertices in a graph, we say that the edges between these two sets form a matching if each vertex in X has at most one neighbor in Y and vice versa (if each vertex has exactly one such neighbor, we say that the matching is *perfect*). Similarly, the edges between these sets form a comatching if each vertex in X has at most one nonneighbor in Y and vice versa. Also note that when describing a set as being a clique or an independent set, we allow the case where this set is empty.

LEMMA 8. *Let G be a $(2P_1 + P_3, \overline{2P_1 + P_3})$-free graph whose vertex set can be partitioned into two sets X and Y, each of which is a clique or an independent set. Then by deleting at most one vertex from each of X and Y, it is possible to obtain subsets such that the edges between them form a matching or a comatching.*

Proof. Given two disjoint sets of vertices, we say that with respect to these sets, a vertex is *full* if it is adjacent to all but at most one vertex in the other set, and it is *empty* if it is adjacent to at most one vertex in the other set. If every vertex in the two sets is full, then the edges between the two sets form a comatching, and if every vertex in the two sets is empty, then the edges between them form a matching.

Claim 1. *Each vertex in X and Y is either full or empty.*
If a vertex in, say, X is neither full nor empty, then it has two neighbors and two nonneighbors in Y, and these five vertices induce a $2P_1 + P_3$ if Y is an independent set, or a $\overline{2P_1 + P_3}$ if Y is a clique. This completes the proof of the claim.

To prove the lemma, we must show that, after discarding at most one vertex from each of X and Y, we have a pair of sets such that every vertex is full or every vertex is empty with respect to this pair. We note that if a vertex is full (or empty) with respect to X and Y then it is also full (or empty) with respect to any pair of subsets of X and Y, respectively, so if we establish or assume fullness (or emptiness) before discarding a vertex, then it still holds afterwards.

We consider a number of cases.

Case 1. *Neither X nor Y contains two full vertices, or neither X nor Y contains two empty vertices.*
By deleting at most one vertex from each of X and Y, we can obtain a pair of sets where either every vertex is full or every vertex is empty. This completes the proof of Case 1.

Case 2. *$|X| \leq 2$ or $|Y| \leq 2$.*
By symmetry we may assume that $|X| \leq 2$. If X is empty or contains exactly one vertex, the lemma is immediate, so we may assume that X contains exactly two vertices, say x and x'. Consider the pair of sets $\{x\}$ and Y. Every vertex in Y is both full and empty with respect to $\{x\}$ and Y, and, by Claim 1, x is either full or empty with respect to $\{x\}$ and Y. This completes the proof of Case 2.

Case 3. *There are vertices $x_1, x_2 \in X$ and $y_1 \in Y$ such that x_1 and x_2 are complete to $Y \setminus \{y_1\}$.*
In this case, every vertex in $Y \setminus \{y_1\}$ is adjacent to both x_1 and x_2, so it cannot be empty with respect to X and Y. By Claim 1, it follows that every vertex in $Y \setminus \{y_1\}$ is full. We may assume that $|Y| \geq 3$ (otherwise we apply Case 2). Let y_2 and y_3 be vertices in $Y \setminus \{y_1\}$. As y_2 and y_3 are both full with respect to X and $Y \setminus \{y_1\}$, all but at most two vertices of X are adjacent to both y_2 and y_3. Note that if a vertex x is adjacent to both y_2 and y_3 then it must be full with respect to X and $Y \setminus \{y_1\}$. If at most one vertex of X is empty with respect to X and $Y \setminus \{y_1\}$ then by discarding this vertex (if it exists) from X and discarding y_1 from Y, we are done.

So we may assume that X contains exactly two vertices x_3 and x_4 that are not full with respect to X and $Y \setminus \{y_1\}$ and thus are empty. Suppose that $|Y| \geq 4$. Then there are three full vertices in $Y \setminus \{y_1\}$ that must each be adjacent to at least one of x_3 and x_4. Thus at least one of x_3 and x_4 has at least two neighbors in $Y \setminus \{y_1\}$ contradicting the fact that they are both empty with respect to X and $Y \setminus \{y_1\}$.

Thus we may now assume that $|Y| = 3$, so $Y \setminus \{y_1\} = \{y_2, y_3\}$. By assumption, x_3 and x_4 are not full with respect to X and $\{y_2, y_3\}$, so they must have two nonneighbors in $\{y_2, y_3\}$, i.e., they must be anticomplete to $\{y_2, y_3\}$. Thus y_2 has two nonneighbors in X, so it is empty with respect to X and Y. Since $|X| \geq 4$, this means that y_2 is not full with respect to X and Y, a contradiction. This completes the proof of Case 3.

We note that if, in Case 3, we swap X and Y, or write anticomplete instead of complete, we obtain further cases with essentially the same proof. We now assume that neither these cases, nor Cases 1 and 2, hold.

Claim 2. *If there are two full vertices $x_1, x_2 \in X$, then they have distinct nonneighbors in Y. If there are two empty vertices $x_1, x_2 \in X$, then they have distinct neighbors in Y.*
We prove the first statement (the second follows by symmetry). If x_1 and x_2 are both complete to Y then Case 3 would apply with any vertex in Y chosen as y_1. Suppose instead that y_1 is the unique nonneighbor of x_1. Then x_2 must have a nonneighbor in Y that is different from y_1, otherwise Case 3 would apply. This completes the proof of the claim.

Claim 3. *There are at least two empty vertices in X and at least two full vertices in Y or vice versa.*
As Case 1 does not apply, we know that one of X and Y contains two empty vertices, and one of X and Y contains two full vertices. We are done unless these two properties belong to the same set. So let us suppose that, without loss of generality, it is X that contains two empty vertices and two full vertices, which we may assume are distinct (if a vertex in X is both full and empty, then $|Y| \leq 2$ and Case 2 applies). By Claim 2, the two empty vertices of X have distinct neighbors y_1 and y_2 in Y. If y_1 and y_2 are both full, we are done. If, say, y_1 is empty, then, as it is adjacent to one of the empty vertices in X, it cannot be adjacent to either of the full vertices in X, contradicting Claim 2. This completes the proof of Claim 3.

We immediately use Claim 3. Let us assume, without loss of generality, that $x_1, x_2 \in X$ are empty and $y_1, y_2 \in Y$ are full with respect to X and Y. Moreover, by Claim 2, we may assume that y_1 is the unique neighbor of x_1 and y_2 is the unique neighbor of x_2 (so x_1 is the unique nonneighbor of y_2 and x_2 is the unique nonneighbor of y_1). Thus every vertex of $X \setminus \{x_1, x_2\}$ is complete to $\{y_1, y_2\}$, and therefore, by Claim 1, full with respect to X and Y. Similarly, every vertex of $Y \setminus \{y_1, y_2\}$ is anticomplete to $\{x_1, x_2\}$, and therefore empty with respect to X and Y.

If $|X| = 3$, then every vertex in $\{x_1, x_2\}$ and Y is empty with respect to $\{x_1, x_2\}$ and Y. Otherwise we can find distinct vertices x_3, x_4 in $X \setminus \{x_1, x_2\}$ which we know are both full and both complete to $\{y_1, y_2\}$. Hence, by Claim 2, there are distinct vertices y_3, y_4 in $Y \setminus \{y_1, y_2\}$ such that y_3 is the unique nonneighbor of x_3 and y_4 is the unique nonneighbor of x_4. If X and Y are independent sets then $G[x_1, y_4, y_2, x_4, y_3]$ is a $2P_1 + P_3$. If X is an independent set and Y is a clique then $G[y_1, y_2, y_3, x_3, x_4]$ is a $\overline{2P_1 + P_3}$. If X is a clique and Y is an independent set then $G[x_3, x_4, x_2, y_1, y_2]$ is a $\overline{2P_1 + P_3}$. Finally if X and Y are cliques then $G[x_3, y_1, y_2, x_1, y_4]$ is a $\overline{2P_1 + P_3}$. This contradiction completes the proof. ∎

LEMMA 9. *Let G be a $(2P_1 + P_3, \overline{2P_1 + P_3})$-free graph whose vertex set can be partitioned into a clique X and an independent set Y. Then by deleting at most three* *vertices from each of X and Y, it is possible to obtain subsets that are either complete or anticomplete to each other.*

Proof. Let G be such a graph. By Lemma 8, by deleting at most one vertex from each of X and Y, we may reduce to the case where the edges between X and Y form a matching or a comatching. (Note that after this we may delete at most two further vertices from each of X and Y.) Complementing the graph if necessary (in which case we also swap X and Y), we may assume that the edges between X and Y form a matching. Let $x_1y_1, \dots, x_iy_i$ be the edges between X and Y, with $x_1, \dots, x_i \in X$ and $y_1, \dots, y_i \in Y$. If $i \geq 4$ then $G[y_1, y_2, y_3, x_3, x_4]$ is a $2P_1 + P_3$, a contradiction. We may therefore assume that $i \leq 3$. Deleting the vertices x_1, x_2, y_3 (if they are present) completes the proof. ∎

LEMMA 10. *The class of those $(2P_1 + P_3, \overline{2P_1 + P_3})$-free graphs whose vertex set can be partitioned into at most three cliques and at most three independent sets has bounded clique-width.*

Proof. Let G be a $(2P_1 + P_3, \overline{2P_1 + P_3})$-free graph whose vertex set can be partitioned into three (possibly empty) cliques K^1, K^2, K^3 and three (possibly empty) independent sets I^1, I^2, I^3. By Lemma 9 and Fact 1, we may delete at most $2 \times 3 \times 3 \times 3 = 54$ vertices, after which every K^i is either complete or anticomplete to every I^j. By Lemma 8 and Fact 1, we may delete at most $2 \times 2 \times \binom{3}{2} = 12$ vertices, after which the edges between any two cliques K^i, K^j and the edges between any two independent sets I^i, I^j either form a matching or a comatching. If the edges form a comatching, then by Fact 3 we may apply a bipartite complementation between these sets. If a set K^i is complete to some set I^j then by Fact 3, we may apply a bipartite complementation between them. Finally, by Fact 2, we may complement every clique K^i. The resulting graph has maximum degree at most 2, and therefore has clique-width at most 4 by Lemma 3. It follows that G has bounded clique-width. ∎

LEMMA 11. *The class of $(2P_1+P_3, \overline{2P_1 + P_3})$-free graphs containing an induced C_5 has bounded clique-width.*

Proof. Suppose G is a $(2P_1 + P_3, \overline{2P_1 + P_3})$-free graph containing an induced cycle C on five vertices, say $v_1, \dots, v_5$ in that order. For $S \subseteq \{1, \dots, 5\}$, let V_S be the set of vertices $x \in V(G) \setminus V(C)$ such that $N(x) \cap V(C) = \{v_i \mid i \in S\}$. We say that a set V_S is *large* if it contains at least five vertices, otherwise it is *small*.

To ease notation, in the following claims, subscripts on vertex sets should be interpreted modulo 5 and whenever possible we will write V_i instead of $V_{\{i\}}$ and $V_{i,j}$ instead of $V_{\{i,j\}}$ and so on.

Claim 1. *We may assume that for $S \subseteq \{1, 2, 3, 4, 5\}$, the set V_S is either large or empty.*
If a set V_S is small, but not empty, then by Fact 1, we may delete all vertices of this set. If later in our proof we delete vertices in some set V_S and in doing so make a large set V_S become small, we may immediately delete the remaining vertices in V_S. The above arguments involve deleting a total of at most $2^5 \times 4$ vertices. By Fact 1, the claim follows.

Claim 2. *For $i \in \{1, 2, 3, 4, 5\}$, $V_\emptyset \cup V_i \cup V_{i+1} \cup V_{i,i+1}$ is a clique.*
Indeed, if $x, y \in V_\emptyset \cup V_1 \cup V_2 \cup V_{1,2}$ are nonadjacent then $G[x, y, v_3, v_4, v_5]$ is a $2P_1 + P_3$, a contradiction. The claim follows by symmetry.

TABLE 1
The correspondence between the sets W_T and the sets V_S.

$V_1 = W_{2,3,4,5}$	$V_2 = W_{1,2,3,5}$	$V_3 = W_{1,3,4,5}$	$V_4 = W_{1,2,3,4}$	$V_5 = W_{1,2,4,5}$
$V_{1,2} = W_{2,3,5}$	$V_{2,3} = W_{1,3,5}$	$V_{3,4} = W_{1,3,4}$	$V_{4,5} = W_{1,2,4}$	$V_{1,5} = W_{2,4,5}$
$V_{1,3} = W_{3,4,5}$	$V_{2,4} = W_{1,2,3}$	$V_{3,5} = W_{1,4,5}$	$V_{1,4} = W_{2,3,4}$	$V_{2,5} = W_{1,2,5}$
$V_{1,2,3} = W_{3,5}$	$V_{2,3,4} = W_{1,3}$	$V_{3,4,5} = W_{1,4}$	$V_{1,4,5} = W_{2,4}$	$V_{1,2,5} = W_{2,5}$
$V_{1,2,4} = W_{2,3}$	$V_{2,3,5} = W_{1,5}$	$V_{1,3,4} = W_{3,4}$	$V_{2,4,5} = W_{1,2}$	$V_{1,3,5} = W_{4,5}$
$V_{1,2,3,4} = W_3$	$V_{1,2,3,5} = W_5$	$V_{1,2,4,5} = W_2$	$V_{1,3,4,5} = W_4$	$V_{2,3,4,5} = W_1$
$V_\emptyset = W_{1,2,3,4,5}$	$V_{1,2,3,4,5} = W_\emptyset$			

Claim 3. *For $i \in \{1,2,3,4,5\}$, $G[V_{i,i+2}]$ is P_3-free.*
Indeed, if $G[V_{1,3}]$ contains an induced P_3, say on vertices x, y, z, then $G[v_2, v_4, x, y, z]$ is a $2P_1 + P_3$, a contradiction. The claim follows by symmetry.

Note that since G is a $(2P_1 + P_3, \overline{2P_1 + P_3})$-free graph containing a C_5, it follows that $\overline{G}$ is also a $(2P_1 + P_3, \overline{2P_1 + P_3})$-free graph containing a C_5, namely, on the vertices v_1, v_3, v_5, v_2, v_4, in that order. Let $w_1 = v_1$, $w_2 = v_3$, $w_3 = v_5$, $w_4 = v_2$, and $w_5 = v_4$. For $S \subseteq \{1,2,3,4,5\}$, we say that a vertex x not in C belongs to W_S if $N(x) \cap V(C) = \{w_i \mid i \in S\}$ *in the graph* $\overline{G}$. We define the function $\sigma : \{1,2,3,4,5\} \to \{1,2,3,4,5\}$ as follows: $\sigma(1) = 1$, $\sigma(3) = 2$, $\sigma(5) = 3$, $\sigma(2) = 4$, and $\sigma(4) = 5$. Now for $S, T \subseteq \{1,2,3,4,5\}$, $x \in V_S$ if and only if $x \in W_T$, where $T = \{1,2,3,4,5\} \setminus \{\sigma(i) \mid i \in S\}$. Therefore, we may assume that any claims proved for a set V_S in G also hold for the set W_S in $\overline{G}$.

For convenience we provide Table 1, which lists the correspondence between the sets W_T and the sets V_S.

We therefore get the following two corollaries of Claims 2 and 3, respectively. We include the argument for the first corollary to demonstrate how this "casting to the complement" argument works.

Claim 4. *For $i \in \{1,2,3,4,5\}$, $V_{i,i+1,i+3} \cup V_{i,i+1,i+2,i+3} \cup V_{i,i+1,i+3,i+4} \cup V_{1,2,3,4,5}$ is an independent set.*
Indeed, for $i = 1$, $V_{i,i+1,i+3} \cup V_{i,i+1,i+2,i+3} \cup V_{i,i+1,i+3,i+4} \cup V_{1,2,3,4,5}$ is $V_{1,2,4} \cup V_{1,2,3,4} \cup V_{1,2,4,5} \cup V_{1,2,3,4,5}$, which is equal to $W_{2,3} \cup W_3 \cup W_2 \cup W_\emptyset$. By Claim 2, $W_{2,3} \cup W_3 \cup W_2 \cup W_\emptyset$ is an independent set. The claim follows by complementing and symmetry.

Claim 5. *For $i \in \{1,2,3,4,5\}$, $G[V_{i,i+1,i+2}]$ is $(P_1 + P_2)$-free.*

Claim 6. *We may assume that for distinct $S, T \subseteq \{1,2,3,4,5\}$ if V_S is an independent set and V_T is a clique then V_S is either complete or anticomplete to V_T.*
Let $S, T \subseteq \{1, \dots, 5\}$ be distinct. If V_S is an independent set and V_T is a clique, then by Lemma 9, we may delete at most three vertices from each of these sets, such that in the resulting graph, V_S will be complete or anticomplete to V_T. Doing this for every pair of independent set V_S and a clique V_T we delete at most $\binom{2^5}{2} \times 2 \times 3$ vertices from G. The claim follows by Fact 1.

Claim 7. *We may assume that for distinct $S, T \subseteq \{1,2,3,4,5\}$, if V_S and V_T are both independent sets then the edges between V_S and V_T form a comatching.*
Let $S, T \subseteq \{1, \dots, 5\}$ be distinct. We may assume that V_S and V_T are not empty, in which case they must both be large, i.e., $|V_S|, |V_T| \geq 5$. If V_S and V_T are both independent sets, then by Lemma 8, we may delete at most one vertex from each of these sets, such that in the resulting graph, the edges between V_S and V_T form a matching or a comatching. Note that after this modification we only have the weaker bound $|V_S|, |V_T| \geq 4$ in the resulting graph. Suppose, for contradiction, that the edges

between V_S and V_T form a matching. Without loss of generality assume there is an $i \in T \setminus S$. Since $|V_S| \geq 4$, there must be vertices $x, x' \in V_S$. Since each vertex in V_S has at most one neighbor in V_T and $|V_T| \geq 4$, there must be vertices $y, y' \in V_T$ that are nonadjacent to both x and x'. Then $G[x, x', y, v_i, y']$ is a $2P_1 + P_3$, a contradiction. Therefore the edges between V_S and V_T must indeed form a comatching. The claim follows by Fact 1.

In many cases, we can prove a stronger claim, as follows.

Claim 8. *For distinct $S, T \subseteq \{1, 2, 3, 4, 5\}$, if V_S and V_T are both independent and there is an $i \in \{1, 2, 3, 4, 5\}$ with $i \notin S$ and $i \notin T$ then V_S is complete to V_T.*
Let $S, T \subseteq \{1, \dots, 5\}$ be distinct and suppose there is an $i \in \{1, 2, 3, 4, 5\}$ with $i \notin S$ and $i \notin T$. We may assume V_S and V_T are not empty, so they must be large. By Claim 7, we may assume that the edges between V_S and V_T form a comatching. Suppose, for contradiction that $x \in V_S$ is nonadjacent to $y \in V_T$. Since V_S is large, there must be vertices $x', x'' \in V_S \setminus \{x\}$ and these vertices must be adjacent to y. Now $G[v_i, x, x', y, x'']$ is a $2P_1 + P_3$, a contradiction. Therefore V_S must be complete to V_T. The claim follows.

Casting to the complement we get the following as a corollary to the above two claims.

Claim 9. *We may assume that for distinct $S, T \subseteq \{1, 2, 3, 4, 5\}$, if V_S and V_T are both cliques then the edges between V_S and V_T form a matching.*

Claim 10. *For distinct $S, T \subseteq \{1, 2, 3, 4, 5\}$, if V_S and V_T are both cliques and there is an $i \in \{1, 2, 3, 4, 5\}$ with $i \in V_S$ and $i \in V_T$ then V_S is anticomplete to V_T.*

Claim 11. *For $i \in \{1, \dots, 5\}$, if $\{i, i+1\} \subseteq S \cap T$ and $T \neq S$ then either V_S or V_T is empty.*
Suppose S and T are as described above, but V_S and V_T are both nonempty. By Claim 1, V_S and V_T must be large. Without loss of generality, we may assume that $1, 2 \in S \cap T$ and $3 \in T \setminus S$ or $S = \{1, 2\}$, $T = \{1, 2, 4\}$.

First consider the case where $1, 2 \in S \cap T$ and $3 \in T \setminus S$. If $x \in V_S$ and $y \in V_T$ are adjacent then $G[y, v_2, v_1, v_3, x]$ is a $\overline{2P_1 + P_3}$, a contradiction. Therefore V_S is anticomplete to V_T. Suppose $x, x' \in V_S$ and $y, y' \in V_T$. If x is adjacent to x' then $G[v_1, v_2, x, y, x']$ is a $\overline{2P_1 + P_3}$, a contradiction. If y is adjacent to y' then $G[v_1, v_2, y, x, y']$ is a $\overline{2P_1 + P_3}$, a contradiction. Therefore x must be nonadjacent to x' and y must be nonadjacent to y'. This means that $G[x, x', y, v_3, y']$ is a $2P_1 + P_3$, a contradiction.

Now consider the case where $S = \{1, 2\}$, $T = \{1, 2, 4\}$. Then $V_{1,2}$ is a clique and $V_{1,2,4}$ is an independent set, by Claims 2 and 4, respectively. By Claim 6, $V_{1,2}$ must be complete or anticomplete to $V_{1,2,4}$. Suppose $x, x' \in V_{1,2}$ and $y, y' \in V_{1,2,4}$. If $V_{1,2}$ is anticomplete to $V_{1,2,4}$ then $G[v_1, v_2, x, y, x']$ is a $\overline{2P_1 + P_3}$. If $V_{1,2}$ is complete to $V_{1,2,4}$ then $G[v_3, v_5, y, x, y']$ is a $2P_1 + P_3$. This is a contradiction. The claim follows by symmetry.

Casting Claim 11 to the complement, we obtain the following corollary.

Claim 12. *For $i \in \{1, \dots, 5\}$, if $\{i, i+2\} \cap (S \cup T) = \emptyset$ and $T \neq S$ then V_S or V_T is empty.*

We now give a brief outline of the remainder of the proof. First, in Claims 13–24 we will analyze the edges between different sets V_S and sets of the form $V_{i,i+2}$. Next, in Claim 25 we will consider the case where a set $V_{i,i+2}$ is neither a clique nor an independent set. We will then assume that this case does not hold, in which case every set of the form $V_{i,i+2}$ is either a clique or an independent set. Casting to the complement, we will get the same conclusion for all sets of the form $V_{i,i+1,i+2}$. Combined with Claims 2 and 4, this means that every set V_S is either a clique or an independent set. By Fact 1, we may delete the vertices $v_1, \ldots, v_5$ of the original cycle. By Claim 6, if V_S is a clique and V_T is an independent set, then applying at most one bipartite complementation (which we may do by Fact 3) we can remove all edges between V_S and V_T. It is therefore sufficient to consider the case where all sets V_S are cliques or all sets V_T are independent. If there are at most three large cliques and at most three large independent sets, then by Lemma 10 we can bound the clique-width of the graph induced on these sets. In the proof of Claim 27 we consider the situation where a set of the form $V_{i,i+2}$ is a large clique. Having dealt with this case, we may assume that every set of the form $V_{i,i+2}$ is an independent set (so, casting to the complement, every set of the form $V_{i,i+1,i+2}$ is a clique) and we deal with this case in Claim 29. Finally, we deal with the case where all sets of the form $V_{i,i+2}$ and $V_{i,i+1,i+2}$ are empty.

Claim 13. *For $i \in \{1,2,3,4,5\}$, if $V_{i,i+2}$ and V_{i+1} are large then $V_{i,i+2}$ is either an independent set or a clique.*

Suppose, that both $V_{1,3}$ and V_2 are nonempty. Then, by Claim 1, they must both be large. Suppose that $V_{1,3}$ is not a clique. Then there are $y, y' \in V_{1,3}$ that are nonadjacent. Suppose $x \in V_2$ is nonadjacent to y'. Then $G[v_4, y', v_2, x, y]$ or $G[x, v_4, y, v_1, y']$ is a $2P_1 + P_3$ if x is adjacent or nonadjacent to y, respectively. Therefore x must be complete to $\{y, y'\}$. By Claim 3, $G[V_{1,3}]$ is P_3-free, so it is a disjoint union of cliques. Since y and y' were chosen arbitrarily and $V_{1,3}$ is not a clique, it follows that x must be complete to $V_{1,3}$. Therefore V_2 is complete to $V_{1,3}$. By Claim 2, V_2 is a clique. If $x, x' \in V_2$, $z, z' \in V_{1,3}$ with z adjacent to z' then $G[x, x', z, v_2, z']$ is a $\overline{2P_1 + P_3}$, a contradiction. Therefore if $V_{1,3}$ is not a clique then it must be an independent set. The claim follows by symmetry.

Claim 14. *For $i \in \{1,2,3,4,5\}$, if $V_{i,i+2}$ and $V_{i+1,i+3}$ are both large then either*
- (i) *both $V_{i,i+2}$ and $V_{i+1,i+3}$ are cliques or*
- (ii) *at least one of them is an independent set and the two sets are complete to each other.*

By Claim 3, $G[V_{1,3}]$ and $G[V_{2,4}]$ are P_3-free, so every component in these graphs is a clique. Suppose $G[V_{1,3}]$ is not a clique, so there are nonadjacent vertices $x, x' \in V_{1,3}$. Suppose $y \in V_{2,4}$ is nonadjacent to x'. Then $G[x', v_5, v_2, y, x]$ or $G[y, v_5, x, v_3, x']$ is a $2P_1 + P_3$ if y is adjacent or nonadjacent to x, respectively. This contradiction implies that y is complete to $\{x, x'\}$. Since we assumed that $V_{1,3}$ was not a clique and x and x' were chosen to be arbitrary nonadjacent vertices in $V_{1,3}$, it follows that y must be complete to $V_{1,3}$. Therefore if $V_{1,3}$ is not a clique then $V_{2,4}$ is complete to $V_{1,3}$. Similarly, if $V_{2,4}$ is not a clique then $V_{2,4}$ is complete to $V_{1,3}$.

Now suppose that neither $V_{1,3}$ nor $V_{2,4}$ is an independent set. If they are both cliques, then we are done, so assume for contradiction that at least one of them is not a clique. Then $V_{1,3}$ is complete to $V_{2,4}$. We can find $x, x' \in V_{1,3}$ that are adjacent and $y, y' \in V_{2,4}$ that are adjacent. However, this means that $G[x, x', y, v_1, y']$ is a $\overline{2P_1 + P_3}$, a contradiction. The claim follows by symmetry.

Claim 15. *For $i \in \{1,2,3,4,5\}$ and $S = \{i, i+1\}$ or $S = \{i+1, i+2\}$, if $V_{i,i+2}$ and V_S are large then either $V_{i,i+2}$ is a clique that is anticomplete to V_S or an independent set that is complete to V_S.*

By Claim 2, $V_{1,2}$ is a clique. By Claim 3, $G[V_{1,3}]$ is P_3-free, so it is a disjoint union of cliques. If $y \in V_{1,3}$ is adjacent to $x \in V_{1,2}$, but nonadjacent to $x' \in V_{1,2}$ then $G[v_1, x, x', y, v_2]$ is a $\overline{2P_1 + P_3}$, a contradiction. Therefore every vertex of $V_{1,3}$ is either complete or anticomplete to $V_{1,2}$.

Suppose $y, y' \in V_{1,3}$ are adjacent and suppose $x, x' \in V_{1,2}$. Suppose y is complete to $V_{1,2}$. Then $G[x, x', y, v_2, y']$ or $G[v_1, y, x, y', x']$ is a $\overline{2P_1 + P_3}$, if y' is complete or anticomplete to $V_{1,2}$, respectively. This contradiction implies that if $y, y' \in V_{1,3}$ are adjacent then they are both anticomplete to $V_{1,2}$. It follows that every nontrivial component of $G[V_{1,3}]$ is anticomplete to $V_{1,2}$. In particular, if $V_{1,3}$ is a clique, then it is anticomplete to $V_{1,2}$.

Now suppose that $V_{1,3}$ is not a clique, so there are nonadjacent vertices $y, y' \in V_{1,3}$. Choose a vertex $x \in V_{1,2}$. Suppose y' is anticomplete to $V_{1,2}$. Then $G[v_4, y', y, x, v_2]$ or $G[x, v_5, y, v_3, y']$ is a $2P_1 + P_3$ if y is complete or anticomplete to $V_{1,2}$, respectively. Therefore both y and y' must be complete to $V_{1,2}$. Since every nontrivial component of $G[V_{1,3}]$ is anticomplete to $V_{1,2}$ and $G[V_{1,3}]$ is a disjoint union of cliques, it follows that y and y' must belong to trivial components of $G[V_{1,3}]$. Since y and y' were arbitrary nonadjacent vertices in $V_{1,3}$, it follows that every component of $G[V_{1,3}]$ must be trivial. Therefore if $V_{1,3}$ is not a clique then it is an independent set and it is complete to $V_{1,2}$. The claim follows by symmetry.

Claim 16. *We may assume that for $i \in \{1, 2, 3, 4, 5\}$ if $V_{i,i+2}$ and $V_{i+3,i+4}$ are large then either $V_{i,i+2}$ is a clique or $V_{i,i+2}$ is anticomplete to $V_{i+3,i+4}$.*

If $y \in V_{1,3}$ has two neighbors $x, x' \in V_{4,5}$, then x is adjacent to x' by Claim 2, so $G[x, x', v_4, y, v_5]$ is a $\overline{2P_1 + P_3}$, a contradiction. Therefore every vertex of $V_{1,3}$ has at most one neighbor in $V_{4,5}$.

By Claim 3, $G[V_{1,3}]$ is P_3-free, so it is a disjoint union of cliques. Suppose $V_{1,3}$ is not a clique, so there are nonadjacent vertices $y, y' \in V_{1,3}$. If $x \in V_{4,5}$ is adjacent to y, but nonadjacent to y' then $G[v_2, y', y, x, v_4]$ is a $2P_1 + P_3$, a contradiction. Therefore every vertex of $V_{4,5}$ is complete or anticomplete to $\{y, y'\}$. Since y and y' were arbitrary nonadjacent vertices in $V_{1,3}$ and $V_{1,3}$ is a disjoint union of (at least two) cliques, it follows that every vertex of $V_{4,5}$ is complete or anticomplete to $V_{1,3}$. Since every vertex of $V_{1,3}$ has at most one neighbor in $V_{4,5}$, at most one vertex in $V_{4,5}$ is complete to $V_{1,3}$. If such a vertex exists then by Fact 1, we may delete it. Therefore we may assume that either $V_{1,3}$ is a clique or $V_{1,3}$ is anticomplete to $V_{4,5}$. The claim follows by symmetry.

Claim 17. *We may assume the following: for $i \in \{1, 2, 3, 4, 5\}$, if $V_{i,i+2}$ and $V_{i,i+1,i+2}$ are both large then all of the following statements hold:*

(i) *Either $V_{i,i+2}$ is an independent set or $V_{i,i+1,i+2}$ is a clique.*

(ii) *If $V_{i,i+2}$ is not an independent set then it is anticomplete to $V_{i,i+1,i+2}$.*

(iii) *If $V_{i,i+1,i+2}$ is not a clique then it is complete to $V_{i,i+2}$.*

Suppose $V_{1,3}$ and $V_{1,2,3}$ are large. By Claim 3, $G[V_{1,3}]$ is P_3-free, so it is a disjoint union of cliques. By Claim 5, $G[V_{1,2,3}]$ is (P_1+P_2)-free, so its complement is a disjoint union of cliques. We consider three cases.

Case 1. *$V_{1,2,3}$ is independent.*

We will show that in this case $V_{1,3}$ must be an independent set which is complete to $V_{1,2,3}$. If $x \in V_{1,3}$ is nonadjacent to $y, y' \in V_{1,2,3}$ then $G[x, v_4, y, v_2, y']$ is a $2P_1+P_3$, a contradiction. Therefore every vertex in $V_{1,3}$ has at most one nonneighbor in $V_{1,2,3}$. Suppose $x, x' \in V_{1,3}$ are adjacent. Since $V_{1,2,3}$ is large, there must be a vertex $y \in V_{1,2,3}$ that is adjacent to both x and x'. Then $G[y, v_3, x, v_2, x']$ is a $\overline{2P_1 + P_3}$. Therefore $V_{1,3}$

must be independent. If $x, x' \in V_{1,3}$ are nonadjacent and $y \in V_{1,2,3}$ is adjacent to x, but not to x', then $G[x', v_4, x, y, v_2]$ is a $2P_1 + P_3$, a contradiction. Therefore every vertex of $V_{1,2,3}$ is either complete or anticomplete to $V_{1,3}$. Suppose there is a vertex $y \in V_{1,2,3}$ that is anticomplete to $V_{1,3}$. Since every vertex of $V_{1,3}$ has at most one nonneighbor in $V_{1,2,3}$, there must be a vertex $y' \in V_{1,2,3}$ that is complete to $V_{1,3}$. Now $G[v_4, y', x, y, x']$ is a $2P_1 + P_3$, a contradiction. Therefore $V_{1,2,3}$ is complete to $V_{1,3}$. We conclude that if $V_{1,2,3}$ is an independent set then $V_{1,3}$ must also be an independent set and furthermore $V_{1,3}$ must be complete to $V_{1,2,3}$. By symmetry, if $V_{i,i+1,i+2}$ is an independent set then statements (i)–(iii) of the claim hold.

Case 2. *$V_{1,3}$ is a clique.*
Casting to the complement as before, the clique $V_{1,3}$ in G becomes the independent set $W_{3,4,5}$ in $\overline{G}$ and the set $V_{1,2,3}$ in G becomes the set $W_{3,5}$ in $\overline{G}$. By the above argument, this means that in $\overline{G}$, $W_{3,5}$ must be an independent set and it must be complete to $W_{3,4,5}$. Therefore in G the set $V_{1,2,3}$ must be a clique and it must be anticomplete to $V_{1,3}$. By symmetry, if $V_{i,i+2}$ is a clique then statements (i)–(iii) of the claim hold.

Case 3. *$V_{1,2,3}$ is not independent and $V_{1,3}$ is not a clique.*
If $x, x' \in V_{1,3}$ are nonadjacent and $y \in V_{1,2,3}$ is adjacent to x, but not x' then $G[v_4, x', x, y, v_2]$ is a $2P_1 + P_3$, a contradiction. Since $G[V_{1,3}]$ is a disjoint union of (at least two) cliques, it follows that every vertex of $V_{1,2,3}$ is complete or anticomplete to $V_{1,3}$. If $y, y' \in V_{1,2,3}$ are adjacent and $x \in V_{1,3}$ is adjacent to y, but not y', then $G[y, v_1, y', x, v_2]$ is a $\overline{2P_1 + P_3}$, a contradiction. Since $G[V_{1,2,3}]$ is the complement of a disjoint union of (at least two) cliques, it follows that every vertex of $V_{1,3}$ is complete or anticomplete to $V_{1,2,3}$. We conclude that $V_{1,3}$ is complete or anticomplete to $V_{1,2,3}$.

Suppose for contradiction that $V_{1,3}$ is not an independent set and $V_{1,3}$ is complete to $V_{1,2,3}$. Choose adjacent vertices $x, x' \in V_{1,3}$ and adjacent vertices $y, y' \in V_{1,2,3}$. Then $G[y, y', x, v_2, x']$ is a $\overline{2P_1 + P_3}$, a contradiction. Therefore either $V_{1,3}$ is independent or it is anticomplete to $V_{1,2,3}$. By symmetry statement (ii) of the claim holds.

Suppose for contradiction that $V_{1,2,3}$ is not a clique and $V_{1,2,3}$ is anticomplete to $V_{1,3}$. Choose nonadjacent vertices $y, y' \in V_{1,2,3}$ and nonadjacent vertices $x, x' \in V_{1,3}$. Then $G[x, x', y, v_2, y']$ is a $2P_1 + P_3$, a contradiction. Therefore either $V_{1,2,3}$ is a clique or it is complete to $V_{1,2,3}$. By symmetry statement (iii) of the claim holds.

Note that if $V_{1,3}$ is not independent then it is anticomplete to $V_{1,2,3}$ and that if $V_{1,2,3}$ is not a clique then it is complete to $V_{1,3}$. Since $V_{1,3}$ and $V_{1,2,3}$ are large, it follows that either $V_{1,2,3}$ is an independent set or $V_{1,3}$ is a clique. By symmetry statement (i) of the claim holds. This completes the proof of Claim 17.

Claim 18. *We may assume that for $i \in \{1, 2, 3, 4, 5\}$ and $S \in \{\{i+1, i+2, i+3\}, \{i, i+1, i+4\}\}$, if $V_{i,i+2}$ and V_S are large then one of the following cases holds:*
(i) *$V_{i,i+2}$ and V_S are cliques and $V_{i,i+2}$ is anticomplete to V_S.*
(ii) *$V_{i,i+2}$ is independent and complete to V_S.*

Suppose $V_{1,3}$ and $V_{2,3,4}$ are large. By Claim 3, $G[V_{1,3}]$ is P_3-free, so it is a disjoint union of cliques. By Claim 5, $G[V_{1,2,3}]$ is (P_1+P_2)-free, so its complement is a disjoint union of cliques.

First suppose that $V_{1,3}$ is not a clique. Let $x, x' \in V_{1,3}$ be nonadjacent and suppose $y \in V_{2,3,4}$ is nonadjacent to x'. Then $G[x', v_5, x, y, v_2]$ or $G[x, x', v_2, y, v_4]$ is a $2P_1 + P_3$ if x is adjacent or nonadjacent to y, respectively. Since $G[V_{1,3}]$ is a disjoint union of (at least two) cliques, this contradiction implies that $V_{2,3,4}$ is complete to $V_{1,3}$. If $x, x' \in V_{1,3}$ are adjacent and $y \in V_{2,3,4}$ then $G[y, v_3, x, v_2, x']$ is a $\overline{2P_1 + P_3}$, a contradiction. Therefore $V_{1,3}$ must be independent, so statement (ii) of the claim holds.

Now suppose that $V_{1,3}$ is a clique. Again, if $y \in V_{2,3,4}$ is adjacent to $x, x' \in V_{1,3}$ then $G[y, v_3, x, v_2, x']$ is a $\overline{2P_1 + P_3}$, a contradiction. Therefore every vertex of $V_{2,3,4}$ has at most one neighbor in $V_{1,3}$. Suppose $y, y' \in V_{2,3,4}$ are nonadjacent. Since $V_{1,3}$ is large, there must be a vertex $x \in V_{1,3}$ that is nonadjacent to both y and y'. Now $G[x, v_5, y, v_2, y']$ is a $2P_1 + P_3$, a contradiction. Therefore $V_{2,3,4}$ must be a clique. Suppose $x \in V_{1,3}$ and $y, y' \in V_{2,3,4}$ with x adjacent to y, but not to y'. Then $G[y, v_3, y', x, v_2]$ is a $\overline{2P_1 + P_3}$, a contradiction. Therefore every vertex of $V_{1,3}$ is either complete or anticomplete to $V_{2,3,4}$. Since every vertex of $V_{2,3,4}$ has at most one neighbor in $V_{1,3}$, at most one vertex of $V_{1,3}$ is complete to $V_{2,3,4}$. If such a vertex exists then by Fact 1, we may delete it. Therefore we may assume that $V_{1,3}$ is anticomplete to $V_{2,3,4}$, so statement (i) of the claim holds. The claim follows by symmetry.

Claim 19. *We may assume that for $i \in \{1, 2, 3, 4, 5\}$ and $S \in \{\{i + 2, i + 3, i + 4\}, \{i, i + 3, i + 4\}\}$, if $V_{i,i+2}$ and V_S are large then one of the following cases holds:*

(i) *$V_{i,i+2}$ and V_S are independent and $V_{i,i+2}$ is complete to V_S.*

(ii) *V_S is a clique and anticomplete to $V_{i,i+2}$.*

By symmetry we only need to prove the claim for the case where $i = 1$ and $S = \{3, 4, 5\}$. In this case the sets $V_{i,i+2}$ and V_S are $V_{1,3}$ and $V_{3,4,5}$, respectively, which are equal to $W_{3,4,5}$ and $W_{1,4}$, respectively (see also Table 1). The claim follows by casting to the complement and applying Claim 18.

Claim 20. *For $i \in \{1, 2, 3, 4, 5\}$ and $S \in \{\{i, i + 1, i + 3\}, \{i + 1, i + 2, i + 4\}\}$, if $V_{i,i+2}$ and V_S are large then $V_{i,i+2}$ is independent and it is complete to V_S.*

By Claim 4, $V_{1,2,4}$ is independent. Suppose $x \in V_{1,3}$ and $y, y' \in V_{1,2,4}$ with x nonadjacent to y'. Then $G[y', v_5, v_3, x, y]$ or $G[v_5, x, y, v_2, y']$ is a $2P_1 + P_3$ if x is adjacent or nonadjacent to y, respectively. Therefore $V_{1,3}$ is complete to $V_{1,2,4}$. If $x, x' \in V_{1,3}$ are adjacent and $y \in V_{1,2,4}$ then $G[v_1, y, x, v_2, x']$ is a $\overline{2P_1 + P_3}$, a contradiction. Therefore $V_{1,3}$ is independent. The claim follows by symmetry.

Claim 21. *For $i \in \{1, 2, 3, 4, 5\}$, if $V_{i,i+2}$ and $V_{i+1,i+3,i+4}$ are large then $G[V_{i,i+2} \cup V_{i+1,i+3,i+4}]$ has bounded clique-width.*

Suppose $V_{1,3}$ and $V_{2,4,5}$ are large. By Claim 3, $G[V_{1,3}]$ is P_3-free, so it is a disjoint union of cliques. By Claim 4, $V_{2,4,5}$ is independent. If $x \in V_{1,3}$ is nonadjacent to $y, y' \in V_{2,4,5}$ then $G[y, y', v_1, x, v_3]$ is a $2P_1 + P_3$, a contradiction. Therefore every vertex of $V_{1,3}$ has at most one nonneighbor in $V_{2,4,5}$. If $x, x' \in V_{1,3}$ are nonadjacent and $y \in V_{2,4,5}$ is nonadjacent to x and x' then $G[x, x', v_2, y, v_4]$ is a $2P_1 + P_3$, a contradiction. Therefore every vertex of $V_{2,4,5}$ is complete to all but at most one component of $V_{1,3}$. Let G' be the graph obtained from $G[V_{1,3} \cup V_{2,4,5}]$ by applying a bipartite complementation between $V_{1,3}$ and $V_{2,4,5}$. By Fact 3, $G[V_{1,3} \cup V_{2,4,5}]$ has bounded clique-width if and only if every component of G' has bounded clique-width. Now consider a component $C^{G'}$ of G'. We will prove that $C^{G'}$ has bounded clique-width. We first note that $C^{G'}$ consists of either a single vertex (in which case it has clique-width 1), or a clique in $V_{1,3}$ together with an independent set in $V_{2,4,5}$, no two vertices of which have a common neighbor in the clique. By Fact 2, we may complement the clique $V(C^{G'}) \cap V_{1,3}$ in $C^{G'}$. The resulting graph $C'^{G'}$ is a disjoint union of stars, which has clique-width at most 2. We conclude that $C^{G'}$ has bounded clique-width.

Claim 22. *For $i \in \{1,2,3,4,5\}$ and $S \in \{\{i,i+1,i+2,i+3\},\{i,i+1,i+2,i+4\}\}$, if $V_{i,i+2}$ and V_S are large then $V_{i,i+2}$ is independent and it is complete to V_S.*
Suppose $V_{1,3}$ and $V_{1,2,3,4}$ are large. By Claim 3, $G[V_{1,3}]$ is P_3-free, so it is a disjoint union of cliques. By Claim 4, $V_{1,2,3,4}$ is independent.

Suppose $x \in V_{1,3}$ has two nonneighbors $y,y' \in V_{1,2,3,4}$. Then $G[x,v_5,y,v_2,y']$ is a $2P_1+P_3$, a contradiction. It follows that every vertex of $V_{1,3}$ has at most one nonneighbor in $V_{1,2,3,4}$.

Suppose, for contradiction, that $V_{1,3}$ is not an independent set. Let $x,x' \in V_{1,3}$ be adjacent vertices. If $y \in V_{1,2,3,4}$ is adjacent to both x and x' then $G[y,v_3,x,v_2,x']$ is a $\overline{2P_1+P_3}$, a contradiction. Therefore every vertex of $V_{1,2,3,4}$ has at most one neighbor in $\{x,x'\}$. Since $V_{1,2,3,4}$ is large, there must be two vertices $y',y'' \in V_{1,2,3,4}$ that are nonadjacent to the same vertex in $\{x,x'\}$. This is a contradiction since every vertex of $V_{1,3}$ has at most one nonneighbor in $V_{1,2,3,4}$. It follows that $V_{1,3}$ is an independent set. Since $5 \notin \{1,3\} \cup \{1,2,3,4\}$, Claim 8 implies that $V_{1,3}$ is complete to $V_{1,2,3,4}$. The claim follows by symmetry.

Claim 23. *We may assume that for $i \in \{1,2,3,4,5\}$ and $S \in \{\{i,i+1,i+3,i+4\}, \{i+1,i+2,i+3,i+4\},\{1,2,3,4,5\}\}$, if $V_{i,i+2}$ and V_S are large then one of the following holds:*
(i) *$V_{i,i+2}$ is an independent set or*
(ii) *$V_{i,i+2}$ is the disjoint union of a (possibly empty) clique that is anticomplete to V_S and a (possibly empty) independent set that is complete to V_S.*

Suppose $V_{1,3}$ and V_S are large for $S \in \{\{1,2,4,5\},\{1,2,3,4,5\}\}$ (the $S=\{2,3,4,5\}$ case is symmetric). By Claim 3, $G[V_{1,3}]$ is P_3-free, so it is a disjoint union of cliques. By Claim 4, V_S is independent.

If $x,x' \in V_{1,3}$ are nonadjacent and $y \in V_S$ is anticomplete to $\{x,x'\}$ then $G[x,x',v_2,y,v_4]$ is a $2P_1+P_3$, a contradiction. Since $G[V_{1,3}]$ is a disjoint union of cliques, it follows that every vertex of V_S is complete to all but at most one component of $G[V_{1,3}]$. If $x,x' \in V_{1,3}$ are adjacent and $y \in V_S$ is complete to $\{x,x'\}$ then $G[v_1,y,x,v_2,x']$ is a $\overline{2P_1+P_3}$, a contradiction. Therefore no vertex of V_S has two neighbors in the same component of $G[V_{1,3}]$. It follows that $G[V_{1,3}]$ contains at most one nontrivial component. In other words, either $V_{1,3}$ is an independent set or the disjoint union of a clique and an independent set.

Suppose that $V_{1,3}$ is not an independent set. Then $G[V_{1,3}]$ contains a nontrivial component C'. We may assume C' contains at least three vertices, otherwise we may delete it by Fact 1. No vertex of V_S can have two neighbors in C' and every vertex of V_S is complete to all but at most one component of $G[V_{1,3}]$. Therefore every vertex of V_S is complete to the independent set $V_{1,3} \setminus V(C')$. Suppose $x \in V_S$ has a neighbor $y \in V(C')$. Since $V(C')$ contains at least three vertices, and every vertex of V_S has at most one neighbor in $V(C')$, we can find vertices $y',y'' \in V(C')$ that are nonadjacent to x. Now $G[v_1,y,y',x,y'']$ is a $\overline{2P_1+P_3}$, a contradiction. Therefore V_S is anticomplete to $V(C')$.

We conclude that either $V_{1,3}$ is an independent set or it is the disjoint union of an independent set that is complete to V_S and a clique that is anticomplete to V_S. The claim follows by symmetry.

Claim 24. *For $i \in \{1,2,3,4,5\}$, if $V_{i,i+2}$ and $V_{i,i+2,i+3,i+4}$ are large then $V_{i,i+2}$ is an independent set that is complete to $V_{i,i+2,i+3,i+4}$.*
Suppose $V_{1,3}$ and $V_{1,3,4,5}$ are large. By Claim 3, $G[V_{1,3}]$ is P_3-free, so it is a disjoint union of cliques. By Claim 4, $V_{1,3,4,5}$ is independent.

Suppose $x \in V_{1,3}$ is nonadjacent to $y, y' \in V_{1,3,4,5}$. Then $G[v_2, x, y, v_4, y']$ is a $2P_1 + P_3$, a contradiction. Therefore every vertex of $V_{1,3}$ has at most one nonneighbor in $V_{1,3,4,5}$. Suppose $x, x' \in V_{1,3}$ are adjacent. Since $V_{1,3,4,5}$ is large, there must be a vertex $y \in V_{1,3,4,5}$ that is adjacent to both x and x'. Now $G[y, v_1, x, v_5, x']$ is an $\overline{2P_1 + P_3}$, a contradiction. Therefore $V_{1,3}$ must be an independent set. If $x, x' \in V_{1,3}$ and $y \in V_{1,3,4,5}$ is adjacent to x, but not to x' then $G[x', v_2, x, y, v_4]$ is a $2P_1 + P_3$, a contradiction. Therefore every vertex of $V_{1,3,4,5}$ must be either complete or anticomplete to $V_{1,3}$. Since every vertex of $V_{1,3}$ has at most one nonneighbor in $V_{1,3,4,5}$, it follows that at most one vertex of $V_{1,3,4,5}$ may be anticomplete to $V_{1,3}$. Suppose $x, x' \in V_{1,3}$. If $y \in V_{1,3,4,5}$ is anticomplete to $V_{1,3}$ and $y' \in V_{1,3,4,5}$ is complete to $V_{1,3}$ then $G[y, v_2, x, y', x']$ is a $2P_1 + P_3$. We conclude that $V_{1,3}$ is complete to $V_{1,3,4,5}$. The claim follows by symmetry.

The next two claims will allow us to assume that every set V_S is either a clique or an independent set.

Claim 25. *For $i \in \{1, 2, 3, 4, 5\}$, if $V_{i,i+2}$ is large then we may assume it is an independent set or a clique.*
Suppose $V_{1,3}$ is large and that it is not a clique or an independent set.

By Claim 12, if V_S is large for some $S \subseteq \{1, 2, 3, 4, 5\}$ with $S \neq \{1, 3\}$ then $S \cap \{2, 4\} \neq \emptyset$ and $S \cap \{2, 5\} \neq \emptyset$. It follows that $V_\emptyset$, V_4, V_5, V_1, $V_{1,4}$, $V_{1,5}$, V_3, $V_{3,4}$, $V_{3,5}$, $V_{1,3,4}$, and $V_{1,3,5}$ are empty. V_2 is empty by Claim 13. $V_{1,2}$ and $V_{2,3}$ are empty by Claim 15. $V_{2,3,4}$ and $V_{1,2,5}$ are empty by Claim 18. $V_{1,2,3,4}$ and $V_{1,2,3,5}$ are empty by Claim 22. $V_{1,3,4,5}$ is empty by Claim 24. $V_{1,2,4}$ and $V_{2,3,5}$ are empty by Claim 20. Therefore in addition to $V_{1,3}$, only the following sets may be nonempty: $V_{4,5}$, $V_{2,4}$, $V_{2,5}$, $V_{1,2,3}$, $V_{3,4,5}$, $V_{1,4,5}$, $V_{2,4,5}$, $V_{1,2,4,5}$, $V_{2,3,4,5}$, and $V_{1,2,3,4,5}$.

Suppose $V_{2,4,5} = W_{1,2}$ is large. By Claim 12 if a set V_S is large for some $S \subseteq \{1, 2, 3, 4, 5\}$ with $S \neq \{2, 4, 5\}$ then $S \cap \{1, 3\} \neq \emptyset$. It follows that $V_{4,5}$, $V_{2,4}$, and $V_{2,5}$ are empty. By Claim 11 if V_S is large for some $S \subseteq \{1, 2, 3, 4, 5\}$ with $S \neq \{2, 4, 5\}$ then $\{4, 5\} \not\subseteq S$. It follows that $V_{3,4,5}$, $V_{1,4,5}$, $V_{1,2,4,5}$, $V_{2,3,4,5}$, and $V_{1,2,3,4,5}$ are empty. Therefore, apart from $V_{1,3}$ and $V_{2,4,5}$, only the set $V_{1,2,3}$ can be large. By Claim 17, if $V_{1,2,3}$ is large then it is a clique that is anticomplete to $V_{1,3}$. Casting to the complement (see also Table 1), since $V_{1,2,3}$ is a clique in G, it follows that $W_{3,5} = V_{1,2,3}$ is an independent set in $\overline{G}$, so by Claim 16, $W_{3,5}$ is anticomplete to $W_{1,2} = V_{2,4,5}$ in $\overline{G}$. Therefore in the graph G, $V_{1,2,3} = W_{3,5}$ is a clique that is complete to $V_{2,4,5} = W_{1,2}$ and anticomplete to $V_{1,3}$. By Fact 1, we may delete the five vertices in the original cycle C. By Fact 3, we may apply a bipartite complementation between $V_{2,4,5}$ and $V_{1,2,3}$. This separates the graph into two parts: $G[V_{1,3} \cup V_{2,4,5}]$, which has bounded clique-width by Claim 21 and $G[V_{1,2,3}]$, which is a clique and so has clique-width at most 2. Therefore if $V_{2,4,5}$ is large then G has bounded clique-width. Thus we may assume that $V_{2,4,5} = \emptyset$.

We will now show how to disconnect $V_{1,3}$ from the rest of the graph. Note that $V_{4,5}$ is anticomplete to $V_{1,3}$ by Claim 16. $V_{1,2,3}$ is anticomplete to $V_{1,3}$ by Claim 17. $V_{3,4,5}$ and $V_{1,4,5}$ are anticomplete to $V_{1,3}$ by Claim 19. $V_{2,4}$ and $V_{2,5}$ are complete to $V_{1,3}$ by Claim 14. By Fact 3, we may apply a bipartite complementation between $V_{1,3}$ and $\{v_1, v_3\} \cup V_{2,4} \cup V_{2,5}$. By Claim 23, for $S \in \{\{1, 2, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}\}$, either V_S is empty or $V_{1,3}$ is the disjoint union of a clique C' that is anticomplete to V_S and an independent set I that is complete to V_S. If $V_{1,3}$ does have this form, then by Fact 3, we may apply a bipartite complementation between I and $V_{1,2,4,5} \cup V_{2,3,4,5} \cup V_{1,2,3,4,5}$. Doing this removes all edges from $V_{1,3}$ to vertices not in $V_{1,3}$. By Claim 3, $G[V_{1,3}]$ is a P_3-free graph, so it is a disjoint union of cliques and thus has clique-width at most 2.

We conclude that if $V_{1,3}$ is large, but is neither a clique nor an independent set, then $G[V_{1,3}]$ has bounded clique-width and we can remove all edges from $V_{1,3}$ to vertices not in $V_{1,3}$. We may therefore remove all vertices in $V_{1,3}$ from the graph. The claim follows by symmetry.

Claim 26. *For $i \in \{1, 2, 3, 4, 5\}$, if $V_{i,i+1,i+2}$ is large then we may assume it is an independent set or a clique.*
This follows from Claim 25 by casting to the complement (see also Table 1).

Note that by Claims 2, 4, 25, and 26, we may assume that every large set V_S is either a clique or an independent set.

Claim 27. *For $i \in \{1, 2, 3, 4, 5\}$, if $V_{i,i+2}$ is large then we may assume it is an independent set.*
Suppose that $V_{1,3}$ is large, but not an independent set. By Claim 25, we may assume that it is a clique. We will show how to disconnect $V_{1,3}$ (or a part of the graph that contains $V_{1,3}$ and has bounded clique-width) from the rest of the graph. First, by Fact 1, we may delete the five vertices of the original cycle C. Let $G' = G[\bigcup V_S \mid V_S \text{ is a clique}]$ and let $G'' = G[\bigcup V_S \mid V_S \text{ is an independent set}]$. By Claim 6, if V_S is a clique and V_T is an independent set, then V_S is either complete or anticomplete to V_T. If V_S is complete to V_T, by Fact 3, we may apply a bipartite complementation between these sets. Doing so for every pair of a clique V_S and an independent set V_T that are complete to each other, we disconnect G' from G''. Since our aim is to show how to remove the clique $V_{1,3}$ from G, it is therefore sufficient to show how to remove it from G'. In other words, we may assume that if V_T is an independent set then $V_T = \emptyset$. That is, we may assume that every set V_S is a (possibly empty) clique.

By Claim 4, $V_{1,2,4}$, $V_{2,3,5}$, $V_{1,3,4}$, $V_{2,4,5}$, $V_{1,3,5}$, $V_{1,2,3,4}$, $V_{1,2,3,5}$, $V_{1,2,4,5}$, $V_{1,3,4,5}$, $V_{2,3,4,5}$, and $V_{1,2,3,4,5}$ are independent sets, so we may assume that they are empty. Since $V_{1,3}$ is a large, by Claim 12 if V_S is large for some $S \subseteq \{1, 2, 3, 4, 5\}$ with $S \neq \{1, 3\}$ then $S \cap \{2, 4\} \neq \emptyset$ and $S \cap \{2, 5\} \neq \emptyset$. It follows that $V_\emptyset$, V_4, V_5, V_1, $V_{1,4}$, $V_{1,5}$, V_3, $V_{3,4}$, $V_{3,5}$ are empty. This means that apart from $V_{1,3}$, only the following sets can be large: $V_{1,2}$, $V_{2,3}$, $V_{1,2,3}$, $V_{2,3,4}$, $V_{3,4,5}$, $V_{1,4,5}$, $V_{1,2,5}$, V_2, $V_{4,5}$, $V_{2,4}$, $V_{2,5}$ and recall that all these sets are (possibly empty) cliques by assumption (see also Figure 4). For two of these sets, if there is an $i \in S \cap T$ then V_S is anticomplete to V_T by Claim 10. Since $\{1, 3\} \cap (\{2\} \cup \{4, 5\} \cup \{2, 4\} \cup \{2, 5\}) = \emptyset$, at most one of the sets V_2, $V_{4,5}$, $V_{2,4}$, and $V_{2,5}$ is large by Claim 12. We consider several cases.

Case 1. *$V_{2,4}$ or $V_{2,5}$ is large.*
By symmetry, we may assume $V_{2,4}$ is large. Then V_2, $V_{4,5}$, and $V_{2,5}$ are empty, as stated above. Also, $V_{1,3}$ and $V_{2,4}$ are anticomplete to $V_{1,2}$, $V_{2,3}$, $V_{1,2,3}$, $V_{2,3,4}$, $V_{3,4,5}$, $V_{1,4,5}$, and $V_{1,2,5}$ by Claim 10. This means that $G[V_{1,3} \cup V_{2,4}]$ is disconnected from the rest of G'. By Lemma 10, $G[V_{1,3} \cup V_{2,4}]$ has bounded clique-width. This completes the case.

Case 2. *V_2 is large.*
Then $V_{4,5}$, $V_{2,4}$, and $V_{2,5}$ are empty, as stated above. Since $\{3, 5\} \notin \{2\} \cup \{1, 2\}$ and $\{1, 4\} \notin \{2\} \cup \{2, 3\}$, Claim 12 implies that $V_{1,2}$ and $V_{2,3}$ are empty. Now $V_{1,3}$, $V_{1,2,3}$, $V_{2,3,4}$, $V_{3,4,5}$, $V_{1,4,5}$, and $V_{1,2,5}$ are pairwise anticomplete by Claim 10. By Claim 9, the edges between V_2 and each of $V_{1,3}$, $V_{1,2,3}$, $V_{2,3,4}$, $V_{3,4,5}$, $V_{1,4,5}$, and $V_{1,2,5}$ form matchings. By Fact 2, we can complement all of the large sets. We obtain a graph which

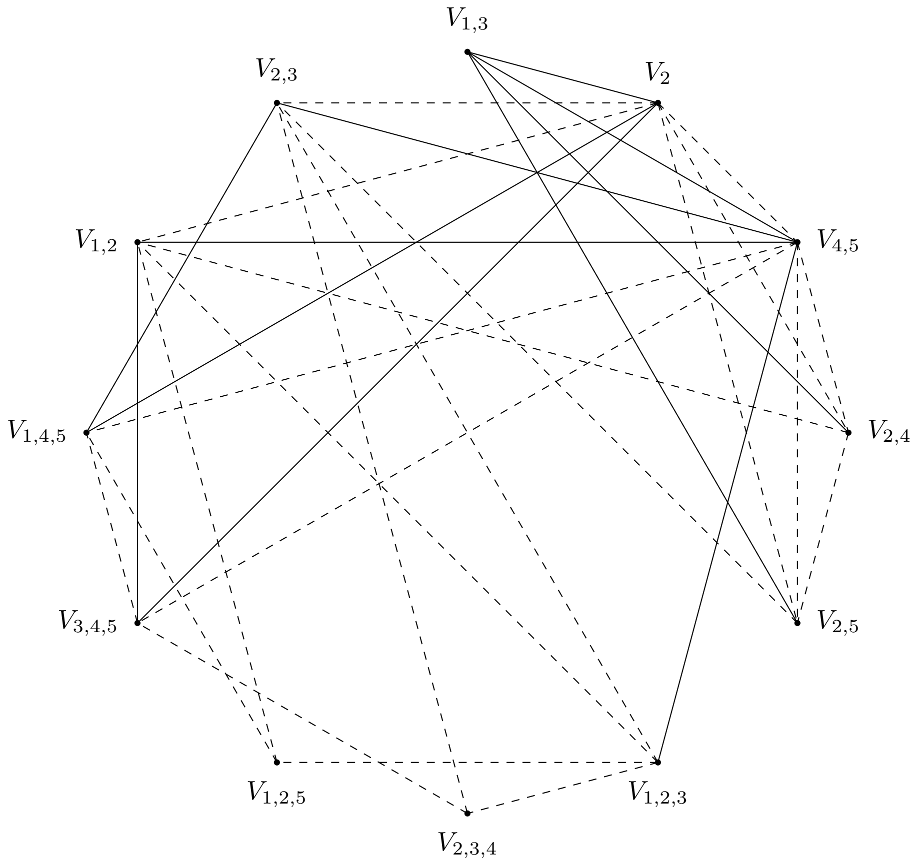

FIG. 4. *The set of possible cliques when $V_{1,3}$ is a clique. Two sets are joined by a line if the edges between them form a matching (recall that a matching may contain no edges, in which case the two sets are anticomplete to each other). Two sets are joined by a dashed line if at most one of them is large and the other is empty. Two sets are not joined by a line if they are anticomplete to each other. These properties follow from Claims* 9, 10, 11, *and* 12.

is a disjoint union of stars, which have clique-width at most 2. It follows that $G' = G[V_{1,3} \cup V_{1,2,3} \cup V_{2,3,4} \cup V_{3,4,5} \cup V_{1,4,5} \cup V_{1,2,5}]$ has bounded clique-width. This completes the case.

Case 3. *$V_{4,5}$ is large.*
Then V_2, $V_{2,4}$, and $V_{2,5}$ are empty, as stated above. Since $V_{4,5}$ is large and $\{4,5\} \subseteq \{3,4,5\}, \{1,4,5\}$, Claim 11 implies that $V_{3,4,5}$ and $V_{1,4,5}$ are empty. Now $V_{1,3}$, $V_{1,2}$, $V_{2,3}$, $V_{1,2,3}$, $V_{2,3,4}$, and $V_{1,2,5}$ are pairwise anticomplete by Claim 10. By Claim 9, the edges between $V_{4,5}$ and each of $V_{1,3}$, $V_{1,2}$, $V_{2,3}$, $V_{1,2,3}$, $V_{2,3,4}$, and $V_{1,2,5}$ form matchings. By Fact 2, we can complement all of the large sets. We obtain a graph which is a disjoint union of stars, which have clique-width at most 2. It follows that $G' = G[V_{1,3} \cup V_{4,5} \cup V_{1,2} \cup V_{2,3} \cup V_{1,2,3} \cup V_{2,3,4} \cup V_{1,2,5}]$ has bounded clique-width. This completes the case.

Case 4. *V_2, $V_{4,5}$, $V_{2,4}$, and $V_{2,5}$ are empty.*
$V_{1,3}$ is anticomplete to $V_{1,2}$, $V_{2,3}$, $V_{1,2,3}$, $V_{2,3,4}$, $V_{3,4,5}$, $V_{1,4,5}$, and $V_{1,2,5}$ by Claim 10. Therefore $G'[V_{1,3}] = G[V_{1,3}]$ is disconnected from the rest of G'. Since $V_{1,3}$ is a clique,

$G[V_{1,3}]$ has clique-width at most 2. We may therefore remove $V_{1,3}$ from the graph. This completes the case.

Since one of the above cases must hold by Claim 1, this completes the proof of the claim when $i = 1$. The claim follows by symmetry.

Claim 28. *For* $i \in \{1, 2, 3, 4, 5\}$*, if* $V_{i,i+1,i+2}$ *is large then we may assume it is a clique.* This follows from Claim 27 by casting to the complement (see also Table 1).

Claim 29. *For* $i \in \{1, 2, 3, 4, 5\}$*, we may assume* $V_{i,i+2}$ *is empty.*
Suppose that $V_{1,3}$ is large. By Claim 27, we may assume that it is an independent set. We will show how to disconnect $V_{1,3}$ (or a part of the graph that contains $V_{1,3}$ and has bounded clique-width) from the rest of the graph. First, by Fact 1, we may delete the five vertices of the original cycle C. Let $G' = G[\bigcup V_S \mid V_S \text{ is a clique}]$ and let $G'' = G[\bigcup V_S \mid V_S \text{ is an independent set}]$. By Claim 6, if V_S is a clique and V_T is an independent set, then V_S is either complete or anticomplete to V_T. If V_S is complete to V_T, by Fact 3, we may apply a bipartite complementation between these sets. Doing so for every pair of a clique V_S and an independent set V_T that are complete to each other, we disconnect G' from G''. Since our aim is to show how to remove the independent set $V_{1,3}$ from G, it is therefore sufficient to show how to remove it from G''. In other words, we may assume that if V_S is a clique then $V_S = \emptyset$. That is, we may assume that every set V_T is a (possibly empty) independent set.

By Claim 2, $V_\emptyset$, V_1, V_2, V_3, V_4, V_5, $V_{1,2}$, $V_{2,3}$, $V_{3,4}$, $V_{4,5}$, and $V_{1,5}$ are cliques, so we may assume that they are empty. By Claim 28, $V_{1,2,3}$, $V_{2,3,4}$, $V_{3,4,5}$, $V_{1,4,5}$, and $V_{1,2,5}$ are cliques, so we may assume that they are empty.

Since $V_{1,3}$ is large, by Claim 12, if V_S is large for some $S \subseteq \{1, 2, 3, 4, 5\}$ with $S \neq \{1, 3\}$ then $S \cap \{2, 4\} \neq \emptyset$ and $S \cap \{2, 5\} \neq \emptyset$. It follows that $V_{1,4}$, $V_{3,5}$, $V_{1,3,4}$, and $V_{1,3,5}$ are empty.

This means that apart from $V_{1,3}$, only the following sets can be large: $V_{2,4}$, $V_{2,5}$, $V_{1,2,4}$, $V_{2,3,5}$, $V_{1,2,3,4}$, $V_{1,2,3,5}$, $V_{1,3,4,5}$, $V_{2,4,5}$, $V_{1,2,4,5}$, $V_{2,3,4,5}$, and $V_{1,2,3,4,5}$ and note that they are all (possibly empty) independent sets by assumption (see also Figure 5). For two of these sets, if there is an $i \in \{1, 2, 3, 4, 5\}$ such that $i \notin S$ and $i \notin T$ then V_S is complete to V_T by Claim 8.

Since $\{4, 5\} \subseteq \{1, 2, 3, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 4, 5\}, \{2, 4, 5\}$, at most one of the sets $V_{1,2,3,4,5}$, $V_{2,3,4,5}$, $V_{1,2,4,5}$, and $V_{2,4,5}$ is large by Claim 11. We consider several cases.

Case 1. $V_{1,2,3,4,5}$ *is large.*
Then, since
- $\{1, 2\} \subseteq \{1, 2, 4\}$,
- $\{2, 3\} \subseteq \{2, 3, 5\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}$, and
- $\{4, 5\} \subseteq \{1, 3, 4, 5\}, \{2, 4, 5\}, \{1, 2, 4, 5\}, \{2, 3, 4, 5\}$,

Claim 11 implies that $V_{1,2,4}$, $V_{2,3,5}$, $V_{1,2,3,4}$, $V_{1,2,3,5}$, $V_{1,3,4,5}$, $V_{2,4,5}$, $V_{1,2,4,5}$, and $V_{2,3,4,5}$ are empty. Now $\{1, 3\} \cap (\{2, 4\} \cup \{2, 5\}) = \emptyset$, so Claim 12 implies that either $V_{2,4}$ or $V_{2,5}$ is empty. By symmetry we may assume that $V_{2,5}$ is empty. This means that only the sets $V_{1,3}$, $V_{1,2,3,4,5}$, and $V_{2,4}$, are large. By Lemma 10, it follows that G'' has bounded clique-width. This completes the case.

Case 2. $V_{2,3,4,5}$ *or* $V_{1,2,4,5}$ *is large.*
By symmetry, we may assume that $V_{2,3,4,5}$ is large. Then, since
- $\{2, 3\} \subseteq \{2, 3, 5\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}$ and
- $\{4, 5\} \subseteq \{1, 3, 4, 5\}, \{2, 4, 5\}, \{1, 2, 4, 5\}, \{1, 2, 3, 4, 5\}$,

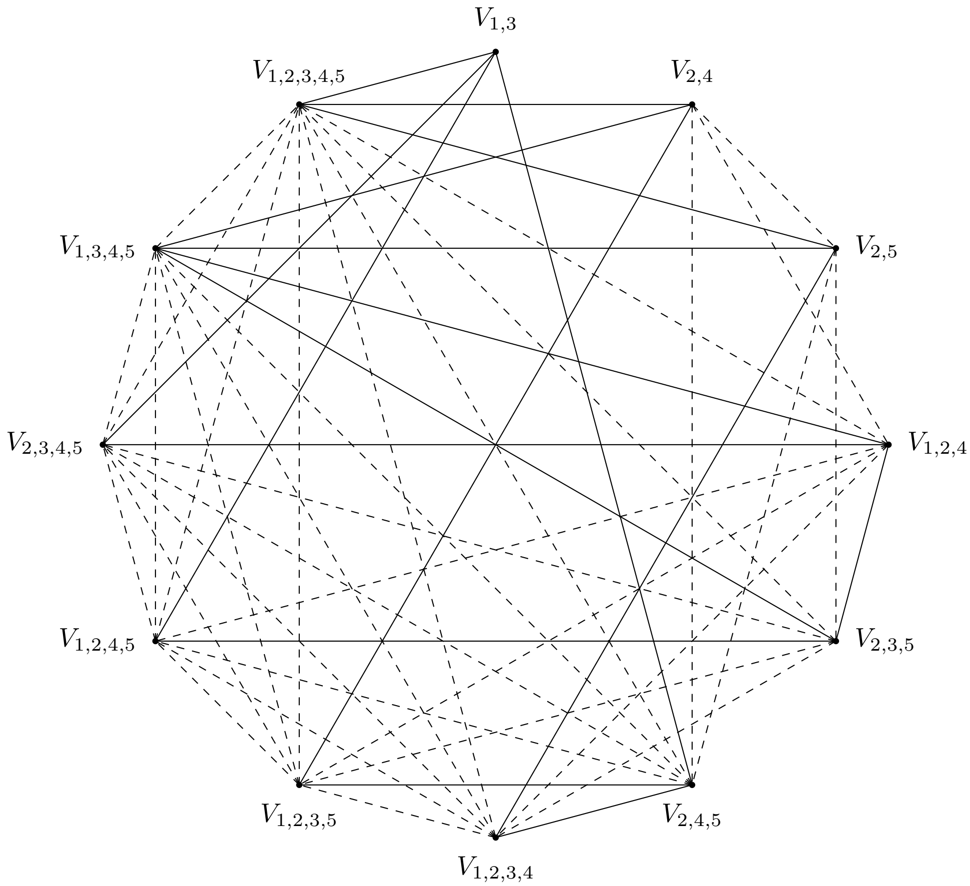

FIG. 5. *The set of possible independent sets when $V_{1,3}$ is an independent set. Two sets are joined by a line if the edges between them form a comatching. Two sets are joined by a dashed line if at most one of them is large. Two sets are not joined by a line if they are complete to each other. These properties follow from Claims* 7, 8, 11, *and* 12.

Claim 11 implies that $V_{2,3,5}$, $V_{1,2,3,4}$, $V_{1,2,3,5}$, $V_{1,3,4,5}$, $V_{2,4,5}$, $V_{1,2,4,5}$, and $V_{1,2,3,4,5}$ are empty. This means that apart from $V_{1,3}$ and $V_{2,3,4,5}$, only the sets $V_{2,4}$, $V_{2,5}$, and $V_{1,2,4}$ can be large. Now $4 \notin \{2,5\} \cup \{1,3\}$, $1 \notin \{2,5\} \cup \{2,3,4,5\}$, and $3 \notin \{2,5\} \cup \{2,4\} \cup \{1,2,4\}$, so by Claim 8, $V_{2,5}$ is complete to all the other large sets. By Fact 3, we may apply a bipartite complementation between $V_{2,5}$ and $V_{1,3} \cup V_{2,3,4,5} \cup V_{2,4} \cup V_{1,2,4}$. This will disconnect $G[V_{2,5}]$ from the rest of G''. Since $V_{2,5}$ is an independent set, $G[V_{2,5}]$ has clique-width at most 1. We may therefore assume that $V_{2,5}$ is empty. Since $\{3,5\} \cap (\{2,4\} \cup \{1,2,4\}) = \emptyset$, Claim 12 implies that either $V_{2,4}$ or $V_{1,2,4}$ is empty. This means that only at most three sets V_S are large: $V_{1,3}$, $V_{2,3,4,5}$, and either $V_{2,4}$ or $V_{1,2,4}$. By Lemma 10, it follows that G'' has bounded clique-width. This completes the case.

Case 3. $V_{2,4,5}$ *is large.*
Then, since

- $\{4,5\} \subseteq \{1,3,4,5\}, \{1,2,4,5\}, \{2,3,4,5\}, \{1,2,3,4,5\}$,

Claim 11 implies that $V_{1,3,4,5}$, $V_{1,2,4,5}$, $V_{2,3,4,5}$, and $V_{1,2,3,4,5}$ are empty. Since $\{1,3\} \cap (\{2,4\} \cup \{2,4,5\}) = \emptyset$ and $\{1,3\} \cap (\{2,5\} \cup \{2,4,5\}) = \emptyset$, Claim 12 implies that $V_{2,4}$ and $V_{2,5}$ are empty. This means that apart from $V_{1,3}$ and $V_{2,4,5}$, only the following sets may be large: $V_{1,2,4}$, $V_{2,3,5}$, $V_{1,2,3,4}$, and $V_{1,2,3,5}$. Since $\{1,2\} \subseteq \{1,2,4\},\{1,2,3,4\},\{1,2,3,5\}$, Claim 11 implies that at most one of $V_{1,2,4}$, $V_{1,2,3,4}$, and $V_{1,2,3,5}$ is large. Since $\{2,3\} \subseteq \{2,3,5\},\{1,2,3,4\},\{1,2,3,5\}$, Claim 11 implies that at most one of $V_{2,3,5}$, $V_{1,2,3,4}$, and $V_{1,2,3,5}$ is large. Since $5 \notin \{1,2,4\} \cup \{1,3\}$, $3 \notin \{1,2,4\} \cup \{2,4,5\}$, $4 \notin \{2,3,5\} \cup \{1,3\}$, $1 \notin \{2,3,5\} \cup \{2,4,5\}$, Claim 8 implies that $V_{1,3}$ and $V_{2,4,5}$ are both complete to both $V_{1,2,4}$ and $V_{2,3,5}$. Therefore, if $V_{1,2,4}$ or $V_{2,3,5}$ are large, then $V_{1,2,3,4}$ and $V_{1,2,3,5}$ are empty and by Fact 3 we can apply a bipartite complementation between $V_{1,3} \cup V_{2,4,5}$ and $V_{1,2,4} \cup V_{2,3,5}$. This will disconnect $G[V_{1,3} \cup V_{2,4,5}]$ from the rest of the graph. By Claim 21, $G[V_{1,3} \cup V_{2,4,5}]$ has bounded clique-width. We may therefore assume that $V_{1,2,4}$ and $V_{2,3,5}$ are empty. This means that at most three sets V_S are large: $V_{1,3}$, $V_{2,4,5}$, and either $V_{1,2,3,4}$ or $V_{1,2,3,5}$. By Lemma 10, it follows that G'' has bounded clique-width. This completes the case.

Case 4. *$V_{1,2,3,4,5}$, $V_{2,3,4,5}$, $V_{1,2,4,5}$, and $V_{2,4,5}$ are empty.*
The only sets apart from $V_{1,3}$ that can be large are $V_{2,4}$, $V_{2,5}$, $V_{1,2,4}$, $V_{2,3,5}$, $V_{1,2,3,4}$, $V_{1,2,3,5}$, and $V_{1,3,4,5}$. Since $4 \notin \{1,3\} \cup \{2,5\},\{2,3,5\},\{1,2,3,5\}$, $5 \notin \{1,3\} \cup \{2,4\} \cup \{1,2,4\} \cup \{1,2,3,4\}$, and $2 \notin \{1,3\} \cup \{1,3,4,5\}$, Claim 8 implies that $V_{1,3}$ is complete to all the other large sets. Applying a bipartite complementation between $V_{1,3}$ and $V_{2,4} \cup V_{2,5} \cup V_{1,2,4} \cup V_{2,3,5} \cup V_{1,2,3,4} \cup V_{1,2,3,5} \cup V_{1,3,4,5}$ disconnects $G[V_{1,3}]$ from the rest of G''. Since $V_{1,3}$ is an independent set, $G[V_{1,3}]$ has clique-width at most 1. Therefore, by Fact 3, we may delete $V_{1,3}$ from the graph. This completes the case.

Since one of the above cases must hold, this completes the proof of the claim when $i = 1$. The claim follows by symmetry.

Claim 30. *For $i \in \{1,2,3,4,5\}$, we may assume $V_{i,i+1,i+2}$ is empty.*
This follows from Claim 29 by casting to the complement (see also Table 1).

We are now ready to complete the proof of the lemma. First, by Fact 1, we may delete the five vertices of the original cycle C. Let $G' = G[\bigcup V_S \mid V_S \text{ is a clique}]$ and let $G'' = G[\bigcup V_S \mid V_S \text{ is an independent set}]$. By Claim 6, if V_S is a clique and V_T is an independent set, then V_S is either complete or anticomplete to V_T. If V_S is complete to V_T, by Fact 3, we may apply a bipartite complementation between these sets. Doing so for every pair of a clique V_S and an independent set V_T that are complete to each other, we disconnect G' from G''. By Fact 3 it is sufficient to show that G' and G'' have bounded clique-width. In fact, it is sufficient to show that G' has bounded clique-width, since then we can obtain the same result for G'' by casting to the complement (see also Table 1) and applying Fact 2. In the remainder of the proof, we show that G' has bounded clique-width.

Note that only the following sets V_S can remain: $V_\emptyset$, V_1, V_2, V_3, V_4, V_5, $V_{1,2}$, $V_{2,3}$, $V_{3,4}$, $V_{4,5}$, and $V_{1,5}$. Note that all of these sets are cliques by Claim 2 and by Claim 9 the edges between any two of these sets form a matching.

If $V_\emptyset$ is large then, since $\{1,3\} \cap (\emptyset \cup \{4\} \cup \{4,5\}) = \emptyset$, Claim 12 implies that V_4 and $V_{4,5}$ are empty. Similarly, every set apart from $V_\emptyset$ is empty, so G' is a complete graph and therefore has clique-width 2. We may therefore assume that $V_\emptyset$ is empty.

Suppose that V_1 is large. Since $\{2,5\} \cap (\{1\} \cup \{3\} \cup \{4\} \cup \{3,4\}) = \emptyset$, $\{2,4\} \cap (\{1\} \cup \{5\} \cup \{1,5\}) = \emptyset$ and $\{3,5\} \cap (\{1\} \cup \{2\} \cup \{1,2\}) = \emptyset$, Claim 12 implies that V_3, V_4, $V_{3,4}$, V_5, $V_{1,5}$, V_2, and $V_{1,2}$ are empty. Therefore only V_1, $V_{2,3}$, and $V_{4,5}$ can be large. Hence by Lemma 10 the graph G' has bounded clique-width. We may therefore assume that V_1 is empty. By symmetry, we may assume that V_i is empty for all $i \in \{1,2,3,4,5\}$.

Now by Claim 10 if $j \in \{i+1, i-1\}$ then $V_{i,i+1}$ is anticomplete to $V_{j,j+1}$. For every $i \in \{1,2,3,4,5\}$, by Claim 9 the edges between $V_{i,i+1}$ and $V_{i+2,i+3}$ form a matching. By Fact 2, we may apply a complementation to each set $V_{i,i+1}$. We obtain a graph of maximum degree at most 2, which therefore has clique-width at most 4 by Lemma 3. This completes the proof of the lemma. ☐

LEMMA 12. *The class of* $(2P_1+P_3, \overline{2P_1+P_3})$*-free graphs containing an induced* C_6 *has bounded clique-width.*

Proof. Suppose G is a $(2P_1+P_3, \overline{2P_1+P_3})$-free graph containing an induced cycle C on six vertices, say $v_1, \ldots, v_6$ in order. By Lemma 11, we may assume that G is C_5-free. For $S \subseteq \{1, \ldots, 6\}$, let V_S be the set of vertices $x \in V(G) \setminus V(C)$ such that $N(x) \cap V(C) = \{v_i \mid i \in S\}$. We say that a set V_S is *large* if it contains at least two vertices, otherwise it is *small*.

To ease notation, in the following claims, subscripts on vertex sets should be interpreted modulo 6 and whenever possible we will write V_i instead of $V_{\{i\}}$ and $V_{i,j}$ instead of $V_{\{i,j\}}$ and so on.

Claim 1. *We may assume that for* $S \subseteq \{1, \ldots, 6\}$*, the set* V_S *is either large or empty.*
If a set V_S is small, but not empty, then by Fact 1, we may delete all vertices of this set. If later in our proof we delete vertices in some set V_S and in doing so make a large set V_S become small, we may immediately delete the remaining vertices in V_S. The above arguments involve deleting a total of at most 2^6 vertices. By Fact 1, the claim follows.

Claim 2. *For* $S \subseteq \{1, \ldots, 6\}$ *if* $|S| \leq 1$ *then* $V_S = \emptyset$.
If $x \in V_\emptyset \cup V_2$ then $G[x, v_1, v_3, v_4, v_5]$ is a $2P_1 + P_3$, a contradiction. The claim follows by symmetry.

Claim 3. *For* $i \in \{1, \ldots, 6\}$*,* $V_{i,i+1}$ *is a clique.*
Suppose that $x, x' \in V_{1,2}$ are nonadjacent. Then $G[x, x', v_3, v_4, v_5]$ is a $2P_1 + P_3$, a contradiction. The claim follows by symmetry.

Claim 4. *For* $i \in \{1, \ldots, 6\}$*,* $V_{i,i+1,i+2}$ *is a clique.*
Suppose that $x, x' \in V_{1,2,3}$ are nonadjacent. Then $G[x, x', v_4, v_5, v_6]$ is a $2P_1 + P_3$, a contradiction. The claim follows by symmetry.

Claim 5. *For* $i \in \{1, \ldots, 6\}$*,* $V_{i,i+2}$ *is empty.*
If $x \in V_{1,3}$ then $G[x, v_2, v_4, v_5, v_6]$ is a $2P_1 + P_3$, a contradiction. The claim follows by symmetry.

Claim 6. *For* $i \in \{1, \ldots, 6\}$*,* $V_{i,i+3} \cup V_{i,i+1,i+3} \cup V_{i,i+2,i+3} \cup V_{i,i+1,i+2,i+3}$ *is empty.*
If $x \in V_{1,4} \cup V_{1,2,4} \cup V_{1,3,4} \cup V_{1,2,3,4}$ then $G[x, v_4, v_5, v_6, v_1]$ is a C_5, a contradiction. The claim follows by symmetry.

Claim 7. *For* $i \in \{1, 2\}$*,* $G[V_{i,i+2,i+4}]$ *is* P_3*-free.*
Suppose $x, x', x'' \in V_{1,3,5}$ are such that $G[x, x', x'']$ is a P_3. Then $G[v_2, v_4, x, x', x'']$ is a $2P_1 + P_3$, a contradiction. The claim follows by symmetry.

Claim 8. *For* $i \in \{1, \ldots, 6\}$*,* $V_{i,i+1,i+2,i+4}$ *is empty.*
Suppose for contradiction that $V_{1,2,3,5}$ is not empty. By Claim 1, there are two vertices $x, x' \in V_{1,2,3,5}$. If x is adjacent to x' then $G[x, x', v_1, v_5, v_2]$ is a $\overline{2P_1 + P_3}$. If x is nonadjacent to x' then $G[v_4, v_6, x, v_2, x']$ is a $2P_1 + P_3$. This contradiction implies that $V_{1,2,3,5}$ is empty. The claim follows by symmetry.

Claim 9. *For* $i \in \{1,\ldots,6\}$*,* $V_{i,i+1,i+3,i+4} \cup V_{i,i+1,i+2,i+3,i+4} \cup V_{i,i+1,i+3,i+4,i+5} \cup V_{i,i+1,i+2,i+3,i+4,i+5}$ *is an independent set.*
Suppose that $x, x' \in V_{1,2,4,5} \cup V_{1,2,3,4,5} \cup V_{1,2,4,5,6} \cup V_{1,2,3,4,5,6}$ are adjacent. Then the graph $G[x, x', v_1, v_4, v_2]$ is a $\overline{2P_1 + P_3}$, a contradiction. The claim follows by symmetry.

By Claims 1–9, only the following sets can be nonempty:

- $V_{i,i+1}$ for $i \in \{1,\ldots,6\}$, which are cliques;
- $V_{i,i+1,i+2}$ for $i \in \{1,\ldots,6\}$, which are cliques;
- $V_{i,i+2,i+4}$ for $i \in \{1,2\}$, which induce P_3-free graphs in G;
- $V_{i,i+1,i+3,i+4}$ for $i \in \{1,2,3\}$, which are independent sets;
- $V_{i,i+1,i+2,i+3,i+4}$ for $i \in \{1,\ldots,6\}$, which are independent sets; and
- $V_{1,2,3,4,5,6}$, which is an independent set.

In the remainder of the proof, we will prove a number of claims. First, we will show that we can remove sets of the form $V_{i,i+1}$ (Claim 14), and of the form $V_{i,i+1,i+2}$ (Claim 18) from the graph. Then we will show that for $T \subseteq \{1,\ldots,6\}$ with $|T| \geq 4$ we can remove V_T from the graph (Claims 22 and 23). This will leave only the sets $V_{1,3,5}$ and $V_{2,4,6}$ and the last stage will be to deal with these sets.

Claim 10. *We may assume that for distinct* $S, T \subseteq \{1,\ldots,6\}$ *if* V_S *is an independent set and* V_T *is a clique then* V_S *is either complete or anticomplete to* V_T*.*
Let $S, T \subseteq \{1,\ldots,6\}$ be distinct. If V_S is an independent set and V_T is a clique, then by Lemma 9, we may delete at most three vertices from each of these sets, such that in the resulting graph, V_S will be complete or anticomplete to V_T. Doing this for every pair of an independent set V_S and a clique V_T we delete at most $\binom{2^6}{2} \times 2 \times 3$ vertices from G. The claim follows by Fact 1.

Claim 11. *For* $i \in \{1,2\}$ *and* $j \in \{1,\ldots,6\}$*, if* $V_{i,i+2,i+4}$ *is large then* $V_{j,j+1}$ *is empty.*
Suppose, for contradiction, that $x \in V_{1,3,5}$ and $y \in V_{1,2}$. Then $G[v_4, v_6, v_2, y, x]$ or $G[y, v_6, x, v_3, v_4]$ is a $2P_1 + P_3$ if x is adjacent or nonadjacent to y, respectively. This is a contradiction. The claim follows by symmetry.

Claim 12. *For* $i \in \{1,2,3\}$*, either* $V_{i,i+1}$ *or* $V_{i+3,i+4}$ *is empty.*
Suppose $x \in V_{1,2}$ and $y \in V_{4,5}$. If x is adjacent to y then $G[x, v_2, v_3, v_4, y]$ is a C_5. If x is nonadjacent to y then $G[y, v_3, x, v_1, v_6]$ is a $2P_1 + P_3$, a contradiction. The claim follows by symmetry.

Claim 13. *For* $i, j \in \{1,\ldots,6\}$*,* $V_{i,i+1,i+2}$ *is anticomplete to* $V_{j,j+1}$*.*
Let $i = 1$, $j \in \{2,3,4\}$ (the other cases follow by symmetry). If $V_{1,2,3}$ and $V_{j,j+1}$ are not empty then by Claim 1 they must be large. Suppose $x, x' \in V_{1,2,3}$ and $y \in V_{j,j+1}$ with x adjacent to y. By Claim 4, x must be adjacent to x'. If $j = 2$ then $G[x, v_2, v_3, v_1, y]$ is a $\overline{2P_1 + P_3}$. If $j = 3$ then $G[x, x', v_1, y, v_2]$ or $G[v_3, x, x', y, v_2]$ is a $\overline{2P_1 + P_3}$ if x' is adjacent or nonadjacent to y, respectively. If $j = 4$ then $G[y, v_5, v_6, v_1, x]$ is a C_5. This is a contradiction. The claim follows by symmetry.

Claim 14. *We may assume that* $V_{i,i+1}$ *is empty for all* $i \in \{1,\ldots,6\}$*.*
Let G' be the graph induced by the sets of the form $V_{i,i+1}$. We will show how to disconnect G' from the rest of G and then show that G' has bounded clique-width.

We may assume that at least one set of the form $V_{i,i+1}$ is nonempty (in which case it must be large by Claim 1), otherwise we are done. By Fact 3, for each $i \in \{1,\ldots,6\}$ we may apply a bipartite complementation between $V_{i,i+1}$ and $\{v_i, v_{i+1}\}$.

By Claim 13, for $i, j \in \{1, \dots, 6\}$ there are no edges between $V_{i,i+1}$ and $V_{j,j+1,j+2}$. By Claim 11, $V_{1,3,5}$ and $V_{2,4,6}$ are empty. By Claim 3, all sets of the form $V_{i,i+1}$ are cliques. By Claim 9, if V_T is large with $|T| \geq 4$ then V_T is an independent set. Therefore, by Claim 10, for all $i \in \{1, \dots, 6\}$ and all $T \subseteq \{1, \dots, 6\}$ with $|T| \geq 4$, $V_{i,i+1}$ is either complete or anticomplete to V_T. If $V_{i,i+1}$ is complete to V_T then by Fact 3 we may apply a bipartite complementation between these two sets. This removes all edges from vertices in G' to vertices outside G'.

It remains to show that G' has bounded clique-width. By Claim 12, $V_{1,2}$ or $V_{4,5}$ is empty, $V_{2,3}$ or $V_{5,6}$ is empty, and $V_{3,4}$ or $V_{1,6}$ is empty. This means that at most three sets of the form $V_{i,i+1}$ can be large. By Claim 3, every set of the form $V_{i,i+1}$ induces a clique in G (and therefore in G'). By Lemma 10, it follows that G' has bounded clique-width.

We conclude that we can remove all sets of the form $V_{i,i+1}$ from G, that is, we may assume that these sets are empty. This completes the proof of the claim.

Claim 15. *For $i \in \{1, 2\}$ and $j \in \{i+1, i+3, i+5\}$, $V_{i,i+2,i+4}$ is complete to $V_{j,j+1,j+2}$.*
If $x \in V_{1,3,5}$ is nonadjacent to $y \in V_{2,3,4}$ then $G[x, v_6, v_2, y, v_4]$ is a $2P_1 + P_3$, a contradiction. The claim follows by symmetry.

Claim 16. *For $i \in \{1, 2\}$ and $j \in \{i, i+2, i+4\}$, $V_{i,i+2,i+4}$ is anticomplete to $V_{j,j+1,j+2}$.*
If $x \in V_{1,3,5}$ is adjacent to $y \in V_{1,2,3}$ then $G[v_4, v_6, v_2, y, x]$ is a $2P_1 + P_3$, a contradiction. The claim follows by symmetry.

Claim 17. *For $i \in \{1, \dots, 6\}$ either $V_{i,i+1,i+2}$ or $V_{i+1,i+2,i+3}$ is empty.*
Suppose that $V_{1,2,3}$ and $V_{2,3,4}$ are both nonempty. Then by Claim 1 they must both be large and by Claim 4, they must both be cliques. If $x \in V_{1,2,3}$ is adjacent to $y \in V_{2,3,4}$ then $G[x, v_2, y, v_1, v_3]$ is a $\overline{2P_1 + P_3}$, a contradiction. Therefore $V_{1,2,3}$ is anticomplete to $V_{2,3,4}$. If $x \in V_{1,2,3}$ and $y, y' \in V_{2,3,4}$ then $G[v_2, v_3, y, x, y']$ is a $\overline{2P_1 + P_3}$, a contradiction. The claim follows by symmetry.

Claim 18. *We may assume that $V_{i,i+1,i+2}$ is empty for all $i \in \{1, \dots, 6\}$.*
Let G' be the graph induced by the sets of the form $V_{i,i+1,i+2}$. We will show how to disconnect G' from the rest of G and then show that G' has bounded clique-width.

We may assume that at least one set of the form $V_{i,i+1,i+2}$ is nonempty (in which case it must be large by Claim 1), otherwise we are done. By Fact 3, for each $i \in \{1, \dots, 6\}$ we may apply a bipartite complementation between $V_{i,i+1,i+2}$ and $\{v_i, v_{i+1}, v_{i+2}\}$. By Claims 15 and 16 for $i \in \{1, \dots, 6\}$ and $j \in \{1, 2\}$, $V_{i,i+1,i+2}$ is either complete or anticomplete to $V_{j,j+2,j+4}$; if it is complete then by Fact 3, we may apply a bipartite complementation between these two sets. By Claim 4, all sets of the form $V_{i,i+1,i+2}$ are cliques. By Claim 9, if V_T is large with $|T| \geq 4$ then V_T is an independent set. Therefore, for all $i \in \{1, \dots, 6\}$ and all $T \subseteq \{1, \dots, 6\}$ with $|T| \geq 4$, $V_{i,i+1,i+2}$ is either complete or anticomplete to V_T. If $V_{i,i+1,i+2}$ is complete to V_T then by Fact 3 we may apply a bipartite complementation between these two sets. This removes all edges from vertices in G' to vertices outside G'.

It remains to show that G' has bounded clique-width. By Claim 17, $V_{1,2,3}$ or $V_{2,3,4}$ is empty, $V_{3,4,5}$ or $V_{4,5,6}$ is empty, and $V_{1,5,6}$ or $V_{1,2,6}$ is empty. This means that at most three sets of the form $V_{i,i+1,i+2}$ can be large. By Claim 4, every set of the form $V_{i,i+1,i+2}$ induces a clique in G (and therefore in G'). By Lemma 10, it follows that G' has bounded clique-width.

We conclude that we can remove all sets of the form $V_{i,i+1,i+2}$ from G, that is, we may assume that these sets are empty. This completes the proof of the claim.

Claim 19. *Suppose $S, T \subseteq \{1, \dots, 6\}$ are distinct with $|S| \geq 4$ and $|T| \geq 5$. Then V_S or V_T is empty.*
Suppose the claim is false for some S and T. By Claim 1, we may assume V_S and V_T are large. Without loss of generality, we may assume that $|T| \geq |S|$. Without loss of generality, we may assume that $\{1, 2, 4, 5\} \subseteq S$. The vertices of V_T are nonadjacent to at most one vertex of the original cycle C, so without loss of generality, we may assume that $\{1, 2, 4\} \subseteq S \cap T$. If $x, x' \in V_S \cup V_T$ are adjacent then $G[x, x', v_1, v_4, v_2]$ is a $\overline{2P_1 + P_3}$, so $V_S \cup V_T$ is an independent set. Suppose $x, x \in V_T$, $y, y' \in V_S$, and $i \in T \setminus S$ (which exists since $|T| \geq |S|$ and $S \neq T$). Then $G[y, y', x, v_i, x']$ is a $2P_1 + P_3$, a contradiction. The claim follows by symmetry.

Claim 19 implies that, in addition to the sets $V_{1,3,5}$ and $V_{2,4,6}$, which may be large, exactly one of the following must hold:

- Exactly one, two, or three sets V_S with $|S| = 4$ are large and all sets V_T with $|T| \in \{5, 6\}$ are empty.
- Exactly one set V_S with $|S| = 5$ is large and all sets V_T with $|T| \in \{4, 6\}$ are empty.
- $V_{1,2,3,4,5,6}$ is large and all sets V_T with $|T| \in \{4, 5\}$ are empty.
- All sets V_T with $|T| \geq 4$ are empty.

Claim 20. *Let $i \in \{1, 2\}$ and $T \subseteq \{1, \dots, 6\}$ with $|T| \geq 4$ such that there is a $j \in \{1, \dots, 6\}$ with $j \notin \{i, i+2, i+4\}$, $j \notin T$. If $V_{i,i+2,i+4}$ and V_T are large then $V_{i,i+2,i+4}$ is an independent set that is complete to V_T.*
By symmetry, we need only prove the claim in the case where $i = 1$ and $T \in \{\{1, 2, 4, 5\}, \{1, 2, 3, 4, 5\}\}$, in which case $j = 6$. Suppose $x \in V_{1,3,5}$ is nonadjacent to $y \in V_T$. Then $G[x, v_6, v_2, y, v_4]$ is a $2P_1 + P_3$, a contradiction. Therefore $V_{1,3,5}$ is complete to V_T. If $x, x' \in V_{1,3,5}$ with x adjacent to x' and $y \in V_T$ then $G[v_1, y, x, v_2, x']$ is a $\overline{2P_1 + P_3}$, a contradiction. Therefore $V_{1,3,5}$ is an independent set. The claim follows by symmetry.

Claim 21. *For $i \in \{1, 2\}$ and $T \subseteq \{1, \dots, 6\}$ with $|T| \geq 4$ and $\{i, i+2, i+4\} \cup T = \{1, \dots, 6\}$. If $V_{i,i+2,i+4}$ and V_T are large then we may assume $V_{i,i+2,i+4}$ is the disjoint union of a (possibly empty) clique and a (possibly empty) independent set.*
Suppose $i = 1$ and $T \in \{\{1, 2, 3, 4, 6\}, \{1, 2, 3, 4, 5, 6\}\}$. Suppose $V_{1,3,5}$ and V_T are large. If $x \in V_T$ and $y, y' \in V_{1,3,5}$ with x, y, y' pairwise nonadjacent then $G[y, y', v_2, x, v_4]$ is a $2P_1 + P_3$, a contradiction. Therefore every vertex of V_T is complete to all but at most one component of $G[V_{1,3,5}]$. If $x \in V_T$ and $y, y' \in V_{1,3,5}$ with x, y, y' pairwise adjacent then $G[x, v_1, y, v_2, y']$ is a $\overline{2P_1 + P_3}$, a contradiction. Therefore every vertex of V_T has at most one neighbor in each component of $G[V_{1,3,5}]$. It follows that $G[V_{1,3,5}]$ has at most one nontrivial component. The claim follows by symmetry.

Claim 22. *For $i \in \{1, 2, 3\}$, we may assume that $V_{i,i+1,i+3,i+4}$ is empty.*
Suppose that $V_{i,i+1,i+3,i+4}$ is not empty for some $i \in \{1, 2, 3\}$. Then by Claim 1, this set must be large. By Claim 19, in this case only the following sets can be large: $V_{1,3,5}$, $V_{2,4,6}$, $V_{1,2,4,5}$, $V_{2,3,5,6}$, and $V_{3,4,6,1}$. By Claim 20, $V_{1,3,5}$ and $V_{2,4,6}$ are complete to $V_{1,2,4,5}$, $V_{2,3,5,6}$, and $V_{3,4,6,1}$. By Fact 3, we may apply a bipartite complementation between $V_{1,3,5} \cup V_{2,4,6}$ and $V_{1,2,4,5} \cup V_{2,3,5,6} \cup V_{3,4,6,1}$. By Fact 3, we can also apply bipartite complementations between $\{v_1, v_2, v_4, v_5\}$ and $V_{1,2,4,5}$, between $\{v_2, v_3, v_5, v_6\}$ and $V_{2,3,5,6}$ and between $\{v_3, v_4, v_6, v_1\}$ and $V_{3,4,6,1}$. This disconnects $G' = G[V_{1,2,4,5} \cup V_{2,3,5,6} \cup V_{3,4,6,1}]$ from the rest of the graph.

It remains to show that G' has bounded clique-width. By Claim 9, $V_{1,2,4,5}$, $V_{2,3,5,6}$, and $V_{3,4,6,1}$ are independent sets. By Lemma 10, it follows that G' has bounded clique-width. We may therefore assume that $V_{1,2,4,5} \cup V_{2,3,5,6} \cup V_{3,4,6,1} = \emptyset$. The claim follows.

Claim 23. *For $T \subseteq \{1, \ldots, 6\}$ with $|T| \geq 5$ we may assume that V_T is empty.*
Suppose there is a $T \subseteq \{1, \ldots, 6\}$ with $|T| \geq 5$ such that V_T is large. By Claim 19, only the following sets can be large: $V_{1,3,5}$, $V_{2,4,6}$, and V_T. By Claims 20 and 21, each of $V_{1,3,5}$ and $V_{2,4,6}$ is the union of a (possibly empty) clique and a (possibly empty) independent set. By Claim 9, V_T is an independent set. By Fact 1, we may delete the vertices of the original cycle C. We obtain a graph whose vertex set can be partitioned into at most two cliques and at most three independent sets, so G has bounded clique-width by Lemma 10. The claim follows.

Only two sets V_S remain that may be nonempty, namely, $V_{1,3,5}$ and $V_{2,4,6}$. If one of these sets is empty, then by Fact 1 we may delete the vertices of the original cycle C. Claim 7 implies that the resulting graph is a disjoint union of cliques, and so has clique-width at most 2. We may therefore assume that both $V_{1,3,5}$ and $V_{2,4,6}$ are nonempty, in which case they are both large by Claim 1.

Suppose $x \in V_{1,3,5}$ and $y, y' \in V_{2,4,6}$ and these three vertices are pairwise nonadjacent. Then $G[y, y', v_1, x, v_3]$ is a $2P_1 + P_3$, a contradiction. Therefore each vertex of $V_{1,3,5}$ is complete to all but at most one component of $G[V_{2,4,6}]$. Similarly, each vertex of $V_{2,4,6}$ is complete to all but at most one component of $G[V_{1,3,5}]$.

First consider the case where $G[V_{2,4,6}]$ contains at least three nontrivial components. Every vertex in $V_{1,3,5}$ must be complete to at least two of these components. If $x, x' \in V_{1,3,5}$ are adjacent then they must both be complete to a common nontrivial component C' of $G[V_{2,4,6}]$. Let $y, y' \in V(C')$. Then $G[x, x', y, v_1, y']$ is a $\overline{2P_1 + P_3}$, a contradiction. It follows that $V_{1,3,5}$ is an independent set. Note that this implies that every vertex of $V_{2,4,6}$ has at most one nonneighbor in $V_{1,3,5}$. By Fact 3, we may apply a bipartite complementation between $V_{1,3,5}$ and $V_{2,4,6}$. Let G' be the resulting graph. In G', every vertex in $V_{1,3,5}$ has neighbors in at most one component of $G[V_{2,4,6}]$ and each vertex of $V_{2,4,6}$ has at most one neighbor in $V_{1,3,5}$. This means that every component of $G[V_{2,4,6}]$ lies in a different component of G'. It suffices to show that the components of G' have bounded clique-width. Let C'' be such a component of G'. By Fact 2, we may apply a complementation to $V_{2,4,6} \cap V(C'')$. We obtain a disjoint union of stars (some of which may be isolated vertices). Since stars have clique-width at most 2, we have shown that in this case G has bounded clique-width.

We may therefore assume that $V_{2,4,6}$ contains at most two nontrivial components. By symmetry, we may also assume that $V_{1,3,5}$ contains at most two nontrivial components. This means that each of these sets consists of the disjoint union of at most two cliques and at most one independent set. Let K^1 and K^2 be the two cliques and I^1 be the independent set in $V_{1,3,5}$. Let K^3 and K^4 be the two cliques and I^2 be the independent set in $V_{2,4,6}$. (We allow the case where some of the sets K^i or I^j are empty.) Also note that in G, K^1 is anticomplete to K^2 and K^3 is anticomplete to K^4. By Fact 1 and Lemma 9, we may assume that each clique K^i is either complete or anticomplete to each independent set I^j. If a clique K^i is complete to I^j then by Fact 3 we may apply a bipartite complementation between these sets. This removes all edges between $K^1 \cup K^2 \cup K^3 \cup K^4$ and $I^1 \cup I^2$. Now by Fact 1 and Lemma 8 we may assume that the edges between each pair of K^1, K^2, K^3, and K^4 and the edges between I^1 and I^2 either form a matching or a comatching. If the edges between two such sets form a comatching, by Fact 3 we may apply a bipartite complementation between these sets.

Finally, by Fact 2, we may complement each clique K^i. Let G'' be the resulting graph and note that in this graph it is still the case that $K^1 \cup K^2 \cup K^3 \cup K^4$ is anticomplete to $I^1 \cup I^2$. In G'' the edges between I^1 and I^2 form a matching, so $G''[I^1 \cup I^2]$ has maximum degree at most 1 and thus clique-width at most 2. In G'' the sets K^1, K^2, K^3, and K^4 are independent and the edges between each pair of these sets forms a matching. In fact, K^1 is anticomplete to K^2 and K^3 is anticomplete to K^4. Therefore $G''[K^1 \cup K^2 \cup K^3 \cup K^4]$ has maximum degree at most 2, and therefore clique-width at most 4 by Lemma 3. It follows that G'' has bounded clique-width and therefore G also has bounded clique-width. This completes the proof of the lemma. ∎

LEMMA 13. *Every prime $(2P_1 + P_3, \overline{2P_1 + P_3}, C_6, \overline{C_6})$-free graph is either K_7-free or $\overline{K_7}$-free.*

Proof. Let G be a prime $(2P_1 + P_3, \overline{2P_1 + P_3}, C_6, \overline{C_6})$-free graph. Suppose, for contradiction, that G contains an induced K_7 and an induced $\overline{K_7}$. We will show that in this case the graph G is not prime. Note that any induced K_7 and induced $\overline{K_7}$ in G can share at most one vertex. We may therefore assume that G contains a clique C on at least six vertices and a vertex-disjoint independent set I on at least six vertices. Furthermore, we may assume that C is a maximum clique in $G \setminus I$ and I is a maximum independent set in $G \setminus C$ (if not, then replace C or I with a bigger clique or independent set, respectively).

By Lemma 9, there exist sets $R_1 \subset C$ and $R_2 \subset I$ each of size at most 3 such that $C' = C \setminus R_1$ is either complete or anticomplete to $I' = I \setminus R_2$. Without loss of generality, we may assume that R_1 and R_2 are minimal, in the sense that the above property does not hold if we remove any vertex from R_1 or R_2. Note that the class of prime $(2P_1 + P_3, \overline{2P_1 + P_3}, C_6, \overline{C_6})$-free graphs containing an induced K_7 and an induced $\overline{K_7}$ is closed under complementation. Therefore, complementing G if necessary (in which case the sets I and C will be swapped, and the sets R_1 and R_2 will be swapped), we may assume that C' is anticomplete to I'.

Claim 1. $|R_1| \leq 1$ *and* $|R_2| \leq 1$.
By construction, R_1 and R_2 each contain at most three vertices and I' and C' each contain at least three vertices. Since R_1 (resp., R_2) is minimal, every vertex of R_1 (resp., R_2) has at least one neighbor in I' (resp., C').

Choose $i_1, i_2 \in I'$ arbitrarily and suppose, for contradiction, that $y \in R_2$ is not complete to C'. Then y must have a neighbor $c_1 \in C'$ and a nonneighbor $c_2 \in C'$, so $G[i_1, i_2, y, c_1, c_2]$ is a $2P_1 + P_3$, a contradiction. Therefore R_2 is complete to C'. If $y, y' \in R_2$ then for arbitrary $c_1 \in C'$, the graph $G[i_1, i_2, y, c_1, y']$ is a $2P_1 + P_3$, a contradiction. It follows that $|R_2| \leq 1$.

Choose $c_1, c_2 \in C'$ arbitrarily. Suppose, for contradiction, that $x \in R_1$ has two nonneighbors $i_1, i_2 \in I'$. Recall that x must have a neighbor $i_3 \in I'$, so $G[i_1, i_2, i_3, x, c_1]$ is a $2P_1 + P_3$, a contradiction. Therefore every vertex of R_1 has at most one nonneighbor in I'. Suppose, for contradiction, that $x, x' \in R_1$. Since I' contains at least three vertices, there must be a vertex $i_1 \in I'$ that is a common neighbor of x and x'. Now $G[x, x', c_1, i_1, c_2]$ is a $\overline{2P_1 + P_3}$, a contradiction. It follows that $|R_1| \leq 1$. This completes the proof of Claim 1.

Note that Claim 1 implies that $|C'| \geq 5$ and $|I'| \geq 5$. Let A be the set of vertices in $V \setminus (C \cup I)$ that are complete to C'. If $x \in A$ is adjacent to $y \in R_1$ then by Claim 1 $C \cup \{x\}$ is a bigger clique than C, contradicting the maximality of C. It follows that A is anticomplete to R_1. If $x, y \in A$ are adjacent then by Claim 1, $(C \cup \{x, y\}) \setminus R_1$ is a bigger clique than C, contradicting the maximality of C. It follows that A is an independent set. Furthermore, by the maximality of I and the definition of A, every vertex in $V \setminus (C \cup I \cup A)$ has a neighbor in I and nonneighbor in C'.

Claim 2. *Let x be a vertex in $V \setminus (C \cup I \cup A)$. Then either x is complete to I', or x has exactly one neighbor in I.*
Suppose, for contradiction, that x has a nonneighbor z in I', and two neighbors $y, y' \in I$. Now x cannot have another nonneighbor $z' \in I \setminus \{z\}$, otherwise $G[z, z', y, x, y']$ would be a $2P_1 + P_3$. Therefore x must be complete to $I \setminus \{z\}$. In particular, since $|I'| \geq 5$, this means that x has two neighbors in I', say y_1 and y_2 (not necessarily distinct from y and y'). Recall that x must have a nonneighbor $c_1 \in C'$. Now $G[c_1, z, y_1, x, y_2]$ is a $2P_1 + P_3$. This contradiction completes the proof of Claim 2.

By Claim 2 we can partition the vertex set $V \setminus (C \cup I \cup A)$ into subsets $V_{I'}$ and V_x for every $x \in I$, where $V_{I'}$ is the set of vertices that are complete to I', and V_x is the set of vertices whose unique neighbor in I is x. Let $U_x = V_x \cup \{x\}$.

Claim 3. *For all $x \in I'$, U_x is anticomplete to C'.*
Suppose $x \in I'$. Clearly x is anticomplete to C'. Suppose, for contradiction, that $y \in U_x \setminus \{x\} = V_x$ has a neighbor $z \in C'$ and choose $u, v \in I' \setminus \{x\}$. Then $G[u, v, x, y, z]$ is a $2P_1 + P_3$. This contradiction completes the proof of Claim 3.

Claim 4. *For every $x \in I$, the set U_x is a clique.*
Note that $x \in I$ is adjacent to all other vertices of U_x, by definition. If $y, z \in V_x$ are nonadjacent then $(I \setminus \{x\}) \cup \{y, z\}$ would be a bigger independent set than I. This contradiction completes the proof of Claim 4.

Claim 5. *If $x, y \in I$ are distinct, then U_x is anticomplete to U_y.*
Clearly x is anticomplete to U_y and y is anticomplete to U_x. Suppose, for contradiction, that $x' \in U_x \setminus \{x\}$ is adjacent to $y' \in U_y \setminus \{y\}$. Choose $u, v \in I \setminus \{x, y\}$. Then $G[u, v, x, x', y']$ is a $2P_1 + P_3$. This contradiction completes the proof of Claim 5.

Claim 6. *For every $x \in I'$, the set U_x is complete to $V_{I'}$.*
Clearly x is complete to $V_{I'}$, by definition. Suppose, for contradiction that $x' \in U_x \setminus \{x\}$ is nonadjacent to $y \in V_{I'}$. Since $y \notin A$, the vertex y must have a nonneighbor $c_1 \in C'$ and note that x' is nonadjacent to c_1 by Claim 3. Choose $u, v \in I' \setminus \{x\}$. Then $G[c_1, x', u, y, v]$ is a $2P_1 + P_3$. This contradiction completes the proof of Claim 6.

Suppose $x \in I'$. Claim 4 implies that U_x is a clique, Claim 3 implies that U_x is anti-complete to C' and Claim 6 implies that U_x is complete to $V_{I'}$. Furthermore for all $y \in I \setminus \{x\}$, Claim 5 implies that U_x is anti-complete to U_y. We conclude that given any two vertices $x, y \in I'$, no vertex in $V \setminus (A \cup R_1 \cup U_x \cup U_y)$ can distinguish the set $U_x \cup U_y$. In the remainder of the proof, we will show that there exist $x, y \in I'$ such that no vertex of $A \cup R_1$ distinguishes the set $U_x \cup U_y$, meaning that $U_x \cup U_y$ is a nontrivial module, contradicting the assumption that G is prime.

Claim 7. *If $u \in A \cup R_1$ then either u is anticomplete to U_x for all $x \in I'$ or else u is complete to U_x for all but at most one $x \in I'$.*
Suppose, for contradiction, that the claim does not hold for a vertex $u \in A \cup R_1$. Then u must have a neighbor $x' \in U_x$ for some $x \in I'$ and must have nonneighbors $y' \in U_y$ and $z' \in U_z$ for some $y, z \in I'$ with $y \neq z$. Since $|I'| \geq 5$, we may also assume that $x \notin \{y, z\}$. Choose $c_1 \in C'$ arbitrarily. By Claim 3, c_1 is nonadjacent to x', y', and z'. It follows that $G[y', z', c_1, u, x']$ is a $2P_1 + P_3$. This contradiction completes the proof of Claim 7.

Let A^* denote the set of vertices in $A \cup R_1$ that have a neighbor in U_x for some $x \in I'$.

Claim 8. *The set A^* is complete to all, except possibly two, sets $U_x, x \in I'$.*
Suppose, for contradiction, that there are three different vertices $x, y, z \in I'$ such that A^* is complete to none of the sets U_x, U_y, and U_z. By Claim 7 and the definition of A^*, every vertex in A^* is complete to at least two of the sets U_x, U_y, U_z. Therefore there exist three vertices $u, v, w \in A^*$ such that

- u is not adjacent to some vertex $x' \in U_x$, but is complete to U_y and U_z;
- v is not adjacent to some vertex $y' \in U_y$, but is complete to U_x and U_z;
- w is not adjacent to some vertex $z' \in U_z$, but is complete to U_x and U_y.

Therefore $G[u, y', w, x', v, z']$ is a C_6. This contradiction completes the proof of Claim 8.

Now, since $|I'| \geq 5$, Claims 7 and 8 imply that there exist two distinct vertices $x, y \in I'$ such that every vertex of $A \cup R_1$ is either complete or anticomplete to $U_x \cup U_y$. It follows that $U_x \cup U_y$ is a nontrivial module in G, contradicting the fact that G is prime. This completes the proof. ∎

Let G be a graph. The *chromatic number* $\chi(G)$ of G is the minimum positive integer k such that G is k-colorable. The *clique number* $\omega(G)$ of G is the size of a largest clique in G. A hereditary class $\mathcal{C}$ of graphs is χ-bounded if there is a function f such that $\chi(G) \leq f(\omega(G))$ for all $G \in \mathcal{C}$.

Lemma 14 ([34]). *For every natural number k the class of P_k-free graphs is χ-bounded.*

Lemma 15. *For every fixed $k \geq 1$, the class of $(K_k, 2P_1+P_3, \overline{2P_1 + P_3})$-free graphs has bounded clique-width.*

Proof. Fix a constant $k \geq 1$ and let G be a $(K_k, 2P_1 + P_3, \overline{2P_1 + P_3})$-free graph. By Lemma 12, we may assume that G is C_6-free. Since G is $(2P_1 + P_3)$-free, it is P_7-free, so by Lemma 14 it has chromatic number at most ℓ for some constant ℓ. This means that we can partition the vertices of G into ℓ independent sets $V_1, \dots, V_\ell$ (some of which may be empty). Since G is $(2P_1 + P_3, \overline{2P_1 + P_3})$-free, by Lemma 8, deleting finitely many vertices (which we may do by Fact 1), we may assume that for all distinct $i, j \in \{1, \dots, \ell\}$, the edges between V_i and V_j form a matching or a comatching. Since G is C_6-free, if the vertices between V_i and V_j form a comatching, this comatching can contain at most two nonedges. Therefore, by deleting finitely many vertices (which we may do by Fact 1), we may assume that the edges between V_i and V_j form a matching or V_i and V_j are complete to each other. By deleting finitely many vertices (which we may do by Fact 1), we may assume that each set V_i is either empty or contains at least five vertices.

Suppose the edges from V_i to V_j and the edges from V_i to V_h form a matching and that there is a vertex $x \in V_i$ that has a neighbor $y \in V_j$ and a neighbor $z \in V_h$. Then y must be adjacent to z, otherwise for $x', x'' \in V_i \setminus \{x\}$ the graph $G[x', x'', y, x, z]$ would be a $2P_1 + P_3$, a contradiction. If V_j is complete to V_h then for $y', y'' \in V_j$, $z' \in V_h$ and $x', x'' \in V_i \setminus (N(y') \cup N(y'') \cup N(z'))$ (such vertices exist since each of y', y'', and z' have at most one neighbor in V_i and V_i contains at least five vertices) we have $G[x', x'', y', z', y'']$ is a $2P_1 + P_3$, a contradiction. Therefore the edges between V_j and V_h form a matching.

Now for each $i, j \in \{1, \dots, \ell\}$ with $i < j$, if V_i is complete to V_j, we apply a bipartite complementation between V_i and V_j (which we may do by Fact 3). Let G'

be the resulting graph. The previous paragraph implies that if x has two neighbors y and z in G' then y is adjacent to z in G, so G' is P_3-free. Therefore G' is a disjoint union of cliques, so it has clique-width at most 2. ∎

We are now ready to prove that $(2P_1+P_3, \overline{2P_1+P_3})$-free graphs have bounded clique-width.

THEOREM 4. *The class of $(2P_1+P_3, \overline{2P_1+P_3})$-free graphs has bounded clique-width.*

Proof. Let G be a $(2P_1+P_3, \overline{2P_1+P_3})$-free graph. By Lemma 6, we may assume that G is prime. If G contains an induced C_6 then we are done by Lemma 12. If G contains an induced $\overline{C_6}$ then we are done by Lemma 12 and Fact 2. We may therefore assume that G is also $(C_6, \overline{C_6})$-free. By Lemma 13, we may assume that G is either K_7-free or $\overline{K_7}$-free. By Fact 2, we may assume that G is K_7-free. Lemma 15 completes the proof. ∎

Theorem 2 follows from the summary of [18] for the clique-width of bigenic graph classes (see also [21]) after updating it with our new result for $(2P_1+P_3, \overline{2P_1+P_3})$-free graphs (Theorem 4), the result of Dabrowski, Lozin, and Paulusma [22] that $(P_1+P_3, \overline{P_2+P_4})$-free graphs have bounded clique-width and the recent result of Bonamy et al. [4], who proved that the class of $(P_1+P_4, \overline{P_1+2P_2})$-free graphs has unbounded clique-width. We first present the updated summary and then explain how Theorem 2 follows from it.

THEOREM 5. *Let $\mathcal{G}$ be a class of graphs defined by two forbidden induced subgraphs. Then*

1. *$\mathcal{G}$ has bounded clique-width if it is equivalent[3] to a class of (H_1, H_2)-free graphs such that one of the following holds:*
   - (i) *H_1 or $H_2 \subseteq_i P_4$;*
   - (ii) *$H_1 = sP_1$ and $H_2 = K_t$ for some s, t;*
   - (iii) *$H_1 \subseteq_i P_1+P_3$ and $\overline{H_2} \subseteq_i K_{1,3}+3P_1,\ K_{1,3}+P_2,\ P_1+P_2+P_3,\ P_1+P_5,\ P_1+S_{1,1,2},\ P_2+P_4,\ P_6,\ S_{1,1,3}$, or $S_{1,2,2}$;*
   - (iv) *$H_1 \subseteq_i 2P_1+P_2$ and $\overline{H_2} \subseteq_i P_1+2P_2,\ 3P_1+P_2$, or P_2+P_3;*
   - (v) *$H_1 \subseteq_i P_1+P_4$ and $\overline{H_2} \subseteq_i P_1+P_4$ or P_5;*
   - (vi) *$H_1, \overline{H_2} \subseteq_i K_{1,3}$;*
   - (vii) *$H_1, \overline{H_2} \subseteq_i 2P_1+P_3$.*
2. *$\mathcal{G}$ has unbounded clique-width if it is equivalent to a class of (H_1, H_2)-free graphs such that one of the following holds:*
   - (i) *$H_1 \notin \mathcal{S}$ and $H_2 \notin \mathcal{S}$;*
   - (ii) *$\overline{H_1} \notin \mathcal{S}$ and $\overline{H_2} \notin \mathcal{S}$;*
   - (iii) *$H_1 \supseteq_i K_{1,3}$ or $2P_2$ and $\overline{H_2} \supseteq_i 4P_1$ or $2P_2$;*
   - (iv) *$H_1 \supseteq_i 2P_1+P_2$ and $\overline{H_2} \supseteq_i K_{1,3},\ 5P_1,\ P_2+P_4$, or P_6;*
   - (v) *$H_1 \supseteq_i 3P_1$ and $\overline{H_2} \supseteq_i 2P_1+2P_2,\ 2P_1+P_4,\ 4P_1+P_2,\ 3P_2$, or $2P_3$;*
   - (vi) *$H_1 \supseteq_i 4P_1$ and $\overline{H_2} \supseteq_i P_1+P_4$ or $3P_1+P_2$;*
   - (vii) *$H_1 \supseteq_i P_1+P_4$ and $\overline{H_2} \supseteq_i P_1+2P_2$.*

[3]Given four graphs H_1, H_2, H_3, H_4, the class of (H_1, H_2)-free graphs and the class of (H_3, H_4)-free graphs are *equivalent* if the unordered pair H_3, H_4 can be obtained from the unordered pair H_1, H_2 by some combination of the operations (i) complementing both graphs in the pair and (ii) if one of the graphs in the pair is K_3, replacing it with $\overline{P_1+P_3}$ or vice versa. If two classes are equivalent, then one of them has bounded clique-width if and only if the other one does (see [25]).

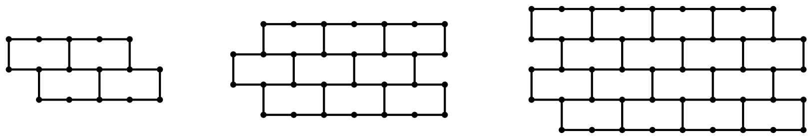

FIG. 6. *Walls of height* 2, 3, *and* 4, *respectively.*

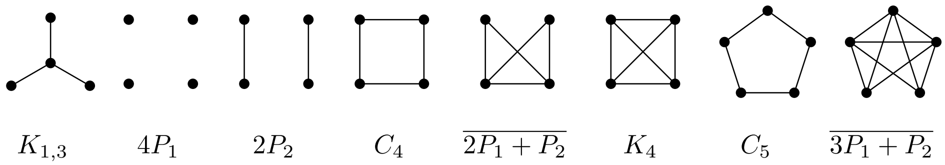

FIG. 7. *Forbidden induced subgraphs from Theorems* 6, 7, *and* 8.

THEOREM 2 (restated). *For a graph H, the class of $(H, \overline{H})$-free graphs has bounded clique-width if and only if H or $\overline{H}$ is an induced subgraph of $K_{1,3}$, $P_1 + P_4$, $2P_1 + P_3$, or sP_1 for some $s \geq 1$.*

Proof. Let H be a graph. It can be readily checked from Theorem 5 that if H or $\overline{H}$ is an induced subgraph of $K_{1,3}$, $P_1 + P_4$, $2P_1 + P_3$, or sP_1 for some $s \geq 1$ then the class of $(H, \overline{H})$-free graphs has bounded clique-width. Suppose this is not the case. If $H \notin \mathcal{S}$ and $\overline{H} \notin \mathcal{S}$, then the class of $(H, \overline{H})$-free graphs has unbounded clique-width by Theorem 5. By Fact 2, we may therefore assume that $H \in \mathcal{S}$. By Lemma 2, H contains $K_{1,3} + P_1$, $2P_2$, $3P_1 + P_2$, or $S_{1,1,2}$ as an induced subgraph. This means that the class of $(H, \overline{H})$-free graphs contains the class $(K_{1,3}, \overline{4P_1})$-free, $(2P_2, \overline{2P_2})$-free, $(4P_1, \overline{3P_1 + P_2})$-free, or $(2P_1 + P_2, \overline{K_{1,3}})$-free graphs, respectively. In each of these cases we apply Theorem 5. ∎

5. Three new classes of unbounded clique-width and the proof of Theorem 3. In this section we first identify three new graph classes of unbounded clique-width. To do so, we will need the notion of a *wall*. We do not formally define this notion, but instead refer to Figure 6, in which three examples of walls of different height are depicted (see, e.g., [14] for a formal definition). The class of walls is well known to have unbounded clique-width; see for example [37]. A *k-subdivided wall* is the graph obtained from a wall after subdividing each edge exactly k times for some constant $k \geq 0$, and the following lemma is well known.

LEMMA 16 ([40]). *For any constant $k \geq 0$, the class of k-subdivided walls has unbounded clique-width.*

In [19], Dabrowski, Golovach, and Paulusma showed that $(4P_1, \overline{3P_1 + P_2})$-free graphs have unbounded clique-width. However, their construction was not C_5-free. We give an alternative construction that neither contains an induced C_5 nor an induced copy of any larger self-complementary graph (see also Figure 7).

THEOREM 6. *Let $\mathcal{F}$ be the set of all self-complementary graphs on at least five vertices that are not equal to the bull. The class of $(\{4P_1, \overline{3P_1 + P_2}\} \cup \mathcal{F})$-free graphs has unbounded clique-width.*

Proof. Consider a wall H (see also Figure 6). Let H' be the graph obtained from H by subdividing every edge once, that is, H' is a 1-subdivided wall. By Lemma 16, such graphs have unbounded clique-width. Let V_1 be the set of vertices in H' that are also present in H. Let V_2 be the set of vertices obtained from subdividing vertical edges in H, and let V_3 be the set of vertices obtained from subdividing horizontal edges. Note that V_1, V_2, and V_3 are independent sets. Furthermore, every vertex in V_1 has at most one neighbor in V_2 and at most two neighbors in V_3, while every vertex in $V_2 \cup V_3$ has at most two neighbors, each of which is in V_1. Let H'' be the graph obtained from H' by applying complementations on V_1, V_2, and V_3. By Fact 2, such graphs have unbounded clique-width.

It remains to show that H'' is $(\{4P_1, \overline{3P_1+P_2}\} \cup \mathcal{F})$-free. Since the vertex set of H'' is the disjoint union of the three cliques V_1, V_2, V_3, we find that H'' is $4P_1$-free. We now show that H'' is $\overline{3P_1+P_2}$-free. Suppose, for contradiction, that H'' contains an induced $\overline{3P_1+P_2}$ with vertex set S. Note that $\overline{3P_1+P_2}$ is the graph obtained from a K_5 after deleting an edge. Since V_1, V_2, and V_3 are cliques in H'', this means that S must have vertices in at least two of these sets. As V_2 is anticomplete to V_3, we find that S contains at least one vertex of V_1. Suppose S has only vertices in V_1 and V_i for some $i \in \{2,3\}$. Then, since $|S| = 5$ and S contains vertices of both V_1 and V_i, it follows that one of V_1, V_i contains either three or four vertices. As each vertex of V_1 has at most two neighbors in V_i and vice versa, this means that in both cases $H''[S]$ is missing at least two edges, which is not possible. Hence S must have at least one vertex in each of V_1, V_2, V_3. As V_2 is anticomplete to V_3 and $H''[S]$ is missing only one edge, S must have exactly one vertex in each of V_2 and V_3, and the remaining three vertices of S must be in V_1. We recall that every vertex in V_2 has at most two neighbors in V_1 and no neighbors in V_3. Hence, the vertex of S that is in V_2 has degree at most 2 in $H''[S]$. This is a contradiction, as $\overline{3P_1+P_2}$ has minimum degree 3. We conclude that H'' is $\overline{3P_1+P_2}$-free.

Next, we show that H'' is X-free for any self-complementary graph X on at least five vertices that is not equal to the bull. First suppose that X has at most seven vertices. Then X must be the C_5. Since the vertex set of H'' is the disjoint union of the three cliques V_1, V_2, V_3, and V_2 is anticomplete to V_3, we find that H'' is C_5-free.

It remains to show that if X is a self-complementary graph on at least eight vertices, then H'' is X-free. Suppose, for contradiction, that H'' contains such an induced subgraph X. Since H'' is $4P_1$-free, X must be $4P_1$-free and therefore K_4-free. Let $U_i = V_i \cap V(F)$ for $i = 1,2,3$. Since each set V_i is a clique and X is K_4-free, each set U_i must be a clique on at most three vertices. Since X contains at least eight vertices, two sets of $\{U_1, U_2, U_3\}$ consist of three vertices and the other set consists of either two or three vertices. This means that at least one of U_2, U_3 contains three vertices (while the other set may contain two vertices). Now U_2 is anticomplete to U_3, so X contains an induced $K_3 + P_2$. Since X is self-complementary, X must contain an induced $K_{2,3}$. Consider an induced $K_{1,3}$ in H''. The three degree-1 vertices of the $K_{1,3}$ must be in different sets V_i. As V_2 is anticomplete to V_3, the central vertex of the $K_{1,3}$ must be in V_1. Since two nonadjacent vertices in a $K_{2,3}$ are the centers of an induced $K_{1,3}$, it follows that both these vertices must be contained in $H''[V_1]$. This is a contradiction, since V_1 is a clique. This completes the proof. ∎

Brandstädt et al. [9] proved that $(C_4, K_{1,3}, K_4, \overline{2P_1+P_2})$-free graphs have unbounded clique-width. In fact, their construction is also C_5-free (see also Figure 7). Note that by Lemma 7, any self-complementary graph on at least five vertices that is not equal to the bull contains an induced subgraph isomorphic to C_4, C_5, or K_4, so such graphs are automatically excluded from the class specified in the following theorem.

THEOREM 7. *The class of* $(C_4, C_5, K_{1,3}, K_4, \overline{2P_1+P_2})$*-free graphs has unbounded clique-width.*

Proof. Consider a wall H (see also Figure 6). Let H' be the graph obtained from H by subdividing every edge once. Let V_1 be the set of vertices in H' that are also present in H and let V_2 be the set of vertices obtained by subdividing edges of H. Note that in H', the neighborhood of every vertex in V_1 is an independent set. For every vertex in V_1, we add edges between its neighbors; this will cause its neighborhood to induce a P_2 or a triangle. Finally, we delete the vertices in V_1 and let H'' be the resulting graph. As the smallest induced cycle in H has length 6, the smallest induced cycle in H' has length 12. Hence, making vertices that are of distance 2 from each other in H' adjacent to each other in H'' does not create any induced C_5, and we find that H'' is C_5-free. Brandstädt et al. proved that such graphs are $(C_4, K_{1,3}, K_4, \overline{2P_1 + P_2})$-free and have unbounded clique-width [9]; the latter fact also follows from [25, Theorem 3]. This completes the proof. ∎

Before proving our third unboundedness result, we will first need to introduce some more terminology and two lemmas. Given natural numbers k, ℓ, let $Rb(k, \ell)$ denote the smallest number such that if every edge of a $K_{Rb(k,\ell),Rb(k,\ell)}$ is colored red or blue then it will contain a monochromatic $K_{k,\ell}$. Beineke and Schwenk [1] showed that $Rb(k, \ell)$ always exists and proved the following result.

LEMMA 17 ([1]). *$Rb(2, 2) = 5$.*

The next lemma was independently proved by Ringel and Sachs.

LEMMA 18 ([45, 46]). *Let G be a self-complementary graph on an odd number of vertices and let $f : V(G) \to V(G)$ be an isomorphism from G to $\overline{G}$. Then there is a unique vertex $v \in V(G)$ such that $f(v) = v$.*

Recall that the clique number $\omega(G)$ of G is the size of a largest clique in G. The next lemma was proved by Sridharan and Balaji.

LEMMA 19 ([47]). *Let G be a self-complementary split graph on n vertices. If n is even then $\omega(G) = \frac{n}{2}$.*

Let $G = (V, E)$ be a split graph. The *independence number* $\alpha(G)$ of G is the size of a largest independent set in G. By definition, G has a *split partition*, that is, a partition of V into two (possibly empty) sets C and I, where C is a clique and I is an independent set. A split graph G may have multiple split partitions. For self-complementary split graphs we can show the following.

LEMMA 20. *Let G be a self-complementary split graph on n vertices.*
(i) *If n is even, then G has a unique split partition and in this partition the clique and independent set are of equal size.*
(ii) *If n is odd, then there exists a vertex v such that $G \setminus v$ is also a self-complementary split graph.*

Proof. First consider the case where n is even and let (C, I) be a split partition of G. Then $\omega(G) \geq |C|$ and $\alpha(G) \geq |I|$. Since G is self-complementary, it follows that $\omega(G) = \alpha(G)$ and, by Lemma 19, $\omega(G) = \frac{n}{2}$. Therefore $n = |C| + |I| \leq \omega(G) + \alpha(G) \leq n$ and so $|I| = |C| = \omega(G) = \alpha(G) = \frac{n}{2}$. Suppose, for contradiction, that there is another split partition (C', I') with $|C'| = |I'|$. Now $I \setminus I' \subseteq C' \cap I$, so $|I \setminus I'| = 1$. Similarly $|I' \setminus I| = 1$. This implies that $I \setminus I' = C' \setminus C = \{u\}$ and $I' \setminus I = C \setminus C' = \{w\}$. Note that both u and w are complete to $C \cap C'$ and anticomplete to $I \cap I'$. Hence if u, w are adjacent, then $\{u, w\} \cup (C \cap C')$ is a clique of size $\frac{n}{2} + 1$, and if u, w are nonadjacent, then $\{u, w\} \cup (I \cap I')$ is an independent set of size $\frac{n}{2} + 1$. In both cases we get a contradiction with the fact that $\omega(G) = \alpha(G) = \frac{n}{2}$. This completes the case where n is even.

Now suppose that n is odd and let f be an isomorphism from G to $\overline{G}$. By Lemma 18, there is a vertex $v \in V(G)$ such that f maps v to v and maps the vertices of $G \setminus \{v\}$ to the vertices of $\overline{G} \setminus \{v\}$. Therefore $G \setminus \{v\}$ is self-complementary. □

THEOREM 8. *Let $\mathcal{F}$ be the set of all self-complementary graphs on at least five vertices that are not equal to the bull. The class of $(\{C_4, 2P_2\} \cup \mathcal{F})$-free graphs has unbounded clique-width.*

Proof. First note that the only self-complementary graph on five vertices apart from the bull is the C_5. Since $C_5 \in \mathcal{F}$, the class of $(\{C_4, 2P_2\} \cup \mathcal{F})$-free graphs is a subclass of the class of split graphs by Lemma 1. Hence we may remove all graphs that are not split graphs from $\mathcal{F}$, apart from C_5; in particular, this means that we remove $X_4, \ldots, X_{10}$ from $\mathcal{F}$ (see also Figure 3). By Lemma 20, if $G \in \mathcal{F}$ has an odd number of vertices, but is not equal to C_5, then $G \setminus v \in \mathcal{F}$ for some vertex $v \in V(G)$. Let $\mathcal{F}'$ be the set of self-complementary split graphs on at least eight vertices that have an even number of vertices. It follows that the class of $\mathcal{F}'$-free split graphs is equal to the class of $(\{C_4, 2P_2\} \cup \mathcal{F})$-free graphs.

Consider a 2-subdivided wall H and note that it is (C_4, C_8)-free; recall that 2-subdivided walls have unbounded clique-width by Lemma 16. Note that H is a bipartite graph, and fix a bipartition (A, B) of H. Let H' be the graph obtained from H by applying a complementation to A and note that H' is a split graph with split partition (A, B).

Note that in H', every vertex in B has a nonneighbor in A and every vertex in A has a neighbor in B. We claim that (A, B) is the unique split partition of H'. Indeed, suppose (A', B') is a split partition of H'; we will show that $A' = A$ and $B' = B$. First suppose, for contradiction, that B' contains a vertex $a \in A$. There exists a vertex $b \in B$ that is adjacent to a and therefore $b \notin B'$. Similarly, every vertex of $(A \setminus \{a\})$ is adjacent to a, so $(A \setminus \{a\}) \cap B' = \emptyset$. Therefore every vertex of $(A \setminus \{a\}) \cup \{b\}$ must lie in A', which is a clique. It follows that b is complete to A, a contradiction. Therefore B' cannot contain a vertex of A. Similarly, A' cannot contain a vertex of B. We conclude that $(A', B') = (A, B)$, so the split partition of H' is indeed unique.

Now, by Fact 2, the class of graphs H' produced in the above way also has unbounded clique-width. It remains to show that H' is $\mathcal{F}'$-free.

First note that X_1 (see also Figure 3) is the graph obtained from the bipartite graph C_8 by complementing one of the independent sets in the bipartition. Since H is C_8-free and X_1 has a unique split partition (by Lemma 20), it follows that H' is X_1-free.

Note that H is C_4-free and so H' does not contain two vertices x, x' in the clique A and two vertices y, y' in the independent set B such that $\{x, x'\}$ is complete to $\{y, y'\}$. Now suppose $G \in \mathcal{F}' \setminus \{X_1\}$. Recall that by Lemma 20, G has a unique split partition (C, I), and this partition has the property that $|C| = |I|$. Therefore, if we can show that G contains two vertices $x, x' \in C$ and two vertices $y, y' \in I$ with $\{x, x'\}$ complete to $\{y, y'\}$ then H' must be G-free and the proof is complete. It is easy to verify that this is the case if $G \in \{X_2, X_3\}$ (see also Figure 3 and recall that $X_4, \ldots, X_{10} \notin \mathcal{F}'$). Otherwise, G has at least ten vertices so $|C|, |I| \geq 5$. By Lemma 17, there must be two vertices $x, x' \in C$ and two vertices $y, y' \in I$ with $\{x, x'\}$ either complete or anticomplete to $\{y, y'\}$. In the first case we are done. In the second case we note that complementing G will swap the sets C and I and make $\{x, x'\}$ complete to $\{y, y'\}$, returning us to the previous case.

We conclude that H' is indeed $\mathcal{F}'$-free. This completes the proof. □

We are now ready to prove Theorem 3. Note that this theorem holds even if $\mathcal{F}$ is infinite.

THEOREM 3 (restated). *Let $\mathcal{F}$ be a set of self-complementary graphs on at least five vertices not equal to the bull. For a graph H, the class of $(\{H,\overline{H}\}\cup\mathcal{F})$-free graphs has bounded clique-width if and only if H or $\overline{H}$ is an induced subgraph of $K_{1,3}$, P_1+P_4, $2P_1+P_3$, or sP_1 for some $s\geq 1$.*

Proof. Let H be a graph. By Theorem 2, if H or $\overline{H}$ is an induced subgraph of $K_{1,3}$, P_1+P_4, $2P_1+P_3$, or sP_1 for some $s\geq 1$, then the class of $(\{H,\overline{H}\}\cup\mathcal{F})$-free graphs has bounded clique-width.

Consider a graph $F\in\mathcal{F}$. Since F contains at least five vertices and is not isomorphic to the bull, Lemma 7 implies that F contains an induced subgraph isomorphic to C_4, C_5, or K_4, and so $F\notin\mathcal{S}$. Therefore, the class of $(\{H,\overline{H}\}\cup\mathcal{F})$-free graphs contains the class of $(H,\overline{H},C_4,C_5,K_4)$-free graphs. If $H\notin\mathcal{S}$ and $\overline{H}\notin\mathcal{S}$, then the class of $(H,\overline{H},C_4,C_5,K_4)$-free graphs has unbounded clique-width by Lemma 5. By Fact 2, we may therefore assume that $H\in\mathcal{S}$. By Lemma 2, we may assume H contains $K_{1,3}+P_1$, $2P_2$, $3P_1+P_2$, or $S_{1,1,2}$ as an induced subgraph, otherwise we are done. In this case, the class of $(\{H,\overline{H}\}\cup\mathcal{F})$-free graphs contains the class of $(K_{1,3},K_4,C_4,C_5)$-free, $(\{2P_2,C_4\}\cup\mathcal{F})$-free, $(\{4P_1,\overline{3P_1+P_2}\}\cup\mathcal{F})$-free, or $(K_{1,3},\overline{2P_1+P_2},C_4,C_5,K_4)$-free graphs, respectively. These classes have unbounded clique-width by Theorems 7, 8, 6, and 7, respectively. This completes the proof. ∎

6. Conclusions. We classified the boundedness of clique-width for the class of $\mathcal{H}$-free graphs for every set $\mathcal{H}$ of self-complementary graphs (Theorem 1). Afterwards, we did the same for the class of $(H,\overline{H})$-free graphs for every graph H (Theorem 2). In particular, we proved that the class of $(2P_1+P_3,\overline{2P_1+P_3})$-free graphs has bounded clique-width (Theorem 4). We then proved that for a set $\mathcal{F}$ of self-complementary graphs on at least five vertices, the classification of the boundedness of clique-width for $(\{H,\overline{H}\}\cup\mathcal{F})$-free graphs coincides with the one for the $|\mathcal{H}|=2$ case if and only if $\mathcal{F}$ does not include the bull. As future work, we aim to continue our study of boundedness of clique-width for graph classes closed under complementation (Theorem 3). In particular, to complete the classification for $\mathcal{H}$-free graphs when $|\mathcal{H}|=3$, we still need to determine those graphs H for which $(H,\overline{H},\text{bull})$-free graphs have bounded clique-width, and several such cases remain open. Our results also have a number of algorithmic and structural consequences, which we discuss below.

As explained in section 1, the (un)boundedness of clique-width for bigenic graph classes was determined in a sequence of other papers for all but six cases. As we have proved that the class of $(2P_1+P_3,\overline{2P_1+P_3})$-free graphs has bounded clique-width, five cases remain to be solved in order to complete the classification in Theorem 5.

OPEN PROBLEM 1. *Does the class of (H_1,H_2)-free graphs have bounded or unbounded clique-width when*

(i) $H_1=3P_1$ *and* $\overline{H_2}\in\{P_1+S_{1,1,3},S_{1,2,3}\}$;
(ii) $H_1=2P_1+P_2$ *and* $\overline{H_2}\in\{P_1+P_2+P_3,P_1+P_5\}$;
(iii) $H_1=P_1+P_4$ *and* $\overline{H_2}=P_2+P_3$.

The COLORING problem takes as input a graph $G=(V,E)$ and an integer $k\geq 1$ and asks whether there exists a mapping (coloring) $c:V\to\{1,2,\ldots,k\}$ such that $c(u)\neq c(v)$ whenever $uv\in E$. By combining a result of Kobler and Rotics [38] with a result of Oum and Seymour [42], the COLORING problem is polynomial-time solvable for any graph class of bounded clique-width. Hence, our result that the class of $(2P_1+P_3,\overline{2P_1+P_3})$-free graphs has bounded clique-width implies that COLORING is polynomial-time solvable for this graph class. This result was used by Blanché et al. to prove the following theorem.

THEOREM 9 ([2]). *Let $H, \overline{H} \notin \{(s+1)P_1 + P_3, sP_1 + P_4 \mid s \geq 2\}$. Then* COLORING *is polynomial-time solvable for $(H, \overline{H})$-free graphs if H or $\overline{H}$ is an induced subgraph of $K_{1,3}$, $P_1 + P_4$, $2P_1 + P_3$, $P_2 + P_3$, P_5, or $sP_1 + P_2$ for some $s \geq 0$ and it is* NP*-complete otherwise.*

Comparing Theorems 2 and 9 shows that there are graph classes of unbounded clique-width that are closed under complementation, but for which COLORING is still polynomial-time solvable. Nevertheless, on many graph classes, polynomial-time solvability of NP-hard problems stems from the underlying property of having bounded clique-width. The present paper illustrates this for the COLORING problem, since Theorem 4 implies that COLORING is solvable in polynomial time on $(2P_1+P_3, \overline{2P_1+P_3})$-free graphs. In particular, by updating the summary of [18] (see also [31]) with the results of [4] and this paper, we identify the following ten classes of (H_1, H_2)-free graphs, for which COLORING could still potentially be solved in polynomial time by showing that their clique-width is bounded.

OPEN PROBLEM 2. *Is* COLORING *polynomial-time solvable for (H_1, H_2)-free graphs when*

(i) $\overline{H_1} \in \{3P_1, P_1 + P_3\}$ *and* $H_2 \in \{P_1 + S_{1,1,3}, S_{1,2,3}\}$;

(ii) $H_1 = 2P_1 + P_2$ *and* $\overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + P_5\}$;

(iii) $H_1 = \overline{2P_1 + P_2}$ *and* $H_2 \in \{P_1 + P_2 + P_3, P_1 + P_5\}$;

(iv) $H_1 = P_1 + P_4$ *and* $\overline{H_2} = P_2 + P_3$;

(v) $\overline{H_1} = P_1 + P_4$ *and* $H_2 = P_2 + P_3$.

Finally, we note that it may also be worthwhile to consider the consequences of our research for the boundedness of variants of clique-width, such as linear clique-width [35] and power-bounded clique-width [5].

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