Contests with draws: axiomatization and equilibrium*

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Abstract

We introduce and axiomatize a class of single-winner contest success functions that embody the possibility of a draw. We then analyze the game of contest that our success functions induce, having different prizes delivered in the occurrence of a win and a draw. We identify conditions for the existence and uniqueness of a symmetric interior Nash equilibrium and show that equilibrium efforts and equilibrium rent dissipation can be larger than in a Tullock contest (with no possibility of a draw) due to increased competition even if the draw-prize is null. These results suggest that a contest designer may profit from introducing the possibility of a draw. Finally, we show that this approach naturally extends to multi-prize contests with multiple draws across different subsets of the set of players.

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1 Introduction

In a contest, participants compete for a prize or a number of prizes by exerting costly efforts. Many competitive environments have the structure of a contest, and

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the model has been applied in a variety of fields in economics and related disciplines.[1] A crucial element of a contest model is the contest success function (*success function* hereinafter), which defines how the contended resources are distributed across players depending on their efforts. Contests are typically modeled as exhaustive, in the sense that all contended resources are allocated to participants.[2] In a Tullock contest, a single prize is always allocated to a single winner and a success function defines a probability distribution on the identity of the winner as a function of efforts, which is the probabilistic version of the aforementioned exhaustive property. However, in many applications it is possible for a contest to have no clear winner, i.e., there is a *draw* (or an impasse).[3] In this paper we consider this possibility and address the problem along three directions.

First, we generalize a well-known class of exhaustive success functions for contests with single winner (including the popular functional forms in Tullock, 1980 and Hirshleifer, 1989) by introducing the possibility of a draw, justifying this novel class of success functions via an axiomatic characterization along the lines of Skaperdas (1996) and related approaches in the literature. Second, we apply these success functions to a basic contest game which delivers different prizes depending on the outcome being a win or a draw, showing that a contest designer may profit from introducing the possibility of a draw in a single-winner contest.[4] Third, we show that our approach naturally extends to multi-prize contests by introducing the possibility of multiple draws across different subsets of the set of players.

Let us start by introducing our novel class of success functions for single-winner contests with the possibility of a draw. Our fundamental axiom in the characterization, *weak exhaustivity*, requires the winning probabilities to be a convex transformation of an exhaustive success function. The basic idea is that there is an underlying 'ghost contest' without draw (abbreviated for "without the possibility of a draw") that indirectly determines players' chances of victory in the contest with draw (abbreviated for "with the possibility of a draw") via a transforming technology or institution. Our approach is particularly suited for situations where participants must outperform rivals in multiple tasks to succeed, or for settings where the success of a contestant is indirectly determined by her control of scarce resources of strategic value such as ball possession in soccer, territory in warfare, audience's attention in litigation or marketing, etc. To see an example of the former (multiple tasks), consider a promotion contest between two candidates (candidate 1 and

[1] See, e.g., military conflicts (Hirshleifer, 1991), litigation (Robson and Skaperdas, 2008), sports (Szymanski, 2003), marketing (Schmalensee, 1976), lobbying and rent-seeking (Nitzan, 1994) among others. Konrad (2009) provides an introduction to this literature.

[2] An exception may occur when all efforts are zero so that no contestant effectively participates.

[3] In sports such as soccer, chess, and cricket the possibility of a draw is incorporated by design. For rent-seeking, this possibility may emanate from lack of commitment of the prize-granting authority, see, e.g., Kahana and Nitzan (1999). In military conflicts, Garfinkel and Skaperdas (2007) argue that it is not uncommon to have contenders keeping the status-quo bargaining position they had before the outbreak of war.

[4] An earlier version of the basic model and the core equilibrium results can be found in Yildizparlak (2014).

candidate 2) whose relative performance is measured in two different tasks (task A and task B). Supremacy in each task is governed by an identical success function that is exhaustive, and each candidate exerts an effort that is common to the two tasks (e.g., her overall dedication to promotion). In this contest, we have a winner whenever a candidate prevails in both tasks, while we have a draw whenever a different candidate prevails in each task (see Table 1 below). This contest satisfies weak exhaustivity as a candidate's probability of winning the contest is the exhaustive success function squared (i.e., the product of the probabilities of prevailing in each task).[5] To exemplify the latter (scarce resources), consider a soccer match between

	1 wins B	2 wins B
1 wins A	1 wins the contest	Draw
2 wins A	Draw	2 wins the contest

TABLE 1. Each of the four entries in the table indicates the outcome of the joint realization of the events in the corresponding column and row.

two teams (team 1 and team 2) where s and $(1-s)$ denote their respective shares of ball possession time. In this setting, it seems natural that each team's winning probability is an increasing function of its ball possession time, which we denote by $p(s)$ and $p(1-s)$ for Team 1 and Team 2 respectively. As soccer matches can end in a draw, the function p must be strictly convex so that $p(s)+p(1-s)<1$. Then, if we assume that ball possession time is the outcome of an exhaustive contest, the soccer match is a contest that satisfies weak exhaustivity.[6] Note that as p is strictly convex the probability of a draw is maximal whenever the two teams have equal ball possession time, which is a fundamental property of our class of success functions. Figure 1 illustrates this via an example.

Ours is not the first model of a contest with the possibility of a draw. Within the literature, the class of success functions introduced by Loury (1979) and axiomatized in Blavatskyy (2010) is perhaps the most widely known (the Loury-Blavatskyy model hereinafter), and there are a handful of other models that we discuss in detail in Section 2. A crucial feature of the Loury-Blavatskyy model is that the probability of a draw decreases in the sum of contestants' efforts. While this may seem plausible for many environments, there are cases for which it may not hold. For instance, Peeters and Szymanski (2014) question its plausibility in sports, as one expects two teams to draw more often when they have a similar investment in talent, not when they aggregately have a large investment. Note that, as exemplified in Figure 1, according to our framework the probability of a draw is maximal when efforts are

[5] In this example the ghost contest is the competition in each single task, while the transforming technology requires supremacy in both tasks to win the contest. We refer to Kovenock and Roberson (2012) for a review of different approaches to contests with multiple tasks or battlefields.

[6] In this example the ghost contest is the competition for ball possession time, while the transforming technology implies increasing returns of the winning probability of a team in its ball possession time.

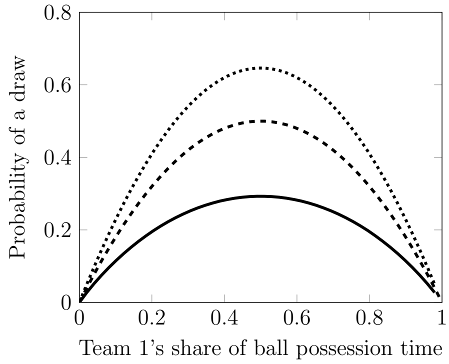

FIGURE 1. The solid (——), dashed (- - - -) and dotted (······) lines correspond to the cases $p(s) = s^{1.5}$, $p(s) = s^2$ and $p(s) = s^{2.5}$, respectively.

symmetric and it is independent of the magnitude of such efforts.[7]

After axiomatizing our class of success functions, we apply it to contest games. We consider the basic extension of a single-winner contest where either there is a winner or a draw, so that: if there is a winner she obtains a win-prize and the losers get nothing; if there is a draw all players receive a draw-prize that is non-negative and weakly lower than the win-prize. We identify sufficient conditions for the existence and the uniqueness of a symmetric pure strategy Nash equilibrium and characterize equilibrium efforts. We show that in such equilibrium individual effort is independent of the value of the draw-prize, although a sufficiently high draw-prize can be necessary to guarantee the existence of the equilibrium when the number of players is high. On the other hand, a large draw-prize is detrimental for aggregate effort in an extension of our model where players are asymmetrically constrained in the resources available for the contest. We argue that this second result may explain a change of rule in soccer that took place in the 1980s, which decreased the points for a draw relative to the points for a win with the intention of increasing the competitiveness of soccer matches. Having established these general properties of our model, we compare a contest with draw with the corresponding contest without draw regarding aggregate effort exerted. For the two-player case, we find that equilibrium effort can be larger in the contest with draw, and this holds true even when a draw is equivalent to losing (i.e., when the draw-prize is null), indicating an increase in the level of competition. This result is in sharp contrast to the equilibrium results in Jia (2012) and Deng et al. (2018) using the Loury-Blavatskyy model. Finally, although introducing the draw possibility always reduces equilibrium efforts when the number of players is three or more, we find that equilibrium rent dissipation (i.e., the share of the expected value of the delivered prize that corresponds to the value of the total exerted efforts in equilibrium) is systematically higher in a contest with draw compared to the contest without draw

[7] Yildizparlak (2018) shows that our class of success functions and a success function introduced in Jia (2012) (the one with performance gap in ratio form, see Section 2) give a better fit than the Loury-Blavatskyy model for the empirical analysis of soccer matches.

for a broad set of parameter configurations. All in all, our results suggest that a contest designer may profit from introducing the possibility of a draw associated with a relatively small or null prize. For instance, a practical way to do so is to require contestants to prevail in two tasks as in the example of the promotion contest in Table 1.

We conclude our analysis by showing that our framework can be extended to multi-prize contests with multiple draws that can occur across different subsets of the set of players. To do so, we take as a benchmark the well-known class of success functions for multi-prize contests introduced in Clark and Riis (1996) and axiomatized in Lu and Wang (2015). These success functions can be interpreted as the outcome of a sequential procedure where prizes are allocated from best to worst to the winners of a sequence of exhaustive stage-contests restricted to players that have not been awarded any prize yet. In our extension, we modify each of these stage-contests to allow for the possibility of a draw so that each prize has a probability of not being awarded to any player. We show that this extended model coincides with our baseline model under the prize configuration that maximizes equilibrium efforts, and that all our core intuitions generalize to this broader class of contests with draws. An example of real-life contest (roughly) organized along these lines is the political campaigning across political parties for a runoff election, such as the one to elect the president in many countries.

The rest of the paper is organized as follows. We provide a review of the literature in Section 2, define our model in Section 3 and provide an axiomatic foundation in Section 4. In Section 5 we study the existence, uniqueness, and the properties of a symmetric equilibrium. In Section 6 we extend our framework to multi-prize contests with multiple draws. We conclude with Section 7. All proofs are in Appendix.

2 Literature

In this section, we focus on the narrow body of literature on contests with the possibility of a draw. In particular, as we axiomatize a novel class of models, we give priority to foundational work on success functions for such contests. We also briefly compare our equilibrium analysis with known results in the literature on contests with draw and related models (i.e., tournaments and all-pay auctions). This comparison is far from comprehensive as it aims only at highlighting the crucial differences between our framework and the leading approaches in the literature. We conclude the section by mentioning alternative extensions of our approach to multi-prize contests.

To the best of our knowledge, the only other axiomatic characterization of a class of success functions for contests with draw is Blavatskyy (2010), which provides a foundation for the framework introduced by Loury (1979). As ours, this characterization is along the lines of Skaperdas (1996) and presents an extension of the canonical class of success functions for single-winner contests, however, it differs fundamentally from ours in the way it introduces the possibility of a draw as discussed in the previous section.[8]

The Loury-Blavatskyy model may be considered the most popular in the literature. Other models of contests with draw are: Jia (2012), which presents a stochastic foundation of a broad class of success functions; Cohen and Sela (2007) and Gelder et al. (2015a), which analyze the possibility of a draw in an all-pay auction framework.[9] In the general framework of Jia (2012), the winning probability of a player is given by the chance of outperforming each rival by a sufficiently wide performance gap in all pairwise comparisons. If the performance gap takes the difference form, the model of Jia (2012) leads to the Loury-Blavatskyy model, while if the performance gap takes the ratio form, the model leads to a novel class of success functions. Similar to the framework in Jia (2012), Cohen and Sela (2007) and Gelder et al. (2015a) assume that a draw takes place when the difference of players' efforts is below a given threshold.

The idea of relating the occurrence of a draw to a threshold performance gap, which is common to all aforementioned models, dates back at least to Nalebuff and Stiglitz (1983). Other contributions that follow this approach include Eden (2006) and Imhof and Kräkel (2014b, 2016). The crucial difference between these models and ours is that, while they all introduce a draw when the performance gap falls below an exogenous level, we assume a continuous proportionality between our success function and the success function of a contest without draw, where the latter is scaled down to allow for the draw possibility. So, our approach is substantially different from the literature from a conceptual viewpoint.

As already mentioned in the previous section, equilibrium results of the game of contest using our class of success functions differ remarkably from the results in Jia (2012) and Deng et al. (2018) with the popular Loury-Blavatskyy model. There, individual and aggregate equilibrium efforts decline due to the introduction of a draw for a symmetric two-player contest with null draw-prize, while we show that both can increase under the same conditions with our class of success functions. Deng et al. (2018) also show that introducing the possibility of a draw can increase the winner's effort (but not the aggregate effort) if players' valuations are sufficiently asymmetric (but not if they are symmetric). While these results are not directly comparable to ours, they can be seen as alternative (and equally plausible) reasons for introducing the possibility of a draw in a contest.

A result similar to ours (i.e., that the possibility of a draw can increase equilibrium efforts) is in Nalebuff and Stiglitz (1983) and related contributions, but it relies on

[8]See Corchón and Dahm (2010) and Jia et al. (2013) for reviews on the axiomatic and stochastic foundations of success functions for single-winner contests. For more recent contributions to the axiomatic foundation of success functions for single-winner and multi-prize contests we refer to Lu and Wang (2015, 2016), Cubel and Sanchez-Pages (2016), Vesperoni (2016), Bozbay and Vesperoni (2018) and references therein.

[9]Gelder et al. (2015b) test the latter model experimentally.

a different mechanism. Broadly speaking, while in Nalebuff and Stiglitz (1983) the greater aggregate effort arises due to the constrained effect of luck in producing output, in ours, it takes place as a result of the convex transformation of the contest technology, which leads to increased competition in the sense of a larger incentive to beat the opponent conclusively.

We find two other relevant points concerning our equilibrium analysis in relation to the literature. Firstly, we show that equilibrium effort is invariant to the value of the draw-prize, which is again in contrast with the Loury-Blavatskyy model where (by our own calculations) a higher draw-prize leads to lower equilibrium effort if payoffs are symmetric. Cohen and Sela (2007) and Imhof and Kräkel (2014a) find effects analogous to this invariance result respectively in an all-pay auction context and a moral-hazard exercise, and (by our own calculations) the same effect can be found using the class of success functions in Jia (2012) with performance gap in ratio form when payoffs are symmetric. We stress that this invariance result should hold with any contest model where the probability of a draw is independent of the level of efforts whenever efforts are symmetric. In relevance to this point, we also show that in our setup the value of the draw-prize can nevertheless affect the existence of the equilibrium and that a positive value can be necessary when the number of players is high, a point that has not been highlighted by the literature so far. Secondly, in an extension of our basic model for two players where contestants are asymmetric in their resources available for efforts, i.e., when there is a 'rich' and a 'poor' (resource constrained) contestant, we show that an increase in the value of the draw-prize is detrimental to aggregate efforts. This result is chiefly due to both players desiring to reduce the difference in their efforts (in order to increase the probability of a draw) in response to an increase in the draw-prize, but only the richer player managing to do so due to the binding resource constraint of the poorer player. To the best of our knowledge this result is unique in the literature.

We conclude this section by briefly discussing success functions for multi-prize contests in relation to our framework. The extension of our model to multiple draws is a generalization of the class of success functions for multi-prize contests introduced in Clark and Riis (1996) and axiomatized in Lu and Wang (2015), known in the literature as the best-shot model. While this is by far the most widely applied model in the literature on multi-prize contests, there are two other classes of models that are equally convenient as they deliver explicit functional forms for the probabilities of outcomes: the worst-shot and the pair-swap success functions introduced in Fu et al. (2014) and Vesperoni (2016), respectively. Our approach can be equally applied to these alternative models.

3 Model

A set $N := \{1, \ldots, n\}$ of $n \geq 2$ players compete in a contest that has two possible outcomes: (i) one player wins and all other players lose; (ii) the contest ends in a draw. If a player wins she gets a prize of value 1 and the others get nothing, while if the outcome is a draw each player gets a prize of value $\delta \in [0,1]$.[10] Each player $i \in N$ exerts an effort $x_i \geq 0$, and given any effort profile $\mathbf{x} := (x_1, \ldots, x_n) \in \mathbb{X} := \mathbb{R}^n_+$, the probability of player $i \in N$ being the winner is $p_i(\mathbf{x}) \in [0,1]$. We call $p_i(\mathbf{x})$ the success function. Assuming $\sum_{i \in N} p_i(\mathbf{x}) \leq 1$, we define the probability of a draw as $p_d(\mathbf{x}) := 1 - \sum_{i \in N} p_i(\mathbf{x})$, and we let the payoff of player $i \in N$ be

$$\pi_i(\mathbf{x}) := p_i(\mathbf{x}) + \delta p_d(\mathbf{x}) - x_i \text{ for each } \mathbf{x} \in \mathbb{X}. \tag{1}$$

In this paper we consider the particular class of success functions that, for each $i \in N$ and $\mathbf{x} \in \mathbb{X}$, take the form

$$p_i(\mathbf{x}) = \Gamma(s_i(\mathbf{x})) \tag{2}$$

for some function $\Gamma : [0,1] \to [0,1]$ that is increasing, strictly convex and twice-differentiable with $\Gamma(0) = 0$ and $\Gamma(1) = 1$, and some function $s_i : \mathbb{X} \to [0,1]$ which is twice-differentiable in the interior and satisfies each of the following properties: $\sum_{i \in N} s_i(\mathbf{x}) = 1$ for each $\mathbf{x} \in \mathbb{X}$; it is anonymous (i.e., invariant to permutations of players' identities), increasing, and strictly concave in x_i at each $\mathbf{x} \in \mathbb{X}$ with $x_i, x_j > 0$ for some $j \neq i$; $s_i(\mathbf{x}) = 0$ if and only if $x_j > x_i = 0$ for some $j \neq i$. We refer to Γ as the *transformation function* and to s_i as the *ghost success function*. In our comparative static exercises, for any success function in form (2) we take as its *benchmark* the corresponding ghost success function s_i. Note that the probability of a draw is zero with the benchmark while with form (2) it is always positive in the interior, therefore the benchmark constitutes the obvious comparison to evaluate the effect of introducing the possibility of a draw in a contest while keeping all else equal. One crucial property of form (2) is that the probability of a draw achieves its maximum value $1 - n\Gamma(1/n)$ at any effort profile that is symmetric (i.e., whenever players exert equal effort).

Form (2) is intuitive and appealing but, as we will see later on, it is too general to achieve sharp predictions in equilibrium analysis (e.g., exact conditions for existence and uniqueness) beyond some basic stylized facts. Among the various specifications within class (2), in this paper we mostly focus on the narrower class of success functions that for each $i \in N$ and $\mathbf{x} \in \mathbb{X}$ can be written as

$$p_i(\mathbf{x}) = \begin{cases} \left(f(x_i) / \sum_{j \in A_{\mathbf{x}}} f(x_j) \right)^k & \text{if } i \in A_{\mathbf{x}}, \\ 1/n^k & \text{if } A_{\mathbf{x}} = \emptyset, \\ 0 & \text{if } i \notin A_{\mathbf{x}} \text{ and } A_{\mathbf{x}} \neq \emptyset, \end{cases} \tag{3}$$

[10] All our results equally hold if we restrict δ to take value in $[0, 1/n]$, which is reasonable if δ is interpreted as a way to share a 'materialistic' win-prize in the occurrence of a draw. However, in many applications there is no such materialistic constraint, for instance in soccer games prizes are simply points assigned to teams.

for some function $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$ that is increasing and some parameter $k \geq 1$,[11] where $A_{\mathbf{x}} \subseteq N$ denotes the set of players that are active (i.e., that exert positive effort) at $\mathbf{x}$. Note that class (3) corresponds to the specification $p_i(\mathbf{x}) = \Gamma(s_i(\mathbf{x}))$ with $\Gamma(s_i) = s_i^k$ and $s_i(\mathbf{x}) = f(x_i)/\sum_{j\in A_{\mathbf{x}}} f(x_j)$, where for each k and f the corresponding benchmark belongs to the well-known class of exhaustive success functions à la Skaperdas (1996). In line with the literature we refer to f as the *impact function*, and to k as the *elasticity parameter* as it coincides with the (constant) % increase in p_i relative to the corresponding % increase in s_i, i.e.,

$$k = \Gamma'(s_i)s_i/\Gamma(s_i).$$

First, k can be seen as a quantification of the increasing marginal returns of p_i in s_i. Second, it is easy to show that the probability of a draw increases in k for any given effort profile such that at least two players (or none) are active, so k can also be interpreted as a measure of the tendency to draw in our model. The class of success functions (3) has an intuitive interpretation whenever k is a positive integer, which is the extension of the example on the promotion contest in Table 1 to multiple tasks (where k is the number of tasks). To further motivate our framework, in the next section we provide an axiomatic characterization of (3) valid for any real number $k > 1$ along the lines of the example on the soccer match in Introduction.

4 Axiomatization

We now take a step back from the previous section to provide a foundation for our class of success functions. With some abuse of notation, for each $i \in N$ let $p_i : \mathbb{X} \to \mathbb{R}_+$ be any non-negative real valued function, so that we can more generally interpret $p_i(\mathbf{x})$ as a quantification of the 'success' of player i in the contest (not necessarily a probability), which naturally depends on the effort profile $\mathbf{x} \in \mathbb{X}$.[12] Starting from this broader premise, in this section we show that the function p_i satisfies a set of axioms if and only if it takes the form (3). The first three axioms are straightforward.

A1. **Probability**: Given any $\mathbf{x} \in \mathbb{X}$, $\sum_{i\in N} p_i(\mathbf{x}) \leq 1$.

A2. **Anonymity**: For each $\mathbf{x} \in \mathbb{X}$ and $\rho : N \to N$, let $\mathbf{x}^\rho := (x_{\rho^{-1}(1)}, \ldots, x_{\rho^{-1}(n)})$. Given any $\mathbf{x} \in \mathbb{X}$ and $i \in N$, $p_i(\mathbf{x}) = p_{\rho(i)}(\mathbf{x}^\rho)$ for any ρ that is a bijection.

A3. **Strict monotonicity**: Given any $\mathbf{x}, \mathbf{x}' \in \mathbb{X}$ and $i \in A_{\mathbf{x}}$ with $|A_{\mathbf{x}}| \geq 2$, $p_i(\mathbf{x}) > p_i(\mathbf{x}')$ if $x_i > x_i'$ and $x_j = x_j'$ for all $j \in N\backslash\{i\}$.

By probability, the success of each player takes value in the unit interval and their aggregate success sums up to 1 or less. Then, a player's success can be interpreted as

[11] Note that letting $k < 1$ would lead to $p_d(\mathbf{x}) < 0$, which is at odds with the probabilistic interpretation.

[12] This is the standard premise of the axiomatic foundation of a class of success functions, see e.g., Bozbay and Vesperoni (2018).

the probability of winning a contest where there can be a single winner or a draw. Anonymity and strict monotonicity are standard axioms: the former demands a contest to be a priori fair (as a player's success should be invariant to permutations of players' identities), while the latter requires a player's success to increase in her own effort when she is active and there is at least another active player, everything else equal. Our next axiom specifies what should happen when some players are inactive.

A4. Perfect discrimination at zero: Given any $\mathbf{x} \in \mathbb{X}$ and $i \in N$,

$$p_i(\mathbf{x}) \begin{cases} > 0 & \text{if } i \in A_{\mathbf{x}}, \\ = 0 & \text{if } i \notin A_{\mathbf{x}} \text{ and } A_{\mathbf{x}} \neq \emptyset. \end{cases}$$

Perfect discrimination at zero requires a player's success to be positive if she is active, and to be null if she is inactive and there is at least one active player.[13] The intuition is that active players try to succeed in the contest (which gives all active players a chance) while inactive players do not try and therefore are a priori excluded from success (unless all players are inactive).[14] The next axiom is our crucial axiom, which is a *generalization* of the standard requirement $\sum_{i \in N} p_i(\mathbf{x}) = 1$ that is sometimes known in the literature as the axiom of exhaustivity. To strengthen our characterization, we present this axiom in two alternative (but equivalent) forms.

A5. Weak exhaustivity 1. For some continuous and increasing function $r : \mathbb{R}_+ \to \mathbb{R}_+$,

(*i*) $\sum_{i \in N} r(p_i(\mathbf{x})) = r(1)$ for all $\mathbf{x} \in \mathbb{X}$;

(*ii*) $r(p_i(\mathbf{x}))/r(1) = r(p_i(\mathbf{x})\mu)/r(\mu)$ for all $\mathbf{x} \in \mathbb{X}$, $i \in N$ and $\mu > 0$.

A5'. Weak exhaustivity 2. For some differentiable and increasing function $r : \mathbb{R}_+ \to \mathbb{R}_+$,

(*i*) $\sum_{i \in N} r(p_i(\mathbf{x})) = r(1)$ for all $\mathbf{x} \in \mathbb{X}$;

(*ii*) $\frac{dr(p_i(\mathbf{x}))}{dp_i(\mathbf{x})} \frac{p_i(\mathbf{x})}{r(p_i(\mathbf{x}))} = \frac{dr(p_i(\mathbf{x}'))}{dp_i(\mathbf{x}')} \frac{p_i(\mathbf{x}')}{r(p_i(\mathbf{x}'))}$ for all $\mathbf{x}, \mathbf{x}' \in \mathbb{X}$ with $p_i(\mathbf{x}), p_i(\mathbf{x}') > 0$.

Roughly speaking weak exhaustivity demands that, by some mapping r, a player's success $p_i(\mathbf{x})$ is proportional to an amount of resources $r(p_i(\mathbf{x}))$ under her control from a common budget $r(1)$ shared with other players. In either form (A5 or A5'), Point (*i*) of the axiom demands the budget constraint to be binding, which suggests scarcity of the resources. In form A5, Point (*ii*) requires that player i's fraction of resources $r(p_i(\mathbf{x}))/r(1)$ is independent of the unit of measurement of success μ (so that the interpretation of $p_i(\mathbf{x})$ as a winning probability - instead of, e.g., the share of a pie of total value μ - relies on normalization only), while in form A5' Point (*ii*)

[13]Although perfect discrimination at zero does not impose any restrictions on a success function when all players are inactive, our other axioms (in particular probability, anonymity and weak exhaustivity) are sufficient to characterize our form for the case $A_{\mathbf{x}} = \emptyset$.

[14]This is in line with the idea of sub-contest put forward in Skaperdas (1996).

demands that the % increase in resources $r(p_i(\mathbf{x}))$ relative to the corresponding % increase in success $p_i(\mathbf{x})$ is independent of the level of success (so that the elasticity of resources with respect to success is constant).[15] Details aside, the basic idea of weak exhaustivity is that the success of a player is closely related to her control of scarce resources that are of strategic value.[16] In line with this intuition, the axiom requires a continuous proportionality between these two variables (represented by the mapping r), implying (jointly with other axioms) the crucial feature of our model that the probability of a draw is maximized at symmetric effort profiles. To see this, Figure 2 illustrates an example with three players comparing the distribution of scarce resources and the mapping r for two different effort profiles $\mathbf{x}$ and $\mathbf{x}'$. Note that, as jointly demanded by weak exhaustivity and probability, the mapping r is concave.[17] Given this, as the distribution of success under $\mathbf{x}'$ is more egalitarian than under $\mathbf{x}$, the probability of a draw is higher under $\mathbf{x}'$ than under $\mathbf{x}$, as shown by the dashed lines in the plots on the right-hand side of Figure 2.

Our last axiom is strong, but standard in the literature.

A6. **Effort independence**: Given any $\mathbf{x}, \mathbf{x}' \in \mathbb{X}$ and $i, j \in N$ such that $p_j(\mathbf{x}) > 0$ and $p_j(\mathbf{x}') > 0$, $\frac{p_i(\mathbf{x})}{p_j(\mathbf{x})} = \frac{p_i(\mathbf{x}')}{p_j(\mathbf{x}')}$ if $x_i = x_i'$ and $x_j = x_j'$.

Effort independence demands the success of a player relative to the success of another player (i.e., their ratio) to be independent of the efforts of all other players. Note that, for effort independence to have bite, we should have at least three players. Analogous versions of this axiom, broadly known as independence of irrelevant alternatives, are common in contest theory and probabilistic choice (see Jia et al., 2013 for a review).[18] We are now ready to state our characterization result.

Theorem 1 *A success function satisfies A1-A4, either A5 or A5', and A6 if it takes the form (3), and only if it takes such form given $n \geq 3$.*

Our characterization is tight as all our axioms are independent, which is easily

[15] While Point (*i*) is at the core of the idea of weak exhaustivity, Point (*ii*) can be seen as a useful simplification that allows to quantify the likelihood of a draw in our contest model by the elasticity parameter k. We wish to remark that Point (*ii*) can (and should) be tested empirically in applications, analogously to the well-known tests for the constant elasticity of production/utility functions.

[16] In the example in Introduction, a team's success in a soccer match depends on its share of ball possession time. Similarly, success in military conflict can be related to the share of a given territory that is under control, in litigation to the share of attention of the jury, and in political lobbying to the share of 'controlled' parliament members. In all these examples there is a symbiotic relation between the success of a contestant and her share of the relevant strategic resources.

[17] Weak exhaustivity implies that, if $\sum_{i \in N} p_i(\mathbf{x}) \leq 1$, the function r is concave. Then, axioms probability and weak exhaustivity jointly demand r to be concave. Note that function r coincides with the inverse of Γ within the general class of success functions (2).

[18] A generalized version of this axiom can be found in Vesperoni (2016), where it is argued that effort independence is related to independence of irrelevant alternatives in Luce (1959) and it is implied by the combination of two axioms in Skaperdas (1996), known as sub-contest independence and sub-contest consistency. Note that all analogous versions of this axiom in the literature require at least three players to have bite.

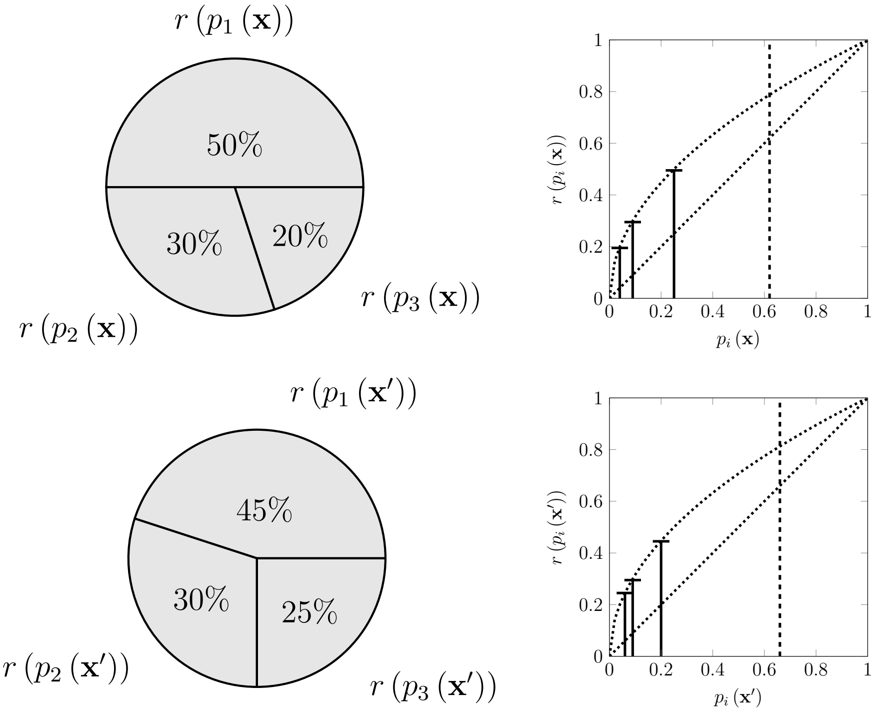

FIGURE 2. In this example, $n = 3$ and $r(z) = \sqrt{z}$. In each row, the pie chart on the left represents the distribution of resources across the three players while the plot on the right depicts the corresponding mapping r, where the two rows correspond to the different effort profiles $\mathbf{x}$ (top row) and $\mathbf{x}'$ (bottom row). In each plot on the right, the dashed line represents value of the probability of a draw (on the horizontal axis), the three solid lines correspond to the success of each player, while the two dotted curves respectively depict the mapping r and the 45° line.

verified as they concern very different aspects of the model. One can show that axioms A1-A4 plus Point (i) of weak exhaustivity are enough to characterize the general form (2),[19] while Point (ii) of weak exhaustivity and effort independence jointly pin down form (3) from (2). In Theorem 1, the case $n = 2$ is special as the 'only if' part of the result does not hold since effort independence has no bite. For the two-player case, one can show that a success function satisfies all our axioms if and only if it takes the form $p_i(\mathbf{x}) = (s_i(\mathbf{x}))^k$, where $k \geq 1$ and s_i is any function that satisfies our general restrictions on the ghost success function.[20] An example of a success function that belongs to this class but not to (3) is the one with ghost success function

$$s_i(\mathbf{x}) = \begin{cases} \dfrac{\exp\left[x_i / \sum_{h \in N} x_h\right]}{\sum\limits_{j \in A_{\mathbf{x}}} \exp\left[x_j / \sum_{h \in N} x_h\right]} & \text{if } i \in A_{\mathbf{x}}, \\ 1/n & \text{if } A_{\mathbf{x}} = \emptyset, \\ 0 & \text{if } i \notin A_{\mathbf{x}} \text{ and } A_{\mathbf{x}} \neq \emptyset. \end{cases}$$

[19]Except for twice-differentiability and strict concavity of s_i, which are not imposed by any axiom.

[20]Except for twice-differentiability and strict concavity, which are not imposed by any axiom.

Clearly, this specification violates effort independence for any $n \geq 3$ but not for $n = 2$.

Before concluding this section, we briefly discuss our axioms in relation to the literature. Probability, anonymity and strict monotonicity are always satisfied by the success functions in Skaperdas (1996), Blavatskyy (2010) and Jia (2012), and perfect discrimination at zero is so whenever these success functions are well-defined when all players are inactive. The success functions in Skaperdas (1996) trivially satisfy weak exhaustivity (in either form) letting $r(z) = z$ for all $z > 0$, while the ones in Blavatskyy (2010) and Jia (2012) always violate it. Finally, effort independence is fulfilled by the success functions in Skaperdas (1996) and Blavatskyy (2010), while the ones in Jia (2012) always violate it.

5 Equilibrium

In this section we study the existence, uniqueness, and the properties of a symmetric equilibrium in the game of contest using our class of success functions (3). In line with the literature, in what follows we assume f to be twice-differentiable, concave, and to satisfy $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to +\infty} f(x) = +\infty$. To illustrate our results, we invoke $f(x) = x^{\alpha}$ with $\alpha \in (0, 1]$ (i.e., the impact function à la Tullock, 1980), whenever stated. While our existence result and major equilibrium properties concern the general case with any number of players, we generally emphasize the two-player contest as we find interesting dynamics and most applications in the literature are concerned with this case. This may seem problematic at first as the 'only if' part of Theorem 1 holds only for $n \geq 3$; however, to circumvent this problem, we may always interpret the two-player case as $n \geq 3$ where $n - 2$ players are inactive, which is the standard (implicit) solution in the literature as most axiomatizations of success functions require $n \geq 3$ while many contest games using these success functions focus on $n = 2$. Finally, while all our theorems and propositions focus on the form (3), we also argue that crucial results extend to many other success functions within the general class (2).

5.1 Existence and uniqueness of the equilibrium

Our theorem below provides a sufficient condition for the existence of an interior equilibrium, showing that there are no other symmetric equilibria and defining the corresponding effort level.

Theorem 2 *Let $g := f/f'$ and let g^{-1} denote the corresponding inverse function.*[21] *A symmetric equilibrium $\mathbf{x}_d^* = (x_d^*, \ldots, x_d^*) \in \mathbb{X}$ exists if (and only if, when f is*

[21] It is straightforward that g is invertible as it is strictly monotonic under our restrictions on f.

linear)

$$\delta > \max\left\{\frac{g^{-1}\left(k\,(n-1)/n^{k+1}\right)n^k - 1}{n\,(n^{k-1}-1)}, 1 - \frac{k+1}{n\,(k-1)}\right\}, \tag{4}$$

it is the only symmetric equilibrium, and x_d^* is implicitly defined by

$$g(x_d^*) = k\,(n-1)/n^{k+1}. \tag{5}$$

Equation (5) uniquely determines the equilibrium effort x_d^*. Moreover, as g is an increasing function by construction, x_d^* decreases in n, it is non-monotonic and strictly concave in k, and it is independent of δ. If we let $f(x) = x^\alpha$, g becomes linear and (5) simplifies to

$$x_d^* = \alpha k\,(n-1)/n^{k+1}, \tag{6}$$

giving an explicit solution for the equilibrium effort. Second, condition (4) defines a lower bound for the value of the draw-prize that guarantees the existence of the single symmetric equilibrium.[22] This lower bound is imposed by the positivity and the local concavity of the equilibrium payoff of each player, corresponding respectively to the first and the second entries in the right-hand side of (4), which we show to be sufficient for global optimality in the proof of Theorem 2.

Letting aside the exact value of the lower bound of δ, one important message of Theorem 2 is that, although equilibrium efforts are independent of the draw-prize, its value can be fundamental for the existence of the equilibrium. On the one hand, one can show that the existence of the equilibrium requires a positive draw-prize if the number of players is sufficiently large. On the other hand, for a broad set of configurations condition (4) holds even if $\delta = 0$, e.g., for the two-player case with $k \leq 2$.[23] As this case will be of special interest for the comparative statics in the next section, we state it formally in the corollary below.

Corollary 1 *Given $n = 2$ and $k \leq 2$, the symmetric equilibrium $\mathbf{x}_d^*$ exists for all $\delta \in [0, 1]$.*

With some caution, we now argue that the main results in Theorem 2 apply to a broader set of success functions that fall into the general class (2). With $p_i(\mathbf{x}) = \Gamma(s_i(\mathbf{x}))$, the first and second order conditions of i's payoff maximization are

$$\partial s_i(\mathbf{x}_d^*)/\partial x_i = 1/\Gamma'(1/n) \text{ and}$$

$$\Gamma'(1/n)\partial^2 s_i(\mathbf{x}_d^*)/\partial x_i^2 + [1 - \delta n/(n-1)]\Gamma''(1/n)\left(\partial s_i(\mathbf{x}_d^*)/\partial x_i\right)^2 < 0,$$

respectively. It is easy to verify that the second order condition always holds for δ sufficiently close to 1, and a higher n is likely to demand a higher δ for this condition

[22] Note that the term $g^{-1}\left(k\,(n-1)/n^{k+1}\right)$ in the first entry of (4) equals x_d^* by (5).

[23] More generally, assuming $f(x) = x^\alpha$ condition (4) holds for all $\delta \in [0, 1]$ if one of the following conditions is true: $k \in (1, (n+1)/(n-1)]$ given $\alpha \in (0, n/(n+1)]$; $k \in (1, n/\alpha(n-1)]$ given $\alpha \in (n/(n+1), 1]$.

to hold since the term $n/(n-1)$ decreases in n. The first order condition leads to a single symmetric equilibrium if the term $\partial s_i(\mathbf{x}_d^*)/\partial x_i$ is decreasing in x_d^*, which is guaranteed by the cross derivatives of s_i satisfying

$$\partial^2 s_i(\mathbf{x}_d^*)/\partial x_j \partial x_i \leq 0 \text{ for all } j \in N\backslash\{i\}. \tag{7}$$

This is a natural property shared by many (exhaustive) success functions requiring that, at any symmetric effort profile, if a single player marginally increases her effort no other player has an incentive to escalate (while instead she may either back down or keep her effort constant). Finally, the first order condition clearly demonstrates that the level of equilibrium effort x_d^* is independent of the value of the draw-prize as δ does not appear in either side of the equation.

Going back to the specific form (3), we conclude this subsection by presenting our results for the uniqueness of the equilibrium below. While Theorem 2 rules out symmetric equilibria other than $\mathbf{x}_d^*$, the following result additionally rules out asymmetric equilibria.

Proposition 1 *The symmetric equilibrium $\mathbf{x}_d^*$ is the unique equilibrium for:*

(i) $n = 2$ *and* $k < 2$*;*

(ii) $n = 2$ *and* $k = 2$ *if* $\delta \neq 0$ *or* $f'' \neq 0$*;*

(iii) $n \geq 2$ *if* $k \leq 1 - f''f/f'^2$.

Proposition 1 identifies sufficient conditions for the uniqueness of the equilibrium identified in Theorem 2, ruling out the possibility of asymmetric equilibria. In Points (*i*) and (*ii*), it shows that the symmetric equilibrium is almost always unique for the two-player case with $k \leq 2$ (our case of interest in Corollary 1), while in Point (*iii*), it further identifies a condition that applies to any number of players and coincides with $k \leq 1/\alpha$ when $f(x_i) = x_i^\alpha$.

5.2 Properties of the symmetric equilibrium

In this subsection we exclusively focus on parameter configurations such that condition (4) holds for all $\delta \in [0,1]$, so that the existence of the symmetric equilibrium is guaranteed for any draw-prize. We are particularly interested in the comparison between the equilibrium efforts in a contest with draw ($k > 1$) and in a contest without draw ($k = 1$) with the same impact function f (i.e., the corresponding benchmark defined in Section 3). In what follows, we denote by x_c^* the equilibrium effort for the contest without draw while we associate x_d^* with any value $k > 1$ of the elasticity parameter unless further restrictions on k are explicitly stated.

Proposition 2 *The symmetric equilibrium $\mathbf{x}_d^*$ satisfies $x_d^* > x_c^*$ ($x_d^* = x_c^*$) if and only if $n = 2$ and $k < 2$ ($k = 2$), and the combination of n and k that maximizes x_d^* is $n = 2$ and $k = 1/\ln 2 \simeq 1.44$.*

At first, Proposition 2 seems counterintuitive: the contest with draw results in greater equilibrium effort even when the draw-prize is null as long as $n = 2$ and k is sufficiently small. Note that when $\delta = 0$ the increase in equilibrium effort cannot be explained by the additional incentive created by the draw-prize as the payoff function of a player for the contest without draw is greater than for the contest with draw for any vector of efforts (by first-order stochastic dominance of the winning probability). Hence, in equilibrium, contestants exert more effort but earn less in expected terms (in Proposition 3, we show that this is a consequence of 'increased competition'). Our result is restricted to $n = 2$ (note the 'only if' statement in Proposition 2) as the growing number of players inflates the equilibrium probability of a draw, hampering the marginal return of effort in the symmetric equilibrium. In addition, Proposition 2 identifies the effort maximizing number of players and elasticity parameter for the contest with draw, which are $n = 2$ and $k \simeq 1.44$.[24]

We now argue that Proposition 2 applies to the broader set of success functions that fall into the general class (2) with $\Gamma(s_i(\mathbf{x})) = s_i(\mathbf{x})^k$ and $s_i(\mathbf{x})$ satisfying condition (7). Note that the left-hand side of the first order condition $\partial s_i(\mathbf{x}_d^*)/\partial x_i = 1/\Gamma'(1/n)$ decreases in x_d^* by (7) and the concavity of s_i in x_i. Under our restrictions, the right-hand side takes value n^{k-1}/k for the contest with draw and 1 for the benchmark. Then, $x_d^* > x_c^*$ if and only if $n^{k-1}/k < 1$, and our results in Proposition 2 easily follow.

Our next proposition demonstrates how the contest with draw yields greater equilibrium effort than the benchmark when $n = 2$ and $k \leq 2$. This is case in Corollary 1 which is of particular interest as, according to Proposition 2, it leads to an equilibrium effort which is higher than (equal to) the one with the benchmark if $k < 2$ ($k = 2$). Let us denote the best-response effort of player $i \in N$ to the rival's effort x_j by $x_{ic}^*(x_j)$ for the contest without draw ($k = 1$), and by $x_{id}^*(x_j)$ for the contest with draw ($k > 1$).

Proposition 3 *Given $n = 2$ and $k \leq 2$, the best-response effort of player $i \in N$ satisfies $x_{id}^*(x_j) > x_{ic}^*(x_j)$ for all $x_j < x_d^*$.*

Proposition 3 clarifies the result obtained in Proposition 2, and Figure 3 below visualizes the argument. Roughly speaking, as long as the rival's effort is less than the equilibrium value, the best-response to that effort is larger when a draw is possible. Intuitively, the marginal return of effort is greater as, besides increasing the probability of winning, higher effort also decreases the chance of a draw when the rival's effort is smaller (as the efforts further diverge), which is always the case in the considered domain.[25] While the first effect is present in the contest without draw, the second is clearly absent. Moreover, the first effect is stronger in the contest with draw as the convex transformation of the success function implies increasing

[24] It is worth mentioning that the estimated values of k in Yildizparlak (2018) are close to 1.44 in all prominent European soccer leagues.

[25] In the proof of Proposition 3, we show that the best-response is above the rival's effort if and only if the rival's effort is below the equilibrium value, that is, $x_{id}^*(x_j) > x_j$ if and only if $x_j < x_d^*$.

returns in effort on the probability of winning the contest. We have already es-

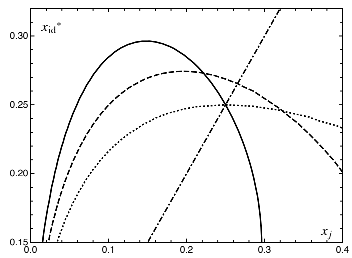

FIGURE 3. Best-response function of $i \in N$ ($n = 2$, $\delta = 0$, and f linear) corresponding to the cases: (——, $k = 2$); (- - - -, $k = 1.4$); (········, $k = 1$). The intersection of each best-response function with the 45° line (— · —) corresponds to the respective equilibrium effort. As we can see, the equilibrium effort for $k = 1.4$ is the greatest among these, while the ones for $k = 1$ and $k = 2$ are equal to each other.

tablished that equilibrium efforts are independent of the draw-prize in Theorem 2. This fundamental property of our model is illustrated in Figure 4 and the intuition is straightforward. We mentioned that with our class of success functions the probability of a draw is maximized at any symmetric effort profile. Then, as the equilibrium effort profile is symmetric, a player's effort has zero marginal impact on the probability of a draw at the equilibrium level, which implies that the value of the draw-prize does not affect the equilibrium effort. Having established this, we now show that in an extension of our basic model, where we introduce upper bounds to effort, a greater draw-prize has an adverse impact on aggregate effort in a two-player contest. Denoting by $\hat{x}_i > 0$ the upper bound to the effort of player $i \in N$, we say that player i is *constrained* if x_d^* exceeds this upper bound.

Proposition 4 *Let $n = 2$, $k \leq 2$ and a single player be constrained. Under the restrictions that guarantee uniqueness of an equilibrium in Points (i) and (ii) of Proposition 1, the aggregate equilibrium effort decreases (increases) with a marginal increase (decrease) in the draw-prize.*

Even though Proposition 4 may seem counterintuitive at first, it has a straightforward explanation: a larger draw-prize results in seeking a draw, which is most likely at symmetric effort profiles. In the initial equilibrium with the smaller draw-prize, the unconstrained player exerts higher effort than the constrained player. Then, in the new equilibrium induced by the marginal increase in the draw-prize, the unconstrained player profits from decreasing the effort gap by reducing her effort compared to the previous equilibrium. On the other hand, the constrained player would like to decrease the effort gap by increasing her effort too, but the constraint renders it impossible and consequently aggregate effort declines. The case for the marginal decrease in the draw-prize is similarly explained. This basic mechanism is illustrated in Figure 4, which shows that for any rival effort $x_j < (>) x_d^*$ the best-response effort $x_{id}^*(x_j)$ decreases (increases) in δ.[26] There is a pertinent application of Proposition 4 to the regulation/design of soccer matches: until the 1980s points in soccer matches

[26] This is shown formally by equation (27) in the proof of Proposition 3.

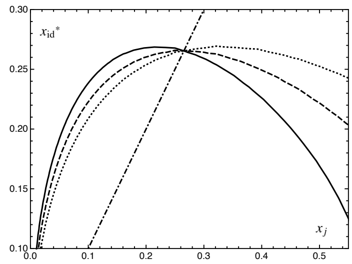

FIGURE 4. Best-response function of $i \in N$ ($n = 2$, $k = 1.4$, and f linear) corresponding to the cases: (——, $\delta = 0.2$); (- - - -, $\delta = 0.5$); (········, $\delta = 0.8$). The intersection of each best-response function with the 45° line (— · —) corresponds to the respective equilibrium effort, which as we can see is independent of δ.

were allocated applying the 'two points for a win' policy, i.e., the winner of a match obtained two points, the loser obtained nothing, and a draw meant one point for each team. However, through the 1980s this rule has been slowly replaced by the 'three points for a win', which increases the win-points to three while keeping the points of the remaining outcomes constant. It has been argued that the motivation for the change was to increase physical competition between teams in soccer matches (see, e.g., Murray and Ingle, 2001; Wilson, 2007). Modeling a soccer match as a two-player contest with draw, we can interpret the policy change as a decrease in the draw-prize δ from 1/2 to 1/3, everything else equal.[27] Then, Proposition 4 predicts this change of rule to induce greater aggregate effort as long as teams are asymmetrically constrained, which seems reasonable given the great inequality in the distribution of talent across teams (see Yildizparlak, 2018).

5.3 Properties of equilibrium rent dissipation

The results we derived so far suggest that, under certain conditions, a contest designer may harvest greater effort by introducing the possibility of a draw in an exhaustive contest. Besides maximizing aggregate effort (which in sports may be interpreted as the entertainment value of a match, in R&D as the total research time put into the project , and so on), a contest designer may be interested in maximizing rent dissipation, which measures aggregate effort as a fraction of the value of the contested prizes that she delivers to the contestants in the occurrence of a victory or a draw. Intuitively, rent dissipation can be interpreted as the rate of return of the contest designer, i.e., the revenues (given by the contestants' aggregate effort) divided by the costs (given by the awarded prizes).[28] Note that, in our setup, the value of the awarded prizes depends on the occurrence of the draw, as it is 1 if there is a winner and $n\delta$ if there is a draw. An obvious way to tackle this issue is to define

[27]Note that points in soccer have relative value only, as all that matters is to accumulate larger points than the opponents. Then, it is legitimate to normalize the value of victory to 1.

[28]Analogous results can be obtained by employing alternative objectives of the designer such as profit (i.e., revenues minus costs).

the value of the prize in expected terms. Given any $\mathbf{x} \in \mathbb{X}$, we define the rate of (expected) rent dissipation by

$$R(\mathbf{x}) := \frac{\sum_{i \in N} x_i}{1 + p_d(\mathbf{x})(n\delta - 1)},$$

which is aggregate effort divided by the expected value of the awarded prize given the effort profile $\mathbf{x}$. By (5), the equilibrium rate of rent dissipation for the contest without draw is $R(\mathbf{x}_c^*) = nx_c^*$, while for the contest with draw it is $R(\mathbf{x}_d^*) = nx_d^*/(1 + p_d(\mathbf{x}_d^*)(n\delta - 1))$. Our next proposition compares the equilibrium rate of rent dissipation for the two contests with any number of players. In order to accommodate for the cases where the prize has economic value (e.g., monetary rewards in promotion contests, as opposed to scores or points in sports) we impose the feasibility constraint $\delta \leq 1/n$, so that the aggregate prize allocated in case of a draw is not larger than the win-prize.

Proposition 5 *Let $f(x) = x^\alpha$ with $\alpha \in (0, 1]$. In a symmetric equilibrium, $R(\mathbf{x}_d^*) \geq R(\mathbf{x}_c^*)$ if and only if*

$$\delta \leq \min \left\{ \frac{k - 1}{n(n^{k-1} - 1)}, \frac{1}{n} \right\}. \tag{8}$$

Roughly speaking, Proposition 5 shows that the contest with draw can lead to higher rent dissipation as long as the draw-prize is not too large (for obvious reasons), where a higher k leads to more rent dissipation due to the increased chances of a draw and the higher equilibrium efforts. On this point, it is noteworthy that if k is too large condition (4) for the existence of the equilibrium may be violated, where the relevant threshold decreases with the number of players n. Figure 5 visualizes the broad set of parameter configurations that guarantee both existence and higher rent dissipation. For instance, this is true given $n = 2$ for all $k \leq 2$ and $\delta \in [0, 1/2]$. All in all, we

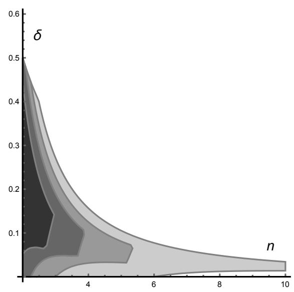

FIGURE 5. Parameter configurations for which rent dissipation is higher for the contest with draw than without draw, given $f(x) = x^\alpha$ with $\alpha = 1$ (for $\alpha < 1$ the corresponding parameter configurations are supersets of these). Note that the horizontal axis starts at $n = 2$. The areas from light-grey to black correspond to $k = 1.2$, $k = 1.5$, $k = 1.8$, and $k = 2.3$, respectively. The darker colored sets are subsets of the lighter colored ones. The boundaries are dictated by condition (8) and the existence condition (4).

conclude that Proposition 5 strengthens the general message that a contest designer may gain from introducing the possibility of a draw in an exhaustive contest, which can be done in a simple and cost-effective way. Let us go back to the example of the promotion contest discussed in Introduction. By Figure 5 we can argue that, by changing the rules of the game so that the two candidates ($n = 2$) are required to prevail in two tasks ($k = 2$) to be promoted and allocating a null draw-prize otherwise ($\delta = 0$), the contest designer extracts greater equilibrium rent dissipation than in the case of a single task ($k = 1$).[29]

6 Extension to multiple draws

In this section we present an extension of our model to allow for multiple draws that can occur across different subsets of the set of players. Generalizing our previous setting, we consider a contest with n win-prizes and n draw-prizes where: (i) in the occurrence of a win, one of the win-prizes is assigned to some player and this player is excluded from receiving any other win-prize; (ii) in the occurrence of a draw, one of the win-prizes is not delivered to any player, one of the draw-prizes is delivered to each of the players involved in the draw, and some player is randomly excluded from receiving any win-prize. Although these modeling choices are quite general and arguably well-suited to represent a broad set of environments (see our discussion later on), they are not completely innocent (in particular the random exclusion condition) and our analysis is far from a general treatment of the subject of multiple draws in multi-prize contests. The aim of this section is only to prove the versatility of the weak exhaustivity approach developed in this paper, showing that it can be applied to much more general settings still maintaining tractability and intuition.

We formalize an outcome of our generalized contest as a mapping $\tau : N \to \{0,1\} \times N$ where, for each $l \in N$ and $i \in N$, $\tau(l) = (1, i)$ indicates that win-prize l is assigned to player i, while $\tau(l) = (0, i)$ indicates that there is a draw corresponding to win-prize l and player i is excluded from receiving any win-prize. Denoting by $\mathcal{T}$ the set of all possible outcomes, a success function $p : \mathcal{T} \times \mathbb{X} \to [0, 1]$ defines a probability distribution over the set of possible outcomes for each effort profile, where $p(\tau, \mathbf{x})$ indicates the probability of outcome $\tau \in \mathcal{T}$ given the effort profile $\mathbf{x} \in \mathbb{X}$. Applying the weak exhaustivity approach, in what follows we focus on a class of success functions that extends our general form (2) to this broader set of contests. The success functions within this class are particularly appealing as they generalize the well-known multi-prize contest model by Clark and Riis (1996), thus being intuitively

[29] It is easy to verify that, if we introduce a budget constraint for the designer so that the sum of the win-prize and the draw-prize is fixed while their values are free to choose, it is always optimal for the maximization of equilibrium rent dissipation to allocate the whole budget to the win-prize as long as the condition for the existence of the equilibrium holds. On the other hand, Figure 5 is suggestive of a trade-off between guaranteeing the existence of the equilibrium (high δ) and maximizing equilibrium rent dissipation conditional on the existence of such equilibrium (low δ). While this trade-off is absent for $n = 2$ and $k \leq 2$, it always kicks in if n or k are large enough so that the existence of the equilibrium requires a positive δ.

interpretable in terms of allocation of prizes by a sequential procedure (we elaborate on this later on).

Let the win-prizes be ranked from best ($l = 1$) to worst ($l = n$). For each $l \in N$ and $\tau \in \mathcal{T}$, denote by $w_{l,\tau} \in N \cup \{0\}$ the player that receives the win-prize ranked l in outcome τ (where $w_{l,\tau} = 0$ indicates the occurrence of a draw for such win-prize), and let $N_{l,\tau} \subseteq N$ be the set of players who in outcome τ do not receive win-prizes ranked above l and are not excluded by draws corresponding to win-prizes ranked above l.[30] We denote by $\mathcal{N}$ the set of all such sets of players. In what follows, we consider the broad class of success functions that take the form

$$p(\tau, \mathbf{x}) = \prod_{l \in \{l' :\, w_{l',\tau} \neq 0\}} \Gamma\left(\sigma_{w_{l,\tau}}(N_{l,\tau}, \mathbf{x})\right) \prod_{l \in \{l' :\, w_{l',\tau} = 0\}} \frac{1}{|N_{l,\tau}|} \left(1 - \sum_{j \in N_{l,\tau}} \Gamma\left(\sigma_j(N_{l,\tau}, \mathbf{x})\right)\right) \quad (9)$$

for some function $\Gamma : [0,1] \to [0,1]$ that is increasing, strictly convex and twice-differentiable with $\Gamma(0) = 0$ and $\Gamma(1) = 1$, and some function $\sigma_i : \mathcal{N} \times \mathbb{X} \to [0,1]$ which is twice-differentiable in each effort in the interior of $\mathbb{X}$ and it satisfies each of the following properties: $\sum_{i \in N_{l,\tau}} \sigma_i(N_{l,\tau}, \mathbf{x}) = 1$ for each $l \in N$, $\tau \in \mathcal{T}$ and $\mathbf{x} \in \mathbb{X}$; it is anonymous (i.e., invariant to permutations of players' identities), increasing, and strictly concave in x_i at each $\mathbf{x} \in \mathbb{X}$ with $i, j \in N_{l,\tau}$ such that $i \neq j$ and $x_i, x_j > 0$; $\sigma_i(N_{l,\tau}, \mathbf{x}) = 0$ if and only if either $i \notin N_{l,\tau}$ or $x_j > x_i = 0$ with $i, j \in N_{l,\tau}$. In line with our interpretation of (2), we call Γ the transformation function and we refer to σ_i as the extended ghost success function. For any given pair of functions Γ and σ_i, the corresponding benchmark is the success function taking the form (9) with $\Gamma(\sigma_i) = \sigma_i$ and the same function σ_i, taking value

$$p(\tau, \mathbf{x}) = \prod_{l \in N} \sigma_{w_{l,\tau}}(N_{l,\tau}, \mathbf{x}) \quad (10)$$

if $w_{l,\tau} \in N$ and $p(\tau, \mathbf{x}) = 0$ if $w_{l,\tau} = 0$, as draws occur with zero probability in the benchmark. Note that, if the extended ghost success function belongs to the class axiomatized in Skaperdas (1996), i.e., $\sigma_i(N_{l,\tau}, \mathbf{x}) = f(x_i)/\sum_{j \in N_{l,\tau}} f(x_j)$ for some positive and increasing function f whenever $i \in N_{l,\tau}$ and $x_i > 0$, (10) coincides with the class of best-shot success functions for multi-prize contests introduced in Clark and Riis (1996) and axiomatized in Lu and Wang (2015).

Let us briefly discuss the intuition behind our model. As pointed out in Clark and Riis (1996), equation (10) can be interpreted as the outcome of a *sequential procedure* where prizes are allocated from best to worst to the winners of a sequence of exhaustive stage-contests restricted to players that have not been awarded any win-prize yet. Our class of success functions (9) is subject to the very same interpretation while relaxing the exhaustivity restriction on the stage-contests, so that each stage-contest can end up in a draw. Specifically, in our extension of the framework we assume that players cannot participate to a stage-contest subsequent to l if they have already received win-prizes ranked weakly above l, or if they have been already

[30]Note that $|N_{l,\tau}| = n + 1 - l$ and $N_{l+1,\tau} \subset N_{l,\tau}$ for each $\tau \in \mathcal{T}$ and $l \in N$.

excluded by the occurrence of draws related to such win-prizes (where such exclusion occurs randomly). To fully grasp the intuition of this sequential procedure, note that with $n = 3$ we have the following three stages: (Stage 1) a single-winner three-player contest with draw possibility that allocates the best win-prize to the winner; (Stage 2) a single-winner two-player contest with draw possibility that allocates the second best win-prize to the winner and it is restricted to the two players that have neither won the first stage nor have been excluded by the occurrence of a draw in the first stage; (Stage 3) the player that has neither won the first or second stage nor has been excluded by the occurrence of a draw in the first or second stage receives the worst win-prize.[31] As mentioned in Introduction, an example of real-life contest organized (roughly) along these lines is the political campaigning across political parties for a runoff election such as the one to elect the president in many countries.[32]

Letting $v_l \geq 0$ and $\delta_l \geq 0$ respectively denote the values of the win-prize and of the draw-prize corresponding to $l \in N$, we can generalize our previous definition of player $i \in N$'s payoff as follows:

$$\pi_i(\mathbf{x}) = \sum_{l \in N} \sum_{\tau \in \left\{\tau' :\, w_{l,\tau'} = i\right\}} p(\tau, \mathbf{x}) v_l + \sum_{l \in N} \sum_{\tau \in \left\{\tau' :\, i \in N_{l,\tau'} \,\wedge\, w_{l,\tau'} = 0\right\}} p(\tau, \mathbf{x}) \delta_l - x_i,$$

where our baseline setting corresponds to $v_1 = 1 \geq \delta_1 \geq v_l = \delta_l = 0$ for all $l \geq 2$.

Although a complete analysis of this model is beyond the scope of this paper, we now show that all our core intuitions extend to this generalization. First, it is straightforward that the existence of a symmetric interior equilibrium is guaranteed under conditions analogous to the ones in Theorem 2 given $v_1 = 1 \geq \delta_1 \geq 0$ as long as $v_l, \delta_l \simeq 0$ for $l \geq 2$. Second, assuming the existence of such equilibrium, the corresponding equilibrium effort (denoted by x_d^*, with some abuse of notation) is determined by the first order condition

$$1 = \sum_{l \in N} \Gamma'(1/(n+1-l))(v_l - v_n) \partial \sigma_i(M_{n+1-l}, \mathbf{x}_d^*)/\partial x_i, \tag{11}$$

where M_{n+1-l} is any set of $n+1-l$ players that contains i and $\mathbf{x}_d^*$ denotes the equilibrium effort profile. Note that (11) is independent of δ_l for all $l \in N$, therefore x_d^* is independent of the values of the draw-prizes. Assuming that $\sigma_n(M_{n+1-l}, \mathbf{x}_d^*)$ satisfies condition (7) on the cross derivatives with respect to efforts, $\partial \sigma_i(M_{n+1-l}, \mathbf{x}_d^*)/\partial x_i$ decreases in x_d^* and there is at most a single symmetric equilibrium.

[31] By the functional form (9), a draw occurs with zero probability if there is only one participant in the stage-contest, therefore the last draw-prize δ_n is never allocated in our model.

[32] In this specific example a draw may occur in the first stage of the election if no candidate achieves an absolute majority, but there is no draw possibility in the second stage as the president is elected by absolute majority between the two candidates that accessed the second stage. To allow for heterogeneity of the likelihood of a draw across different stages, one could generalize equation (9) by allowing a different transformation function Γ_l for each $l \in N$.

As in our previous analysis, we now impose some more restrictions to sharpen our predictions on the comparative statics. Letting $\Gamma(\sigma_i) = \sigma_i^k$ for some $k > 1$ and $\sigma_i(M_{n+1-l}, \mathbf{x}) = f(x_i)/\sum_{j \in M_{n+1-l}} f(x_j)$ if $x_i > 0$ for some function f that is positive, twice-differentiable, increasing and concave, by (11) we obtain

$$g(x_d^*) = k \sum_{l \in N} \frac{(n-l)(v_l - v_n)}{(n+1-l)^{k+1}}, \tag{12}$$

where $g = f/f'$ is the increasing function defined in Theorem 2. Note that x_d^* is decreasing in n and non-monotonic in k as in our previous analysis, and it is decreasing in v_n and increasing in v_l for all $l \neq n$. Moreover, if a contest designer had a fixed budget of value 1 to be distributed across non-negative prizes to provide incentives to maximize equilibrium effort, it is straightforward that she should allocate the whole budget to v_1 since the series $\left\{(n-l)/(n+1-l)^{k+1}\right\}_{l=1}^{n}$ decreases in l and the level of equilibrium effort is independent of the draw-prizes (as long as the conditions for the existence of the equilibrium are met). Note that, with this prize configuration, the general model reduces to the one analyzed in the previous sections and all our previous results equally apply.

7 Conclusion

In this paper we introduce and axiomatize a class of success functions for single-winner contests with draw which generalizes the well-known exhaustive class of success functions axiomatized in Skaperdas (1996). Our characterization is based on five standard axioms plus a crucial one, weak exhaustivity, which requires winning probabilities to be a convex transformation of an exhaustive success function. Our approach differs fundamentally from the literature in the way we introduce the possibility of a draw: while other models incorporate a draw in reference to a performance gap, we do so by scaling down the winning probabilities of an exhaustive contest. Our model is particularly suited to represent contests where winning requires supremacy in multiple tasks, or where the success of a contestant is indirectly determined by a distribution of scarce resources such as ball possession in soccer, territory in warfare, audience's attention in litigation or marketing, etc.

The analysis of a contest game using our class of success functions shows that, under fairly general conditions, there is a unique symmetric interior equilibrium and equilibrium efforts do not depend on the draw-prize, although the value of the draw-prize can affect the existence of the equilibrium when the number of players is high. Particularly for the two-player contest, equilibrium efforts can be greater compared to the contest with no draw even when the draw-prize is null. We show that the difference in effort between the two classes of contests results from higher competition instead of the addition of a draw-prize. We further show that if contestants have asymmetric upper bounds to effort, larger draw-prizes can have adverse effects on the aggregate effort. We argue this may explain a change of rule in soccer in the

1980s that reduced the points of a draw relative to the points of victory to foster competition between teams. Although the introduction of a draw hampers equilibrium efforts when there are more than two contestants in our model, we show that equilibrium rent dissipation is systematically higher in the contest with draw compared to the benchmark case for a broad set of parameter configurations. Then, introducing a draw associated with a relatively small or null prize may enable a contest designer to achieve greater profit, which in practice can be achieved by requiring contestants to prevail in two tasks to avoid the draw. This simple and cost-free change of rule can be seen as complementary to the positive discrimination approach to contest design, where rent dissipation is augmented by biasing the contest in favor of ex-ante weaker players (see, e.g., Epstein et al., 2011; Franke, 2012; Mealem and Nitzan, 2016), or to 'pure' discrimination in environments where biasing the contest in favor of certain participants can induce higher aggregate equilibrium effort even though players are otherwise ex-ante symmetric (see Drugov and Ryvkin, 2017).

Finally, in an extension of our framework we show that our methodology can be extended to multi-prize contests by introducing the possibility of multiple draws across different subsets of the set of players. All our core intuitions on equilibrium behavior extend to this broader class of contests with draws, where the extended model coincides with our baseline model under the prize configuration that maximizes equilibrium efforts. This generalized framework presents broad opportunities for future research in the narrow but growing literature on contests with draws. In particular, one crucial question is how the existence of an equilibrium may be affected by the multiple draw-prizes even though equilibrium efforts are independent of them.

Appendix

In the proofs of Theorem 2 and Propositions 1 and 3, for brevity we suppress the argument of $f(x_i)$ and denote by f_i the impact function of player $i \in N$, also using the simplified notation $F = \sum_{j \in N} f_j$, $F_{\neg i} = \sum_{j \in N \setminus \{i\}} f_j$, and $\hat{F}_{\neg i} = \sum_{j \in N \setminus \{i\}} f_j^k$.

Proof of Theorem 1

It is straightforward that the form (3) satisfies all our axioms for any $n \geq 2$. Let us show that, if a success function satisfies the axioms, it must take the form (3) given $n \geq 3$. Let $\mathbf{x} \in \mathbb{X}$ be any effort profile with $A_{\mathbf{x}} \neq \emptyset$. By perfect discrimination at zero, $p_i(\mathbf{x}) > 0$ if $i \in A_{\mathbf{x}}$. By effort independence, $\frac{p_i(\mathbf{x})}{p_j(\mathbf{x})} = \frac{p_i(\mathbf{x}')}{p_j(\mathbf{x}')}$ for all $i, j \in A_{\mathbf{x}}$ and $\mathbf{x}' \in \mathbb{X}$ such that $x_i = x'_i$ and $x_j = x'_j$. Then, by anonymity

$$\frac{p_i(\mathbf{x})}{p_j(\mathbf{x})} = \phi(x_i, x_j) \text{ for some function } \phi : \mathbb{R}_{++} \to \mathbb{R}_{++}.$$

Let $h \in A_{\mathbf{x}'} \setminus \{i, j\}$ for some $\mathbf{x}' \in \mathbb{X}$ such that $x_i = x'_i$ and $x_j = x'_j$. (This requires

$n \geq 3$.) Then, we can write

$$\phi(x_i, x_j) = \frac{p_i(\mathbf{x})}{p_j(\mathbf{x})} = \left(\frac{p_i(\mathbf{x}')}{p_h(\mathbf{x}')}\right) / \left(\frac{p_j(\mathbf{x}')}{p_h(\mathbf{x}')}\right) = \frac{\phi(x_i, x'_h)}{\phi(x_j, x'_h)}.$$

Note that $\phi(x_i, x'_h) / \phi(x_j, x'_h)$ must be independent of x'_h (as it is equal to $\phi(x_i, x_j)$). Then, defining $\varphi(z) = \phi(z, 1)$ for all $z > 0$, we obtain

$$\frac{p_i(\mathbf{x})}{p_j(\mathbf{x})} = \frac{\varphi(x_i)}{\varphi(x_j)}. \tag{13}$$

Let $\mathbf{x} \in \mathbb{X}$ be such that $A_{\mathbf{x}} = \{i\}$ for some $i \in N$. By Point (ii) of weak exhaustivity 1 there is a continuous and increasing function $r : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $\mu > 0$

$$\frac{r(p_i(\mathbf{x}))}{r(1)} = \frac{r(p_i(\mathbf{x})\mu)}{r(\mu)}, \text{ hence } r(p_i(\mathbf{x})\mu)\, r(1) = r(p_i(\mathbf{x}))\, r(\mu).$$

By perfect discrimination at zero $p_i(\mathbf{x}) > 0$ as we assumed $i \in A_{\mathbf{x}}$. Then, by Theorem 3 at p. 41 of Aczél (1966) the function r must take the form

$$r(z) = \alpha z^{\beta} \text{ for all } z > 0 \text{ and some } \alpha > 0 \text{ and } \beta > 0, \tag{14}$$

and by continuity of r we must have $r(0) = 0$. Note that Point (ii) of weak exhaustivity 2 leads to the very same conclusion.[33] Let $\mathbf{x} \in \mathbb{X}$ be any effort profile. Combining Point (i) of weak exhaustivity (in either form) with (14) and perfect discrimination at zero we obtain

$$\sum_{i \in A_{\mathbf{x}}} p_i(\mathbf{x})^{\beta} = 1 \text{ if } A_{\mathbf{x}} \neq \emptyset \text{ and } \sum_{i \in N} p_i(\mathbf{x})^{\beta} = 1 \text{ if } A_{\mathbf{x}} = \emptyset. \tag{15}$$

By probability $\sum_{i \in N} p_i(\mathbf{x}) \leq 1$, which implies $\beta \in (0, 1]$. Let $i \in A_{\mathbf{x}}$ be any active player. Combining (15) with (13) we get $\sum\limits_{j \in A_{\mathbf{x}}} \left(\frac{\varphi(x_j)}{\varphi(x_i)}\right)^{\beta} = \frac{1}{p_i(\mathbf{x})^{\beta}}$, thus

$$p_i(\mathbf{x}) = \varphi(x_i) / \left(\sum_{j \in A_{\mathbf{x}}} \varphi(x_j)^{\beta}\right)^{1/\beta}, \text{ where } \beta \in (0, 1]. \tag{16}$$

It is easy to verify that, by strict monotonicity and (16), the function φ must be increasing. Let $k := 1/\beta$ and $f(z) := \varphi(z)^{\beta}$. As $\beta \in (0, 1]$ and φ is positive and increasing, then $k \geq 1$ and the function f is positive and increasing. Then, we have shown that $p_i(\mathbf{x})$ must take the form (3) whenever player i is active, that is, for

[33] By Point (ii) of weak exhaustivity 2 there is a differentiable and increasing function $r : \mathbb{R}_+ \to \mathbb{R}_+$ such that $r'(p_i(\mathbf{x}))p_i(\mathbf{x}) / r(p_i(\mathbf{x})) = \beta$ for some $\beta \in \mathbb{R}$. This can be rewritten as $r'(p_i(\mathbf{x}))/r(p_i(\mathbf{x})) = \beta / p_i(\mathbf{x})$, and integrating both sides we obtain $\ln(r(p_i(\mathbf{x}))) = \beta \ln(p_i(\mathbf{x})) + \tilde{\alpha}$ for some $\tilde{\alpha} \in \mathbb{R}$. Then, applying an exponential transformation we have $r(p_i(\mathbf{x})) = \alpha p_i(\mathbf{x})^{\beta}$ where $\alpha = \exp(\tilde{\alpha}) > 0$. Note that $\beta > 0$ as r must be increasing, and by differentiability of r we obtain $r(0) = 0$.

all $\mathbf{x} \in \mathbb{X}$ with $i \in A_{\mathbf{x}}$. To conclude our proof, it remains to be shown that this is true also when i is inactive. Let $\mathbf{x} \in \mathbb{X}$ be such that all players are inactive, so that $x_j = 0$ for all $j \in N$. As $\mathbf{x} = \mathbf{x}^{\rho}$ for any permutation of players' identities ρ, by anonymity $p_j(\mathbf{x}) = p_h(\mathbf{x})$ for all $j, h \in N$. Then,

$$p_i(\mathbf{x}) = (1/n)^k$$

by (15), which implies $p_i(\mathbf{x})$ takes the form (3) for this case. The only remaining case is when player i is inactive and some other player is active. Let $\mathbf{x} \in \mathbb{X}$ be such that $i \notin A_{\mathbf{x}}$ and $A_{\mathbf{x}} \neq \emptyset$. Then, by perfect discrimination at zero $p_i(\mathbf{x}) = 0$, hence $p_i(\mathbf{x})$ takes the form (3). □

Proof of Theorem 2

If the effort profile $\mathbf{x}_d^* \in \mathbb{X}$ is an interior equilibrium, the effort of each player $i \in N$ must satisfy the first order condition (FOC)

$$\frac{\partial \pi_i(\mathbf{x}_d^*)}{\partial x_i} = \frac{k f_i'}{F^{k+1}} \left[(1-\delta) f_i^{k-1} F_{\neg i} + \delta \hat{F}_{\neg i} \right] - 1 = 0, \tag{17}$$

which follows from (1). By symmetry of equilibrium efforts, $x_i = x_d^*$ for all $i \in N$, we obtain

$$g(x_d^*) = f(x_d^*)/f'(x_d^*) = k(n-1)/n^{k+1}. \tag{18}$$

Recall that by assumption $f(x_i)$ is increasing and concave, $\lim_{x_i \to 0} f(x_i) = 0$, and $\lim_{x_i \to +\infty} f(x_i) = +\infty$. Then, as $k(n-1)/n^{k+1} > 0$, there is a unique x_d^* satisfying (18). Hence, there is at most one symmetric interior equilibrium, and by definitions (1) and (3), in such equilibrium $p_i(\mathbf{x}_d^*) = 1/n^k$, $p_d(\mathbf{x}_d^*) = 1 - 1/n^{k-1}$, and

$$\pi_i(\mathbf{x}_d^*) = \frac{1 + \delta n \left(n^{k-1} - 1\right)}{n^k} - g^{-1}\left(k(n-1)/n^{k+1}\right), \tag{19}$$

where g^{-1} is the inverse function of g. It is easy to verify that the symmetric effort profile where all players are inactive cannot be an equilibrium, as any player has an incentive to deviate by exerting an arbitrarily small effort. So, we conclude that there is at most one symmetric equilibrium and that this equilibrium must be in the interior. Having established this, we now identify conditions for the existence of such equilibrium, that is, for the effort level x_d^* to be a global maximizer of the payoff function of each player given that other players' efforts are equal to x_d^*.[34] For any effort profile $\mathbf{x} \in \mathbb{X}$ in the interior, the second derivative of the payoff function of player $i \in N$ is

$$\begin{aligned} \frac{\partial^2 \pi_i(\mathbf{x})}{\partial x_i^2} =& \frac{(1-\delta) F_{\neg i}}{F^{k+2}} \left[f_i'' f_i^{k-1} F + f_i'^2 \left(f_i^{k-2} (k-1) F_{\neg i} - 2 f_i^{k-1} \right) \right] \\ &+ \frac{\delta \hat{F}_{\neg i}}{F^{k+2}} \left[f_i'' F - (k+1) f_i'^2 \right], \end{aligned} \tag{20}$$

[34] As the payoff function (1) is not necessarily concave globally, the FOC may not identify a global maximizer even if the second order condition $\partial^2 \pi_i(\mathbf{x}_d^*)/\partial x_i^2 < 0$ holds.

which evaluated at the symmetric effort profile $x_j = x_d^*$ for all $j \in N$ renders the second order condition (SOC)

$$nf_i'' + \frac{f_i'^2}{f_i}\left[(1-\delta)\left((k-1)(n-1)-2\right)-\delta(k+1)\right] < 0. \tag{21}$$

Note that in equation (20):

- the terms $(1-\delta)F_{\neg i}/F^{k+2}$ and $\delta\hat{F}_{\neg i}/F^{k+2}$ are positive and decrease at the same rate in x_i;

- the term $f_i'^2\left(f_i^{k-2}(k-1)F_{\neg i} - 2f_i^{k-1}\right)$ is negative for sufficiently large x_i;

- the terms $f_i'' f_i^{k-1}F$ and $\left[f_i''F - (k+1)f_i'^2\right]$ are negative.

Given $x_j = x_d^*$ for all $j \neq i$ one can show that, on the interval of x_i's values where i's payoff is non-negative, if (20) changes sign it does so at most once (from positive to negative), and the negativity of (20) at the point where the FOC is satisfied (i.e., the SOC) is necessary and sufficient for the global optimality of x_d^* in the interior.[35] In (21), the term nf_i'' is weakly negative while the term $f_i'^2/f_i$ is positive. Thus, a sufficient (and necessary, if f is linear) condition for the SOC to hold is the negativity of the term in squared parenthesis, which with some algebra reduces into

$$\delta > 1 - (k+1)/n(k-1). \tag{22}$$

Last, for x_d^* to be an equilibrium strategy we need $\pi_i\left(x_{id}^*, \mathbf{x}_{\neg id}^*\right) > \pi_i\left(0, \mathbf{x}_{\neg id}^*\right)$, where $\mathbf{x}_{\neg id}^*$ denotes the profile of equilibrium efforts of all players but i. Then, using (19) we get

$$\delta > \frac{g^{-1}\left(\frac{k(n-1)}{n^{k+1}}\right)n^k - 1}{n\left(n^{k-1}-1\right)}, \tag{23}$$

and the combination of (22) and (23) concludes our proof. □

Proof of Proposition 1

Points (i) and (ii): Let $n = 2$. It is easy to verify that there are no equilibria where some players are inactive, as any inactive player has an incentive to deviate. So, an equilibrium is necessarily in the interior. By the FOC in (17), in any interior equilibrium we must have

$$f_i'\left[f_i^{k-1}f_j + \delta f_j\left(f_j^{k-1} - f_i^{k-1}\right)\right] = f_j'\left[f_j^{k-1}f_i + \delta f_i\left(f_i^{k-1} - f_j^{k-1}\right)\right],$$

[35] For the values of x_i such that i's payoff is non-negative, it is easy to verify that there is a threshold $t \geq 0$ such that (20) is negative if and only if $x_i > t$. Depending on the exact parameter configuration and the third derivative of f, there can be an interval of values of x_i in a neighborhood of zero for which the payoff of a player is negative. On this interval, the second derivative (20) evaluated at $x_j = x_d^*$ for all $j \neq i$ can change sign from negative to positive with increasing x_i, but this is irrelevant for our purposes as none of such x_i's values can be a global maximum since $x_i = 0$ delivers a higher payoff than any of them.

which can be rewritten as

$$f_i f_j \left(f_i' f_i^{k-2} - f_j' f_j^{k-2}\right) = \delta \left(f_i^{k-1} - f_j^{k-1}\right)(f_i - f_j). \tag{24}$$

Condition (24) holds if $f_i = f_j$ since both sides of (24) are zero. For a contradiction, suppose that $f_i > f_j$. If $k < 2$, the left-hand side of (24) is strictly negative as $f_i' \le f_j'$ by the concavity of f and $f_i^{k-2} < f_j^{k-2}$, whereas the right-hand side of it is non-negative, leading to a contradiction.

If $k = 2$, both sides of (24) are zero also whenever $\delta = 0$ and $f'' = 0$. So, there may exist asymmetric equilibria if both these conditions hold. Let at least one of them not to hold, so that given $f_i \ne f_j$ both sides of (24) are never zero simultaneously. Under this restriction, we want to show that there are no asymmetric equilibria as $f_i \ne f_j$ is incompatible with (24). Supposing $f_i > f_j$ again, the right-hand side of (24) is non-negative for all $\delta \in [0, 1]$, and the left-hand side is non-positive as we have $f_i' \le f_j'$ by the concavity of f, and $f_i^{k-2} = f_j^{k-2}$. As the sides of the equation (24) cannot be simultaneously zero under our restrictions, we then have a contradiction.

As we have already proven $\mathbf{x}_d^*$ to be the single symmetric equilibrium in Theorem 2, this concludes the proof of Points (i) and (ii).

Point (iii): Let $i, j \in N$ be any pair of players, $F_{\neg\{i,j\}} := \sum_{\ell \ne i,j} f_\ell$, and $\hat{F}_{\neg\{i,j\}} := \sum_{\ell \ne i,j} f_\ell^k$. By the FOC in (17), in any interior equilibrium we must have

$$(1-\delta)\left[F_{\neg\{i,j\}}\left(f_i' f_i^{k-1} - f_j' f_j^{k-1}\right) + f_i f_j \left(f_i' f_i^{k-2} - f_j' f_j^{k-2}\right)\right]$$
$$= \delta \left[\left(f_j' f_i^k - f_i' f_j^k\right) + \hat{F}_{\neg\{i,j\}}\left(f_j' - f_i'\right)\right]. \tag{25}$$

Define $\nu(x) := f' f^{k-1}$ and note that

$$\nu'(x) = f'' f^{k-1} + (k-1) f'^2 f^{k-2} \le 0 \Leftrightarrow k \le 1 - f'' f / f'^2. \tag{26}$$

The term $\left(f_i' f_i^{k-2} - f_j' f_j^{k-2}\right)$ in the square brackets on the left-hand side of (25) equals $\nu(x_i)/f_i - \nu(x_j)/f_j$, where $\nu(x_i)/f_i$ is strictly decreasing if ν is weakly decreasing as

$$d[\nu(x_i)/f_i]/dx_i = \left[\nu'(x_i) f_i - \nu(x_i) f_i'\right]/f_i^2.$$

Suppose $f_i > f_j$ and $\delta \in (0, 1)$. The right-hand side of (25) is strictly positive as f is concave. However, if (26) holds, the left-hand side is strictly negative by the same token. Given this, it is straightforward that (25) cannot hold also for $f_i > f_j$ and $\delta \in \{0, 1\}$. Thus, the equilibrium is unique for any n given (26). □

Proof of Proposition 2

It is straightforward that (18) implies the necessary and sufficient condition

$$x_d^* > (=) x_c^* \Leftrightarrow k^{\frac{1}{k-1}} > (=) n.$$

Note that on the domain $(1, +\infty)$ the function $k^{1/(k-1)}$ is continuous, decreasing, and satisfies $\lim_{k\to 1} k^{1/(k-1)} = e \simeq 2.72$ and $\lim_{k\to +\infty} k^{1/(k-1)} = 1$. Thus, the condition above is violated for any $n \geq 3$. Assuming $n = 2$, the unique elasticity parameter satisfying $k^{1/(k-1)} = 2$ is $k = 2$. Thus, $x_d^* > (=) x_c^*$ whenever $k < (=) 2$. We now identify the effort maximizing n and k. As we already know that x_d^* decreases in n, we can set $n = 2$. By (5) we have

$$\partial g(x_d^*)/\partial k = (1 - k \ln 2)/2^{k+1} = 0 \Leftrightarrow k = 1/\ln 2,$$

and it is easy to show that $\partial^2 g(x_d^*)/\partial k^2 < 0$. Then, as the function g is increasing (since f is increasing and concave), the value $k = 1/\ln 2$ of the elasticity parameter is the unique maximizer of the effort level in the considered equilibrium. $\square$

Proof of Proposition 3

Let $n = 2$ and take any $k \in (1, 2]$, $\delta \in [0, 1]$ and $i \in N$. Firstly, using implicit differentiation of the FOC (17), we obtain

$$\partial x_{id}^*(x_j)/\partial \delta = \lambda \left(f_j^{k-1} - f_i^{k-1}\right)/(-S),$$

where $\lambda := k f_j f_i' / F^{k+1} > 0$ and $S := \partial^2 \pi_i(\mathbf{x})/\partial x_i^2 < 0$. It follows that

$$\partial x_{id}^*(x_j)/\partial \delta < 0 \Leftrightarrow x_{id}^*(x_j) > x_j, \tag{27}$$

i.e., the best-response function of i is a decreasing function of δ if and only if the value of the best-response function evaluated at x_j is greater than x_j.

Secondly, we now show that $x_{id}^*(x_j) > x_j$ for all $x_j < x_d^*$, which combined with (27) implies $\partial x_{id}^*(x_j)/\partial \delta < 0$ if $x_j < x_d^*$. Using implicit differentiation of FOC (17) once more,

$$\frac{\partial x_{id}^*(x_j)}{\partial x_j} = \frac{\gamma}{-S}\left[f_i^{k-1}(f_i - k f_j) - \delta\left(f_i^k + f_j^k - k\left(f_i^{k-1} f_j + f_i f_j^{k-1}\right)\right)\right], \tag{28}$$

where $\gamma := k f_i' f_j' / F^{k+2} > 0$. In a symmetric equilibrium (28) reduces to

$$\left.\frac{\partial x_{id}^*(x_j)}{\partial x_j}\right|_{x_i = x_j = x_d^*} = \frac{-(k-1)(1-2\delta)}{-2f'' f / f'^2 + \delta(k+1) + (1-\delta)(3-k)}. \tag{29}$$

The denominator of (29) is positive as f is concave and $k \in (1, 2]$. Thus, expression (29) is non-positive if $\delta \in [0, 1/2]$ and simple algebra shows that (29) is positive but strictly less than 1 if $\delta \in (1/2, 1]$. As we know from Theorem 2 that there is a single symmetric equilibrium under our restrictions, these results imply that $x_{id}^*(x_j)$ crosses the 45° line only once, from above, and at $x_j = x_d^*$. Therefore, $x_{id}^*(x_j) > x_j$ if $x_j < x_d^*$ and we can conclude that

$$\partial x_{id}^*(x_j)/\partial \delta < 0 \text{ if } x_j < x_d^*. \tag{30}$$

Finally, we now prove our desired result $x_{id}^*(x_j) > x_{ic}^*(x_j)$ given $x_j < x_d^*$. To do so for any $\delta \in [0,1]$, it is sufficient to show that the result holds for $\delta = 1$, as we have already shown in (30) that $x_{id}^*(x_j)$ is decreasing in δ when $x_j < x_d^*$. Let $p_{ic} := f_i/F$. For $\delta = 1$, the payoff function (1) for the contest with draw can be rewritten as $\pi_{id} := p_{ic}^k + p_d - x_i = 1 - p_{jc}^k - x_i$, while the payoff function for the contest without draw is $\pi_{ic} := p_{ic} - x_i = 1 - p_{jc} - x_i$. Combining the FOCs of these payoffs we obtain

$$kp_{jc}^{k-1} = \frac{\partial p_{jc}\left(x_{ic}^*(x_j), x_j\right)/\partial x_i}{\partial p_{jc}\left(x_{id}^*(x_j), x_j\right)/\partial x_i}.$$

As f is concave p_{jc} is decreasing and convex in x_i, $kp_{jc}^{k-1} < 1 \Leftrightarrow x_{id}^*(x_j) > x_{ic}^*(x_j)$. Since $x_j < x_d^*$ if and only if $p_{jc} < 1/2$ (this is easily verified by the anonymity and strict monotonicity properties of the success function, and by the fact that in the benchmark winning probabilities sum up to 1), it follows that $kp_{jc}^{k-1} < 1$ for all $k \in (1,2]$. Thus, we have $x_{id}^*(x_j) > x_{ic}^*(x_j)$ given $x_j < x_d^*$, which completes our proof. □

Proof of Proposition 4

Suppose $n = 2$ and player $i \in N$ is unconstrained, while player $j \neq i$ is constrained. Denote the constrained best-response of player j by

$$\hat{x}_{jd}^*(x_i) := \begin{cases} x_{jd}^*(x_i) & \text{if } x_{jd}^*(x_i) < \hat{x}_j, \\ \hat{x}_j & \text{otherwise.} \end{cases}$$

Since j is constrained, $\hat{x}_j < x_d^*$. Then, as shown in the proof of Proposition 3, we must have $x_{id}^*(\hat{x}_j) > \hat{x}_j$, which implies $\hat{x}_{jd}^*(x_{id}^*(\hat{x}_j)) = \hat{x}_j$. Suppose there is an equilibrium $\hat{\mathbf{x}}_{\mathbf{d}}^* := (\hat{x}_{id}^*, \hat{x}_{jd}^*)$ where $\hat{x}_{id}^* := x_{id}^*(\hat{x}_j) > \hat{x}_{jd}^* := \hat{x}_j$. Then, given our restrictions, by FOC (17) we must have $\partial\pi_i(\hat{\mathbf{x}}_{\mathbf{d}}^*)/\partial x_i = 0$ and $\partial\pi_j(\hat{\mathbf{x}}_{\mathbf{d}}^*)/\partial x_j > 0$ (note that i plays an unconstrained best-response while j's constraint is binding), which implies that $\hat{\mathbf{x}}_{\mathbf{d}}^*$ is an equilibrium. Moreover, $\hat{\mathbf{x}}_{\mathbf{d}}^*$ is the unique equilibrium as there is no equilibrium within the set of effort profiles where j's constraint is not binding by Points (*i*) and (*ii*) of Proposition 1 given our restrictions. Using (27), the unconstrained best-responses must satisfy

$$\partial x_{id}^*(\hat{x}_{jd}^*)/\partial\delta < 0 < \partial x_{jd}^*(\hat{x}_{id}^*)/\partial\delta.$$

As $\hat{x}_{jd}^* = \hat{x}_j$ for all $\delta \in [0,1]$, the equilibrium effort of j is unaffected by the change in δ. On the other hand, $\hat{x}_{id}^*$ decreases in δ. Thus, the aggregate effort $\hat{x}_{id}^* + \hat{x}_{jd}^*$ always decreases (increases) due to an increase (decrease) in the draw prize. □

Proof of Proposition 5

For the contest without draw, the equilibrium rent dissipation is

$$R(\mathbf{x}_c^*) = nx_c^* = \alpha(n-1)/n,$$

which is obtained by evaluating (6) at $k = 1$. For the contest with draw, the expected prize is

$$1 + p_d(\mathbf{x}_d)(n\delta - 1) = \frac{1}{n^{k-1}} + \delta n \frac{n^{k-1} - 1}{n^{k-1}},$$

using (1), and the aggregate equilibrium effort is $nx_d^* = \alpha k (n-1)/n^k$. Hence,

$$R(\mathbf{x}_d^*) = \frac{\alpha k (n-1)}{n\left[1 + \delta n \left(n^{k-1} - 1\right)\right]}.$$

Then, by straightforward algebra we get

$$R(\mathbf{x}_d^*) \geq R(\mathbf{x}_c^*) \Leftrightarrow \delta \leq \frac{k-1}{n\left(n^{k-1} - 1\right)}.$$

The second term in the curly brackets of (8) follows from the feasibility constraint $\delta \leq 1/n$. □

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