SIAM J. MATH. ANAL.
Vol. 50, No. 6, pp. 5825–5856




# EXTREME LOCALIZATION OF EIGENFUNCTIONS TO ONE-DIMENSIONAL HIGH-CONTRAST PERIODIC PROBLEMS WITH A DEFECT*

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**Abstract.** Following a number of recent studies of resolvent and spectral convergence of nonuniformly elliptic families of differential operators describing the behavior of periodic composite media with high contrast, we study the corresponding one-dimensional version that includes a "defect": an inclusion of fixed size with a given set of material parameters. It is known that the spectrum of the purely periodic case without the defect and its limit, as the period $\varepsilon$ goes to zero, has a band-gap structure. We consider a sequence of eigenvalues $\lambda_\varepsilon$ that are induced by the defect and converge to a point $\lambda_0$ located in a gap of the limit spectrum for the periodic case. We show that the corresponding eigenfunctions are "extremely" localized to the defect, in the sense that the localization exponent behaves as $\exp(-\nu/\varepsilon)$, $\nu > 0$, which has not been observed in the existing literature. In two- and three-dimensional configurations, whose one-dimensional cross sections are described by the setting considered, this implies the existence of propagating waves that are localized to a vicinity of the defect. We also show that the unperturbed operators are norm-resolvent close to a degenerate operator on the real axis, which is described explicitly.

**Key words.** high-contrast homogenization, wave localization, spectrum, decay estimates

**AMS subject classifications.** 34L05, 47A10, 34E10, 34C25

**DOI.** 10.1137/17M112261X

## 1. Introduction.

The question of whether a macroscopic perturbation of material properties in a periodic medium or structure (periodic composite) induces the existence of a localized solution (bound state) to the time-harmonic version of the equations of motion is of special importance from the physics, engineering, and mathematical points of view. Depending on the application context, such a solution can have either an advantageous or undesirable effect on the behavior of systems containing the related composite medium as a component. For example, in the context of photonic (phononic) crystal fibers, perturbations of this kind have been exploited for the transport of electromagnetic (elastic) energy over large distances with little loss into the surrounding space; see, e.g., [14], [17]. In the mathematics literature, proofs of the existence or nonexistence of such a localized solution have been carried out using the tools of the classical asymptotic analysis of the governing equations and spectral analysis of operators generated by the governing equations in various "natural" function spaces. The choice of the concrete class of equations and functions under study is usually motivated by the applications in mind, and several works that

*Received by the editors April 4, 2017; accepted for publication (in revised form) August 22, 2018; published electronically November 20, 2018.
http://www.siam.org/journals/sima/50-6/M112261.html
**Funding:** The work of the second author was supported by the Engineering and Physical Sciences Research Council grant EP/L018802/2 "Mathematical foundations of metamaterials: homogenisation, dissipation and operator theory." The work of the third author was supported by the Engineering and Physical Sciences Research Council grant EP/M017281/1 "Operator asymptotics, a new approach to length-scale interactions in metamaterials."
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have marked the development of the related analytical techniques cover a wide range of operators and their relatively compact perturbations, e.g., [20], [3], [2], [10].

The present work is a study of localization properties for a class of composite media that has been the subject of increasing interest in the mathematics and physics literature recently, in view of its relation to the so-called metamaterials, e.g., manufactured composites possessing negative refraction properties. It has been shown in [8] that the spectrum of a stratified high-contrast composite, represented mathematically by a one-dimensional periodic second-order differential equation, has an infinitely increasing number of gaps (lacunae) opening in the spectrum, in the limit of the small ratio $\varepsilon$ between the period and the overall size of the composite. This analytical feature, analogous to the spectral property of multidimensional high-contrast periodic composites shown in [22], provides a mathematical recipe for the use of such materials in the physics context or technologies where the presence of localized modes (generated by defects in the medium) has important practical implications. In the physical context of photonic crystal fibers and within the mathematical setup of multidimensional high-contrast media, this link has been studied in [12], [5], [6]. In the paper [12], a two-scale asymptotics for eigenfunctions of a high-contrast second-order elliptic differential operator with a finite-size perturbation (defect) was derived. It was shown that for eigenvalues $\lambda$ in gaps of the spectrum of the (two-scale) operator representing the leading-order term of this asymptotics, there are sequences of eigenvalues of the finite-period problems that converge to $\lambda$ as $\varepsilon \to 0$. The subsequent works [5], [6] developed a multiscale version of Agmon's approach [1] and proved that the corresponding eigenfunctions of the limit operator decay exponentially fast away from the defect. An important technical assumption in all these works is that the low-modulus inclusions in the composite have a positive distance to the boundary of the period cell, which is not possible to satisfy in one dimension.

In the more recent paper [8], a family of nonuniformly elliptic periodic one-dimensional problems with high contrast was studied, which in practically relevant situations corresponds to a stratified composite with alternating layers of homogeneous media with highly contrasting material properties. It was shown that the spectra of the corresponding operators converge, as $\varepsilon \to 0$, to the band-gap spectrum of a two-scale operator described explicitly in terms of the original material parameters. Introducing a finite-size defect $D$ into the setup of [8], one is led to consider the operator

$$-\big(a_D^\varepsilon u'\big)', \qquad a_D^\varepsilon > 0,$$

where $a_D^\varepsilon$ takes values of order one on $D$ and is $\varepsilon$-periodic ($\varepsilon > 0$) in $\mathbb{R} \setminus D$ with alternating values of order one and $\varepsilon^2$. As was mentioned in [8, section 5.1], a formal analysis suggests that the rate of decay of eigenfunctions localized in the vicinity of the perturbation $D$ is "accelerated exponential," rather than just exponential as in [6], in the sense that the decay exponent increases in absolute value as $\varepsilon \to 0$. The goal of the present work is to provide a rigorous proof of this property, formulated below as Theorem 2.4. In view of the above discussion, this new localization property can be seen as a consequence of two features of the underlying periodic composite: loss of uniform ellipticity (via the presence of soft inclusions in a moderately stiff material) and the one-dimensional nature of the problem.

In addition to our main result, we formulate (section 3) a new characterization of the limit spectrum for the unperturbed family of problems in the whole space discussed in [8] and strengthen (section 6) the result of [8] by proving an order-sharp norm-resolvent convergence estimate for this family (Theorem 2.2). In particular, this new estimate implies order-sharp uniform asymptotic estimates, as $\varepsilon \to 0$, for the related family of evolution semigroups; cf., e.g. [23] for a strong-convergence version of this kind of result.

## 2. Problem formulation and main results.

For $\varepsilon, h \in (0,1)$, we introduce the sets

$$\Omega_0^\varepsilon := \bigcup_{z\in\mathbb{Z}} (\varepsilon z, \varepsilon z + \varepsilon h) \qquad \text{and} \qquad \Omega_1^\varepsilon := \bigcup_{z\in\mathbb{Z}} (\varepsilon z + \varepsilon h, \varepsilon z + \varepsilon) = \mathbb{R} \setminus \overline{\Omega_0^\varepsilon}$$

and denote $Y_0 := (0,h)$, $Y_1 := (h,1)$, $Y := (0,1)$. We define the $\varepsilon$-periodic functions

$$a^\varepsilon(x) := \begin{cases} \varepsilon^2 a_0(\frac{x}{\varepsilon}), & x \in \Omega_0^\varepsilon, \\ a_1(\frac{x}{\varepsilon}), & x \in \Omega_1^\varepsilon, \end{cases} \qquad \rho^\varepsilon(x) = \rho(\tfrac{x}{\varepsilon}), \qquad \rho(y) := \begin{cases} \rho_0(y), & y \in Y_0, \\ \rho_1(y), & y \in Y_1 \end{cases} \tag{2.1}$$

for $a_j, a_j^{-1}, \rho_j, \rho_j^{-1} \in L^\infty(Y_j)$, $j = 0,1$, periodic with period 1. It is convenient to set $a_0 \equiv 0$ on $Y_1$ and $a_1 \equiv 0$ on $Y_0$; thus we can write, for example, $a^\varepsilon(x) = \varepsilon^2 a_0(x/\varepsilon) + a_1(x/\varepsilon)$. We denote $\Omega_0 := \bigcup_{z\in\mathbb{Z}}(Y_0 + z)$, $\Omega_1 := \bigcup_{z\in\mathbb{Z}}(Y_1+z)$ and reserve the notation $z$ for an integer, unless stated otherwise. We will refer to the sets $\Omega_0^\varepsilon, \Omega_0$ and $\Omega_1^\varepsilon, \Omega_1$ as the soft and stiff component, respectively.

For a positive Lebesgue-measurable function $w$ on a Borel set $B \subset \mathbb{R}$, such that $w, w^{-1} \in L^\infty(B)$, we employ the notation $L^2_w(B)$ to indicate that the space $L^2(B)$ is equipped with the inner product

$$(u,v)_w := \int_B w u \overline{v}, \qquad u, v \in L^2(B).$$

For a closed and semibounded sesquilinear form $\beta : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{C}$, the (self-adjoint) operator $A$ associated to $\beta$ is densely defined in $L^2_w(\mathbb{R})$ by the action $Au = f$, where for a given $f \in L^2_w(\mathbb{R})$, the function $u \in H^1(\mathbb{R})$ is the solution to the integral identity

$$\beta(u,v) = \int_{\mathbb{R}} w f \overline{v} \qquad \forall v \in H^1(\mathbb{R}).$$

Henceforth, all function spaces we employ consist of complex-valued functions and are over $\mathbb{C}$.

For the sesquilinear form

$$\beta^\varepsilon(u,v) := \int_{\mathbb{R}} a^\varepsilon u' \overline{v'}, \qquad u, v \in H^1(\mathbb{R}),$$

we consider $A^\varepsilon$, the operator defined in $L^2_{\rho^\varepsilon}(\mathbb{R})$ and associated to $\beta^\varepsilon$. The spectrum $\sigma(A^\varepsilon)$ of $A^\varepsilon$ is absolutely continuous and, by introducing the rescaled Floquet–Bloch transform $\mathcal{U}_\varepsilon$ (see (6.1)), we note that $\sigma(A^\varepsilon)$ admits the representation

$$\sigma(A^\varepsilon) = \bigcup_{\theta \in [0,2\pi)} \sigma(A_\theta^\varepsilon),$$

where $\sigma(A_\theta^\varepsilon)$ is the spectrum of the $L^2_\rho(Y)$ densely defined self-adjoint operator $A_\theta^\varepsilon$ associated to the form

$$\beta_\theta^\varepsilon(u,v) := \int_Y \big(a_0 + \varepsilon^{-2} a_1\big) u' \overline{v'},$$

acting in the space $H^1_\theta(Y)$ of functions $u \in H^1(Y)$ that are $\theta$-quasiperiodic, i.e., such that $u(y) = \exp(\mathrm{i}\theta y)v(y)$, $y \in Y$, for some 1-periodic function $v \in H^1(Y)$. For each $\varepsilon$, $\theta$, the operator $A^\varepsilon_\theta$ has compact resolvent and consequently its spectrum $\sigma(A^\varepsilon_\theta)$ is discrete.

Consider the space

$$V_\theta := \left\{ u \in H^1_\theta(Y) : u' = 0 \text{ on } Y_1 \right\} \tag{2.2}$$

and its closure in $L^2_\rho(Y)$, which we denote by $\overline{V_\theta}$, which we also equip with the norm of $L^2_\rho(Y)$. We introduce the densely defined operators $A_\theta$ in $\overline{V_\theta}$ given by $A_\theta u = f$, where for all $f \in \overline{V_\theta}$ the function $u$ in the domain of $A_\theta$ is such that

$$\int_{Y_0} a_0 u' \overline{v'} = \int_Y \rho f \overline{v} \qquad \forall v \in V_\theta. \tag{2.3}$$

For each $\theta$, the operator $A_\theta$ has compact resolvent, and so $\sigma(A_\theta)$ is discrete. In a recent work [8] (see also section 6 of the present manuscript), the spectrum $\sigma(A^\varepsilon)$ was shown to converge in the Hausdorff sense to the union of the spectra of the operators $A_\theta$, i.e.,

$$\lim_{\varepsilon \to 0} \sigma(A^\varepsilon) = \bigcup_{\theta \in [0,2\pi)} \sigma(A_\theta). \tag{2.4}$$

*Remark* 2.1. $\bigcup_{\theta \in [0,2\pi)} \sigma(A_\theta)$ can be seen as the spectrum of a certain operator $A^0$ unitary equivalent to the direct integral of operators $\int^\oplus A_\theta$; see Appendix A for the details.

In section 6, we construct infinite-order asymptotics (as $\varepsilon \to 0$) for the resolvents of $A^\varepsilon_\theta$, uniform in $\theta$, with respect to the $H^1$-norm and, in particular, prove the following refinement of the result established in [8].

THEOREM 2.2. *The operator $A^\varepsilon_\theta$ norm-resolvent converges to $A_\theta$, uniformly in $\theta$, at the rate $\varepsilon^2$. More precisely, there exists a constant $C > 0$ such that*

$$\left\| (A^\varepsilon_\theta + 1)^{-1} f - (A_\theta + 1)^{-1} P_\theta f \right\|_{L^2_\rho(Y)} \leq C\varepsilon^2 \|f\|_{L^2_\rho(Y)} \qquad \forall \theta \in [0, 2\pi), \quad f \in L^2_\rho(Y),$$

*where $P_\theta$ is the orthogonal projection of $L^2_\rho(Y)$ onto $\overline{V_\theta}$.*

Consequently, since the spectra $\sigma(A^\varepsilon_\theta)$ and $\sigma(A_\theta)$ are discrete, we have the following result: for each $n \in \mathbb{N}$ there exists a constant $c_n > 0$ such that

$$\left| \lambda^\varepsilon_n(\theta) - \lambda_n(\theta) \right| \leq c_n \varepsilon^2 \qquad \forall \theta \in [0, 2\pi), \quad \varepsilon \in (0, 1).$$

Here, $\{\lambda^\varepsilon_n(\theta)\}_{n \in \mathbb{N}}$, $\{\lambda_n(\theta)\}_{n \in \mathbb{N}}$ are the eigenvalue sequences of $A^\varepsilon_\theta$, $A_\theta$, respectively, labeled in the increasing order.[1] It follows that for sufficiently small $\varepsilon$, the spectrum $\sigma(A^\varepsilon)$ has a gap if the set $\bigcup_\theta \sigma(A_\theta)$ contains a gap. In section 3 we give an example of a class of coefficients for which this set contains infinitely many gaps. Furthermore, we demonstrate that $\lambda \in \bigcup_\theta \sigma(A_\theta)$ if and only if the inequality

$$\left| v_1(h) + (a_0 v_2')(h) - \lambda v_2(h) \int_{Y_1} \rho_1 \right| \leq 2$$

[1]Notice that all the eigenvalues are simple due to the one-dimensional nature of the corresponding problem.

holds. Here $v_1$ and $v_2$ are the ($\lambda$-dependent) solutions of

$$-(a_0 v_j')' = \lambda \rho_0 v_j \quad \text{on } Y_0, \qquad j = 1, 2,$$

subject to the conditions

$$\begin{pmatrix} v_1(0) & v_2(0) \\ (a_0 v_1')(0) & (a_0 v_2')(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

*Remark* 2.3. Note that any solution $u$ of $-(a_0 u')' = \lambda \rho_0 u$ is absolutely continuous and so is its coderivative $a_0 u'$. Hence, their value at any point $y$ is well defined (unlike the value of $a_0$ or $u'$ in general). This explains the use of notation $(a_0 v_j')(y)$, which we will hold to throughout the paper.

Next, we introduce $d_-, d_+ \in \mathbb{R}$, and on the set $D = (d_-, d_+)$ replace the coefficients (2.1) by some uniformly positive and bounded functions $a_D$, $\rho_D$; namely, we consider

$$a_D^\varepsilon(x) := \begin{cases} a_D(x), & x \in D, \\ a_1(\frac{x}{\varepsilon}), & x \in \Omega_1^\varepsilon \backslash D, \\ \varepsilon^2 a_0(\frac{x}{\varepsilon}), & x \in \Omega_0^\varepsilon \backslash D, \end{cases} \qquad \rho_D^\varepsilon(x) := \begin{cases} \rho_D(x), & x \in D, \\ \rho_1(\frac{x}{\varepsilon}), & x \in \Omega_1^\varepsilon \backslash D, \\ \rho_0(\frac{x}{\varepsilon}), & x \in \Omega_0^\varepsilon \backslash D. \end{cases}$$

We shall study the spectrum of the operator $A_D^\varepsilon$ defined in $L^2_{\rho_D^\varepsilon}(\mathbb{R})$ and associated to the form

$$\beta_D^\varepsilon(u, v) := \int_{\mathbb{R}} a_D^\varepsilon u' \overline{v'}, \qquad u, v \in H^1(\mathbb{R}). \tag{2.5}$$

As this operator arises from a compact perturbation of the coefficients of $A^\varepsilon$, it is well-known that the essential spectra of $A_D^\varepsilon$ and $A^\varepsilon$ coincide; see, e.g., [10]. For eigenvalues situated, for small values of $\varepsilon$, in the gaps of the essential spectrum of $A_D^\varepsilon$ (equivalently, in the gaps of the essential spectrum of $A^\varepsilon$), we expect the eigenfunctions to be localized around the defect. In view of the above observation about the spectra of $A^\varepsilon$ and $A_\theta$, $\theta \in [0, 2\pi)$, we are therefore interested in the analysis of eigenfunctions of $A_D^\varepsilon$ corresponding to eigenvalues that are located in the gaps of the limit spectrum $\bigcup_\theta \sigma(A_\theta)$.

Consider the operator $A_{\mathrm{N},D}$ defined in $L^2_{\rho_D}(D)$ and associated to the form

$$\beta_{\mathrm{N},D}(u, v) := \int_D a_D u' \overline{v'}, \qquad u, v \in H^1(D), \tag{2.6}$$

acting in $H^1(D)$. The functions from the domain of $A_{\mathrm{N},D}$ satisfy the Neumann condition on the boundary of $D$. We show that if the defect $D$ is chosen so that the spectrum $\sigma(A_{\mathrm{N},D})$ has a nonempty intersection with $\mathbb{R} \backslash \bigcup_\theta \sigma(A_\theta)$, then for sufficiently small $\varepsilon$ the operator $A_D^\varepsilon$ has nonempty point spectrum. Notice that we can always choose $a_D$, $\rho_D$, $d_-$ and $d_+$ such that this is true. Moreover, we demonstrate that for eigenvalue sequences that converge to a point in $\mathbb{R} \backslash \bigcup_\theta \sigma(A_\theta)$ the corresponding eigenfunctions are localized to a small neighborhood of the defect. Namely, the eigenfunctions $u_\varepsilon$ exhibit accelerated exponential decay outside the defect in the sense that the function $\exp\bigl(\mathrm{dist}(x, D)\nu/\varepsilon\bigr) u_\varepsilon(x)$, $x \in \mathbb{R}$, is an element of $L^2(\mathbb{R} \backslash D)$ for sufficiently small $\varepsilon$, where the value $\nu$ is determined by the distance of the limit point of $\lambda_\varepsilon$ to the set $\bigcup_\theta \sigma(A_\theta)$. These results are collated in the following theorem, which we prove in sections 4 and 5.

THEOREM 2.4.

1. *For every $\lambda_0 \in \sigma(A_{\mathrm{N},D})\backslash\bigcup_\theta \sigma(A_\theta)$ (which is always simple) there exists a sequence of simple eigenvalues $\lambda_\varepsilon$ of $A^\varepsilon_D$ converging to $\lambda_0$ and constants $C_1, C_2 > 0$ such that*

$$\begin{aligned} &\big|\lambda_\varepsilon - \lambda_0\big| \le C_1\varepsilon^{1/2},\\ &\Big\|\sum_{j\in J_\varepsilon} c^\varepsilon_j u_{\varepsilon,j} - u_0\Big\|_{L^2(D)} \le C_2\varepsilon^{1/2}, \qquad (2.7)\\ &\Big\|\sum_{j\in J_\varepsilon} c^\varepsilon_j u_{\varepsilon,j}\Big\|_{L^2(\mathbb{R}\backslash D)} \le C_2\varepsilon^{1/2}, \end{aligned}$$

*where $u_0$ is a normalized eigenfunction of $A_{\mathrm{N},D}$ corresponding to the eigenvalue $\lambda_0$, the set*

$$J_\varepsilon := \big\{ j : |\lambda_{\varepsilon,j} - \lambda_0| \le C_2\varepsilon^{1/2} \big\} \qquad (2.8)$$

*is finite, and for each $j \in J_\varepsilon$ the function $u_{\varepsilon,j}$ is the $L^2_{\rho^\varepsilon_D}(\mathbb{R})$-normalized eigenfunction of $A^\varepsilon_D$ with eigenvalue $\lambda_{\varepsilon,j}$.*

2. *Suppose that $\lambda_\varepsilon$ is an eigenvalue of $A^\varepsilon_D$ for each $\varepsilon$ and that $\lambda_\varepsilon \to \lambda_0 \notin \lim_{\varepsilon\to 0}\sigma(A^\varepsilon) = \bigcup_\theta \sigma(A_\theta)$. Then the $L^2(\mathbb{R})$-normalized eigenfunctions $u_\varepsilon$ of $A^\varepsilon_D$ corresponding to the eigenvalues $\lambda_\varepsilon$ are localized in the following sense.*
*For $\nu > 0$, let $g_{\nu/\varepsilon}$ denote the exponentially growing function*

$$g_{\nu/\varepsilon}(x) := \begin{cases} 1, & x \in D,\\ \exp\Big(\dfrac{\nu}{\varepsilon}\,\mathrm{dist}(x, D)\Big), & x \in \mathbb{R}\setminus D, \end{cases} \qquad (2.9)$$

*and take $\mu_1$ to be the smallest by the absolute value root of the quadratic function*

$$q(\mu) := \mu^2 - \Big(v_1(h) + (a_0 v_2')(h) - \lambda_0 v_2(h)\int_{Y_1}\rho_1\Big)\mu + 1. \qquad (2.10)$$

*Then, for sufficiently small values of $\varepsilon$, the function $g_{\nu/\varepsilon}\,u_\varepsilon$ is an element of $L^2(\mathbb{R})$ for all $\nu < \big|\ln|\mu_1|\big|$.*

*Remark* 2.5. One can improve the eigenvalues convergence rate at least to $|\lambda_\varepsilon - \lambda_0| \le \widetilde{C}\varepsilon$ and in a rather generic case even to $|\lambda_\varepsilon - \lambda_0| \le \widetilde{C}\varepsilon^2$ for some $\widetilde{C} > 0$ (improving accordingly the convergence estimate for the eigenfunctions), by "attaching" the periodic structure to the defect $D$ in a "correct" way; see the end of section 4 and Theorem 4.2 for the details.

**3. The limit spectrum of the unperturbed operator.** Here we quantitatively characterize the limit spectrum (cf. (2.4))

$$\bigcup_{\theta\in[0,2\pi)} \sigma(A_\theta)$$

and establish criteria for the existence of spectral gaps. To this end we consider the eigenvalue problem: find $\lambda \in [0,\infty)$ and $u \in V_\theta = \big\{ v \in H^1_\theta(Y) : v' \equiv 0 \text{ on } Y_1 \big\}$ such that

$$\int_0^h a_0 u' \overline{v'} = \lambda \int_0^1 \rho u \overline{v} \qquad \forall v \in V_\theta. \tag{3.1}$$

By taking test functions $v \in C_0^\infty(Y_0)$ we deduce that $u|_{Y_0}$ is a weak solution to the equation

$$-(a_0 u')' = \lambda \rho_0 u \tag{3.2}$$

on $Y_0$. For $L^\infty$-functions $a_0$ and $\rho_0$, (3.2) holds pointwise almost everywhere, and by integrating by parts in (3.1) we deduce that

$$(a_0 u')(h^-)\,\overline{v(h)} - (a_0 u')(0^+)\,\overline{v(0)} = \lambda \int_{Y_1} \rho_1 u \overline{v} \qquad \forall v \in V_\theta.$$

Here $f(z^+) := \lim_{x \searrow z} f(x)$, and $f(z^-) := \lim_{x \nearrow z} f(x)$ for a function $f$, whenever the corresponding limit exists. Since any element $v \in V_\theta$ satisfies $v(y) = e^{\mathrm{i}\theta} v(0)$, $y \in Y_1$, the above observations imply that $u$ satisfies (3.1) if and only if $w = u|_{Y_0} \in H^1(Y_0)$ is a weak solution of the problem

$$\begin{cases} -(a_0 u')' = \lambda \rho_0 u \qquad \text{in } Y_0, \\ u(h) = e^{\mathrm{i}\theta} u(0), \\ e^{-\mathrm{i}\theta}(a_0 u')(h^-) - (a_0 u')(0^+) = \lambda u(0) \int_{Y_1} \rho_1. \end{cases} \tag{3.3}$$

We now describe the solutions to (3.3), equivalently (3.1).

**3.1. Representation via a fundamental system.** Due to the existence and uniqueness theorem for linear first-order systems with locally integrable coefficients (see, e.g., [21]), for all $\lambda \in \mathbb{R}$ the first-order system

$$U' = AU, \qquad A := \begin{pmatrix} 0 & a_0^{-1} \\ -\lambda \rho_0 & 0 \end{pmatrix}, \tag{3.4}$$

has a fundamental solution system

$$V(\lambda, \cdot) := \begin{pmatrix} v_1(\lambda, \cdot) & v_2(\lambda, \cdot) \\ (a_0 v_1')(\lambda, \cdot) & (a_0 v_2')(\lambda, \cdot) \end{pmatrix},$$

so that any solution to $-(a_0 u')' = \lambda \rho_0 u$ in $Y_0$ is a linear combination of $v_1(\lambda, \cdot)$ and $v_2(\lambda, \cdot)$, and

$$V(\lambda, 0) = \begin{pmatrix} v_1(\lambda, 0) & v_2(\lambda, 0) \\ (a_0 v_1')(\lambda, 0) & (a_0 v_2')(\lambda, 0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \tag{3.5}$$

cf. Remark 2.3. Note that the associated Wronskian of the system is constant:

$$\det V(\lambda, y) = v_1(\lambda, y)(a_0 v_2')(\lambda, y) - v_2(\lambda, y)(a_0 v_1')(\lambda, y) = 1 \qquad \forall\, y \in Y_0, \quad \lambda \in \mathbb{R}. \tag{3.6}$$

It follows from the above that all solutions $u$ to (3.3) are of the form

$$u = c_1 v_1 + c_2 v_2 \tag{3.7}$$

for some $c_1, c_2 \in \mathbb{C}$. Substituting the representation (3.7) into the second and third equations of (3.3) leads to the system

$$\begin{pmatrix} v_1(\lambda, h) - e^{\mathrm{i}\theta} & v_2(\lambda, h) \\ (a_0 v_1')(\lambda, h) - e^{\mathrm{i}\theta}\lambda \int_{Y_1} \rho_1 & (a_0 v_2')(\lambda, h) - e^{\mathrm{i}\theta} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.8}$$

For the existence of a nontrivial solution $(c_1, c_2)$ to (3.8), and therefore nontrivial $u$ in (3.3), the value $\lambda$ must necessarily solve the equation

$$2\cos\theta = D(\lambda), \qquad D(\lambda) := v_1(\lambda, h) + (a_0 v_2')(\lambda, h) - \lambda v_2(\lambda, h) \int_{Y_1} \rho_1. \tag{3.9}$$

Hence, the set (cf. (2.4))

$$\bigcup_{\theta \in [0, 2\pi)} \sigma(A_\theta)$$

consists of all nonnegative $\lambda$ such that

$$\bigl|D(\lambda)\bigr| \leq 2. \tag{3.10}$$

From the relation (3.9) we can deduce much more about the limit spectrum. Setting $\lambda_k(\theta)$, $k \in \mathbb{N}$, $\theta \in [0, 2\pi)$, to the $k$th eigenvalue of $A_\theta$ labeled according to the min-max principle, we define $E_k : [0, 2\pi) \to [0, \infty)$ to be the $k$th spectral band function given by $\theta \mapsto \lambda_k(\theta)$. The name "spectral band function" comes from the (clear) characterization:

$$\bigcup_{\theta \in [0, 2\pi)} \sigma(A_\theta) = \bigcup_{k \in \mathbb{N}} \mathrm{Ran} E_k.$$

We shall prove below the following result about the nature of the spectral band functions.

THEOREM 3.1.
1. *The functions $E_k$, $k \in \mathbb{N}$, are continuous and even around $\theta = \pi$: $E_k(\theta) = E_k(2\pi - \theta)$, $\theta \in [0, \pi]$.*
2. *The functions $E_{2m-1}(\cdot)$, $m \in \mathbb{N}$, are strictly increasing on $(0, \pi)$.*
3. *The functions $E_{2m}(\cdot)$, $m \in \mathbb{N}$, are strictly decreasing on $(0, \pi)$.*
4. *The spectral bands are given by the following intervals:*

$$\mathrm{Ran}\, E_{2m-1} = \bigl[\lambda_{2m-1}(0), \lambda_{2m-1}(\pi)\bigr], \qquad \mathrm{Ran}\, E_{2m} = \bigl[\lambda_{2m}(\pi), \lambda_{2m}(0)\bigr], \quad m \in \mathbb{N}.$$

Let us focus on claim 4 of the above theorem. It informs us that the interval $\bigl(\lambda_{2m-1}(\pi), \lambda_{2m}(\pi)\bigr)$ (respectively, $\bigl(\lambda_{2m}(0), \lambda_{2m+1}(0)\bigr)$) is a spectral gap if and only if $\lambda = \lambda_{2m-1}(\pi)$ (respectively, $\lambda = \lambda_{2m}(0)$) is a simple eigenvalue of the antiperiodic (respectively, periodic) limit problem (3.2).[2] We now characterize when the eigenvalues of the periodic, antiperiodic problems are degenerate, i.e., have multiplicity two, in terms of the fundamental system $(v_1, v_2)$. At such points $\lambda$ the spectral bands touch and there are no gaps.

[2]It is straightforward to argue (cf. [18, Theorem XIII.89 (c)]) that each eigenvalue $\lambda_k(\theta)$, $k \in \mathbb{N}$, is always simple when $\theta \in (0, \pi)$.

PROPOSITION 3.2 (condition for the absence of spectral gaps). *Fix $n = 2m$, $m \in \mathbb{N}$. Then $\lambda_n(0) = \lambda_{n+1}(0)$ if and only if*

$$v_2(\lambda_n(0), h) = 0, \qquad (a_0 v_1')(\lambda_n(0), h) = \lambda_n(0) v_1(\lambda_n(0), h)\langle \rho_1 \rangle, \qquad \langle \rho_1 \rangle := \int_{Y_1} \rho_1. \tag{3.11}$$

*Similarly, if $n = 2m-1$, $m \in \mathbb{N}$, then $\lambda_n(\pi) = \lambda_{n+1}(\pi)$ if and only if*

$$v_2(\lambda_n(\pi), h) = 0, \qquad (a_0 v_1')(\lambda_n(\pi), h) = \lambda_n(\pi) v_1(\lambda_n(\pi), h)\langle \rho_1 \rangle. \tag{3.12}$$

*Remark* 3.3. Note that (3.11) is equivalent to

$$\begin{aligned} v_2(\lambda_n(0), h) &= 0, \quad (a_0 v_2')(\lambda_n(0), h) = 1, \\ v_1(\lambda_n(0), h) &= 1, \quad (a_0 v_1')(\lambda_n(0), h) = \lambda_n(0)\langle \rho_1 \rangle. \end{aligned} \tag{3.13}$$

and that (3.12) is equivalent to

$$\begin{aligned} v_2(\lambda_n(\pi), h) &= 0, \quad (a_0 v_2')(\lambda_n(\pi), h) = -1, \\ v_1(\lambda_n(\pi), h) &= -1, \quad (a_0 v_1')(\lambda_n(\pi), h) = -\lambda_n(\pi)\langle \rho_1 \rangle. \end{aligned} \tag{3.14}$$

Sufficiency is obvious for both cases. For necessity, we shall consider $\lambda = \lambda_n(0)$. The case $\lambda = \lambda_n(\pi)$ is similar. We shall drop the argument $(\lambda_n(0), h)$ to neaten up the derivation. By (3.6) and (3.11) we have

$$1 = v_1 a_0 v_2' - v_2 a_0 v_1' = v_1 a_0 v_2'.$$

Furthermore, since $D(\lambda_n(0)) = 2$ (cf. (3.9)), then multiplying (3.9) by $a_0 v_2'$ (which is seen to be nonzero from the above equation) we compute

$$\begin{aligned} 2a_0 v_2' &= a_0 v_2' v_1 + (a_0 v_2')^2 - a_0 v_2' \lambda_n(0) v_2 \langle \rho_1 \rangle \\ &= 1 + (a_0 v_2')^2. \end{aligned}$$

Equivalently, $(a_0 v_2' - 1)^2 = 0$. Hence, $(a_0 v_2')(\lambda_n(0), h) = 1$, and therefore $v_1(\lambda_n(0), h) = 1$, and (3.13) follows.

*Proof of Proposition* 3.2. Let us consider $\lambda_n(0)$, the case $\lambda_n(\pi)$ is similar. Necessity follows by noting that if $\lambda = \lambda_n(0)$ has multiplicity 2 then $v_i(\lambda, \cdot)$, $i = 1, 2$, are linear combinations of the orthogonal periodic eigenfunctions to the limit problem (3.11). In particular the conditions $v_i(\lambda, h) = v_i(\lambda, 0)$ and $(a_0 v_i')(\lambda, h) - (a_0 v_i')(\lambda, 0) = \lambda v_i(\lambda, 0)\langle \rho_1 \rangle$, $i = 1, 2$, hold. Then (3.11) follows from the initial values $v_2(\lambda, 0) = 0$ and $(a_0 v_1')(\lambda, 0) = 0$.

For sufficiency, by (3.13) both the fundamental solutions satisfy the limit spectral problem; i.e., they are linearly independent eigenfunctions of $\lambda = \lambda_n(0)$. ∎

The proof of Theorem 3.1 readily follows, by arguing, for example, as in [9, Chapter 2.3], from the following further analysis on the function $D$ given by (3.9).

LEMMA 3.4. *The function $D$ is analytic. Furthermore, the following assertions hold.*

(a) $D'(\lambda) \neq 0$ *when* $|D(\lambda)| < 2$.
(b) $D'\big(\lambda_n(0)\big) = 0$ *if and only if* (3.11) *holds.*
(c) *If* $D'\big(\lambda_n(0)\big) = 0$, *then* $D''\big(\lambda_n(0)\big) < 0$.
(d) $D'\big(\lambda_n(\pi)\big) = 0$ *if and only if* (3.12) *holds.*
(e) *If* $D'\big(\lambda_n(\pi)\big) = 0$, *then* $D''\big(\lambda_n(\pi)\big) > 0$.

*Proof.* The analyticity of $D$ follows from that of $\lambda \mapsto V(\lambda, h)$, which is well-known to hold; see, for example, [21, Chapter 2].

*Proof of* (a). Differentiating $D$ (cf. (3.9)) gives

$$D'(\lambda) = \partial_\lambda v_1(\lambda, h) + \partial_\lambda (a_0 v_2')(\lambda, h) - v_2(\lambda, h)\langle \rho_1 \rangle - \lambda \partial_\lambda v_2(\lambda, h)\langle \rho_1 \rangle. \tag{3.15}$$

Now if we differentiate both sides of (3.4) with respect to $\lambda$ we get

$$\partial_\lambda u' = A \partial_\lambda u + (0, -\rho_0 u_1)^\top.$$

Furthermore, if $u = (v_i, a_0 v_i')^\top$, then $\partial_\lambda u(\lambda, 0) = 0$. Therefore by the variation of constants method we deduce (recalling $V$ is the fundamental system of (3.4)) that

$$\begin{pmatrix} \partial_\lambda v_i(\lambda, h) \\ \partial_\lambda (a_0 v_i')(\lambda, h) \end{pmatrix} = \begin{pmatrix} \int_0^h \rho_0(s) v_i(\lambda, s) \big[ v_1(\lambda, h) v_2(\lambda, s) - v_2(\lambda, h) v_1(\lambda, s) \big] \, \mathrm{d}s \\ \int_0^h \rho_0(s) v_i(\lambda, s) \big[ (a_0 v_1')(\lambda, h) v_2(\lambda, s) - (a_0 v_2')(\lambda, h) v_1(\lambda, s) \big] \, \mathrm{d}s \end{pmatrix}. \tag{3.16}$$

Therefore, from (3.15), (3.16), after some algebra, we deduce that

$$\begin{aligned} D'(\lambda) = {} & \alpha(\lambda, h) \int_0^h \rho_0(s) v_1(\lambda, s) v_2(\lambda, s) \, \mathrm{d}s - v_2(\lambda, h) \int_0^h \rho_0(s) v_1^2(\lambda, s) \, \mathrm{d}s \\ & + \big[ (a_0 v_1')(\lambda, h) - \lambda \langle \rho_1 \rangle v_1(\lambda, h) \big] \int_0^h \rho_0(s) v_2^2(\lambda, s) \, \mathrm{d}s - v_2(\lambda, h)\langle \rho_1 \rangle, \end{aligned} \tag{3.17}$$

and, from (3.6), (3.9), we readily compute

$$D(\lambda)^2 = 4 + \alpha(\lambda, h)^2 + 4\big( v_2 (a_0 v_1') - \lambda v_1 v_2 \langle \rho_1 \rangle \big)(\lambda, h), \tag{3.18}$$

$$\alpha(\lambda, s) := \big( v_1 - a_0 v_2' + \lambda v_2 \langle \rho_1 \rangle \big)(\lambda, s). \tag{3.19}$$

Multiplying (3.17) by $4v_2(\lambda, h)$ and using (3.18), after some more algebra, we determine that

$$\begin{aligned} 4v_2(\lambda, h) D'(\lambda) = {} & -\big(4 - D(\lambda)^2\big) \int_0^h \rho_0(s) v_2^2(\lambda, s) \, \mathrm{d}s - 4v_2^2(\lambda, h)\langle \rho_1 \rangle \\ & - \int_0^h \rho_0(s) \big( \alpha(\lambda, h) v_2(\lambda, s) - 2 v_2(\lambda, h) v_1(\lambda, s) \big)^2 \, \mathrm{d}s. \end{aligned} \tag{3.20}$$

It follows from (3.20) that if $|D(\lambda)| < 2$, then $4v_2(\lambda, h) D'(\lambda) < 0$. In particular, $D'(\lambda) \neq 0$, i.e., (a) holds.

*Proof of* (b). Henceforth, we shall use the notation $f$ for the value of a function $f(\lambda, s)$ at $(\lambda_n(0), h)$. Suppose that (3.11) holds. By Remark 3.3, equivalently (3.13) holds. Then, by (3.19) we compute $\alpha = 0$ and, therefore, from (3.17) it follows that $D'(\lambda_n(0)) = 0$.

Next, we suppose that $D'(\lambda_n(0)) = 0$ and prove that (3.11) holds. As $\lambda_n(0)$ is a root of $D(\cdot)^2 - 4$, we deduce from (3.20) that

$$0 = -4v_2^2 \langle \rho_1 \rangle - \int_0^h \rho_0(s) \big( \alpha v_2(\lambda_n(0), s) - 2 v_2 v_1(\lambda_n(0), s) \big)^2 \, \mathrm{d}s.$$

Therefore $v_2^2 = v_2(\lambda_n(0), h)^2 = 0$, and

$$\rho_0(s)\big(\alpha v_2(\lambda_n(0), \cdot) - 2v_2 v_1(\lambda_n(0), \cdot)\big)^2 = 0.$$

Setting $v_2 = 0$, above gives $\alpha(\lambda_n(0), h) = 0$. It remains to show that $\big[(a_0 v_1')(\lambda_n(0), h) - \lambda_n(0)\langle\rho_1\rangle v_1(\lambda_n(0), h)\big] = 0$. Upon setting $v_2 = \alpha = 0$ and $\lambda = \lambda_n(0)$ in (3.17) we conclude

$$0 = \big[(a_0 v_1')(\lambda_n(0), h) - \lambda_n(0)\langle\rho_1\rangle v_1(\lambda_n(0), h)\big] \int_0^h \rho_0(s) v_2^2(\lambda_n(0), s)\,\mathrm{d}s.$$

Since $v_2(\lambda_n(0), \cdot) \neq 0$ on $(0, h)$ we deduce $\big[(a_0 v_1')(\lambda_n(0), h) - \lambda_n(0)\langle\rho_1\rangle v_1(\lambda_n(0), h)\big] = 0$, i.e., (3.11) holds.

*Proof of* (c). We shall prove that

$$\frac{1}{2}D''(\lambda_n(0)) \leq \left(\int_0^h \rho_0(s) v_1(\lambda_n(0), s) v_2(\lambda_n(0), s)\,\mathrm{d}s\right)^2 - \left(\int_0^h \rho_0(s) v_1^2(\lambda_n(0), s)\,\mathrm{d}s\right)\left(\int_0^h \rho_0(s) v_2^2(\lambda_n(0), s)\,\mathrm{d}s\right). \tag{3.21}$$

Then, as $v_1(\lambda_n(0), \cdot)$ and $v_2(\lambda_n(0), \cdot)$ are linearly independent it follows from (3.21) and the Hölder inequality that

$$\frac{1}{2}D''(\lambda_n(0)) < 0.$$

Let us prove (3.21). First we make some preliminary calculations. By (3.16) and (3.13), which holds since $D'(\lambda_n(0)) = 0$ (cf. (b)) and Remark 3.3, we compute (dropping the argument $(\lambda_n(0), h)$ as above)

$$\begin{aligned}
\partial_\lambda v_1 &= \int_0^h \rho_0(s) v_1(\lambda_n(0), s) v_2(\lambda_n(0), s)\,\mathrm{d}s,\\
\partial_\lambda v_2 &= \int_0^h \rho_0(s) v_2^2(\lambda_n(0), s)\,\mathrm{d}s,\\
\partial_\lambda (a_0 v_1') &= \lambda_n(0)\langle\rho_1\rangle \partial_\lambda v_1 - \int_0^h \rho_0(s) v_1^2(\lambda_n(0), s)\,\mathrm{d}s,\\
\partial_\lambda (a_0 v_2') &= \lambda_n(0)\langle\rho_1\rangle \partial_\lambda v_2 - \partial_\lambda v_1.
\end{aligned} \tag{3.22}$$

We now are going to differentiate both sides of (3.17) with respect to $\lambda$. Half of the terms will immediately been seen to be zero. Indeed, if we take the first term in the right-hand side of (3.17), differentiate it, and evaluate it at $(\lambda_n(0), h)$, then because $\alpha = 0$ (cf. (3.19) and (3.13)) we deduce that

$$\partial_\lambda \left(\alpha(\lambda, h) \int_0^h \rho_0(s) v_1(\lambda, s) v_2(\lambda, s)\,\mathrm{d}s\right) = \partial_\lambda \alpha \int_0^h \rho_0(s) v_1(\lambda_n(0), s) v_2(\lambda_n(0), s)\,\mathrm{d}s.$$

The same is true for all the other terms. Therefore, differentiating (3.17) and bearing in mind (3.22), we compute (after a little bit more algebra) that

$$D''(\lambda_n(0)) = \partial_\lambda \alpha \partial_\lambda v_1 - 2\partial_\lambda v_2 \left(\int_0^h \rho_0(s) v_1^2(\lambda_n(0), s)\,\mathrm{d}s + \langle\rho_1\rangle\right). \tag{3.23}$$

From (3.22) we see that $\partial_\lambda v_2 \geq 0$ and so it follows from (3.23) that

$$D''(\lambda_n(0)) \leq \partial_\lambda \alpha \partial_\lambda v_1 - 2\partial_\lambda v_2 \int_0^h \rho_0(s) v_1^2(\lambda_n(0), s)\,\mathrm{d}s.$$

We complete the proof of (3.21) (cf. (3.22)) if we can prove that

$$\partial_\lambda \alpha = 2\partial_\lambda v_1. \tag{3.24}$$

Multiplying (3.18) by $v_1(\lambda, s)$ and utilizing (3.6) gives

$$v_1(\lambda, s)\alpha(\lambda, s) = v_1^2(\lambda, s) - 1 - v_2(\lambda, s)(a_0 v_1')(\lambda, s) + \lambda v_1(\lambda, s) v_2(\lambda, s)\langle \rho_1 \rangle.$$

Then, differentiating both sides of the above equation with respect to $\lambda$ and evaluating at $(\lambda_n(0), h)$ gives

$$\begin{aligned}\alpha \partial_\lambda v_1 + v_1 \partial_\lambda \alpha \\ = 2v_1 \partial_\lambda v_1 - (a_0 v_1')\partial_\lambda v_2 - v_2 \partial_\lambda (a_0 v_1') + v_1 v_2 \langle \rho_1 \rangle + \lambda_n(0)\langle \rho_1 \rangle (v_2 \partial_\lambda v_1 + v_1 \partial_\lambda v_2).\end{aligned}$$

Upon utilizing (3.13), we deduce that (3.24) holds, and the proof of (c) follows.

The proofs of (d) and (e) are similar to that of (b) and (c). ◻

*Example* 3.5. We end the subsection with the following simple example. Suppose that $a_0, \rho_0$ and $\rho_1$ are equal to unity on their support; then $v_1(\lambda, y) = \cos(\sqrt{\lambda} y)$, $v_2(\lambda, y) = (1/\sqrt{\lambda})\sin(\sqrt{\lambda} y)$, and

$$D(\lambda) = 2\cos(\sqrt{\lambda} h) - \sqrt{\lambda}\sin(\sqrt{\lambda} h)(1 - h).$$

In particular, we see that conditions (3.11) and (3.12) never hold and consequently the (infinitely many) intervals

$$\big(\lambda_{2m-1}(\pi), \lambda_{2m}(\pi)\big), \qquad \big(\lambda_{2m}(0), \lambda_{2m+1}(0)\big), \qquad m \in \mathbb{N}$$

are gaps. Here $\lambda_k(0)$ (respectively, $\lambda_k(\pi)$) is the $k$th zero of $D-2$ (respectively, $D+2$). These gaps become wider as $k \to \infty$.

**3.2. Representation via a spectral decomposition.** Consider the operator $\tilde{A}_\theta$ defined on $L^2_{\rho_0}(Y_0)$ and associated to the form

$$\tilde{\beta}_\theta(u, v) := \int_{Y_0} a_0 u' \overline{v'}, \qquad u, v \in H^1_\theta(Y_0)$$

in the sense of procedure described in section 2. By virtue of the fact that the operator $\tilde{A}_\theta$ has compact resolvent, its $L^2_{\rho_0}(Y_0)$-orthonormal sequence of eigenfunctions $\{\Phi_\theta^{(n)}\}_{n\in\mathbb{N}}$ is complete in the space $L^2_{\rho_0}(Y_0)$. We denote by $\mu_n(\theta)$, $n \in \mathbb{N}$, the eigenvalues of $\Phi_\theta^{(n)} \in H^1_\theta(Y_0)$:

$$\int_{Y_0} a_0 \big(\Phi_\theta^{(n)}\big)' \overline{v'} = \mu_n(\theta) \int_{Y_0} \rho_0 \Phi_\theta^{(n)} \overline{v} \qquad \forall v \in H^1_\theta(Y_0). \tag{3.25}$$

Multiplying the first equation in (3.3) by $\overline{\Phi_\theta^{(n)}}$ and integrating by parts we have

$$\begin{aligned}\lambda\int_{Y_0}\rho_0 u\overline{\Phi_\theta^{(n)}} &= -\int_{Y_0}(a_0u')'\overline{\Phi_\theta^{(n)}}\\ &= -\big((a_0u')(h^-)\overline{\Phi_\theta^{(n)}(h)}-(a_0u')(0^+)\overline{\Phi_\theta^{(n)}(0)}\big)+\int_{Y_0}a_0u'\,\overline{\big(\Phi_\theta^{(n)}\big)'}\\ &= -\left(e^{-\mathrm{i}\theta}(a_0u')(h^-)-(a_0u')(0^+)\right)\overline{\Phi_\theta^{(n)}(0)}+\mu_n(\theta)\int_{Y_0}\rho_0 u\,\overline{\Phi_\theta^{(n)}}.\end{aligned}$$

The third equation in (3.3) implies

$$\big(\mu_n(\theta)-\lambda\big)\int_{Y_0}\rho_0 u\,\overline{\Phi_\theta^{(n)}}=\lambda u(0)\overline{\Phi_\theta^{(n)}(0)}\int_{Y_1}\rho_1.$$

Therefore, upon performing a spectral decomposition of $u$ in terms of $\Phi_\theta^{(n)}$, i.e., setting

$$u=\sum_{n\in\mathbb{N}}\zeta_n\Phi_\theta^{(n)},\qquad \zeta_n=\int_{Y_0}\rho_0 u\,\overline{\Phi_\theta^{(n)}},$$

we see that

$$\zeta_n=\frac{\lambda}{\mu_n(\theta)-\lambda}u(0)\overline{\Phi_\theta^{(n)}(0)}\int_{Y_1}\rho_1,\qquad n\in\mathbb{N}.$$

In particular, one has $u(0)=\sum_{n\in\mathbb{N}}\zeta_n\Phi_\theta^{(n)}(0)$. Thus, we arrive at the statement: if $\lambda\in\bigcup_\theta\sigma(A_\theta)$, then there exists $\theta\in[0,2\pi)$ such that

$$\sum_{n\in\mathbb{N}}\frac{\lambda}{\mu_n(\theta)-\lambda}\big|\Phi_\theta^{(n)}(0)\big|^2=\left(\int_{Y_1}\rho_1\right)^{-1}. \tag{3.26}$$

The converse statement is also true. Indeed, suppose that for some $\theta\in[0,2\pi)$ the value $\lambda$ satisfies (3.26), then we find that

$$\zeta_n:=\frac{\lambda}{\mu_n(\theta)-\lambda}u(0)\overline{\Phi_\theta^{(n)}(0)}\int_{Y_1}\rho_1,\qquad n\in\mathbb{N}$$

satisfy

$$\limsup_n\frac{|\zeta_n|^2}{b_n}=0,\qquad 0\le\limsup_n\mu_n(\theta)\frac{|\zeta_n|^2}{b_n}<\infty,\qquad b_n:=\frac{\lambda}{\mu_n(\theta)-\lambda}\big|\Phi_\theta^{(n)}(0)\big|^2.$$

Since, by assumption,

$$\sum_n b_n=\left(\int_{Y_1}\rho_1\right)^{-1},$$

it follows that

$$\sum_n|\zeta_n|^2<\infty,\qquad \sum_n\mu_n(\theta)|\zeta_n|^2<\infty,$$

that is, the function $u=\sum_{n\in\mathbb{N}}\zeta_n\Phi_\theta^{(n)}$ belongs to $H^1_\theta(Y_0)$ and, consequently, to $V_\theta$ when extended by the constant $u(h)$ into $Y_1$. Moreover, direct calculation shows that $\lambda$ and $u$ satisfy (3.1). Hence, we have shown that $\lambda\in\bigcup_\theta\sigma(A_\theta)$.

**4. Asymptotics of the defect eigenvalue problem.** Suppose $\lambda_\varepsilon$, $u_\varepsilon$ is an eigenvalue-eigenfunction pair for the defect problem, that is,

$$-(a_D^\varepsilon u_\varepsilon')' = \lambda_\varepsilon \rho_D^\varepsilon u_\varepsilon \quad \text{on } \mathbb{R}, \tag{4.1}$$

where $u_\varepsilon$ is continuous, subject to the interface conditions

$$a_D u_\varepsilon'\big|_D = a_D^\varepsilon u_\varepsilon'\big|_{\mathbb{R}\setminus D} \quad \text{on } \{d_-, d_+\} \tag{4.2}$$

and

$$a_1 u_\varepsilon'\big|_{\Omega_1^\varepsilon\setminus D} = \varepsilon^2 a_0 u_\varepsilon'\big|_{\Omega_0^\varepsilon\setminus D} \quad \text{on } \big\{x\in\mathbb{R}\setminus D : x = \varepsilon(z+h) \text{ or } x = \varepsilon z \text{ for some } z\in\mathbb{Z}\big\}. \tag{4.3}$$

In this section we study the behavior with respect to $\varepsilon$ of the eigenvalues $\lambda_\varepsilon$ and eigenfunctions $u_\varepsilon$, using asymptotic expansions. We show that, up to the leading order, the values of $\lambda_\varepsilon$ are described by an eigenvalue of the weighted Neumann–Laplacian on the defect $D$; see (4.6) below. More precisely, we show that for each eigenvalue $\lambda_0$ of (4.6) in a gap of $\bigcup_\theta \sigma(A_\theta)$, there exists a sequence of eigenvalues $\lambda_\varepsilon$ of (4.1) converging to $\lambda_0$. However, it remains unclear whether every accumulation point of $\lambda_\varepsilon$ inside a gap of $\bigcup_\theta \sigma(A_\theta)$ belongs to the spectrum of (4.6).

We seek asymptotic expansions for the eigenvalues $\lambda_\varepsilon$ and eigenfunctions $u_\varepsilon$ of (4.1)–(4.3) in the form

$$\lambda_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \dots, \tag{4.4}$$

with

$$u_\varepsilon(x) = \begin{cases} u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + \dots, & x\in(d_-, d_+),\\ w_0(\tfrac{x}{\varepsilon}) + \varepsilon^2 w_2(\tfrac{x}{\varepsilon}) + \dots, & x\in(-\infty, d_-)\cup(d_+,\infty). \end{cases} \tag{4.5}$$

We assume that functions $w_i$, $u_i$, $i = 0, 1, 2, \dots$, are continuous.

**4.1. Governing equations.** Substituting (4.4), (4.5) into (4.1) and (4.2) and equating the $\varepsilon^0$-coefficient on the defect gives

$$\begin{cases} -(a_D u_0')' = \lambda_0 \rho_D u_0 & \text{on } (d_-, d_+),\\ \ a_D u_0'|_D = 0 & \text{on } \{d_-, d_+\}, \end{cases} \tag{4.6}$$

that is, $\lambda_0$ is an eigenvalue of the weighted Neumann–Laplace operator $A_{\mathrm{N},D}$ on the defect, cf. (2.6). Note that this is true regardless of whether $d_-$, $d_+$ belong to $\Omega_1^\varepsilon$ or $\Omega_0^\varepsilon$. We fix $u_0$ by setting $\|u_0\|_{L^2_{\rho_D}(D)} = 1$.

For $c\in\mathbb{R}$, let $\lfloor c\rfloor_\varepsilon$ and $\lceil c\rceil_\varepsilon$ denote the largest integer $z$ such that $\varepsilon z \le c$ and the smallest integer $z$ such that $c \le \varepsilon z$, respectively. Substituting (4.4), (4.5) into (4.1), (4.3) and comparing the coefficients for different powers of $\varepsilon$ in the resulting expression yields

$$\begin{cases} -(a_1 w_0')' = 0 & \text{on } Y_1 + z,\\ (a_1 w_0')((z+h)^+) = 0,\\ (a_1 w_0')((z+1)^-) = 0 \end{cases} \tag{4.7}$$

and

$$\begin{cases} -(a_0 w_0')' = \lambda_0 \rho_0 w_0 & \text{on } Y_0 + z, \\ -(a_1 w_2')' = \lambda_0 \rho_1 w_0 & \text{on } Y_1 + z, \\ (a_1 w_2')((z+h)^+) = (a_0 w_0')((z+h)^-), \\ (a_1 w_2')((z+1)^-) = (a_0 w_0')((z+1)^+) \end{cases} \tag{4.8}$$

for all

$$z \in \mathcal{I}_\varepsilon := \left\{ z \in \mathbb{Z} : z \geq \lceil d_+ \rceil_\varepsilon \text{ or } z \leq \lfloor d_- \rfloor_\varepsilon - 1 \right\} \tag{4.9}$$

(that is, $z \in \mathcal{I}_\varepsilon$ if and only if the intersection of $\varepsilon(Y + z)$ and $D$ is empty). The assertion (4.7) implies that $a_1 w_0' \equiv 0$ on $Y_1 + z$ and therefore $w_0$ is constant on each such interval. By the second equation of (4.8) and the fact $w_0$ is constant on each interval $Y_1 + z$, the function $a_1 w_2'$ has the form

$$(a_1 w_2')(y) = (a_1 w_2')((z+h)^+) - \lambda_0 w_0(z+h) \int_{z+h}^{y} \rho_1, \qquad y \in Y_1 + z. \tag{4.10}$$

Combining (4.10), the fact that $w_0$ is constant on $Y_1+z$ and the first and last equations of (4.8) implies that for all $z \in \mathcal{I}_\varepsilon$, one has

$$\begin{cases} -(a_0 w_0')' = \lambda_0 \rho_0 w_0 & \text{on } Y_0 + z, \\ w_0 \equiv w_0(z+h) = w_0(z+1) & \text{on } Y_1 + z, \\ (a_0 w_0')\big((z+1)^+\big) - (a_0 w_0')\big((z+h)^-\big) = -\lambda_0 w_0(z+h) \int_{Y_1} \rho_1. \end{cases} \tag{4.11}$$

The problem (4.11) fully governs the behavior of $w_0$ in $\mathbb{R} \backslash (\lfloor d_- \rfloor_\varepsilon - 1, \lceil d_+ \rceil_\varepsilon)$. We can utilize the fundamental system $(v_1, v_2)$ from section 3.1 to quantitatively characterize $w_0$. Indeed, since in each cell $Y + z$ any solution to the first equation in (4.11) is a linear combination of $v_1$ and $v_2$, one has

$$w_0(y) = \begin{cases} l_z v_1(y - z) + m_z v_2(y - z), & y \in Y_0 + z, \\ l_z v_1(h) + m_z v_2(h), & y \in Y_1 + z \end{cases} \tag{4.12}$$

for constants $l_z, m_z$, $z \in \mathcal{I}_\varepsilon$, where the expression on $Y_1 + z$ follows from the second condition in (4.11). Using (3.5), the continuity of $w_0$ and the jump of the coderivative condition from (4.11), it is not difficult to derive the following recurrence relation:

$$\begin{pmatrix} l_{z+1} \\ m_{z+1} \end{pmatrix} = \begin{pmatrix} v_1(h) & v_2(h) \\ (a_0 v_1')(h) - \lambda_0 v_1(h) \int_{Y_1} \rho_1 & (a_0 v_2')(h) - \lambda_0 v_2(h) \int_{Y_1} \rho_1 \end{pmatrix} \begin{pmatrix} l_z \\ m_z \end{pmatrix}. \tag{4.13}$$

Now, recalling the Wronskian property (3.6), we find that the characteristic polynomial $q$ of the matrix in (4.13) is (cf. (2.10))

$$q(\mu) = \mu^2 - \left( v_1(h) + (a_0 v_2')(h) - \lambda_0 v_2(h) \int_{Y_1} \rho_1 \right) \mu + 1. \tag{4.14}$$

The roots $\mu_1$, $\mu_2$ of $q$ satisfy the identity $\mu_1\mu_2 = 1$ and the nature of $w_0$ as it varies from one period to the next is determined by the quantity $v_1(h)+(a_0v_2')(h)-\lambda_0 v_2(h)\int_{Y_1}\rho_1$. Namely, if (cf. (3.10))

$$\left|v_1(h) + (a_0v_2')(h) - \lambda_0 v_2(h)\int_{Y_1}\rho_1\right| \le 2,$$

then the roots $\mu_1$, $\mu_2$ are complex conjugate with $|\mu_1| = |\mu_2| = 1$ and solutions $w_0$ are described by the linear span of two quasi-periodic functions with phase difference $\pi$. In section 3 we demonstrated that $\lambda_0$ satisfies this constraint if and only if $\lambda_0$ belongs to the limit spectrum

$$\lim_{\varepsilon\to 0}\sigma(A^\varepsilon) = \bigcup_\theta \sigma(A_\theta).$$

**4.2. Construction of an approximate solution.** For $\lambda_0$ in the gaps of this limit spectrum, i.e., when $\lambda_0$ satisfies the inequality

$$\left|v_1(h) + (a_0v_2')(h) - \lambda_0 v_2(h)\int_{Y_1}\rho_1\right| > 2,$$

the roots $\mu_1, \mu_2$ of $q$ (see (4.14)) satisfy $|\mu_1| < 1$ and $|\mu_2| > 1$. For such $\lambda_0$, we can construct "unstable" solutions, one of which decays at $+\infty$ and the other at $-\infty$. Indeed, denoting by $\varkappa_1$ and $\varkappa_2$ the eigenvectors of the matrix in (4.13) corresponding to $\mu_1$ and $\mu_2$, respectively, we find in the interval $[\lceil d_+\rceil_\varepsilon, \infty)$ that $w_0$ given by (4.12), (4.13) satisfies $w_0(y+1) = \mu_j w_0(y)$ if $(l_{\lceil d_+\rceil_\varepsilon}, m_{\lceil d_+\rceil_\varepsilon})^\top = \varkappa_j$, $j = 1, 2$. Similarly, in the interval $(-\infty, \lfloor d_-\rfloor_\varepsilon]$, one has $w_0(y) = \mu_j w_0(y-1)$ if $(l_{\lfloor d_-\rfloor_\varepsilon - 1}, m_{\lfloor d_-\rfloor_\varepsilon - 1})^\top = \varkappa_j$, $j = 1, 2$. For $w_0$ to decay to the left and right of the defect, we set $(l_{\lceil d_+\rceil_\varepsilon}, m_{\lceil d_+\rceil_\varepsilon})^\top = \varkappa_1$ and $(l_{\lfloor d_-\rfloor_\varepsilon - 1}, m_{\lfloor d_-\rfloor_\varepsilon - 1})^\top = \varkappa_2$. In this way we ensure that

$$\begin{aligned} &w_0(y+1) = \mu_1 w_0(y) && \text{for } y \in [d_+/\varepsilon, \infty),\\ &w_0(y-1) = \mu_2^{-1} w_0(y) = \mu_1 w_0(y) && \text{for } y \in (-\infty, d_-/\varepsilon], \end{aligned} \tag{4.15}$$

where we have extended $w_0$ to the intervals $[d_+/\varepsilon, \lceil d_+\rceil_\varepsilon)$ and $(\lfloor d_-\rfloor_\varepsilon, d_-/\varepsilon]$ by the formulae $w_0(y) = \mu_1^{-1} w_0(y+1)$ and $w_0(y) = \mu_1^{-1} w_0(y-1)$, respectively.

The function $w_0$ to the right and to the left from the defect is defined up to multiplication by a constant. The next natural step is to "attach" both parts of $w_0$ to the solution $u_0$ on the defect choosing the aforementioned constants appropriately. However, there is a possibility that for particular values $\varepsilon$ one has $w_0(d_+/\varepsilon) = 0$ or $w_0(d_-/\varepsilon) = 0$. This requires that $w_0$ be redefined near the boundary of $D$, which we do next.

On each side of the defect $D$, there are two possibilities on the stiff component $\Omega_1 \cap [d_+/\varepsilon, \infty)$ (or $\Omega_1 \cap (-\infty, d_-/\varepsilon]$): either $w_0$ does not vanish or $w_0 \equiv 0$. In the latter case, since $w_0$ is not identically zero on the whole interval $[d_+/\varepsilon, \infty)$ (or $(-\infty, d_-/\varepsilon]$), it necessarily has an extremum inside each soft interval $Y_0 + z$ and $a_0 w_0' = 0$ at the points of extrema. If $w_0$ does not vanish on the stiff component, we set

$$w_0(y) = w_0(\lceil d_+\rceil_\varepsilon + h), \qquad y \in [d_+/\varepsilon, \lceil d_+\rceil_\varepsilon + h], \tag{4.16a}$$

$$w_0(y) = w_0(\lfloor d_-\rfloor_\varepsilon), \qquad y \in [\lfloor d_-\rfloor_\varepsilon, d_-/\varepsilon], \tag{4.16b}$$

while in the case when $w_0 \equiv 0$ on the stiff component there exist $y_+^*, y_-^* \in Y_0$, independent of $\varepsilon$, such that

$$\left|w_0(\lceil d_+\rceil_\varepsilon + y_+^*)\right| = \max_{Y_0+\lceil d_+\rceil_\varepsilon} |w_0| > 0,$$
$$\left|w_0(\lfloor d_-\rfloor_\varepsilon - 1 + y_-^*)\right| = \max_{Y_0+\lfloor d_-\rfloor_\varepsilon - 1} |w_0| > 0,$$

and we set

$$w_0(y) = w_0(\lceil d_+\rceil_\varepsilon + y_+^*), \qquad y \in [d_+/\varepsilon, \lceil d_+\rceil_\varepsilon + y_+^*], \tag{4.17a}$$

$$w_0(y) = w_0(\lfloor d_-\rfloor_\varepsilon - 1 + y_-^*), \qquad y \in [\lfloor d_-\rfloor_\varepsilon - 1 + y_-^*, d_-/\varepsilon]. \tag{4.17b}$$

We then choose $\varkappa_1$ and $\varkappa_2$ so that the modified $w_0$ matches the value of $u_0$ at the end-points of $D$:

$$w_0(d_+/\varepsilon) = u_0(d_+), \quad w_0(d_-/\varepsilon) = u_0(d_-).$$

Putting together (4.12), (4.16a)–(4.17b), it follows that the vectors $\varkappa_1$, $\varkappa_2$ do not depend on $\varepsilon$. Hence, it is not difficult to see that so constructed $w_0$ is bounded in $L^2_\rho$ uniformly in $\varepsilon$,

$$\|w_0\|_{L^2_\rho(\mathbb{R}\setminus\varepsilon^{-1}D)} \leq C. \tag{4.18}$$

Next, we construct the corrector $w_2$, treating first the right side of the defect. According to the two possibilities above, we start by assuming that (4.16a) holds. The second and the third equations in (4.8) determine $w_2$ up to an arbitrary additive constant in each interval

$$Y_1 + z \subset \big[\lceil d_+\rceil_\varepsilon + 1 + h, +\infty\big). \tag{4.19}$$

The choice of these constants is not important, except that $w_2$ should remain "controlled," and in what follows, for simplicity, we set

$$w_2(z+h) = 0. \tag{4.20}$$

Note that the existence of $w_2$ satisfying the second and third equations in (4.8) follows from the last identity in (4.11). On the interval $Y_1 + \lceil d_+\rceil_\varepsilon$ we only require $w_2$ to satisfy the following conditions at its boundary:

$$(a_1 w_2')\big((\lceil d_+\rceil_\varepsilon + 1)^-\big) = (a_0 w_0')\big((\lceil d_+\rceil_\varepsilon + 1)^+\big),$$
$$(a_1 w_2')\big((\lceil d_+\rceil_\varepsilon + h)^+\big) = w_2(\lceil d_+\rceil_\varepsilon + h) = 0.$$

To this end we fix two smooth functions $f_1$ and $f_2$ such that $0 \leq f_1 \leq 1$, $f_1(1) = 1$, $f_1(y) = 0$ for $y \in [h, (h+1)/2]$, $f_2(y) = 0$ for $y = h$, and $y \in [(h+1)/2, 1]$, $f_2(y) < 0$ for $y \in (h, (h+1)/2)$. We define $w_2$ on $Y_1 + \lceil d_+\rceil_\varepsilon$ by

$$w_2(\lceil d_+\rceil_\varepsilon + y) := -(a_0 w_0')\big((\lceil d_+\rceil_\varepsilon + 1)^+\big) \int_y^1 a_1^{-1}(f_1 + c f_2), \quad y \in Y_1, \tag{4.21}$$

choosing the constant $c$ so that $w_2(\lceil d_+\rceil_\varepsilon + h) = 0$. Moreover, we have $w_2(\lceil d_+\rceil_\varepsilon + 1) = 0$. We set $w_2 = 0$ on $[d_+/\varepsilon, \lceil d_+\rceil_\varepsilon + h] \cup [\lceil d_+\rceil_\varepsilon + 1, \lceil d_+\rceil_\varepsilon + 1 + h]$. Finally, in the intervals $Y_0 + z \subset [\lceil d_+\rceil_\varepsilon + 2, +\infty)$ we do not require $w_2$ to satisfy any equation. Instead we make a specific choice of $w_2$ as follows. For a nonnegative function $f \in C_0^\infty(Y_0)$, $f \not\equiv 0$, we define

$$w_2(z+y) := w_2(z) + c_z \int_0^y \frac{f}{a_0}, \quad y \in Y_0, \quad z \geq \lceil d_+\rceil_\varepsilon + 2, \tag{4.22}$$

where the coefficient $c_z$ is chosen so that $w_2$ is continuous on $[d_+/\varepsilon, +\infty)$, namely,

$$c_z = -w_2(z)\left(\int_{Y_0} \frac{f}{a_0}\right)^{-1}. \tag{4.23}$$

In particular, we have

$$(a_0 w_2')(z^+) = (a_0 w_2')((z+h)^-) = 0.$$

Moving on to the second possibility, we assume that (4.17a) holds. Then on the intervals $Y_1 + z \subset [\lceil d_+ \rceil_\varepsilon + h, +\infty)$ we choose $w_2$ to satisfy the second and third equations in (4.8) and the condition $w_2(z+h) = 0$. We extend $w_2$ by zero on $[d_+/\varepsilon, \lceil d_+ \rceil_\varepsilon + h)$, and on the intervals $Y_0 + z \subset [\lceil d_+ \rceil_\varepsilon + 1, +\infty)$ we define $w_2$ as in (4.22).

We define $w_2$ to the left of the defect in a similar way. Namely, we assume first that (4.16b) holds and define $w_2$ according to (4.8) in the intervals $Y_1+z \subset (-\infty, \lfloor d_- \rfloor_\varepsilon - 1]$, requiring $w_2(z+1) = 0$. On $[\lfloor d_- \rfloor_\varepsilon - 1 + h, \lfloor d_- \rfloor_\varepsilon]$ we define $w_2$ by a formula analogous to (4.21) so that it satisfies the conditions

$$(a_1 w_2')\big((\lfloor d_- \rfloor_\varepsilon - 1 + h)^+\big) = (a_0 w_0')\big((\lfloor d_- \rfloor_\varepsilon - 1 + h)^-\big),$$
$$(a_1 w_2')\big((\lfloor d_- \rfloor_\varepsilon)^-\big) = w_2(\lfloor d_- \rfloor_\varepsilon) = 0.$$

We then extend $w_2$ by zero on $(\lfloor d_- \rfloor_\varepsilon, d_-/\varepsilon]$ and define it on the intervals $Y_0 + z \subset (-\infty, \lfloor d_- \rfloor_\varepsilon - 1 + h]$ according to (4.22).

Finally, if (4.17b) holds we define $w_2$ according to (4.8) on the intervals $Y_1 + z \subset (-\infty, \lfloor d_- \rfloor_\varepsilon - 1]$, additionally requiring that $w_2(z+1) = 0$, extend $w_2$ by zero into $(\lfloor d_- \rfloor_\varepsilon - 1, d_-/\varepsilon]$, and use (4.22) to define $w_2$ on the intervals $Y_0 + z \subset (-\infty, \lfloor d_- \rfloor_\varepsilon - 1 - h]$.

**4.3. Justification of asymptotics.** First we estimate the term $w_2$. Assume that (4.16a) holds, and consider $w_2$ on $[d_+/\varepsilon, +\infty)$. A straightforward calculation gives (cf. (4.8), (4.20))

$$w_2(z+y) = (a_0 w_0')((z+h)^-) \int_h^y a_1^{-1} - \lambda_0 w_0(z+h) \int_h^y \left(a_1^{-1}(\cdot) \int_h^{\cdot} \rho_1\right), \tag{4.24}$$
$$z \geq \lceil d_+ \rceil_\varepsilon + 1, \quad y \in Y_1.$$

It follows from (4.22), (4.23), (4.24), and (4.15) that

$$w_2(y+1) = \mu_1 w_2(y) \quad \text{for } y \in [\lceil d_+ \rceil_\varepsilon + 1 + h, +\infty),$$

and, thereupon,

$$((a_0 + a_1) w_2')'(y+1) = \mu_1 ((a_0 + a_1) w_2')'(y) \quad \text{for } y \in [\lceil d_+ \rceil_\varepsilon + 1 + h, +\infty).$$

With $w_2$ and $((a_0+a_1)w_2')'$ clearly bounded in $L^2_\rho(d_+/\varepsilon, \lceil d_+ \rceil_\varepsilon + 1 + h)$ independently of $\varepsilon$ (cf. (4.21)), we conclude that

$$\|((a_0 + a_1) w_2')'\|_{L^2_\rho(d_+/\varepsilon, +\infty)} + \|w_2\|_{L^2_\rho(d_+/\varepsilon, +\infty)} \leq C$$

for a suitable constant $C$ independent of $\varepsilon$.

In an analogous way we derive the estimates for $w_2$ on $[d_+/\varepsilon, +\infty)$ in the case (4.17a) and on $(-\infty, d_-/\varepsilon]$ in the cases (4.16b), (4.17b). Thus we assert

$$\|((a_0+a_1)w_2')'\|_{L^2_\rho(\mathbb{R}\setminus\varepsilon^{-1}D)} + \|w_2\|_{L^2_\rho(\mathbb{R}\setminus\varepsilon^{-1}D)} \le C. \tag{4.25}$$

Suppose now that $\lambda_0 \in \sigma\big(A_{\mathrm{N},D}\big)\setminus\big(\bigcup_\theta \sigma(A_\theta)\big)$. The construction described above guarantees that the function

$$u_{\varepsilon,\mathrm{ap}}(x) := \begin{cases} u_0(x), & x \in D, \\ w_0(x/\varepsilon) + \varepsilon^2 w_2(x/\varepsilon), & x \in \mathbb{R}\setminus D \end{cases} \tag{4.26}$$

is continuous and has a continuous coderivative $a^\varepsilon_D u'_{\varepsilon,\mathrm{ap}}$, implying that $u_{\varepsilon,\mathrm{ap}}$ belongs to the domain of the operator $A^\varepsilon_D$.

It follows from the spectral theorem for self-adjoint operators (see, e.g., [4]) that for all functions $f \in \mathrm{dom}\big(A^\varepsilon_D\big) \subset L^2_{\rho^\varepsilon_D}(\mathbb{R})$ such that $\|f\|_{L^2_{\rho^\varepsilon_D}(\mathbb{R})} = 1$, one has

$$\mathrm{dist}\big(\lambda_0, \sigma\left(A^\varepsilon_D\right)\big) \le \big\|(A^\varepsilon_D - \lambda_0)f\big\|_{L^2_{\rho^\varepsilon_D}(\mathbb{R})}.$$

Straightforward calculations show that, except for the small regions near the boundary of the defect, we have

$$\begin{aligned} &\rho^\varepsilon_D(A^\varepsilon_D-\lambda_0)u_{\varepsilon,\mathrm{ap}} \\ &\quad= \begin{cases} 0, & x\in D, \\ -\varepsilon^2(a_0w_2')'(x/\varepsilon) - \varepsilon^2\lambda_0\rho_0(x/\varepsilon)w_2(x/\varepsilon), & x \in \Omega^\varepsilon_0 \setminus \varepsilon[\lfloor d_-\rfloor_\varepsilon - 1, \lceil d_+\rceil_\varepsilon + 1], \\ -\varepsilon^2\lambda_0\rho_1(x/\varepsilon)w_2(x/\varepsilon), & x \in \Omega^\varepsilon_1 \setminus \varepsilon[\lfloor d_-\rfloor_\varepsilon - 1, \lceil d_+\rceil_\varepsilon + 1]. \end{cases} \end{aligned} \tag{4.27}$$

Near the boundary we need to consider each of the cases (4.16a)–(4.17b) separately. Assume first that (4.16a) holds. Then we have

$$\begin{aligned} &\rho^\varepsilon_D(A^\varepsilon_D-\lambda_0)u_{\varepsilon,\mathrm{ap}} \\ &\quad= \begin{cases} -\lambda_0\rho(x/\varepsilon)w_0(x/\varepsilon), & x/\varepsilon \in [d_+/\varepsilon, \lceil d_+\rceil_\varepsilon + h], \\ -\lambda_0\rho_1(x/\varepsilon)w_0(x/\varepsilon) - (a_1w_2')'(x/\varepsilon) & \\ \qquad\qquad - \varepsilon^2\lambda_0\rho_1(x/\varepsilon)w_2(x/\varepsilon), & x/\varepsilon \in [\lceil d_+\rceil_\varepsilon + h, \lceil d_+\rceil_\varepsilon + 1]. \end{cases} \end{aligned}$$

By construction, $w_0, w_2$ and $(a_1w_2')'$ are bounded continuous functions independent of $\varepsilon$. Hence $\rho^\varepsilon_D(A^\varepsilon_D-\lambda_0)u_{\varepsilon,\mathrm{ap}}$ is bounded in $L^\infty(d_+, \varepsilon(\lceil d_+\rceil_\varepsilon + 1))$. Performing direct calculations and applying a similar argument in the three remaining cases, we conclude that

$$\big\|\rho^\varepsilon_D(A^\varepsilon_D-\lambda_0)u_{\varepsilon,\mathrm{ap}}\big\|_{L^\infty(\varepsilon[\lfloor d_-\rfloor_\varepsilon - 1, \lceil d_+\rceil_\varepsilon + 1]\setminus D)} \le C.$$

Since the size of the region $\varepsilon[\lfloor d_-\rfloor_\varepsilon - 1, \lceil d_+\rceil_\varepsilon + 1] \setminus D$ is of order $\varepsilon$, the latter inequality together with (4.27) and (4.25) readily implies that

$$\big\|(A^\varepsilon_D - \lambda_0)u_{\varepsilon,\mathrm{ap}}\big\|_{L^2_{\rho^\varepsilon_D}(\mathbb{R})} \le C\varepsilon^{1/2} \tag{4.28}$$

for some constant $C > 0$.

We establish the following result, which implies claim 1 of Theorem 2.4. In particular, the second and third estimates in (2.7) follow from (4.18), (4.25), (4.26), the estimate (4.30) below and the identity

$$\big\|w_i(\cdot/\varepsilon)\big\|_{L^2_{\rho^\varepsilon_D}(\mathbb{R}\setminus D)} = \varepsilon^{1/2}\|w_i\|_{L^2_\rho(\mathbb{R}\setminus\varepsilon^{-1}D)},\ i=0,2. \tag{4.29}$$

THEOREM 4.1. *Suppose that* $\lambda_0\in\sigma\big(A_{\mathrm{N},D}\big)\setminus\big(\bigcup_\theta\sigma(A_\theta)\big)$.

1. *There exists* $C_1>0$*, independent of* $\varepsilon$*, such that*
$$\operatorname{dist}\big(\lambda_0,\sigma(A^\varepsilon_D)\big)\le C_1\varepsilon^{1/2}.$$
2. *For sufficiently small* $\varepsilon$ *there exist (simple) eigenvalues* $\lambda_\varepsilon$ *of* $A^\varepsilon_D$ *such that* $|\lambda_\varepsilon-\lambda_0|\le C_1\varepsilon^{1/2}$.
3. *For sufficiently small* $\varepsilon$ *the function* $u_{\varepsilon,\mathrm{ap}}$ *is an approximate eigenfunction of* $A^\varepsilon_D$*, in the sense that there exists an* $\varepsilon$*-independent constant* $C_2>0$ *and* $c^\varepsilon_j\in\mathbb{R}$ *such that*
$$\Big\|u_{\varepsilon,\mathrm{ap}}-\sum_{j\in J_\varepsilon}c^\varepsilon_j u_{\varepsilon,j}\Big\|_{L^2_{\rho^\varepsilon_D}(\mathbb{R})}\le C_2\varepsilon^{1/2}, \tag{4.30}$$
*where the set* $J_\varepsilon$ *is defined by* (2.8)*, and* $u_{\varepsilon,j}$ *are appropriate eigenfunctions of* $A^\varepsilon_D$.

*Proof.* Claim 1 of the theorem follows from (4.28) and the fact that

$$\lim_{\varepsilon\to0}\|u_{\varepsilon,\mathrm{ap}}\|_{L^2_{\rho^\varepsilon_D}(\mathbb{R})}=\|u_0\|_{L^2_{\rho_D}(D)}=1,$$

due to (4.18), (4.25), and (4.29). Claim 2 follows by noting that the essential spectra of $A^\varepsilon_D$ and $A^\varepsilon$ coincide and that $\sigma(A^\varepsilon)=\sigma_{\mathrm{ess}}(A^\varepsilon)$ converges, as $\varepsilon\to0$, to $\bigcup_\theta\sigma(A_\theta)$, which $\lambda_0$ does not belong to. To prove claim 3, one can argue as in [19], or [11, section 11.1], using (4.28) and a spectral decomposition of $u_{\varepsilon,\mathrm{ap}}$ with respect to the operator $A^\varepsilon_D$. ∎

**4.4. Improvement of the error bound.** It is clear from the construction of $u_{\varepsilon,\mathrm{ap}}$ that the main error term of order $\varepsilon^{1/2}$ comes from what is conventionally called boundary layer, near the endpoints of the defect $D$. In fact, one can improve the error bound (4.28) by "attaching" the $\varepsilon$-periodic structure to the defect in an appropriate way, thereby preventing the appearance of the boundary effect. Our approach is based on the behavior of the function $w_0$; see the observation made in the beginning of section 4.2 preceding the adjustment of $w_0$. We provide the detailed construction only at the right end of the defect $D$. The construction at the left end is completely analogous.

First, let us assume that $w_0$ in (4.15) has no extrema inside the soft intervals $[d_+/\varepsilon,\infty)\cap\Omega_0$. Then $w_0$ does not vanish on the stiff component $[d_+/\varepsilon,\infty)\cap\Omega_1$. In this case we attach the periodic structure to $D$ so that it touches the soft component, i.e., we define the soft and stiff components to the right of $D$ via $\Omega_0^+:=\bigcup_z(Y_0+z+d_+/\varepsilon)$ and $\Omega_1^+:=\bigcup_z(Y_1+z+d_+/\varepsilon)$, respectively, $z=0,1,\dots$. The definition of the relevant notation, such as coefficients $a^\varepsilon_D$, $\rho^\varepsilon_D$, etc., should be adjusted in an obvious way, however, we will not dwell on this. We define $w_0$ on $[d_+/\varepsilon,\infty)$ according to (4.11), (4.15), requiring $w_0(d_+/\varepsilon)=u_0(d_+)$. We construct $w_2$ on $[d_+/\varepsilon+h,\infty)$ according to (4.8) and (4.22), requiring $w_2(d_+/\varepsilon+h+z)=0$, $z=0,1,\dots$, and set $w_2\equiv0$ on $Y_0+d_+/\varepsilon$. Now the coderivative of $w_0(x/\varepsilon)+\varepsilon^2w_2(x/\varepsilon)$ at $d_+$ is equal to $\varepsilon(a_0w_0')((d_+/\varepsilon)^+)$. We define a corrector $\varepsilon u_1$ in $D$ by setting (cf. (4.21))

$$u_1(x) := -(a_0 w_0')\big((d_+/\varepsilon)^+\big) \int_x^{d_+} a_D^{-1}(f_1 + cf_2), \quad x \in D.$$

Here the smooth functions $f_1$ and $f_2$ are chosen in the following way. Let the points $d_1$ and $d_2$ be such that $(d_- + d_+)/2 \le d_1 < d_2 < d_+$. We require that $0 \le f_1 \le 1$, $f_1(d_+) = 1$, $f_1(x) = 0$ for $x \in [d_-, d_2]$, $f_2(x) = 0$ for $x \in [d_-, d_1] \cup [d_2, d_+]$, $f_2(x) < 0$ for $x \in (d_1, d_2)$. Finally, the constant $c$ is chosen so that $u_1 \equiv 0$ on $[d_-, d_1]$.

Now we assume that $w_0$ in (4.15) has extrema inside the soft component $[d_+/\varepsilon, \infty) \cap \Omega_0$, i.e., there exists a point $y_+^* \in Y_0$ such that $|w_0(y_+^* + z)| = \max_{Y_0+z} |w_0|$ and $(a_0 w_0')(y_+^* + z) = 0$ for all $z$ satisfying $Y_0 + z \subset [d_+/\varepsilon, \infty)$ (note that $w_0$ may or may not vanish on the stiff component—it is not important). This situation is rather generic, for example, for constant $a_0$ and $\rho_0$ this assumption is true for any $\lambda_0 > a_0\pi^2/(\rho_0 h^2)$. In this case we attach the periodic structure to $D$ at the point where $|w_0|$ attains its maximum: we define the soft and stiff components to the right of $D$ via $\Omega_0^+ := \bigcup_z (Y_0 - y_+^* + z + d_+/\varepsilon) \cap (d_+/\varepsilon, \infty)$, and $\Omega_1^+ := \bigcup_z (Y_1 - y_+^* + z + d_+/\varepsilon)$ respectively, $z = 0, 1, \ldots$. Analogously to the above, $w_0$ is defined according to (4.8), (4.15), requiring $w_0(d_+/\varepsilon) = u_0(d_+)$, and $w_2$ is defined according to (4.11) and (4.22) on $[d_+/\varepsilon + h - y_+^*, \infty)$, requiring $w_2(d_+/\varepsilon + h - y_+^* + z) = 0$, $z = 0, 1, \ldots$, and we set $w_2 \equiv 0$ on $[0, h - y_+^*] + d_+/\varepsilon$. Since the coderivative of $w_0(x/\varepsilon) + \varepsilon^2 w_2(x/\varepsilon)$ vanishes at $d_+$ by construction, we do not need any corrector term in $D$ contrary to the previous case. Thus we set $u_1 \equiv 0$ in this case.

One can perform analogous construction on the left of $D$. In any case the new approximate solution

$$u_{\varepsilon,\mathrm{ap}}(x) := \begin{cases} u_0(x) + \varepsilon u_1(x), & x \in D, \\ w_0(x/\varepsilon) + \varepsilon^2 w_2(x/\varepsilon), & x \in \mathbb{R} \setminus D \end{cases}$$

belongs to the domain of $A_D^\varepsilon$ and satisfies

$$\rho_D^\varepsilon (A_D^\varepsilon - \lambda_0) u_{\varepsilon,\mathrm{ap}} = \begin{cases} -\varepsilon (a_D u_1')'(x) - \varepsilon \lambda_0 \rho_D u_1(x), & x \in D, \\ -\varepsilon^2 (a_0 w_2')'(x/\varepsilon) - \varepsilon^2 \lambda_0 \rho_0(x/\varepsilon) w_2(x/\varepsilon), & x \in \Omega_0^\varepsilon \setminus D, \\ -\varepsilon^2 \lambda_0 \rho_1(x/\varepsilon) w_2(x/\varepsilon), & x \in \Omega_1^\varepsilon \setminus D. \end{cases}$$

(We remind the reader that the notation $a_D^\varepsilon$, $\rho_D^\varepsilon$, $\Omega_0^\varepsilon$ and $\Omega_1^\varepsilon$ has to be redefined accordingly to the above construction in each case).

In the case if $w_0$ has no extrema inside the soft component at least on one of the intervals $(-\infty, d_-/\varepsilon]$ or $[d_+/\varepsilon, \infty)$, the term $u_1$ is nonzero. Then, similarly to (4.28), we obtain an improved estimate

$$\big\|(A_D^\varepsilon - \lambda_0) u_{\varepsilon,\mathrm{ap}}\big\|_{L^2_{\rho_D^\varepsilon}(\mathbb{R})} \le C\varepsilon.$$

However, if $w_0$ has extrema inside the soft component on each of the intervals $(-\infty, d_-/\varepsilon]$ and $[d_+/\varepsilon, \infty)$, the term $u_1 \equiv 0$, and we have

$$\big\|(A_D^\varepsilon - \lambda_0) u_{\varepsilon,\mathrm{ap}}\big\|_{L^2_{\rho_D^\varepsilon}(\mathbb{R})} \le C\varepsilon^2.$$

The improved estimates for the error term immediately imply the following statement.

THEOREM 4.2. *Suppose that* $\lambda_0 \in \sigma\big(A_{\mathrm{N},D}\big)\backslash\big(\bigcup_\theta \sigma(A_\theta)\big)$ *and additionally the solution* $w_0$ *in* (4.15) *has extrema inside the soft component on each of the intervals* $(-\infty, d_-/\varepsilon]$ *and* $[d_+/\varepsilon, \infty)$ *(respectively, has no extrema inside the soft component at least on one of the intervals* $(-\infty, d_-/\varepsilon]$ *or* $[d_+/\varepsilon, \infty)$*). Then one can attach composite structures on both sides of the defect* $D$ *and define the approximate solution* $u_{\varepsilon,\mathrm{ap}}$ *to the eigenvalue problem in a specific way described above, so that the assertions of Theorem* 4.1 *hold with the improved estimates*

$$
\begin{aligned}
&\operatorname{dist}\big(\lambda_0, \sigma(A_D^\varepsilon)\big) \le C_1\varepsilon^2 \text{ (resp., } \le C_1\varepsilon),\\
&\big|\lambda_\varepsilon - \lambda_0\big| \le C_1\varepsilon^2 \text{ (resp., } \le C_1\varepsilon),\\
&\Big\|u_{\varepsilon,\mathrm{ap}} - \sum_{j\in J_\varepsilon} c_j^\varepsilon u_{\varepsilon,j}\Big\|_{L^2_{\rho_D^\varepsilon}(\mathbb{R})} \le C_2\varepsilon^2 \text{ (resp., } \le C_2\varepsilon)
\end{aligned}
\tag{4.31}
$$

*for some* $C_1$, $C_2 > 0$.

*Remark* 4.3. In the above theorem the attached structures do not need to be periodic extensions of each other. In case of "nonmatching" periodic structures on each side of the defect the essential spectrum of the resulting operator is exactly the same as in the purely periodic case without the defect. This can easily be seen by considering Weyl's sequences in each of the cases.

**5. Extreme localization of defect eigenfunctions.** The method of asymptotic expansions allows us to show that for any eigenvalue $\lambda_0$ of $A_{\mathrm{N},D}$ (cf. (2.6), (4.6)) in a gap of $\bigcup_\theta \sigma(A_\theta)$ there exists a sequence of eigenvalues of $A_D^\varepsilon$ converging to $\lambda_0$. In this section we provide a statement on the rate of decay of eigenfunctions of $A_D^\varepsilon$ outside the defect. Namely, the fact that one-dimensional problems admit an explicit form of solutions in terms of the fundamental system allows us to show that the eigenfunctions $u_\varepsilon$ decay at an accelerated exponential rate outside of the defect, which is claim 2 of Theorem 2.4.

We assume a sequence of eigenvalues $\lambda_\varepsilon$ of $A_D^\varepsilon$ converges to $\lambda_0 \in \mathbb{R}\backslash\bigcup_\theta \sigma(A_\theta)$ as $\varepsilon \to 0$, and consider the corresponding sequence $u_\varepsilon$ of $L^2(\mathbb{R})$-normalized eigenfunctions, i.e.,

$$
\int_{\mathbb{R}} a_D^\varepsilon u_\varepsilon' \varphi' = \lambda_\varepsilon \int_{\mathbb{R}} \rho_D^\varepsilon u_\varepsilon \varphi \qquad \forall \varphi \in H^1(\mathbb{R}).
$$

Recalling the unitary operator $\mathcal{R}_\varepsilon : L^2_{\rho_\varepsilon}(\mathbb{R}) \to L^2_\rho(\mathbb{R})$ given by $\mathcal{R}_\varepsilon(f)(y) = \varepsilon^{1/2} f(\varepsilon y)$, we note that for all $z \in \mathcal{I}_\varepsilon$ (see (4.9)), the function $\tilde{u}_\varepsilon := \mathcal{R}_\varepsilon u_\varepsilon$ solves

$$
-(a_0\tilde{u}_\varepsilon')' = \lambda_\varepsilon \rho_0 \tilde{u}_\varepsilon \qquad \text{on } Y_0 + z, \tag{5.1}
$$

$$
-\varepsilon^{-2}(a_1\tilde{u}_\varepsilon')' = \lambda_\varepsilon \rho_1 \tilde{u}_\varepsilon \qquad \text{on } Y_1 + z \tag{5.2}
$$

and satisfies the interface conditions

$$
\begin{aligned}
\tilde{u}_\varepsilon|_{Y_0+z}(z+h) = \tilde{u}_\varepsilon|_{Y_1+z}(z+h), \qquad (a_0\tilde{u}_\varepsilon')\big((z+h)^-\big) = \varepsilon^{-2}(a_1\tilde{u}_\varepsilon')\big((z+h)^+\big),\\
\tilde{u}_\varepsilon|_{Y_0+z+1}(z+1) = \tilde{u}_\varepsilon|_{Y_1+z}(z+1), \qquad (a_0\tilde{u}_\varepsilon')\big((z+1)^+\big) = \varepsilon^{-2}(a_1\tilde{u}_\varepsilon')\big((z+1)^-\big).
\end{aligned}
\tag{5.3}
$$

There exist solutions $v_1^\varepsilon, v_2^\varepsilon$ to the equation $-(a_0u')' = \lambda_\varepsilon\rho_0 u$, on $Y_0$, and solutions $w_1^\varepsilon, w_2^\varepsilon$ to the equation $-\varepsilon^{-2}(a_1u')' = \lambda_\varepsilon\rho_1 u$, on $Y_1$, such that

(5.4)
$$\left(\begin{array}{cc} v_1^\varepsilon & v_2^\varepsilon \\ a_0 v_1^{\varepsilon\prime} & a_0\, v_2^{\varepsilon\prime}\end{array}\right)\Bigg|_{y=0} = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1\end{array}\right), \qquad \left(\begin{array}{cc} w_1^\varepsilon & w_2^\varepsilon \\ a_1 w_1^{\varepsilon\prime} & a_1\, w_2^{\varepsilon\prime}\end{array}\right)\Bigg|_{y=h} = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1\end{array}\right).$$

The solution $\tilde{u}_\varepsilon$ to (5.1), (5.2), $z \in \mathcal{I}_\varepsilon$, admits the representation

$$\tilde{u}_\varepsilon(y) = \begin{cases} a_z^\varepsilon v_1^\varepsilon(y-z) + b_z^\varepsilon v_2^\varepsilon(y-z), & y \in Y_0 + z, \\ c_z^\varepsilon w_1^\varepsilon(y-z) + d_z^\varepsilon w_2^\varepsilon(y-z), & y \in Y_1 + z. \end{cases} \tag{5.5}$$

For all $\varepsilon$, the coefficients $a_z^\varepsilon, b_z^\varepsilon$ $c_z^\varepsilon$ and $d_z^\varepsilon$, $z \in \mathcal{I}_\varepsilon$, are related to each other by the conditions (5.3), as follows:

$$c_z^\varepsilon = a_z^\varepsilon v_1^\varepsilon(h) + b_z^\varepsilon v_2^\varepsilon(h), \qquad \varepsilon^{-2} d_z^\varepsilon = a_z^\varepsilon (a_0 v_1^{\varepsilon\prime})(h) + b_z^\varepsilon (a_0 v_2^{\varepsilon\prime})(h),$$
$$a_{z+1}^\varepsilon = c_z^\varepsilon w_1^\varepsilon(1) + d_z^\varepsilon w_2^\varepsilon(1), \qquad \varepsilon^2 b_{z+1}^\varepsilon = c_z^\varepsilon (a_1 w_1^{\varepsilon\prime})(1) + d_z^\varepsilon (a_1 w_2^{\varepsilon\prime})(1).$$

Eliminating $c_z^\varepsilon$ and $d_z^\varepsilon$ gives the iterative system

$$\left(\begin{array}{c} a_{z+1}^\varepsilon \\ b_{z+1}^\varepsilon \end{array}\right) = M_\varepsilon \left(\begin{array}{c} a_z^\varepsilon \\ b_z^\varepsilon \end{array}\right), \tag{5.6}$$

where the matrix $M_\varepsilon$ is given by

(5.7)
$$M_\varepsilon = \left(\begin{array}{cc} v_1^\varepsilon(h)w_1^\varepsilon(1) + \varepsilon^2 (a_0 v_1^{\varepsilon\prime})(h) w_2^\varepsilon(1) & v_2^\varepsilon(h)w_1^\varepsilon(1) + \varepsilon^2 (a_0 v_2^{\varepsilon\prime})(h) w_2^\varepsilon(1) \\ \varepsilon^{-2} v_1^\varepsilon(h)(a_1 w_1^{\varepsilon\prime})(1) + (a_0 v_1^{\varepsilon\prime})(h)(a_1 w_2^{\varepsilon\prime})(1) & \varepsilon^{-2} v_2^\varepsilon(h)(a_1 w_1^{\varepsilon\prime})(1) + (a_0 v_2^{\varepsilon\prime})(h)(a_1 w_2^{\varepsilon\prime})(1) \end{array}\right).$$

It follows from the property that the modified Wronskian is constant,

$$\det\left(\begin{array}{cc} v_1^\varepsilon & v_2^\varepsilon \\ a_0 v_1^{\varepsilon\prime} & a_0\, v_2^{\varepsilon\prime}\end{array}\right) \equiv 1, \qquad \det\left(\begin{array}{cc} w_1^\varepsilon & w_2^\varepsilon \\ a_1 w_1^{\varepsilon\prime} & a_1\, w_2^{\varepsilon\prime}\end{array}\right) \equiv 1$$

that the characteristic polynomial of $M_\varepsilon$ is given by

(5.8)
$$\det(M_\varepsilon - \mu I) = \mu^2 - \mu h_\varepsilon + 1,$$
$$h_\varepsilon = v_1^\varepsilon(h) w_1^\varepsilon(1) + \varepsilon^2 (a_0 v_1^{\varepsilon\prime})(h) w_2^\varepsilon(1) + \varepsilon^{-2} v_2^\varepsilon(h)(a_1 w_1^{\varepsilon\prime})(1) + (a_0 v_2^{\varepsilon\prime})(h)(a_1 w_2^{\varepsilon\prime})(1).$$

Recalling, from section 3.1, the fundamental solutions $v_1$, $v_2$ of (cf. (3.5))

$$-(a_0 u')' = \lambda_0 \rho_0 u \quad \text{in } Y_0,$$

satisfying

$$\left(\begin{array}{cc} v_1(0) & v_2(0) \\ (a_0 v_1')(0) & (a_0 v_2')(0)\end{array}\right) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1\end{array}\right),$$

we shall prove in the second half of this section the following property.

LEMMA 5.1. *The following convergence holds:*

$$\lim_{\varepsilon\to 0} h_\varepsilon = v_1(h) + (a_0 v_2')(h) - \lambda_0 v_2(h)\int_{Y_1}\rho_1. \tag{5.9}$$

Assuming that (5.9) holds, since $\lambda_0 \in \mathbb{R}\backslash\bigcup_\theta \sigma(A_\theta)$, or equivalently (see section 3.1) $\lambda_0$ is such that (cf. (3.10))

$$\left|v_1(h) + (a_0 v_2')(h) - \lambda_0 v_2(h)\int_{Y_1}\rho_1\right| > 2,$$

for sufficiently small $\varepsilon$ we find that $|h_\varepsilon| > 2$.

As per the discussion in section 4, the eigenvalues $\mu_1^\varepsilon, \mu_2^\varepsilon$ of the matrix $M_\varepsilon$ satisfy the identity $\mu_1^\varepsilon\mu_2^\varepsilon = 1$ and the nature of $\tilde{u}_\varepsilon$ away from the defect is determined by the coefficient $h_\varepsilon$. In particular, if $|h_\varepsilon| > 2$, then the roots $\mu_1^\varepsilon, \mu_2^\varepsilon$ are such that $|\mu_1^\varepsilon| < 1$ and $|\mu_2^\varepsilon| > 1$ and there exist linearly independent functions $v_{\rm g}, v_{\rm d}$ on $\mathbb{R}\backslash\big(\lfloor d_-\rfloor_\varepsilon, \lceil d_+\rceil_\varepsilon\big)$ that grow and decay, respectively. In this case, for $u_\varepsilon$ to be an element of $L^2(\mathbb{R})$ it is necessary that $u_\varepsilon$ is proportional to the decaying solution $v_{\rm d}$, which takes the form

$$v_{\rm d}(x) = \begin{cases} \exp\left(\dfrac{\ln|\mu_1^\varepsilon|}{\varepsilon}\,{\rm dist}(x,D)\right) p_1^\varepsilon(x/\varepsilon), & x\in[d_+,\infty),\\[2mm] \exp\left(\dfrac{\ln|\mu_1^\varepsilon|}{\varepsilon}\,{\rm dist}(x,D)\right) p_2^\varepsilon(x/\varepsilon), & x\in(-\infty,d_-]\end{cases}$$

for some periodic (respectively, antiperiodic) functions $p_1^\varepsilon$, $p_2^\varepsilon$, when $h_\varepsilon > 2$ (respectively, when $h_\varepsilon < -2$). Therefore, for any $\nu$ satisfying $\nu < -\ln|\mu_1^\varepsilon| = \big|\ln|\mu_1^\varepsilon|\big|$ the product $g_{\nu/\varepsilon}u_\varepsilon$ is in $L^2(\mathbb{R})$, where $g_{\nu/\varepsilon}$ is defined by (2.9). Then the third claim of Theorem 2.4 follows by noticing that by (5.9) $\mu_1^\varepsilon$ converges to $\mu_1$, the smallest by absolute value root of $\mu^2 - h\mu + 1$, where

$$h := v_1(h) + (a_0 v_2')(h) - \lambda_0 v_2(h)\int_{Y_1}\rho_1,$$

as $\varepsilon \to 0$.

It remains to prove the convergence (5.9).

*Proof of Lemma* 5.1. The vector field

$$\eta_j^\varepsilon := \begin{pmatrix} v_j^\varepsilon - v_j \\ a_0 {v_j^\varepsilon}' - a_0 {v_j}' \end{pmatrix}, \qquad j = 1,2 \tag{5.10}$$

solves the initial-value problem

$${\eta_j^\varepsilon}' = \Phi^\varepsilon \eta_j^\varepsilon + \Psi_j^\varepsilon \quad \text{in } Y_0, \qquad \eta_j^\varepsilon(0) = 0, \quad j = 1,2 \tag{5.11}$$

for the matrix $\Phi^\varepsilon$ and vector $\Psi_j^\varepsilon$, $j = 1,2$, given by

$$\Phi^\varepsilon = \begin{pmatrix} 0 & a_0^{-1} \\ -\lambda_\varepsilon\rho_0 & 0 \end{pmatrix}, \qquad \Psi_j^\varepsilon = \begin{pmatrix} 0 \\ (\lambda_0 - \lambda_\varepsilon)\rho_0 v_j \end{pmatrix}, \qquad j = 1,2.$$

Since $\lambda_\varepsilon \to \lambda_0$ the solutions to (5.11) converge uniformly on $Y_0$ to the trivial solution of

$$\eta' = \Phi\eta \quad \text{in } Y_0, \qquad \eta(0) = 0,$$

where $\Phi$ is the limit of $\Phi_\varepsilon$, as $\varepsilon \to 0$ (see, e.g., [21, Theorem 1.6.1]). Namely, we have

$$\left|\eta_j^\varepsilon(y)\right| = \left|\eta_j^\varepsilon(y) - \eta(y)\right| \le C\left|\lambda_\varepsilon - \lambda_0\right|, \quad j = 1, 2,$$

for some constant $C$ independent of $\varepsilon$. In particular, (5.10) implies that

$$\lim_{\varepsilon\to 0} v_j^\varepsilon(h) = v_j(h), \qquad \lim_{\varepsilon\to 0}(a_0 {v_j^\varepsilon}')(h) = (a_0 v_j')(h), \qquad j = 1, 2. \tag{5.12}$$

Similarly, it is easy to see that $w_j^\varepsilon$ and $a_1 {w_j^\varepsilon}'$ converge uniformly on $Y_1$ to $w_j$ and $a_1 w_j'$, where $w_j$, $j = 1, 2$, are the solutions of $(a_1 w')' = 0$ satisfying

$$\begin{pmatrix} w_1(h) & w_2(h) \\ (a_1 w_1')(h) & (a_1 w_2')(h) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $w_1 \equiv 1$ and $a_1 w_2' \equiv 1$ on $Y_1$ we see that

$$\begin{pmatrix} w_1^\varepsilon & w_2^\varepsilon \\ a_1 {w_1^\varepsilon}' & a_1 {w_2^\varepsilon}' \end{pmatrix} \to \begin{pmatrix} 1 & \int_h^y a_1^{-1} \\ 0 & 1 \end{pmatrix} \text{ uniformly on } Y_1 \text{ as } \varepsilon \to 0. \tag{5.13}$$

Furthermore, by the fundamental theorem of calculus and the fact $-\varepsilon^{-2}(a_1 {w_1^\varepsilon}')' = \lambda_\varepsilon \rho_1 w_1^\varepsilon$, we have

$$\varepsilon^{-2}(a_1 {w_1^\varepsilon}')(1) - \varepsilon^{-2}(a_1 {w_1^\varepsilon}')(h) = -\lambda_\varepsilon \int_h^1 \rho_1 w_1^\varepsilon,$$

and since

$$\int_h^1 \rho_1 w_1^\varepsilon - w_1^\varepsilon(h)\int_h^1 \rho_1 = \int_h^1 \rho_1\big(w_1^\varepsilon - w_1^\varepsilon(h)\big) = \int_{Y_1} \rho_1(y)\left(\int_h^y {w_1^\varepsilon}'\right) \mathrm{d}y,$$

it follows that

$$\begin{aligned}\left|\varepsilon^{-2}(a_1 {w_1^\varepsilon}')(1) - \varepsilon^{-2}(a_1 {w_1^\varepsilon}')(h) + \lambda_\varepsilon w_1^\varepsilon(h)\int_h^1 \rho_1\right| &= \left|\lambda_\varepsilon \int_{Y_1} \rho_1(y)\left(\int_h^y {w_1^\varepsilon}'\right) \mathrm{d}y\right| \\ &\le |\lambda_\varepsilon| ||\rho_1||_{L^\infty} ||{w_1^\varepsilon}'||_{L^\infty},\end{aligned}$$

which together with (5.13) implies

$$\lim_{\varepsilon\to 0}\left|\varepsilon^{-2}(a_1 {w_1^\varepsilon}')(1) - \varepsilon^{-2}(a_1 {w_1^\varepsilon}')(h) + w_1^\varepsilon(h)\lambda_\varepsilon \int_h^1 \rho_1\right| = 0.$$

Taking into account the initial conditions (5.4) we obtain

$$\lim_{\varepsilon\to 0} \varepsilon^{-2}\big(a_1 {w_1^\varepsilon}'\big)(1) = -\lambda_0 \int_{Y_1} \rho_1. \tag{5.14}$$

Finally, assertions (5.12), (5.13), and (5.14) imply (5.9), as required. ∎

**6. Resolvent estimates for the problem without defect.** In this section we study the behavior of the unperturbed periodic operator $A^\varepsilon$ in the operator norm as $\varepsilon \to 0$. In particular, we construct a full asymptotic expansions for the resolvent of $A^\varepsilon$ using a version of the asymptotic framework developed in [7]; see Theorem 6.2 below. This directly implies the order-sharp operator norm resolvent convergence estimate, uniform in $\theta$, formulated in Theorem 2.2. The latter, in turn, implies the uniform in $\theta$ convergence, as $\varepsilon \to 0$, of the spectral band functions $\lambda_n^\varepsilon(\theta)$ to $\lambda_n(\theta)$, $n \in \mathbb{N}$, which is also order-sharp.

Recall the operator $A^\varepsilon$ in $L^2_{\rho^\varepsilon}(\mathbb{R})$ associated with the sesquilinear form

$$\beta^\varepsilon(u,v) = \int_{\Omega_1^\varepsilon} a_1(\tfrac{\cdot}{\varepsilon})u'\overline{v'} + \int_{\Omega_0^\varepsilon} \varepsilon^2 a_0(\tfrac{\cdot}{\varepsilon})u'\overline{v'}, \qquad u,v \in H^1(\mathbb{R}).$$

By a scaled version of the Floquet–Bloch transform,[3] which is given as the continuous extension of the following action on, e.g., continuous functions with compact support

$$(6.1) \qquad (\mathcal{U}_\varepsilon f)(\theta,y) = \sqrt{\frac{\varepsilon}{2\pi}}\sum_{z\in\mathbb{Z}} f\big(\varepsilon(y-z)\big)\mathrm{e}^{\mathrm{i}\theta z}, \qquad y \in Y,\ \theta \in [0,2\pi),$$

we see that $\mathcal{U}_\varepsilon$ unitarily maps $L^2_{\rho_\varepsilon}(\mathbb{R})$ to the Bochner space $L^2\big(0,2\pi; L^2_\rho(Y)\big)$ and $\mathcal{U}_\varepsilon A^\varepsilon f(\theta,\cdot) = A^\varepsilon_\theta \mathcal{U}_\varepsilon f(\theta,\cdot)$. Here, $A^\varepsilon_\theta$ is the operator defined in $L^2_\rho(Y)$ and associated with the form

$$\beta^\varepsilon_\theta(u,v) := \int_{Y_0} a_0 u'\overline{v'} + \varepsilon^{-2}\int_{Y_1} a_1 u'\overline{v'}, \qquad u,v \in H^1_\theta(Y).$$

We recall that $H^1_\theta(Y)$ is the complex Hilbert space of $H^1(Y)$-functions that are $\theta$-quasiperiodic. We equip the space $H^1_\theta(Y)$ with the graph norm

$$(6.2) \qquad |||u||| := \sqrt{\int_{Y_0} a_0|u'|^2 + \int_{Y_1} a_1|u'|^2 + \int_Y \rho|u|^2}$$

and consider the subspace

$$V_\theta := \big\{v \in H^1_\theta(Y) : v' \equiv 0 \text{ in } Y_1\big\}$$

and its orthogonal complement $V_\theta^\perp$ in $H^1_\theta$ with respect to the inner product associated with $|||\cdot|||$. The following result, established in [8], is of fundamental importance in studying the asymptotics of $A^\varepsilon$, equivalently $A^\varepsilon_\theta$.

LEMMA 6.1. *There exists a constant $C_{\mathrm{P}} > 0$, independent of $\theta$, such that*

$$(6.3) \qquad |||P_\theta^\perp u||| \le C_{\mathrm{P}}||\sqrt{a_1}u'||_{L^2(Y_1)} \qquad \forall u \in H^1_\theta(Y),$$

*where $P_\theta^\perp$ is the orthogonal projection of $H^1_\theta(Y)$ onto $V_\theta^\perp$.*

For $\theta \in [0,2\pi)$ and all $f \in L^2_\rho(Y)$, we consider the resolvent problem

$$(6.4) \qquad -\big((\varepsilon^{-2}a_1 + a_0){u^\varepsilon_\theta}'\big)' + \rho u^\varepsilon_\theta = \rho f \qquad \text{on } (0,1).$$

We look for an asymptotic expansion of $u^\varepsilon_\theta$ in the form

$$(6.5) \qquad u^\varepsilon_\theta = \sum_{n=0}^\infty \varepsilon^{2n}u_\theta^{(2n)}, \qquad u_\theta^{(2n)} \in H^1_\theta(Y) \quad \forall n \in \mathbb{N}.$$

The following result holds.

[3]See Appendix A below for further information on the Floquet–Bloch transform.

THEOREM 6.2. *For each* $\theta \in [0, 2\pi)$ *and* $f \in L^2_\rho(Y)$, *consider the unique solution* $u_\theta^{(0)} \in V_\theta$ *to the problem*

$$\int_{Y_0} a_0 (u_\theta^{(0)})' \overline{\varphi'} + \int_Y \rho u_\theta^{(0)} \overline{\varphi} = \int_Y \rho f \overline{\varphi} \qquad \forall \varphi \in V_\theta,$$

*and for all* $n \in \mathbb{N}$ *consider the unique solution* $u_\theta^{(2n)} \in V_\theta^\perp$ *to*

$$-\Big(a_1 \big(u_\theta^{(2n)}\big)'\Big)' = \Big(a_0 \big(u_\theta^{(2(n-1))}\big)'\Big)' - \rho u_\theta^{(2(n-1))} + \delta_{1n} \rho f,$$

*where* $\delta_{1n}$ *is the Kronecker delta function. Then, for each* $N \in \mathbb{N}$ *the sum*

$$U_\theta^{(N)} := \sum_{n=0}^{N} \varepsilon^{2n} u_\theta^{(2n)}$$

*approximates the solution* $u_\theta^\varepsilon$ *to* (6.4) *in the following sense:*

$$|||u_\theta^\varepsilon - U_\theta^{(N)}||| \le C_{\mathrm{P}}^{2(N+1)} \varepsilon^{2(N+1)} \big\| f \big\|_{L^2_\rho(Y)}.$$

*Remark* 6.3. By an application of the min-max principle, Theorem 6.2 implies that the $n$th eigenvalue $\lambda_n^\varepsilon(\theta)$ of the operator $A_\theta^\varepsilon$ is $\varepsilon^2$-close, uniformly in $\theta$, to the $n$th eigenvalue $\lambda_n(\theta)$ of $A_\theta$, i.e., for each $n \in \mathbb{N}$ there exists a constant $c_n > 0$ such that

$$\big|\lambda_n^\varepsilon(\theta) - \lambda_n(\theta)\big| \le c_n \varepsilon^2 \qquad \forall \theta \in [0, 2\pi).$$

In particular, this indirectly implies, since $\lambda_n$ is the uniform limit of continuous functions, that $\lambda_n$ is continuous in $\theta$. A direct proof of this fact can be arrived at by the definition of the operators $A_\theta$ and the continuity properties (in the Hausdorff sense) of their domains $D(A_\theta)$; see [8, Appendix B].

*Proof.* Substituting (6.5) into (6.4) and equating powers of $\varepsilon$ yields a system of recurrence relations for the functions $u_\theta^{(2n)}$, $n \in \mathbb{N}$. The first equation in this system, which corresponds to $\varepsilon^{-2}$, is

$$-\Big(a_1 \big(u_\theta^{(0)}\big)'\Big)' = 0 \qquad \text{on} \ \ (0,1), \tag{6.6}$$

which implies that $u_\theta^{(0)} \in V_\theta = \big\{ v \in H^1_\theta(Y) : v' \equiv 0 \text{ on } Y_1 \big\}$ (recall that $a_1 \equiv 0$ on $Y_0$). The remaining equations, obtained by considering the terms of order $\varepsilon^{2j}$, $j = 0, 1, 2, \ldots$, are

$$-\Big(a_1 \big(u_\theta^{(2n)}\big)'\Big)' = \Big(a_0 \big(u_\theta^{(2(n-1))}\big)'\Big)' - \rho u_\theta^{(2(n-1))} + \delta_{1n} \rho f \qquad \text{on } (0,1), \quad n \in \mathbb{N}, \tag{6.7}$$

where, as before, $\delta_{in}$ denotes the Kronecker delta function. The existence of solutions to differential equations with degenerate coefficients such as (6.7) was first studied in [13] for the case $\theta = 0$, and it was shown therein that existence is guaranteed by inequalities of the type (6.3). By following this general framework, and it can be readily shown that (6.3) implies the following result.

LEMMA 6.4. *For a given $F \in H^{-1}_\theta(Y)$, the dual space of $H^1_\theta(Y)$, there exist (infinitely many) solutions $u \in H^1_\theta(Y)$ to the problem*

$$\int_{Y_1} a_1 u' \overline{\varphi'} = {}_{H^{-1}_\theta(Y)}\langle F, \varphi\rangle_{H^1_\theta(Y)} \qquad \forall \varphi \in H^1_\theta(Y),$$

*if and only if $F$ satisfies the condition*

$${}_{H^{-1}_\theta(Y)}\langle F, v\rangle_{H^1_\theta(Y)} = 0 \qquad \forall v \in V_\theta.$$

*Such solutions are unique in $V_\theta^\perp$, i.e., for any two solutions $u_1$, $u_2$ one has $u_1 - u_2 \in V_\theta$.*

Consequently, the system (6.7) is solvable if and only if the conditions

$$\int_{Y_0} a_0 (u_\theta^{(2n)})' \overline{\varphi'} + \int_Y \rho u_\theta^{(2n)} \overline{\varphi} = \delta_{0n} \int_Y \rho f \overline{\varphi} \quad \forall \varphi \in V_\theta, \qquad n+1 \in \mathbb{N} \tag{6.8}$$

hold. The equation for $n = 0$ uniquely determines $u_\theta^{(0)}$ and for $n \geq 1$, due to the choice (6.2) of the norm on $H^1_\theta(Y)$, demonstrates that $u_\theta^{(2n)} \in V_\theta^\perp$. Substituting $\varphi = u_\theta^{(0)}$ into the identity (6.8) for $n = 0$, recalling (6.2), the fact that $a_1(u_\theta^{(0)})' \equiv 0$ and using the Cauchy–Schwarz inequality, we obtain

$$|||u_\theta^{(0)}|||^2 \leq \|f\|_{L^2_\rho(Y)} \big\|u_\theta^{(0)}\big\|_{L^2_\rho(Y)}.$$

Hence, $u_\theta^{(0)}$ satisfies the bound

$$|||u_\theta^{(0)}||| \leq \|f\|_{L^2_\rho(Y)} \qquad \forall \theta \in [0, 2\pi). \tag{6.9}$$

By Lemmas 6.1 and 6.4, the solution $u_\theta^{(2n)} \in V_\theta^\perp$ to (6.7) is unique and

$$|||u_\theta^{(2n)}||| \leq C_{\rm P} \big\|\sqrt{a_1}(u_\theta^{(2n)})'\big\|_{L^2(Y_1)}. \tag{6.10}$$

Equations (6.7) and the orthogonality of $V_\theta$ and $V_\theta^\perp$ with respect to the inner product associated with the norm (6.2), in particular, the orthogonality of $u_\theta^{(0)}$ and $u_\theta^{(2)}$, imply that

$$\int_{Y_1} a_1 \Big|\big(u_\theta^{(2n)}\big)'\Big|^2$$
$$= \delta_{1n} \int_Y \rho f \overline{u_\theta^{(2n)}} - (1-\delta_{1n}) \bigg( \int_{Y_0} a_0 \big(u_\theta^{(2(n-1))}\big)' \overline{\big(u_\theta^{(2n)}\big)'} + \int_Y \rho u_\theta^{(2(n-1))} \overline{u_\theta^{(2n)}} \bigg), \quad n \geq 1,$$

and (6.10) yields

$$|||u_\theta^{(2)}||| \leq C_{\rm P}^2 \big\|f\big\|_{L^2_\rho(Y_1)}, \qquad |||u_\theta^{(2n)}||| \leq C_{\rm P}^2 |||u^{(2(n-1))}|||, \quad n \geq 2.$$

By iterating the above inequalities we establish that

$$|||u_\theta^{(2n)}||| \leq C_{\rm P}^{2n} \big\|f\big\|_{L^2_\rho(Y_1)}, \qquad n \geq 1. \tag{6.11}$$

Having identified each term in the expansion, for each $n \in \mathbb{N}$ we define the remainder $R^\varepsilon_\theta$ (dropping the index $N$ for brevity), according to the formula

$$u^\varepsilon_\theta = \sum_{n=0}^{N} \varepsilon^{2n} u^{(2n)}_\theta + \varepsilon^{2N} R^\varepsilon_\theta, \tag{6.12}$$

and find, via (6.6) and (6.7), that $R^\varepsilon_\theta \in H^1_\theta(Y)$ solves the problem

$$-\big((\varepsilon^{-2}a_1 + a_0)(R^\varepsilon_\theta)'\big)' + \rho R^\varepsilon_\theta = \delta_{0N}\rho f + \big(a_0(u^{(2N)}_\theta)'\big)' - \rho u^{(2N)}_\theta \qquad \text{on } (0,1),$$

that is,

$$\begin{aligned}\int_{Y_1} \varepsilon^{-2} a_1 (R^\varepsilon_\theta)'\overline{v'} + \int_{Y_0} a_0 (R^\varepsilon_\theta)'\overline{v'} + \int_Y \rho R^\varepsilon_\theta \overline{v} \\ = \delta_{0N}\int_Y \rho f\overline{v} - \int_{Y_0} a_0 (u^{(2N)}_\theta)'\,\overline{v'} - \int_Y \rho u^{(2N)}_\theta \overline{v} \quad \forall v \in H^1_\theta(Y).\end{aligned}$$

Setting $v \in V_\theta$, recalling the norm (6.2) and (6.8), demonstrates that $R^\varepsilon_\theta \in V_\theta^\perp$. Additionally, setting $v = R^\varepsilon_\theta$ above implies that

$$\varepsilon^{-2}\int_{Y_1} a_1 \big|(R^\varepsilon_\theta)'\big|^2 \le \delta_{0N}\int_Y \rho f \overline{R^\varepsilon_\theta} - \int_{Y_0} a_0 (u^{(2N)}_\theta)'\,\overline{(R^\varepsilon_\theta)'} - \int_Y \rho u^{(2N)}_\theta \overline{R^\varepsilon_\theta},$$

and inequalities (6.3), (6.11), along with another application of the Cauchy–Schwarz inequality yields

$$|||R^\varepsilon_\theta||| \le C_{\mathrm{P}}^{2(N+1)}\varepsilon^2 \|f\|_{L^2_\rho(Y)}.$$

Finally, by combining this inequality with (6.12) we deduce that

$$\Big|\Big|\Big| u^\varepsilon_\theta - \sum_{n=0}^{N} \varepsilon^{2n} u^{(2n)}_\theta \Big|\Big|\Big| \le C^{2(N+1)}\varepsilon^{2(N+1)}\|f\|_{L^2_\rho(Y)}, \qquad \square$$

as required.

## Appendix A. Norm-resolvent asymptotics of $A^\varepsilon$ and the approximating operator $A^0$.

We consider the space $H$ to be the closure in $L^2_\rho(\mathbb{R})$ of

$$H^+ := \Big\{ v \in H^1(\mathbb{R}) : v' \equiv 0 \text{ on } \Omega_1 := \bigcup_{z\in\mathbb{Z}} (Y_1 + z) \Big\}.$$

Both $H$ and $H^+$ are Hilbert spaces when equipped with the inner products inherited from $L^2_\rho(\mathbb{R})$ and $H^1(\mathbb{R})$, respectively, and clearly $H^+$ is dense in $H$ with continuous embedding (recall $\rho$ is taken to be uniformly positive and bounded). The norm of $H^+$, which is the standard $H^1$-norm, is equivalent to the graph norm

$$||\cdot||_{H^+} := \Big( ||\cdot||^2_{L^2_\rho(\mathbb{R})} + \beta^0(\cdot,\cdot) \Big)^{1/2}, \tag{1.1}$$

where $\beta^0$ is the sesquilinear form

$$\beta^0(u,v) := \int_{\Omega_0} a_0 u'\overline{v'}, \qquad u, v \in H^+.$$

We shall henceforth consider $H^+$ to be equipped with the graph norm (1.1), and denote by $H^-$ the dual space consisting of bounded linear functionals on $H^+$. As $\beta^0$ is a nonnegative closed symmetric sesquilinear form, it generates a densely defined nonnegative self-adjoint linear operator $A^0$ in $H$. The domain $D(A^0)$ is the dense subset of $H^+$ consisting of the solutions to the problem: for each $f \in H^+$ we consider $u \in H^+$ the unique solution to the problem

$$\beta^0(u,v) + \int_{\mathbb{R}} \rho u \overline{v} = \int_{\mathbb{R}} \rho f \overline{v} \qquad \forall v \in H^+,$$

and set $A^0 u = f - u$ for $u \in D(A^0)$. The operator $A^0$ is unitarily equivalent to the fiber integral operator $\int^{\oplus} A_\theta$ (cf. Remark 2.1) and the unitary map is given by the continuous extension of the Floquet–Bloch transform $\mathcal{U}$ (cf. [15, section 2.2]) which acts on smooth functions $f$ with compact support as

$$\mathcal{U}f(\theta, y) := \frac{1}{\sqrt{2\pi}} \sum_{z \in \mathbb{Z}} f(y - z) e^{\mathrm{i}\theta z}, \qquad \theta \in [0, 2\pi), \quad y \in Y.$$

Indeed, $\mathcal{U}$ is well known to be a unitary operator between $L^2_\rho(\mathbb{R})$ and the Bochner space $L^2\big(0, 2\pi; L^2_\rho(Y)\big)$, and it is straightforward to see that

$$\mathcal{U}A^0 f(\theta; \cdot) = A_\theta \mathcal{U} f(\theta; \cdot), \quad \forall f \in L^2_\rho(\mathbb{R}),\ \theta \in [0, 2\pi).$$

Furthermore, it is clear that $\mathcal{U}$ unitarily maps $H^+$ to the space $L^2(0, 2\pi; V_\theta)$ (we recall that $V_\theta = \big\{ v \in H^1_\theta(Y) : v' \equiv 0 \text{ on } Y_1 \big\}$). It is easy to verify that the spectrum of $A^0$ coincides with the union of the spectra of $A_\theta$ over all $\theta \in [0, 2\pi)$, i.e.,

$$\sigma(A^0) = \bigcup_\theta \sigma(A_\theta) = \bigcup_{n \in \mathbb{N}} \big[ \min_\theta \lambda_n(\theta), \max_\theta \lambda_n(\theta) \big].$$

Theorem 2.2 implies in particular that $A^\varepsilon$ is order-$O(\varepsilon^2)$ close in the norm-resolvent sense to $A^0$ (up to unitary equivalence), i.e., there exists a constant $C > 0$ such that

$$\big\| \mathcal{R}_\varepsilon (A^\varepsilon + 1)^{-1} \mathcal{R}_\varepsilon^{-1} - (A^0 + 1)^{-1} \big\|_{L^2_\rho(\mathbb{R}) \to L^2_\rho(\mathbb{R})} \le C\varepsilon^2$$

for all $\varepsilon \in (0,1)$, where $\mathcal{R}_\varepsilon : L^2_{\rho^\varepsilon}(\mathbb{R}) \to L^2_\rho(\mathbb{R})$ is the unitary transformation $\mathcal{R}_\varepsilon(f)(y) = \varepsilon^{1/2} f(\varepsilon y)$.

## Appendix B. Spectral decomposition of $A^0$.

As the operator $A^0$ is self-adjoint, it has a spectral decomposition, and we shall now characterize the space $H^+$ and its dual $H^-$ in terms of a realization of this spectral decomposition. For each $\theta$, the self-adjoint operator $A_\theta$ has compact resolvent and for each of its eigenvalues $\lambda_n(\theta)$, $n \in \mathbb{N}$, we denote by $\psi_n(\theta; .)$ the corresponding $L^2_\rho(Y)$-normalized eigenfunction. Then the mapping $\Psi$ given by

$$\Psi f(\theta; \cdot) = \{c_n(\theta)\}_{n \in \mathbb{N}}, \qquad c_n(\theta) := \int_Y \rho(y) f(\theta, y) \overline{\psi_n(\theta; y)}\, \mathrm{d}y$$

unitarily maps $L^2\big(0, 2\pi; L^2_\rho(Y)\big)$ to $\mathfrak{h} := L^2(0, 2\pi; \ell^2)$ so that

$$\Psi\big(\mathcal{U}A^0 f\big)(\theta, n) = \lambda_n(\theta) \Psi\big(\mathcal{U}f\big)(\theta, n),$$

where for $u \in \mathfrak{h}$, we denote by $u(\theta, n)$ is the $n$th element of the sequence $u(\theta)$. It is easy to verify that $\Psi \circ \mathcal{U}$ unitarily maps $H^+$ to

$$\mathfrak{h}^+ := \left\{ u(\theta, n) \in \mathfrak{h} : \left(\lambda_n(\theta) + 1\right)^{1/2} u(\theta, n) \in \mathfrak{h} \right\}.$$

By standard duality arguments (see, for example, [16, Chapter 1) section 6.2], we show that $\Psi \circ \mathcal{U}$ unitarily maps $H^-$, the dual space of bounded linear functionals on $H^+$, to

$$\mathfrak{h}^- := \left\{ f : (0, 2\pi) \to \ell^2 \text{ measurable} : \left(\lambda_n(\theta) + 1\right)^{-1/2} f(\theta, n) \in \mathfrak{h} \right\},$$

in the sense that $F \in H^-$ if and only if there exists $f \in \mathfrak{h}^-$ such that

$$_{H^-}\langle F, v \rangle_{H^+} = \sum_{n \in \mathbb{N}} \int_0^{2\pi} f(\theta, n) \overline{\left(\Psi \mathcal{U}\right) v(\theta, n)} \, \mathrm{d}\theta \qquad \forall v \in H^+,$$

and we have

$$||F||_{H^{-1}} = \sqrt{\sum_{n \in \mathbb{N}} \int_0^{2\pi} \frac{\left|f(\theta, n)\right|^2}{\lambda_n(\theta) + 1} \, \mathrm{d}\theta}.$$

**Acknowledgments.** We are grateful to the referees for a number of helpful comments.

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