One-loop amplitudes in AdS$_5$×S^5 supergravity from $\mathcal{N} = 4$ SYM at strong coupling

F. Aprile,[a] **J.M. Drummond,**[b] **P. Heslop**[c] **and H. Paul**[b]

[a]*Dipartimento di Fisica, Università di Milano-Bicocca & INFN, Sezione di Milano-Bicocca, I-20126 Milano, Italy*

[b]*School of Physics and Astronomy and STAG Research Centre, University of Southampton, Highfield, SO17 1BJ, U.K.*

[c]*Mathematics Department, Durham University, Science Laboratories, South Rd, Durham, DH1 3LE, U.K.*

E-mail: `francesco.aprile1@unimib.it`, `j.m.drummond@soton.ac.uk`, `paul.heslop@durham.ac.uk`, `H.Paul@soton.ac.uk`

ABSTRACT: We explore the structure of maximally supersymmetric Yang-Mills correlators in the supergravity regime. We develop an algorithm to construct one-loop supergravity amplitudes of four arbitrary Kaluza-Klein supergravity states, properly dualised into single-particle operators. We illustrate this algorithm by constructing new explicit results for multi-channel correlation functions, and we show that correlators which are degenerate at tree level become distinguishable at one-loop. The algorithm contains a number of subtle features which have not appeared until now. In particular, we address the presence of non-trivial low twist protected operators in the OPE that are crucial for obtaining the correct one-loop results. Finally, we outline how the differential operators $\widehat{\mathcal{D}}_{pqrs}$ and $\Delta^{(8)}$, which play a role in the context of the hidden 10d conformal symmetry at tree level, can be used to reorganise our one-loop correlators.

Contents

1   Introduction and summary

Recently there has been significant progress in probing the structure of quantum gravity in the context of the AdS/CFT correspondence. This has been achieved by combining the effectiveness of the large N expansion and the power of CFT techniques. In particular the large N expansion in $\mathcal{N} = 4$ super Yang-Mills theory at large 't Hooft coupling has been investigated in [1–12]. Natural objects of study in this context are the four-point functions $\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle$ of half-BPS operators $\mathcal{O}_p$ which are dual to the scattering processes of four supergravity states of type IIB supergravity on the $\text{AdS}_5 \times \text{S}^5$ background.

In [3] we were able to obtain the full $1/N^4$ contribution to the correlator $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle$, i.e. the one-loop contribution to the four-point amplitude of AdS graviton supermultiplets.[1] This was achieved by promoting the leading logarithmic discontinuity to a crossing-invariant function. The leading logarithmic singularity itself was deduced in [2, 3] by the consistency of the operator product expansion (OPE), after resolving the tree-level mixing of long double-trace operators in the singlet $su(4)$ representation [4]. In fact the leading $1/N^2$ corrections to the spectrum of double-trace operators can be completely solved with surprisingly simple rational functions of the quantum numbers [8]. As observed in [8], the spectrum exhibits a partial degeneracy which motivated the discovery of a surprising ten-dimensional conformal symmetry governing tree-level $\text{AdS}_5 \times \text{S}^5$ supergravity [9]. In [7] we were able to perform a similar analysis for the amplitude of two graviton supermultiplets and two Kaluza-Klein states, $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_3 \rangle$. Both cases involved surprisingly simple analytic functions based essentially on the two-loop four-dimensional ladder integral.

The approach outlined above does not make any reference to actual one-loop diagrams of IIB supergravity on $\text{AdS}_5 \times \text{S}^5$, and in fact this computation in the bulk remains very challenging. Instead, scalar theories on AdS at one-loop have been discussed in many references, for example, see [14–19]. Our approach here uses CFT techniques to extract data in the dual theory, $\mathcal{N} = 4$ SYM, and it is complemented with an understanding of the possible analytic structure of the one-loop correlators, as functions in position space. Similar approaches to half-BPS correlators have been applied also in perturbation theory, both from the point of view of particular diagrams (or integrands) e.g. [24–29] and using the analytic structure of explicitly evaluated loop integrals [27, 30]. It is natural to ask therefore if the large N bootstrap can be applied to arbitrary charge half-BPS operators.

In this paper we solve algorithmically the analytic bootstrap program for the four-point one-loop amplitudes of generic single-particle Kaluza-Klein states. This computation

---

[1]Up to a single ambiguity which was fixed recently in [13].

presents itself as a significant challenge compared to our previous constructions in [3] and [7]. Indeed, the one-loop correlators constructed so far had at least two AdS graviton multiplet insertions, and therefore had some built-in physical simplicity, stemming from the fact that the OPE of two graviton multiplets runs over a special set of both protected and long operators. In general, this simplicity is absent and we have to face a network of complications, which we will solve in this paper.

First we recall that the $1/N$ expansion will naturally stratifies the four-point amplitude in powers of logs of the cross-ratio u, and it leads to an expansion of the following form,[2]

$$\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle = \mathcal{G}_{\vec{p};0,0} + \frac{1}{N^2} \sum_{n=0}^{1} (\log u)^n \, \mathcal{G}_{\vec{p};1,n} + \frac{1}{N^4} \sum_{n=0}^{2} (\log u)^n \, \mathcal{G}_{\vec{p};2,n} + \dots \quad (1.1)$$

where

$$\vec{p} = (p_1, p_2, p_3, p_4) \quad (1.2)$$

comprises the external charges. The expansion in (1.1) goes together with an expansion in the large 't Hooft coupling $\lambda = g^2 N$. The string corrections to the above expansion have been addressed recently in a number of papers [10, 31–33] but here we will restrict ourselves to the terms of order λ^0 corresponding to supergravity contributions. In very general terms, the consistency of the OPE places strong constraints on the various different functions $\mathcal{G}_{n,m}$. We shall now explain how this abstract information, embedded in the $1/N$ expansion, can be used in practice to organise our bootstrap program.

Consider the OPE of single-particle operators $(\mathcal{O}_{p_i} \times \mathcal{O}_{p_j})$, it contains superconformal primary operators $\mathcal{O}$ of twist τ, spin l and $su(4)$ representation $[aba]$,

$$\mathcal{O}_{p_i} \times \mathcal{O}_{p_j} = \sum_{\mathcal{O}} C_{p_i p_j}(\mathcal{O}_{\vec{\tau}}) \, \mathcal{O}_{\vec{\tau}} \, , \quad (1.3)$$

where $\vec{\tau} \equiv (\tau, l, [aba])$ is a compact notation for the representation labels. The first general key point is that a four-point function, $\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle$ is determined non-perturbatively by summing over the OPE coefficients $C_{p_1 p_2}(\mathcal{O}_{\vec{\tau}}) C_{p_3 p_4}(\mathcal{O}_{\vec{\tau}})$ of common exchanged operators $\mathcal{O}_{\vec{\tau}}$. The sum over operators is in principle a sum over all operators compatibly with quantum numbers. However, the second general key point is that the $1/N$ expansion of $C_{p_1 p_2}(\mathcal{O}_{\vec{\tau}})$ stratifies how operators contributes in (1.1).

Of particular importance for us will be the exchanged two-particle (or double-trace) operators, which have the schematic form,

$$\mathcal{O}_{pq;\vec{\tau}} = \mathcal{O}_p \Box^{\frac{1}{2}(\tau-p-q)} \partial^l \mathcal{O}_q \Big|_{[aba]} \, . \quad (1.4)$$

Such operators fall into different series according to their quantum numbers. Half-BPS operators have $l = a = 0$ and $\tau = b = p + q$. Semishort operators have $\tau = 2a + b + 2 = p + q$

---

and spin $l \geq 0$. In both these cases the $\square$ is necessarily absent. Long operators will generically obey the unitarity bound $\tau \geq 2a + b + 2$, but long operators of the form (1.4) actually obey $\tau \geq 2a + b + 4$. Notice that τ might be greater than $p + q$ in this case. Finally, for given $\vec{\tau}$ the space of operators has a degeneracy which we will label according to the pairs pq of $\mathcal{O}_{pq;\vec{\tau}}$. Geometrically, it is described by a rectangle R_τ [8]. The formalism established in [8] will be reviewed in section 3.

In a given $su(4)$ representation $[aba]$, we can organise semishort and long operators $\mathcal{O}_{pq;\vec{\tau}}$ into a tower, whose levels are labelled by the twist. The bottom of the tower corresponds to the unitarity bound. For each operator in this tower we now determine the N counting of the three-point couplings $C_{pq}(\mathcal{O}_{p'q';\vec{\tau}})$.

Three-point couplings of the form $C_{pq}(\mathcal{O}_{rs;\vec{\tau}})$ with $p + q = r + s$ have a leading order contribution in the large N expansion arising from Wick contractions in free supergravity. For example $C_{pq}(\mathcal{O}_{pq;\vec{\tau}})$. In the protected sector this information will be enough to determine the N counting of all three-point couplings. In the non protected sector instead, we also have to consider that the true two-particle scaling eigenstates, i.e. the ones with leading order quantum numbers $\vec{\tau}$ and well defined anomalous dimension, will be mixtures of the form

$$\mathcal{K}_{pq} = \sum_{p'q' \in R_\tau} \mathcal{O}_{p'q'} \frac{C^{(0)}_{p'q'\mathcal{K}_{pq}}}{C^{(0)}_{p'q'\mathcal{O}_{p'q'}}} + O(1/N^2) \,, \tag{1.5}$$

containing some contribution from every operator $\mathcal{O}_{rs;\vec{\tau}}$ with $p + q = r + s$, hence these $\mathcal{K}_{pq}$ will have leading order three-point couplings. We conclude that $C_{pq}(\mathcal{O}_{p'q';\vec{\tau}})$, where the two-particle operators $\mathcal{O}_{p'q',\vec{\tau}}$ have twist $\tau \geq p + q$, is also leading order as far as the N counting goes. That such a three point coupling is non vanishing follows from the explicit solution of the mixing problem [4]. The reasoning we are using here implies that exchanged two-particle operators with twist $\tau < p + q$ do not receive any contribution of the form $\mathcal{O}_{rs;\vec{\tau}}$ and thus have $1/N^2$ suppressed three-point couplings. Summarising, a three-point coupling $C_{pq}(\mathcal{O}_{\vec{\tau}})$ has the large N expansion:

$$C_{pq}(\mathcal{O}_{\vec{\tau}}) = C^{(0)}_{pq}(\mathcal{O}_{\vec{\tau}}) + \frac{1}{N^2} C^{(1)}_{pq}(\mathcal{O}_{\vec{\tau}}) + \dots \tag{1.6}$$

where $C^{(0)}_{pq}(\mathcal{O}_{\vec{\tau}}) \neq 0$ only for $\tau \geq p + q$.

The exchange of two particle operators in the common OPE of a four point correlator, gives a contribution of the form $C_{p_1 p_2, \vec{\tau}} C_{p_3 p_4, \vec{\tau}}$ for different values of twists. As before, we associate to each three-point coupling, $C_{p_1 p_2, \vec{\tau}}$ and $C_{p_3 p_4, \vec{\tau}}$, an infinite tower representing the semishort and long operators $\mathcal{O}_{pq;\vec{\tau}}$ in the $su(4)$ representation $[aba]$. Putting together two of these, we obtain a representation of the common OPE coefficient as in figure 1. Referring the N counting of $C_{p_1 p_2, \vec{\tau}} C_{p_3 p_4, \vec{\tau}}$ to figure 1, we read off the following pattern.

For $\tau \geq \tau_{\vec{p}}^{\max} \equiv \max(p_1 + p_2, p_3 + p_4)$, we find exchanged operators for which both three-point couplings are leading order, i.e. $C^{(0)}_{p_1 p_2}$ and $C^{(0)}_{p_3 p_4}$ are both non-zero. In particular, $\tau^{\max}$ is the threshold twist for exchange of two-particle operators in disconnected free theory $\mathcal{G}_{\vec{p};0,0}$. In the window $\tau^{\max} > \tau \geq \tau_{\vec{p}}^{\min} \equiv \min(p_1 + p_2, p_3 + p_4)$, we find exchanged operators
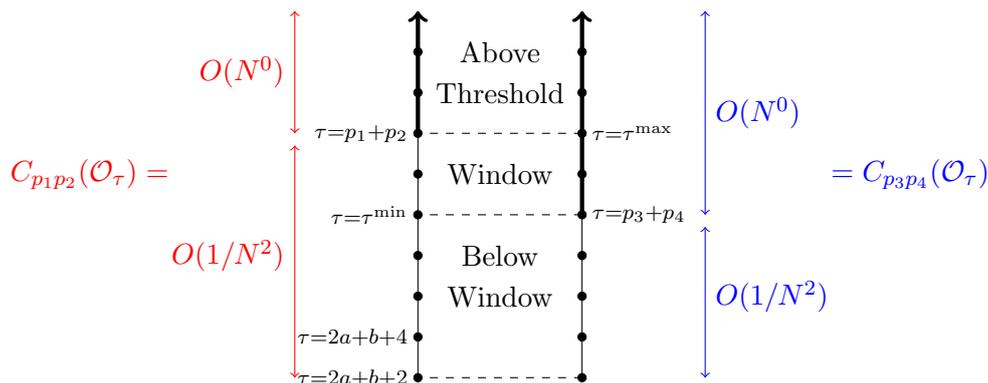
Figure 1. The large N structure of $C_{p_1p_2,\vec{\tau}} C_{p_3p_4,\vec{\tau}}$ for two particle operators $\mathcal{O}_\tau$ in an $su(4)$ representation $[aba]$, and varying twist.

which have leading order three-point couplings with one pair of external operators, but $1/N^2$ suppressed three-point couplings with the other pair of external operators, e.g. we have $C_{p_1p_2}^{(0)} = 0$ but $C_{p_3p_4}^{(0)}$ non-zero. Finally, "below window" will denote the range of twists $2 + 2a + b < \tau < \tau^{\min}$ in which $C_{p_1p_2}^{(0)} = C_{p_3p_4}^{(0)} = 0$ and the OPE contains contributions coming with $1/N^2$ suppressed three-point couplings. These contributions give rise to a novel and genuine $1/N^4$ effect which enters $\mathcal{G}_{2,0}$.

For any arrangement of external charges there is always a threshold twist such that a tower of long operators is exchanged. The window itself might be empty if $\tau^{\min} = \tau^{\max}$.

The location of the unitarity bound in figure 1 depends on the external charges. Generically, the unitarity bound $\tau = 2a+b+2$ is below window, but there are two other situations which do occur. The unitarity bound can coincide with $\tau^{\min}$, i.e $\tau^{\min} = 2a+b+2$, in which case there is no below window region. The unitarity bound can coincide with $\tau^{\max}$, in which case there is an empty window and $\tau^{\max} = \tau^{\min} = 2a + b + 2$.

The strategy followed in [3, 7] to bootstrap the order $1/N^4$ one-loop amplitude was to resolve the mixing problem in the long sector from the knowledge of $\mathcal{G}_{0,0}$ and $\mathcal{G}_{1,1}$ (focusing on the $su(4)$ representations $[000]$ and $[010]$) and thereby obtain explicitly the CFT data needed to bootstrap $\mathcal{G}_{2,2}$. The double logarithmic discontinuity can also be obtained elegantly by using the hidden ten-dimensional conformal symmetry of [9].

To complete the double logarithmic discontinuities into full amplitudes requires additional knowledge about $\mathcal{G}_{2,1}$ and $\mathcal{G}_{2,0}$. The CFT data entering $\mathcal{G}_{2,1}$ is obtained only within the long sector. The CFT data entering $\mathcal{G}_{2,0}$ is instead obtained from the study of both protected semishort and long operators. In both cases, the operators we will consider are two-particle operators.[3] Extracting this information in complete generality is a central new result of this paper. In particular, the study of the protected semishort sector at order $1/N^4$ has never been addressed before, except for the case of $\langle \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \rangle$ in [34].

Let us now project the correlator $\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle$ into an $su(4)$ representation $[aba]$, and distinguish between long and protected sector. Following the logic of figure 1, we now

---

[3]Higher multi-trace operators will also contribute but only at higher orders in the $1/N$ expansion within the ranges of twists we focus on here.

highlight the main inputs of our bootstrap program. These are extensively discussed in section 2 and 3.

The leading logarithmic discontinuity $\mathcal{G}_{2,2}$ (or more generally $\mathcal{G}_{n,n}$ for $n \geq 1$) is only induced by exchanged long two-particle operators with $\tau \geq \max(p_1 + p_2, p_3 + p_4)$. The number of fields exchanged in infinite both in twist and spin. The CFT data entering $\mathcal{G}_{n,n}$ for $n \geq 1$ comprises the $O(1)$ three-point couplings of these long two-particle operators with the external operators, and their $O(1/N^2)$ anomalous dimensions to the power n.

The subleading $\mathcal{G}_{2,1}$ logarithmic discontinuity in the window is also determined only by the long sector. However, there is only a finite number of operators exchanged in the window. For this range of twists $\mathcal{G}_{2,1}$ is essentially given by the product of one power of the anomalous dimension (of the exchanged operators) with the three-point couplings, let's say conventionally, $C^{(1)}_{p_1 p_2, \vec{\tau}}$ and $C^{(0)}_{p_3 p_4, \vec{\tau}}$. The combination $C_{p_1 p_2, \vec{\tau}} C_{p_3 p_4, \vec{\tau}}$ in the window is $O(1/N^2)$ as indicated by the figure. The physical data in the window determines also $\mathcal{G}_{n,n-1}$, with $n = 2$ just the first non trivial case. For generic n, we simply increase the power of the anomalous dimensions to $n - 1$.

The partial degeneracy of the $1/N^2$ anomalous dimensions found in [8] obstructs the explicit determination of $C^{(0)}_{pq, \vec{\tau}}$ in general, and consequently of $C^{(1)}_{pq, \vec{\tau}}$. Nevertheless, we will show in section 3.2 that we can obtain explicit expressions for the superconformal partial wave (SCPW) expansion of $\mathcal{G}_{2,2}$ above threshold and $\mathcal{G}_{2,1}$ in the window, from the analysis of $\mathcal{G}_{0,0}, \mathcal{G}_{1,1}$ and $\mathcal{G}_{1,0}$ of many different correlators. This approach is based on the fact that for a given twist and $su(4)$ representation we know how many two-particle operators there are [8].

Determining $\mathcal{G}_{2,0}$ below the window is more complicated. There are again a finite number of operators exchanged, but there are both protected and long contributions. These are all of the form $C^{(1)}_{p_1 p_2, \vec{\tau}} C^{(1)}_{p_3 p_4, \vec{\tau}}$ for given $\vec{\tau}$ below window. We will show that in the long sector, i.e $\tau \geq 4 + 2a + b$, we can obtain the contribution to $\mathcal{G}_{2,0}$ by rearranging slightly the method used for $\mathcal{G}_{2,1}$. At the unitarity bound, $\tau = 2 + 2a + b$, we will have to use a different approach, which we explain in section 2.4. We will see that the $1/N^4$ semishort contributions to the protected sector can also be determined by using the knowledge of the two-particle operators and various different correlators. In particular, for a given twist $2 + 2a + b$, we will use input from $O(1)$ SCPW coefficients for correlators with $\tau^{\min} = \tau^{\max} = 2 + 2a + b$, as well as input from $O(1/N^2)$ SCPW coefficients for correlators with $\tau^{\max} > \tau^{\min} = 2 + 2a + b$. Finally, we emphasize that multiplet recombination at $O(1/N^4)$ will be very much different from multiplet recombination at $O(1/N^2)$.

The functions $\mathcal{G}_{2,1}$ and $\mathcal{G}_{2,0}$, which we bootstrap starting from the leading logarithmic discontinuity $\mathcal{G}_{2,2}$, should therefore be such that the first can accommodate OPE predictions in the window, and the second can accommodate OPE predictions below window i.e. for $2 + 2a + b \leq \tau < \tau^{\min}$.

Having listed all OPE predictions, we can proceed to assemble the four point correlator at strong coupling. Our field theoretic approach has the partial non-renormalisation theorem [35] as the starting point. This theorem splits the correlator into free theory and a particular form for the dynamical contribution,

$$\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle = \langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle_{\text{free}} + \mathcal{P} \, \mathcal{I}(x, \bar{x}; y, \bar{y}) \mathcal{D}_{\vec{p}}(x, \bar{x}; y, \bar{y}) \,, \qquad (1.7)$$

where $\mathcal{I}(x,\bar{x};y,\bar{y})$ and $\mathcal{P}$ are kinematical factors defined later in (2.9) and (2.20). In the above decomposition of the correlator, free theory is an input, and will be computed by actual Wick contractions in free SU(N) $\mathcal{N}=4$ SYM, having established the precise definition of single particle operators, already given in our previous work [8]. In this line of reasoning, the free theory part of the correlator is obtained independently from Witten diagrams consideration in supergravity.

We will find that the dynamical function in the strong coupling regime has extra structure, inherited from the way OPE predictions naturally organise themselves in supergravity. We will write the result as

$$\mathcal{D}_{\vec{p}}(x,\bar{x};y,\bar{y}) = \mathcal{T}_{\vec{p}} + \frac{1}{N^4}\mathcal{H}_{\vec{p}}^{(2)} + \dots , \tag{1.8}$$

where $\mathcal{H}_{\vec{p}}^{(2)}$ meets all the predictions from the OPE at one loop. The function $\mathcal{T}_{\vec{p}}$ instead is constructed as an exact function of N by requiring that it is tree-like, in the way we specify below. A feature of this function is to cancel stringy states present in free theory but absent in the supergravity spectrum, indeed at any order in N. The leading term of $\mathcal{T}_{\vec{p}}$ coincides with the tree level function of Rastelli and Zhou [1], i.e.

$$\mathcal{T} = \frac{1}{N^2}\mathcal{T}^{(1)} + \frac{1}{N^4}\mathcal{T}^{(2)} + \dots . \tag{1.9}$$

It follows from our construction that the functions $\mathcal{G}_{2,i}$ are given by[4]

$$\mathcal{G}_{\vec{p};2,2} = \mathcal{P}\,\mathcal{I}(x,\bar{x};y,\bar{y})\,\mathcal{H}_{\vec{p}}^{(2)}\Big|_{\log^2 u} \tag{1.10}$$

$$\mathcal{G}_{\vec{p};2,1} = \mathcal{P}\,\mathcal{I}(x,\bar{x};y,\bar{y})\left[\mathcal{T}_{\vec{p}}^{(2)} + \mathcal{H}_{\vec{p}}^{(2)}\right]_{\log^1 u} \tag{1.11}$$

$$\mathcal{G}_{\vec{p};2,0} = \langle\mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle_{\text{free}}\Big|_{\frac{1}{N^4}} + \mathcal{P}\,\mathcal{I}(x,\bar{x};y,\bar{y})\left[\mathcal{T}_{\vec{p}}^{(2)} + \mathcal{H}_{\vec{p}}^{(2)}\right]_{\log^0 u} \tag{1.12}$$

The function $\mathcal{H}_{\vec{p}}^{(2)}$ is what we call the 'minimal' one-loop function. As we shall explain throughout section 3 and section 4 the minimal one-loop function is the unique solution, up to finite spin ambiguities, of our bootstrap algorithm. It matches the constraints from the $1/N^4$ OPE predictions, both in the long sector, and at the unitarity bound, and it is minimal because it can be explicitly separated from $\mathcal{T}_{\vec{p}}$. In section 4, we describe the details of our bootstrap algorithm, we discuss a number of non-trivial examples, and we classify the possible one-loop ambiguities.

The function $\mathcal{T}_{\vec{p}}$ is studied in more detail in section 5. It is a generalisation of the tree-level function of Rastelli and Zhou for all N, and it is defined by the property that, together with connected free theory, it gives empty contributions to any exchanged long operators with twist $2+2a+b \leq \tau < \tau^{\min}$. In this sense, the function $\mathcal{T}_{\vec{p}}$ generalises

---

[4]Free theory and $\mathcal{T}_{\vec{p}}$ can be equivalently expanded in inverse powers of $c = (N^2-1)/4$. The spin and twist dependence of the OPE predictions at one loop is unchanged, because these are obtained by multiplying SCPW data at tree level, and the latter is not sensitive to a shift of the expansion parameter. The minimal one-loop function is thus unchanged. At $O(1/c^2)$ the correlator is given by $\mathcal{H}_{\vec{p}}^{(2)}$ plus the contributions from free theory and $\mathcal{T}_{\vec{p}}$. The $\log u$ stratification has a form similar to that given in (1.11)–(1.12).

the construction of Dolan, Nirschl and Osborn in [36] who obtained tree-level results by precisely demanding such a cancellation of low twist operators against recombined free theory. Because of this property, the minimal loop function $\mathcal{H}_{\vec{p}}^{(2)}$ contains all the dynamical information at $O(1/N^4)$.

Comparing to our previous works in [3, 7], the one-loop functions constructed there coincide with the minimal one-loop function, and $\mathcal{T}_{\vec{p}}$ does not upgrade beyond the function of Rastelli and Zhou. This happens because free theory in those cases is exact in $1/(N^2-1)$. For the special class of correlators denoted as next-to-next-to-extremal we will find a similar situation, see sections 4.6 and 5.3. However, there is no such a simplification in general, and in fact the study of four point correlators with arbitrary external charges, which is the main subject of this paper, requires non trivially both $\mathcal{H}_{\vec{p}}^{(2)}$ and a novel $\mathcal{T}_{\vec{p}}$.

Finally, let us point out a very remarkable feature of $\mathcal{H}_{\vec{p}}^{(2)}$: the three-point couplings $C_{p_1 p_2, \vec{\tau}}$ of exchanged semishort operators, which constrain a piece of $\mathcal{G}_{2,0}$, are obtained only within free theory, since these are not renormalized. At the same time, $\mathcal{G}_{2,2}$ and $\mathcal{G}_{2,1}$ are determined only within the long sector. In this sense, some inputs in $\mathcal{G}_{2,0}$ are obtained in a completely independent way. Nevertheless, $\mathcal{H}_{\vec{p}}^{(2)}$ has to be consistent with all of the $\mathcal{G}_{2,i=0,1,2}$, and the coherence of the whole minimal one-loop function across the various OPE predictions is a non-trivial confirmation of the AdS/CFT correspondence within the $\mathcal{N}=4$ bootstrap program.

2 Free theory of single-particle operators

We are interested in correlation functions of protected half-BPS operators which describe scattering of single-particle states in $\text{AdS}_5 \times \text{S}^5$. The first task is thus to determine the operators dual to single-particle states: these are not simply single-trace operators but can have multi-trace corrections which we must take into account. In [8] we identified the operators dual to single-particle states as those *half-BPS operators which are orthogonal to all multi-trace operators*. In the strict large N limit, our definition reduces to the familiar statement that single-particle states correspond to operators in multiplets whose superconformal primaries are given by single-trace operators in the $[0, p, 0]$ representation of $su(4)$. For finite N instead, our definition automatically picks the correct multi-trace admixtures which is needed to uplift half-BPS single-trace operators to single-particle operators.[5]

Single-trace operators in the $[0, p, 0]$ rep can be given as

$$\text{tr}(\phi^p)(x,y) = y^{R_1} \ldots y^{R_p} \text{tr}(\phi_{R_1} \ldots \phi_{R_p})(x) \tag{2.1}$$

where the fields ϕ_R are the elementary scalars of the $\mathcal{N}=4$ multiplet, and the SO(6) null vector y^R is used to project onto the symmetric traceless representation, $\phi(x,y) = y^R \phi_R(x)$. The $p = 2$ case corresponds to the superconformal primary for the energy-momentum multiplet which is dual to the graviton multiplet in AdS_5. The $p = 3$ case is the first Kaluza-Klein mode arising from reduction of the IIB graviton supermultiplet on S^5. In

---

[5]See also previous discussions in [5, 37, 38] and more recently [39].

these two cases, the single-particle operator equals the single-trace operator, even at finite N, since there are no multi-trace operators of charges $p < 4$ to mix with.

The single-particle operators we consider explicitly in this paper are:

$$\mathcal{O}_2 = \text{tr}(\phi^2)$$
$$\mathcal{O}_3 = \text{tr}(\phi^3)$$
$$\mathcal{O}_4 = \text{tr}(\phi^4) - \frac{2N^2 - 3}{N(N^2 + 1)}\text{tr}(\phi^2)^2\,. \tag{2.2}$$

The coefficients of the higher multi-trace contributions are determined by the orthogonality conditions, according to our definition. For example $\mathcal{O}_4$ is defined by the requirement that it is orthogonal to the double-trace operator $\text{tr}(\phi^2)^2$:

$$\langle \mathcal{O}_4(x_1, y_1)\text{tr}(\phi^2)^2(x_2, y_2)\rangle = 0\,. \tag{2.3}$$

Notice that since all operators involved are half BPS, the two-point functions entering the orthogonality conditions can be computed in free field theory in terms of the elementary propagators

$$\langle (\phi)_r{}^{\bar{r}}(x_1, y_1)(\phi)_s{}^{\bar{s}}(x_2, y_2)\rangle = \left(\delta_r^{\bar{s}}\delta_s^{\bar{r}} - \frac{1}{N}\delta_r^{\bar{r}}\delta_s^{\bar{s}}\right)g_{12}\,, \tag{2.4}$$

where[6]

$$g_{12} = \frac{y_i \cdot y_j}{x_{12}^2}. \tag{2.5}$$

We will now consider four-point correlators of the single-particle half-BPS operators, first in free theory, and then in the interacting regime described by supergravity.

2.1 Free theory four-point functions

Free field four-point functions of single-particle half-BPS operators can be computed simply by performing Wick contractions between the elementary fields. The result is a sum over the different allowed superpropagator structures g_{ij} accompanied by their colour factors. Graphically, the four external operators $\mathcal{O}_{p_i}$ are represented as vertices each with p_i legs, and the propagator g_{ij} is represented as a line between point i and point j. We arrange the four operators at the corners of a square, labelled clockwise from the bottom left.

So for example, for the $\langle \mathcal{O}_3\mathcal{O}_3\mathcal{O}_3\mathcal{O}_3\rangle_{\text{free}}$ correlator

$$\langle \mathcal{O}_3\mathcal{O}_3\mathcal{O}_3\mathcal{O}_3\rangle_{\text{free}} = \quad A_0^0 \,\text{⬭}\;\text{⬭}$$
$$+ A_2^0 \,\text{⬯⬰⬭} + A_2^1 \,\text{▭⬭}$$
$$+ A_4^0 \,\text{⊠} \quad + A_4^1 \,\text{⊠} \quad + A_4^2 \,\text{▯}$$
$$+ A_6^0 \,\text{⊠} \quad + A_6^1 \,\text{⊠} \quad + A_6^2 \,\text{⊠} \quad + A_6^3 \,\text{⬖} \tag{2.6}$$

---

where A_γ^k are the associated colour factors. The subscript γ is the total number of propagators connecting the left half of the graph to the right half, whereas k is the number of propagators along the top edge of the square. Of course many colour factors are equal to each other, where the corresponding graphs are isomorphic. Indeed there are only three independent colour factors in this example and explicit computations of the Wick contractions yields the all orders in N factors

$$A_0^0 = A_6^0 = A_6^3 = \frac{9(N^2-4)^2(N^2-1)^2}{N^2}$$

$$A_2^0 = A_2^1 = A_4^0 = A_4^2 = A_6^1 = A_6^2 = \frac{9}{N^2-1}A_0^0$$

$$A_4^1 = \frac{18(N^2-12)}{(N^2-1)(N^2-4)}A_0^0 . \tag{2.7}$$

For a general free theory correlator, without loss of generality we can arrange the external charges as $p_{43} \geq p_{21} \geq 0$. The general free theory result is then

$$\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle_{\text{free}} = \mathcal{P} \times \sum_{\substack{\gamma = p_{43} \\ \gamma - p_{43} \in 2\mathbb{Z}}}^{\min(p_1+p_2, p_3+p_4)} \left[\left(\frac{g_{13}g_{24}}{g_{12}g_{34}} \right)^{\frac{\gamma - p_{43}}{2}} \sum_{k=0}^{\frac{\gamma - p_{43}}{2}} A_\gamma^k \left(\frac{g_{14}g_{23}}{g_{13}g_{24}} \right)^k \right] \tag{2.8}$$

where A_γ^k are color factors, and we defined the prefactor

$$\mathcal{P} = g_{12}^{\frac{p_1+p_2-p_{43}}{2}} g_{14}^{\frac{p_{43}-p_{21}}{2}} g_{24}^{\frac{p_{43}+p_{21}}{2}} g_{34}^{p_3} . \tag{2.9}$$

Note that the r.h.s. of (2.8) is $\mathcal{P}$ times a function of super cross-ratios. We define space-time cross ratios u, v (equivalently $x, \bar{x}$) and internal cross-ratios $\hat{\sigma}, \hat{\tau}$ (equivalently $y, \bar{y}$) as follows

$$u = x\bar{x} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = (1-x)(1-\bar{x}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2},$$

$$\frac{1}{\hat{\sigma}} = y\bar{y} = \frac{y_1 \cdot y_2 y_3 \cdot y_4}{y_1 \cdot y_3 y_2 \cdot y_4}, \qquad \frac{\hat{\tau}}{\hat{\sigma}} = (1-y)(1-\bar{y}) = \frac{y_1 \cdot y_4 y_2 \cdot y_3}{y_1 \cdot y_3 y_2 \cdot y_4}. \tag{2.10}$$

Inputting the definition of the superpropagator (2.5) we find the super cross-ratios

$$\frac{g_{13}g_{24}}{g_{12}g_{34}} = \frac{x\bar{x}}{y\bar{y}} = u\hat{\sigma}, \qquad \frac{g_{14}g_{23}}{g_{13}g_{24}} = \frac{(1-x)(1-\bar{x})}{(1-y)(1-\bar{y})} = \frac{\hat{\tau}}{v\hat{\sigma}}, \tag{2.11}$$

which we can substitute directly in (2.8)

For single-particle external operators the colour factors of extremal and next-to-extremal correlators vanish identically. These are correlators whose charges satisfy (with our choice of $p_{43} \geq p_{21} \geq 0$)

$$p_4 = p_1 + p_2 + p_3 \qquad \qquad \text{(extremal)},$$

$$p_4 = p_1 + p_2 + p_3 - 2 \qquad \qquad \text{(next-to-extremal)}. \tag{2.12}$$

$\tau^{\max}=p_1+p_2$ • — — — — — — —

Window

• — — — — — — • $\tau^{\min}=p_3+p_4$

$2\kappa=2p_3$ ↕

Below

Window

• — — — — — — • p_4-p_3

p_2-p_1 •

— — — — — — — • $\tau^{\max}=p_3+p_4$

Window

$\tau^{\min}=p_1+p_2$ • — — — — — —

Below

Window

• — — — — — — $2\kappa=p_1+p_2+p_3-p_4$ ↕

• — — — — — • p_4-p_3

p_2-p_1 •
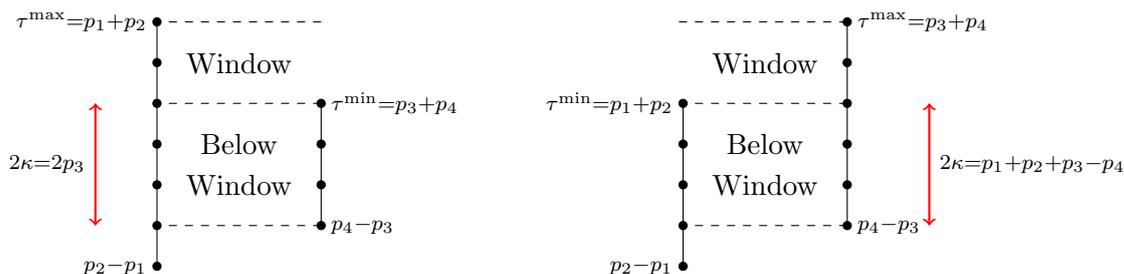
Figure 2. Illustration of the possible $su(4)$ representations exchanged in the overlap of the $(\mathcal{O}_{p_1} \times \mathcal{O}_{p_2})$ and $(\mathcal{O}_{p_3} \times \mathcal{O}_{p_4})$ OPEs. The vertical axis represents the possible values of $b + 2a$.

Notice that extremal and next-to-extremal correlators of half-BPS operator do not vanish for single trace operators but they do for single-particle operators.

The first single-particle correlators that are non-vanishing are next-to-next-to-extremal, with charges obeying

$$p_4 = p_1 + p_2 + p_3 - 4. \tag{2.13}$$

More generally, we define

$$\kappa_{\vec{p}} = \min\left(\frac{1}{2}(p_1 + p_2 + p_3 - p_4), p_3\right) = \text{"degree of extremality"} \tag{2.14}$$

and we say that a correlator is a $N^\kappa E$, according to its degree to extremality. Next-to-next-to-extremal correlators have degree of extremality $\kappa_{\vec{p}} = 2$.

The degree of extremality determines the number of available $su(4)$ representations $[aba]$ in the overlap of the two OPEs $(\mathcal{O}_{p_1} \times \mathcal{O}_{p_2})$ and $(\mathcal{O}_{p_3} \times \mathcal{O}_{p_4})$.[7] For example, N^2E correlators have the feature that the superconformal primaries in the long sector have a single possible $su(4)$ representation. One can visualise the degree of extremality as shown in figure 2. In this figure, the vertical axis represents the possible values of $b + 2a$ in the two OPEs. The degree of extremality $\kappa_{\vec{p}}$ then denotes the size of the overlap in either of the two cases $p_1 + p_2 > p_3 + p_4$ or $p_1 + p_2 < p_3 + p_4$.

Note that, as will be detailed in the next section, the interacting part of the correlator has a universal structure which reduces the range of $su(4)$ structures by 2.

We now review the technology that allows us to perform the superconformal partial wave expansion (SCPW) of a generic $\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle$ correlator. We follow the formalism of [34], which is group theoretic, manifestly unitary, and has the great advantage of dealing with all representations in a uniform way.

2.2 Review of the SCPW expansion

To address the SCPW expansion of $\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle$ we must first describe conformal blocks for all supermultiplets that might be exchanged in the OPE of half-BPS operators.

---

[7]Notice that an $su(4)$ rep $[aba]$ appearing in both OPEs will have $b + 2a$ lying in the same range of values as γ in (2.8). We can then see that degree of extremality $\kappa_{\vec{p}}$ is equal to the number of values of γ in (2.8), minus 1.

Following [34] we label the superconformal primaries $\mathcal{O}_{\gamma,\underline{\lambda}}$ by a number γ and a finite dimensional representation of SL(2|2) which we specify via a Young diagram $\underline{\lambda} \equiv [\lambda_1, \ldots, \lambda_n]$ where λ_i is the length of the ith row.[8]

The Young diagrams do not have an arbitrary shape but have to fit into a 'fat hook' shape, which amounts to the additional constraint that the third row (and hence any subsequent rows) cannot be longer than length two, i.e. $\lambda_3 \leq 2$. The number of rows also satisfies $n \leq (\gamma - p_{43})/2$. For example a generic such diagram has the form

$$= [\lambda_1, \lambda_2, 2^{\mu_2}, 1^{\mu_1}] \qquad (2.15)$$
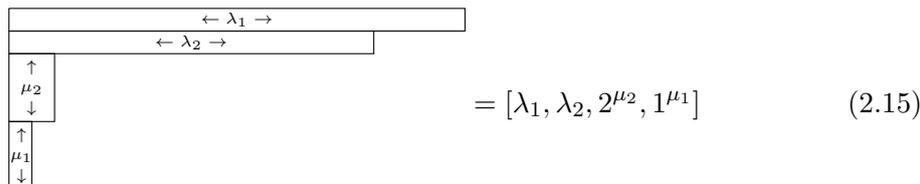
with first row of length λ_1, second row of length λ_2 and then μ_2 rows of length 2 (denoted 2^{μ_2}) and μ_1 rows of length 1 (denoted 1^{μ_1}). Such a generic Young tableau corresponds to a long multiplet.

Short multiplets instead have row 2 of length 1 or 0 and so have the shape of a 'thin hook'. The parameters γ and $\underline{\lambda}$ determine the usual quantum numbers of spin l, dimension Δ (or twist $\tau \equiv \Delta - l$) and $su(4)$ representation, which here always takes the form $[aba]$. The dictionary is summarized by the following table

GL(2|2) rep $\underline{\lambda}$	$\tau = \Delta - l$	l	$su(4)$ labels	multiplet type
$[\emptyset]$	γ	0	$[0, \gamma, 0]$	half BPS
$[1^\mu]$	γ	0	$[\mu, \gamma - 2\mu, \mu]$	quarter BPS
$[\lambda, 1^\mu]$ $(\lambda \geq 2)$	γ	$\lambda - 2$	$[\mu, \gamma - 2\mu - 2, \mu]$	semi-short
$[\lambda_1, \lambda_2, 2^{\mu_2}, 1^{\mu_1}]$ $(\lambda_2 \geq 2)$	$\gamma + 2\lambda_2 - 4$	$\lambda_1 - \lambda_2$	$[\mu_1, \gamma - 2\mu_1 - 2\mu_2 - 4, \mu_1]$	long

$$(2.16)$$

Note that the YT representation of a long multiplet is invariant up to the shift-symmetry,

$$\lambda_1 \to \lambda_1 + 1, \ \lambda_2 \to \lambda_2 + 1, \ \mu_2 \to \mu_2 - 1, \ \gamma \to \gamma - 2, \qquad (2.17)$$

under which twist τ, spin l, and $su(4)$ rep $[aba]$ remain fixed. On the contrary, protected operators require both γ and the Young tableau to be fully specified.

We denote the superconformal block corresponding to the contribution of an operator $\mathcal{O}_{\gamma,\underline{\lambda}}$ to the four-point correlator $\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle$ as

$$\text{superblock}: \qquad \mathbb{S}_{\vec{p};\gamma,\underline{\lambda}}. \qquad (2.18)$$

---

[8]The formalism arises from analytic superspace [40–42] which has SL(2|2) × SL(2|2) × $\mathbb{C}$ isotropy group. A general unitary representation of the $\mathcal{N}=4$ superconformal group is thus specified via two SL(2|2) representations and a weight γ. For four-point functions of half BPS operators, both SL(2|2) representations coincide. Remarkably the SL(2|2) representations are always finite dimensional and the resulting analytic field is unconstrained [43, 44].

Long superblocks (those with $\lambda_2 = 2, 3, \ldots$) will occur often and we will also denote them by $\mathbb{L}_{\vec{p};\vec{\tau}}$. They have the following factorised structure,

$$\mathbb{L}_{\vec{p};\vec{\tau}} \equiv \mathbb{S}_{\vec{p};\gamma,\underline{\lambda}} = \mathcal{P} \times \mathcal{I} \times \widetilde{\mathbb{L}}_{\vec{p};\vec{\tau}} \qquad \widetilde{\mathbb{L}}_{\vec{p};\vec{\tau}} = \frac{\mathcal{B}^{(2+\frac{\tau}{2},l)}}{u^{2+\frac{p_{43}}{2}}} \, \Upsilon_{[aba]} \, , \tag{2.19}$$

where $\mathcal{P}$ is given in (2.9), and $\mathcal{I}$ by

$$\mathcal{I}(x, \bar{x}, y, \bar{y}) = \frac{(x-y)(x-\bar{y})(\bar{x}-y)(\bar{x}-\bar{y})}{(y\bar{y})^2} \, . \tag{2.20}$$

Here $\mathcal{B}^{t,l}$ and $\Upsilon_{[aba]}$ are ordinary bosonic blocks for conformal and internal symmetries. Explicitly,

$$\mathcal{B}^{(t,l)}(x, \bar{x}) = (-1)^l \, u^t \left[\frac{x^{l+1} F_{t+l}(\bar{x}) F_{t-1}(\bar{x}) - (x \leftrightarrow \bar{x})}{x - \bar{x}} \right] \, , \tag{2.21}$$

and

$$\Upsilon_{[aba]}(y, \bar{y}) = -\frac{P_{n+1}(y) P_m(\bar{y}) - (y \leftrightarrow \bar{y})}{y - \bar{y}} \, , \qquad \begin{cases} n = m + a, \\ m = \dfrac{b - p_{43}}{2} \, , \end{cases} \tag{2.22}$$

where

$$F_t(x) = {}_2F_1\left(t - \frac{p_{12}}{2}, t + \frac{p_{34}}{2}, 2t; x\right), \quad P_n(y) = \frac{n! \, y}{(n+1+p_{43})_n} \mathrm{JP}^{(p_{43}-p_{21}|p_{43}+p_{21})}\left(\frac{2}{y} - 1\right) \tag{2.23}$$

The notation JP stands for Jacobi polynomial.

Explicit formulae for semishort, $\frac{1}{4}$-BPS and $\frac{1}{2}$-BPS superblocks were obtained in [34] and can be found in appendix A. Especially in these cases, the superblock formalism naturally provides manifestly unitary representations.

Since the parameters λ_i are defined by a Young diagram, they are a priori integer valued. For long superblocks however in the interacting theory, the scaling dimension Δ (or equivalently the twist τ) of an operator becomes anomalous and hence non-integer. We can thus allow an analytic continuation of λ_1 and λ_2 such that the spin $\lambda_1 - \lambda_2 = l$ remains integer. In such cases we even allow for continuations such that $\lambda_2 < 2$. This means that the labels of such continued long superblocks can coincide with those of short superblocks when $\lambda_2 \to 1, \mu_2 = 0$. To avoid this potential confusion therefore we simply use the notation for long superblocks, $\mathbb{L}_{\vec{p};\vec{\tau}}$, on the l.h.s. of (2.19) and allow $\tau \geq 2a + b + 2$ to be non-integer valued.

When long supermultiplets sit exactly on the unitarity bound, $\tau = 2 + 2a + b$, they become reducible and can be expressed as a sum of short multiplets

$$\mathbb{L}_{\vec{p};\vec{\tau}} = \mathbb{S}_{\vec{p};\tau,[l+2,1^a]} + \mathbb{S}_{\vec{p};\tau+2,[l+1,1^{a+1}]} \qquad \tau = 2 + 2a + b \, . \tag{2.24}$$

The first term on the r.h.s. of (2.24) is a semi-short superblock of spin l while the second is a semi-short superblock of spin $l-1$ or a quarter-BPS superblock (if $l = 0$). We will make use of this reducibility in section 2.4.

2.3 The SCPW expansion of the free theory

The SCPW of free theory correlators naturally stratifies by the label $\gamma = p_{43}, p_{43} + 2, \ldots, \tau^{\min} = \min(p_1 + p_2, p_3 + p_4)$ introduced in (2.8). As mentioned in that context, γ counts the number of propagators connecting operators inserted at points 1 and 2 to operators inserted at points 3 and 4. In the SCPW expansion, γ simply corresponds to the number of fundamental fields appearing in the operator, $\mathcal{O}_{\gamma,\underline{\lambda}}$ being exchanged in the OPE. Note that this is a good quantum number only for free theory, and simply reflects the number of Wick contractions which have occurred in the OPE:

$$
\begin{aligned}
\gamma &= \# \text{ fundamental fields defining } \mathcal{O}_{\gamma,\underline{\lambda}} \\
&= p_1 + p_2 - (\# \text{ Wick contractions in } \mathcal{O}_{p_1}\mathcal{O}_{p_2} \sim \mathcal{O}_{\gamma,\underline{\lambda}} \text{ OPE }) \\
&= p_3 + p_4 - (\# \text{ Wick contractions in } \mathcal{O}_{p_3}\mathcal{O}_{p_4} \sim \mathcal{O}_{\gamma,\underline{\lambda}} \text{ OPE })
\end{aligned}
\tag{2.25}
$$

The general free theory correlator (2.8) then decomposes as

$$
\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4} \rangle_{\text{free}} = \sum_{\substack{\gamma = p_{43} \\ \gamma - p_{43} \in 2\mathbb{Z}}}^{\min(p_1+p_2,p_3+p_4)} \sum_{\underline{\lambda}} A_{\vec{p};\gamma,\underline{\lambda}} \, \mathbb{S}_{\vec{p};\gamma,\underline{\lambda}}
\tag{2.26}
$$

where each term in the sum over γ represents the expansion in SCPW of the analogous terms in (2.8). Furthermore the Young tableau $\underline{\lambda}$ have at most $(\gamma - p_{43})/2$ rows. Note also that in free theory all Young Tableau are proper, having both integer rows and correct shape. Thus the decomposition (2.26) is unambiguous.

But we do not consider the free theory in isolation, rather we will consider it as the limit of the interacting theory as the coupling vanishes. In the interacting theory, the OPE of two half-BPS operators contains both operators in short supermultiplets, whose dimensions are protected, and long operators which have anomalous dimensions. Therefore we will split the SCPW expansion (2.26) accordingly, and we will distinguish between the short sector *which by definition remains short in the interacting theory*, and a *free long* sector which will then acquire an anomalous dimension in the interacting theory. For the short sector we sum over superblocks with the specific form $\mathbb{S}_{\gamma,[\lambda,1^\mu]}$, and for the long sector we sum over superblocks $\mathbb{L}_{\vec{\tau}}$,

$$
\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4} \rangle_{\text{free}} = \langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4} \rangle_{\text{short}} + \langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4} \rangle_{\text{free long}} .
\tag{2.27}
$$

More explicitly, we introduce the SCPW coefficients $S_{\gamma,[\lambda,1^\mu]}$ and $L_{\vec{\tau}}^f$ as follows

$$
\begin{aligned}
\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4} \rangle_{\text{short}} &= \sum_{\substack{\gamma = p_{43} \\ \gamma - p_{43} \in 2\mathbb{Z}}}^{\tau^{\min}} \left[S_{\gamma,\emptyset} \mathbb{S}_{\gamma,\emptyset} + \sum_{\lambda=1}^{\infty} \sum_{\mu=0}^{\frac{1}{2}(\gamma-p_{43})-1} S_{\gamma,[\lambda,1^\mu]} \, \mathbb{S}_{\gamma,[\lambda,1^\mu]} \right] \\
\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4} \rangle_{\text{free long}} &= \sum_{\substack{a,b \in 2\mathbb{Z} \\ 4+b+2a \leq \tau^{\min}}} \sum_{l=0}^{\infty} \sum_{\substack{\tau > 2a+b \\ \tau - b \in 2\mathbb{Z}}} L_{\vec{\tau}}^f \, \mathbb{L}_{\vec{\tau}} .
\end{aligned}
\tag{2.28}
$$

This split is non-trivial due to multiplet recombination; in the free limit a long multiplet whose twist lies on the unitary bound is indistinguishable from the direct sum of certain short multiplets. A consequence of this is the identity of superblocks (2.24). The challenge then is to relate the SCPW coefficients $S_{\gamma,[\lambda,1^\mu]}$ and $L^f_{\vec{\tau}}$ to the original ones $A_{\gamma,\underline{\lambda}}$ in (2.26).

The simplest SCPW coefficients to identify are the coefficients of half BPS ops ($\underline{\lambda} = \emptyset$) which are unchanged. Thus

$$S_{\gamma,\emptyset} = A_{\gamma,\emptyset}\,. \tag{2.29}$$

The next simplest to deal with are the long representations above the unitary bound. Here we take into account the fact that γ ceases to be a good quantum number for long operators. This is because long operators with different numbers of fundamental fields mix. For example $\mathcal{O}_3\mathcal{O}_3$ ($\gamma = 6$) mixes with $\mathcal{O}_2\Box\mathcal{O}_2$ ($\gamma = 4$) which both have twist 6. This is the origin of the ambiguity in the description of long operators (2.17). Thus we need to collect together all SCPW coefficients with the same quantum numbers $\vec{\tau}$ (but different values of γ) using the shift symmetry (2.17). Thus

$$L^f_{\vec{\tau}} = \sum_{\gamma=b+2a+4}^{\min(p_1+p_2,p_3+p_4)} A_{\gamma,[2+\frac{\tau-\gamma}{2}+l,2+\frac{\tau-\gamma}{2},2^{\frac{\gamma-b}{2}-a-2},1^a]}, \qquad \tau \geq 4 + 2a + b\,. \tag{2.30}$$

The most difficult SCPW coefficients to identify in (2.28) are the (non half-BPS) short coefficients $S_{[\lambda,1^\mu]}$ with non-zero λ or μ and the related long coefficients at the unitary bound $L_{\vec{\tau}}$ with $\tau = 2a + b + 2$. This is because as we deform away from the free theory, some semi-short blocks combine to become long (as in (2.24)), whereas others remain semi-short. Thus, a single SCPW coefficient A for a semi-short block at the unitarity bound, can actually contain the contribution of both short and long multiplets of the interacting theory.

Our next task will be to explain how to properly disentangle physical semishort contributions from the SCPW coefficients of free theory, and find $S_{[\lambda,1^\mu]}$. Let us motivate this problem further by mentioning that separating the coefficients S from L at the unitary bound is actually straightforward at $O(1/N^2)$. In particular we will show that apart from the case $S_{\gamma,[\lambda,1^\mu]}$ with $\gamma = \min(p_1 + p_2, p_3 + p_4)$, i.e when $\tau = \tau^{\min}$, all other the coefficients $S_{\gamma,[\lambda,1^\mu]}$ vanish. Thus the values of L will be trivially fixed by multiplet recombination. This feature at $O(1/N^2)$ has lead various people to the assumption that the same would be true for all N (see [45] for a discussion of this point). However, beyond $O(1/N^2)$ the separation of coefficients S from L is a non-trivial problem. We will solve this problem to $O(1/N^4)$ using knowledge about the form of the semi-short operators.

2.4 Multiplet recombination

We now show how to determine, up to order $1/N^4$, the genuine semishort sector of the single particle correlators $\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle$ in the full interacting theory, purely using free theory correlators. In particular we provide formulae for all SCPW coefficients — split according to operators which remain short in the interacting theory and those which are long (2.28) — in terms of the coefficients $A_{\vec{p},\gamma,\underline{\lambda}}$ in (2.26).

Recall that for long blocks at the unitary bound $\tau = 2a + b + 2$ we need to resolve the ambiguity which follows from the reducibility condition (2.24), i.e. that a long SCPW is a sum of two semishort SCPWs

$$\mathbb{L}_{\vec{\tau}} = \mathbb{S}_{\tau,[l+2,1^a]} + \mathbb{S}_{\tau+2,[l+1,1^{a+1}]} \qquad \tau = 2a + b + 2\,. \tag{2.31}$$

Comparing the two pieces of the SCPW expansion (2.28), and equating the coefficient of $\mathbb{S}_{\tau,[l+2,1^a]}$, using (2.31), yields

$$A_{\tau,[l+2,1^a]} = S_{\tau,[l+2,1^a]} + L^f_{\vec{\tau}} + L^f_{\tau-2,l+1,[a-1,b,a-1]} \qquad \tau = 2a + b + 2\,. \tag{2.32}$$

One of the key points allowing us to resolve the ambiguity at the unitarity bound, and correctly distinguish CPW coefficients of long and semi-short operators, is the following (already tacitly assumed in (2.28)): *a long operator at the unitarity bound necessarily has twist less than* $\tau_{min} = \min(p_1 + p_2, p_3 + p_4)$, *i.e.* $L^f_{\vec{\tau}} = 0$ *if* $\tau = 2a + b + 2 \geq \tau^{\min}$. This is a non-perturbative statement, a non-trivial consequence of superconformal symmetry for the corresponding three-point functions [46, 47].

This fact allows us to use equation (2.32) to determine the CPW coefficients of semi-short operators of twist $\tau^{\min} = \min(p_1 + p_2, p_3 + p_4)$ in terms of lower twist coefficients

$$S_{\vec{p};\tau,[l+2,1^a]} = A_{\vec{p};\tau,[l+2,1^a]} - L^f_{\vec{p};\tau-2,l+1,[a-1,b,a-1]} \qquad \text{for} \quad \tau = 2a+b+2 = \tau^{\min}, \quad a \geq 1$$

$$S_{\vec{p};\tau,[l+2]} = A_{\vec{p};\tau,[l+2]} \qquad \qquad \qquad \text{for} \quad \tau = b+2 = \tau^{\min}\,. \tag{2.33}$$

It is useful to understand the $1/N$ expansion[9] of $S_{\vec{p};\tau^{\min},[l+2,1^a]}$ first, since it will play a role in our later formulas. Referring to figure 1, when $\tau^{\min} = 2+2a+b$ two lines coincide, i.e. the lower dashed line sits on top of the middle dashed line, thus we find that $S_{\vec{p};\tau^{\min},[l+2,1^a]}$ in (2.33) is non trivial at $O(1/N^2)$. In particular it gets a contribution from leading order connected propagator structures. In the special case of correlators $\langle \mathcal{O}_p \mathcal{O}_q \mathcal{O}_p \mathcal{O}_q \rangle$, $\tau^{\min} = \tau^{\max}$ and free theory starts with an $O(1)$ contribution from disconnected diagrams. For all representations $[aba]$ such that $\tau^{\min} = 2 + 2a + b$ we find then that all three dashed lines of the figure 1 coincide and $S_{\vec{p};\tau^{\max},[l+2,1^a]}$ indeed has an $O(1)$ contribution from disconnected free theory diagrams.

What about CPW coefficients of semi-short operators of twist less than $\tau^{\min}$? Semishort operators generically will sit in the range of twists $\tau \leq \min(p_1 + p_2, p_3 + p_4)$, therefore at the bottom dashed line in figure 1 below the window. It follows that the corresponding SCPW coefficient is $O(1/N^4)$,

$$S_{\vec{p};\tau,[l+2,1^a]} = O(1/N^4) \qquad \tau = 2a + b + 2 < \tau^{\min}\,. \tag{2.34}$$

This is the well known statement that at $O(1/N^2)$ there are no semishort operators in the spectrum below the window, which implies a cancellation between free theory and the interacting part. Using this information we can solve $S_{\vec{p};\tau^{\min},[l+2,1^a]}$ in (2.33) and $L^f_{\vec{\tau}}$

---

[9]Note that here and below, 'order $1/N^k$', really means $N^{\frac{1}{2}(p_1+p_2+p_3+p_4)}O(1/N^k)$ because we have not normalised our external operators.

in (2.32) explicitly up to order $1/N^2$. First we solve (2.32) recursively, thus obtaining the long SCPW coefficients

$$L_\tau^f = \sum_{k=0}^{a} (-1)^k A_{\tau-2k,[l+2+k,1^{a-k}]} + O(1/N^4) \qquad \tau = 2a+b+2 < \tau^{\min}.$$ (2.35)

Then, we plug this result into (2.33) to give the genuine semi-short coefficients at threshold

$$S_{\vec{p};\tau,[l+2,1^a]} = \sum_{k=0}^{a} (-1)^k A_{\tau-2k,[l+2+k,1^{a-k}]} + O(1/N^4) \qquad \tau = 2a+b+2 = \tau^{\min}.$$ (2.36)

When $a = 0$, we obtain correctly $S_{\vec{p};\tau,[l+2]}$ given above.

Now, can we determine the $1/N^4$ CPW coefficients of semi-short operators of twist less than $\tau^{\min}$? The answer is affirmative. We first need to use some non-trivial information about the spectrum of semi-short operators, and then we can determine these CPW coefficients unambiguously using data from many different correlators!

The key point here is that we know the explicit form of the double trace semi-short operators — or more importantly the number of them. They are twist τ, spin l operators in the $[aba]$ $su(4)$ rep of the form

$$\mathcal{O}_{q\tilde{q}} = \mathcal{O}_q \partial^l \mathcal{O}_{\tilde{q}}$$ (2.37)

as in eq. (1.4) with $\tau = q + \tilde{q} = 2a + b + 2$. For fixed twist and $su(4)$ structure we can enumerate the independent operators as

$$q_r = a+1+r, \qquad \tilde{q}_r = a+1+b-r \qquad r = \delta_{a,0}, \ldots, \mu-1$$ (2.38)

where

$$\mu \equiv \begin{cases} \lfloor \frac{b+2}{2} \rfloor & a+l \text{ even,} \\ \lfloor \frac{b+1}{2} \rfloor & a+l \text{ odd.} \end{cases}$$ (2.39)

Unlike the case of long operators, semishort operators receive no anomalous dimension. The operators enumerated in (2.38) are therefore degenerate and we may freely take the $\mathcal{O}_{q\tilde{q}}$ themselves as our basis. The SCPW coefficients of such operators are then expressed in terms of the products of three-point couplings as follows,

$$S_{\vec{p};\tau,[l+2,1^a]} = \sum_{r,s} C_{p_1 p_2}(\mathcal{O}_{q_r \tilde{q}_r})(M^{-1})_{rs} C_{p_3 p_4}(\mathcal{O}_{q_s \tilde{q}_s}),$$ (2.40)

where M is the matrix of two-point functions (which is diagonal at leading order in large N),

$$M_{rs} = \langle \mathcal{O}_{q_r \tilde{q}_r} \mathcal{O}_{q_s \tilde{q}_s} \rangle = Y_r \delta_{rs} + O(1/N^2).$$ (2.41)

We also recall the fact, discussed in section 1, that the only couplings with a leading order contribution in the large N expansion are the ones of the form $C_{pq}(\mathcal{O}_{pq})$. From this it follows that at leading order in large N we have a diagonal structure for the following three-point couplings,

$$C_{q_r \tilde{q}_r}(\mathcal{O}_{q_s \tilde{q}_s}) = \delta_{rs} X_r + O(1/N^2).$$ (2.42)

Armed with this information we can now predict the CPW coefficients of semishort operators, $S_{\vec{p};\tau,[l+2,1^a]}$, of twist $\tau < \tau^{\min}$ in terms of SCPW coefficients of correlators with either $\tau = \tau^{\min}$. These SCPW are known through (2.36). The formula for $S_{\vec{p};\tau,[l+2,1^a]}$, correct up to and including order $1/N^4$, is given by

$$S_{\vec{p};\tau,[l+2,1^a]} = \sum_{r=0}^{\mu-1} \frac{S_{p_1 p_2 q_r \tilde{q}_r} S_{q_r \tilde{q}_r p_3 p_4}}{S_{q_r \tilde{q}_r q_r \tilde{q}_r}} + O(1/N^6) \qquad \tau = 2a + b + 2 < \tau^{\min}. \qquad (2.43)$$

For simplicity, we have suppressed labels τ and $[l+2,1^a]$ in the SCPW coefficients on the r.h.s. above.

The two factors giving the numerator of (2.43) in the r.h.s. are both $O(1/N^2)$ whereas the factor in the denominator is leading in large N, thus the r.h.s. is $O(1/N^4)$ as we stated already in (2.34). The formula (2.43) may be proven by simply using (2.40) on both sides and then using (2.42) and (2.41) on the r.h.s. to cancel the denominator.

Finally, with the knowledge of (2.43) to hand, we can improve $L_{\vec{\tau}}^f$ in (2.35) and $S_{\tau^{\min},[l+2,1^a]}$ in (2.36) up to order $1/N^4$. The results are

$$L_{\vec{\tau}}^f = \sum_{k=0}^{a} (-1)^k A_{\tau-2k,[l+2+k,1^{a-k}]} - \sum_{k=0}^{a} (-1)^k S_{\tau-2k,[l+2+k,1^{a-k}]} + O(1/N^6)$$
$$\tau = 2a + b + 2 < \tau^{\min}, \qquad (2.44)$$

$$S_{\tau,[l+2,1^a]} = \sum_{k=0}^{a} (-1)^k A_{\tau-2k,[l+2+k,1^{a-k}]} - \sum_{k=1}^{a} (-1)^k S_{\tau-2k,[l+2+k,1^{a-k}]} + O(1/N^6)$$
$$\tau = 2a + b + 2 = \tau^{\min}. \qquad (2.45)$$

Concluding, all SCPW coefficients of $\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle_{\text{short}}$ and $\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle_{\text{long}}$ in (2.28) have been obtained to $O(1/N^4)$ and therefore we have successfully split the free theory correlators into a protected contribution and an unprotected one. In general we can not go further in $1/N$ since to do so would require input from triple-trace (and higher multi-trace) operators.

We conclude this section by illustrating our formulas (2.43) and (2.45) for the semishort sectors of $\langle \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \rangle$, which has been already examined in detail in [34], and $\langle \mathcal{O}_4 \mathcal{O}_4 \mathcal{O}_4 \mathcal{O}_4 \rangle$, which is new.

In the case of $\langle \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \rangle$ we have below threshold twist 2 and 4 semishort predictions. This semishort sector is special because no multi-trace mixing occurs in the large N expansion. Therefore we can give formulas exact in N. Very explicitly we find that,

$$S_{2,[\lambda]}^{\langle 3333 \rangle} = 0$$

$$S_{4,[l+2]}^{\langle 3333 \rangle} = \frac{\left(S_{4,[\lambda]}^{\langle 2233 \rangle}\right)^2}{S_{4,[\lambda]}^{\langle 2222 \rangle}} = \frac{288((l+3)!)^2}{(2l+6)!((l+3)(l+4) + \frac{4}{(N^2-1)})} \frac{A_0^0}{(N^2-1)}$$

$$S_{4,[l+2,1]}^{\langle 3333 \rangle} = \frac{\left(S_{4,[\lambda,1]}^{\langle 2233 \rangle}\right)^2}{S_{4,[\lambda,1]}^{\langle 2222 \rangle}} = \frac{576((l+3)!)^2}{(2l+6)!((l+2)(l+5) - \frac{12}{(N^2-1)})} \frac{A_0^0}{(N^2-1)} \qquad (2.46)$$

where $A_0^0 = (3(N^2-1)(N^2-4)/N)^2$. The structure of the CPW coefficients of operators at threshold, i.e. twist 6, follow straightforwardly by applying (2.45),

$$S_{6,[\lambda]} = A_{6,[\lambda]}$$
$$S_{6,[\lambda,1]} = A_{6,[\lambda,1]} - A_{4,[\lambda+1]} + S_{4,[\lambda+1]} ,$$
$$S_{6,[\lambda,1,1]} = A_{6,[\lambda,1,1]} - A_{4,[\lambda+1,1]} + A_{2,[\lambda+2]} + S_{4,[\lambda+1,1]} . \tag{2.47}$$

In the case of $\langle \mathcal{O}_4 \mathcal{O}_4 \mathcal{O}_4 \mathcal{O}_4 \rangle$ we have twist 2, 4 and 6 predictions. The computations at twist 2 and twist 4 are analogous to the case of $\langle \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \rangle$. We find,

$$S_{2,[\lambda]}^{\langle 4444 \rangle} = 0$$
$$S_{4,[l+2]}^{\langle 4444 \rangle} = \frac{16 \times 1152}{(l+3)(l+4)} \frac{((l+3)!)^2}{(2l+6)!} \frac{1+(-)^l}{2} \frac{1}{N^4},$$
$$S_{4,[l+2,1]}^{\langle 4444 \rangle} = \frac{6 \times 1600}{(l+2)(l+5)} \frac{((l+3)!)^2}{(2l+6)!} \frac{1-(-)^l}{2} \frac{1}{N^4} \tag{2.48}$$

The twist 6 results are new,

$$S_{6,[l+2]}^{\langle 4444 \rangle} = \frac{16 \times 384(29 + 3(2l+9)^2)}{(l+3)(l+6)} \frac{(l+4)!^2}{(2l+8)!} \frac{1+(-)^l}{2} \frac{1}{N^4}$$
$$S_{6,[l+2,1]}^{\langle 4444 \rangle} = \frac{16 \times 72(401 + 174(2l+9)^2 + (2l+9)^4)}{(l+2)(l+4)(l+5)(l+7)} \frac{(l+4)!^2}{(2l+8)!} \frac{1-(-)^l}{2} \frac{1}{N^4}$$
$$S_{6,[l+2,1,1]}^{\langle 4444 \rangle} = \frac{16 \times 2400(l+2)(l+7)}{(l+3)(l+6)} \frac{(l+4)!^2}{(2l+8)!} \frac{1+(-)^l}{2} \frac{1}{N^4} \tag{2.49}$$

We insisted on $\langle \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \rangle$ and $\langle \mathcal{O}_4 \mathcal{O}_4 \mathcal{O}_4 \mathcal{O}_4 \rangle$ since these two correlators capture generic features of our discussion about the semishort sector, and furthermore because they will be investigated in section 4, where we will construct explicitly their one-loop completion. We will see then how crucial it is the information from the semishort sector for our bootstrap program.

3 OPE in Ads$_5 \times$S^5: beyond tree-level

We now turn to the study of correlation functions of single-particle operators in the interacting theory. In particular, we consider $\mathcal{N} = 4$ SYM in the regime of large 't Hooft coupling $\lambda \equiv g_{YM}^2 N$ with $N \gg \lambda$ and λ fixed. In this interacting corner of $\mathcal{N} = 4$ SYM, the theory sits at the boundary of a classical AdS$_5 \times$S^5. The size of the holographic space-time is controlled by $L^4/\alpha'^2 = 4\pi\lambda$, and the action of IIB supergravity is weighted by N^2. Quantum corrections are then organised in a double expansion in $1/N^2$ and $\lambda^{-1/2}$.

The partial non-renormalization theorem [35] is a non perturbative statement about superconformal symmetry, and restricts the most general form of the four-point correlator into the sum of free theory, and a particular form for the dynamical function,

$$\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle = \langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle_{\text{free}} + \mathcal{P} \, \mathcal{I}(x, \bar{x}; y, \bar{y}) \, \mathcal{D}_{p_1 p_2 p_3 p_4}(x, \bar{x}; y, \bar{y}; \lambda) \tag{3.1}$$

where $\mathcal{I}(x, \bar{x}; y, \bar{y})$ is the same rational function characterizing long superblocks in (2.20), and $\mathcal{P}$ is the prefactor (2.9).

Contrary to free theory, the dynamical function depends on both N and λ. Here we will be focussing on the order zero terms in the large λ expansion, and consequently we will drop the λ dependence in our discussion. Stringy corrections have been considered in [10, 11, 31–33].

In the previous section we studied the SCPW decomposition of free theory. In particular we cleanly split the free theory correlator into the piece with only (semi)-short operators in the CPW expansion and a piece with only long operators $\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle_{\text{free long}}$ in (2.27). We will now incorporate the dynamical function, $\mathcal{D}_{p_1 p_2 p_3 p_4}$ and specialise to the long sector. It will be convenient to distinguish the two $1/N$ expansions,

$$\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle_{\text{free long}} = \langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle^{0}_{\text{free long}} + \frac{1}{N^2} \langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle^{(1)}_{\text{free long}} + \dots \quad (3.2)$$

$$\mathcal{D}_{\vec{p}} = \frac{1}{N^2} \mathcal{D}_{\vec{p}}^{(1)} + \frac{1}{N^4} \mathcal{D}_{\vec{p}}^{(2)} + \dots \quad (3.3)$$

The notation we will use to refer to the SCPW expansion of the long sector of $\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle$, (i.e. the long sector of free theory together with the dynamical part) up to order $1/N^4$, is

$$\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle \Big|_{\text{long}} =$$

$$\log^0 u \sum_{\vec{\tau}} \left(L_{\vec{p};\vec{\tau}}^{(0)} + \frac{1}{N^2} L_{\vec{p};\vec{\tau}}^{(1)} + \frac{1}{N^4} L_{\vec{p};\vec{\tau}}^{(2)} \right) \mathbb{L}_{\vec{p};\vec{\tau}} + \dots \quad (3.4)$$

$$\log^1 u \sum_{\vec{\tau}} \left(\frac{1}{N^2} M_{\vec{p};\vec{\tau}}^{(1)} + \frac{1}{N^4} M_{\vec{p};\vec{\tau}}^{(2)} \right) \mathbb{L}_{\vec{p};\vec{\tau}} + \dots \quad (3.5)$$

$$\log^2 u \sum_{\vec{\tau}} \left(\frac{1}{N^4} N_{\vec{p};\vec{\tau}}^{(2)} \right) \mathbb{L}_{\vec{p};\vec{\tau}} \quad (3.6)$$

In the above formulae we are omitting terms which are accompanied by derivatives of the blocks with respect to τ since these are not important for our purpose here.

The $\log^{m \geq 1}$ terms receive contributions only from the dynamical function, $\mathcal{D}$.

The $\log^0$ projection (3.4) is subject to non trivial interplay between free theory and the dynamical function $\mathcal{D}$, since beyond the leading order, both contribute in the $1/N$ expansion,

$$L_{\vec{p};\vec{\tau}}^{(0)} = L_{\vec{p};\vec{\tau}}^{f(0)} \quad (3.7)$$

$$L_{\vec{p};\vec{\tau}}^{(i)} = L_{\vec{p};\vec{\tau}}^{f(i)} + L_{\vec{p};\vec{\tau}}^{\mathcal{D}(i)} \qquad i = 1, \dots \quad (3.8)$$

In (3.4)–(3.6) we clustered together various contributions within each log strata, and we did not specify the range of summation. In fact, understanding the range of summation for different contributions needs extra explanations, which we make precise in sections 3.1 and 3.2. We will summarize all the relevant results in section 3.4 .

Every SCPW coefficient in (3.4)–(3.6), is predicted by the OPE, however in order to have control on these predictions we should first have control on the spectrum of the theory: the spectrum of supergravity consists of single particle half-BPS operators $\mathcal{O}_p$ and multi-particle operators built out of single particle operators. Multi-particles operators can be either protected or long, but regardless of this, multi-particle operators labelled by more than two particles do not have a leading order three-point function with the (normalized) external operators, therefore cannot appear in the leading order OPE. It follows that in the strict large N limit, supergravity describes a free theory of single- and two-particles states of integer twist τ, which we can then classify.

Recall then, that a basis of *long* two-particle superconformal primary operators of twist τ, spin ℓ and $su(4)$ channel $[aba]$, has the schematic form [8],

$$\mathcal{O}_{pq;\vec{\tau}} = \mathcal{O}_p \partial^l \square^{\frac{1}{2}(\tau-p-q)} \mathcal{O}_q, \qquad (p \leq q), \tag{3.9}$$

where the pairs (p,q) are in the set $R_{\vec{\tau}}$

$$R_{\vec{\tau}} := \left\{ (p,q) : \begin{array}{l} p = i + a + 1 + r \\ q = i + a + 1 + b - r \end{array} \quad \text{for} \quad \begin{array}{l} i = 1, \ldots, (t-1) \\ r = 0, \ldots, (\mu-1) \end{array} \right\} \tag{3.10}$$

There are in total $d = \mu(t-1)$ allowed values, where

$$t \equiv \frac{(\tau-b)}{2} - a, \quad \mu \equiv \begin{cases} \lfloor \frac{b+2}{2} \rfloor & a+l \text{ even,} \\ \lfloor \frac{b+1}{2} \rfloor & a+l \text{ odd.} \end{cases} \tag{3.11}$$

Plotting the set $R_{\vec{\tau}}$ in the (p,q) plane helps visualizing that the pairs fill in a rectangle, where the $+\pi/4$ direction contains $(t-1)$ pairs and the $-\pi/4$ direction contains μ pairs. For example, we (re)draw here below $R_{24,2\mathbb{N},0,6}$,

$A = (2,8);$
$B = (5,5);$
$C = (12,12);$
$D = (9,15);$
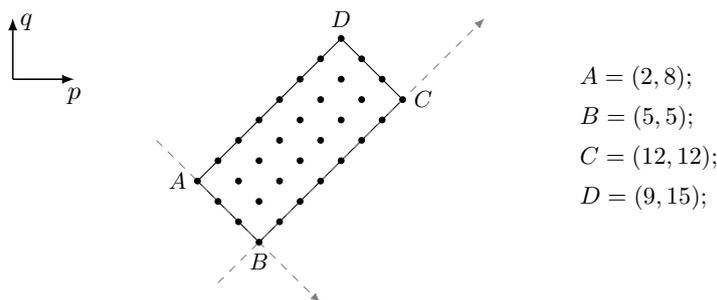
In general the operators $\mathcal{O}_{pq,\vec{\tau}}$ will mix. We will denote the true eigenstates (with well-defined scaling dimensions) by $\mathcal{K}_{pq}$.

Considering now a four point function $\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle$, the OPE predicts the following form of the SCPW coefficients for the indicated ranges of the twist of the exchanged

operators

$$\tau \geq \tau^{\max}: \quad L^{(0)}_{\vec{p};\vec{\tau}} = \sum_{(pq) \in R_{\vec{\tau}}} C^{(0)}_{p_1 p_2 \mathcal{K}_{pq}} C^{(0)}_{p_3 p_4 \mathcal{K}_{pq}} \tag{3.12}$$

$$M^{(1)}_{\vec{p};\vec{\tau}} = \frac{1}{2} \sum_{(pq) \in R_{\vec{\tau}}} C^{(0)}_{p_1 p_2 \mathcal{K}_{pq}} \eta_{\mathcal{K}_{pq}} C^{(0)}_{p_3 p_4 \mathcal{K}_{pq}} \tag{3.13}$$

$$N^{(2)}_{\vec{p};\vec{\tau}} = \frac{1}{8} \sum_{(pq) \in R_{\vec{\tau}}} C^{(0)}_{p_1 p_2 \mathcal{K}_{pq}} \eta^2_{\mathcal{K}_{pq}} C^{(0)}_{p_3 p_4 \mathcal{K}_{pq}} \tag{3.14}$$

$$\tau^{\max} > \tau \geq \tau^{\min}: \quad L^{(1)}_{\vec{p};\vec{\tau}} = \sum_{(pq) \in R_{\vec{\tau}}} C^{(1)}_{p_1 p_2 \mathcal{K}_{pq}} C^{(0)}_{p_3 p_4 \mathcal{K}_{pq}} + C^{(0)}_{p_1 p_2 \mathcal{K}_{pq}} C^{(1)}_{p_3 p_4 \mathcal{K}_{pq}} \tag{3.15}$$

$$M^{(2)}_{\vec{p};\vec{\tau}} = \frac{1}{2} \sum_{(pq) \in R_{\vec{\tau}}} C^{(1)}_{p_1 p_2 \mathcal{K}_{pq}} \eta_{\mathcal{K}_{pq}} C^{(0)}_{p_3 p_4 \mathcal{K}_{pq}} + C^{(0)}_{p_1 p_2 \mathcal{K}_{pq}} \eta_{\mathcal{K}_{pq}} C^{(1)}_{p_3 p_4 \mathcal{K}_{pq}} \tag{3.16}$$

$$\tau < \tau^{\min}: \quad L^{(2)}_{\vec{p};\vec{\tau}} = \sum_{(pq) \in R_{\vec{\tau}}} C^{(1)}_{p_1 p_2 \mathcal{K}_{pq}} C^{(1)}_{p_3 p_4 \mathcal{K}_{pq}} \tag{3.17}$$

where $C_{p_i p_j \mathcal{K}} \equiv C_{p_i p_j}(\mathcal{K})$ denotes the three-point coupling,

$$C_{p_i p_j \mathcal{K}} = C^{(0)}_{p_i p_j \mathcal{K}} + \frac{1}{N^2} C^{(1)}_{p_i p_j \mathcal{K}} + \dots \tag{3.18}$$

of an exchanged pure (and unit normalised) operator $\mathcal{K}_{pq;\vec{\tau}}$ w.r.t. to the external operators $\mathcal{O}_{p_i} \mathcal{O}_{p_j}$, and $\eta_{\mathcal{K}}$ is its anomalous dimensions,

$$\Delta_{\mathcal{K}} = \tau + \ell + \frac{1}{N^2} \eta_{\mathcal{K}} + \dots . \tag{3.19}$$

The pure operators $\mathcal{K}_{pq;\vec{\tau}}$ (i.e. those with well-defined anomalous dimensions) are simply certain linear combinations of the operators $\mathcal{O}_{pq;\vec{\tau}}$ in (3.9). At leading order the explicit change of basis is given in terms of the above leading three-point functions as

$$\mathcal{K}_{pq} = \sum_{(p'q') \in R_{\vec{\tau}}} \mathcal{O}_{p'q'} \frac{C^{(0)}_{p'q' \mathcal{K}_{pq}}}{C^{(0)}_{p'q' \mathcal{O}_{p'q'}}} + O(1/N^2) . \tag{3.20}$$

The normalisation of $\mathcal{K}_{pq}$ is fixed (up to a sign) by insisting on unit two-point function. As before we note that the leading three-point functions with the naive operators are diagonal, i.e. $C^{(0)}_{pq \mathcal{O}_{p'q'}} = 0$ unless $(pq) = (p'q')$, as can be easily verified via explicit Wick contractions.

Our next task is to leverage data mined from tree-level four-point functions, specifically the CPW coefficients $L^{(0)}, M^{(1)}$ and $L^{(1)}$, in order to obtain information about the one-loop four-point function, in particular the entire double log discontinuity, $N^{(2)}$, but also pieces of the single log part $M^{(2)}$ and analytic part $L^{(2)}$. The strategy used in our previous works [3, 4, 7] was to fully solve the mixing problem and obtain complete data for the leading three-point couplings and anomalous dimensions of long operators in $[a, 0, a]$ and $[a, 1, a]$, by considering the leading (i.e. disconnected free theory) four-point functions and the (leading discontinuity of the) $1/N^2$ correction of Rastelli and Zhou [1, 5]. However,

for more general correlators, specifically when $b > 1$, the anomalous dimensions exhibit degeneracy [8]. This degeneracy means it is not possible to fully repeat this program in the same way and determine all leading three-point functions. Nevertheless we will now discuss how to overcome this problem and bootstrap one-loop data from tree-level correlators.

In order to have better control over the various phenomena taking place in (3.12)–(3.17), we will distinguish between three types of contributions:

- Above threshold, $\tau \geq \tau_{\vec{p}}^{\max}$.

  We define the threshold twist $\tau_{\vec{p}}^{\max} \equiv \max(p_1 + p_2, p_3 + p_4)$. The leading log CPW coefficients, $L^{(0)}$, $M^{(1)}$, and $N^{(2)}$, have contributions above the threshold only, and we use $L^{(0)}$ and $M^{(1)}$ (for different correlators) to bootsrap the double discontinuity $N^{(2)}$. This analysis is similar to our previous works [8]. For general correlators the leading log discontinuity alone is not sufficient to fully fix the one-loop dynamical function consistenly. There are also important pieces of the one-loop functions which will be fully determined by data below threshold.

- The window $\tau^{\min} \leq \tau < \tau^{\max}$.

  We define the "window" as the range of twists strictly below $\tau_{\vec{p}}^{\max} = \max(p_1+p_2, p_3+p_4)$ and above $\tau_{\vec{p}}^{\min} = \min(p_1+p_2, p_3+p_4)$. Clearly the window is absent when $p_1 + p_2 = p_3 + p_4$. The significance of this region is that the leading three-point function is absent on one side but not on the other (see diagram below). This allows us to use tree-level SCPW coefficients $L^{(1)}$ to predict part of the single discontinuity of the one-loop correlator, $M^{(2)}$.

- Below window, $\tau < \tau^{\min}$.

  Finally, for each $su(4)$ channel $[aba]$ we define the region "below window", for which the twist is strictly below $\tau_{\vec{p}}^{\min} = \min(p_1+p_2, p_3+p_4)$. For any $su(4)$ rep we also require $\tau \geq 2 + 2a + b$, i.e. the unitarity bound relative to the rep $[aba]$. In this range of twists one can predict a piece of the analytic (in small u) part of the one-loop correlator, $L^{(2)}$, using lower order data, specifically results from $L^{(1)}$ for other correlators.

The three regions described above are shown pictorially in figure 1.

The precise details about how to obtain these predictions are described in the next subsections.

3.1 Predicting the leading log

Let us begin from the $\log^2 u$ discontinuity $N^{(2)}$. We shall use many different correlators and it is thus useful to package together the CPW coefficients, and three-point functions, into matrices. Similarly we want to rewrite the equations (3.12)–(3.17) in matrix form. To this end we view the space of operators with given twist, spin and $su(4)$ labels as a $\mu(t-1)$ dimensional vector space denoted with the (multi-) index $(pq) \in R_{\vec{\tau}}$.

Define the $\mu(t-1) \times \mu(t-1)$ square matrix of leading order three-point functions $\mathbf{C}_{\vec{\tau}}^{(0)}$ which has entries

$$\left(\mathbf{C}_{\vec{\tau}}^{(0)}\right)_{(rs),(pq)} = C_{rs\mathcal{K}_{pq;\vec{\tau}}}^{(0)} \qquad \text{with} \qquad (pq),(rs) \in R_{\vec{\tau}}, \tag{3.21}$$

and the matrix of leading CPW coefficients $\mathbf{L}_{\vec{\tau}}^{(0)}$ which has entries

$$\left(\mathbf{L}_{\vec{\tau}}^{(0)}\right)_{(p_1 p_2),(p_3 p_4)} = L_{\vec{p};\vec{\tau}}^{(0)} \qquad \text{with} \qquad (p_1 p_2),(p_3 p_4) \in R_{\vec{\tau}}. \tag{3.22}$$

As argued in section 1 we only obtain a non-zero result at leading order for twists $\tau \geq \tau^{\max}$, i.e. above threshold.

Then, the equation arising from the OPE and the SCPW expansion in (3.12) is written in matrix form simply as:

$$\mathbf{C}_{\vec{\tau}}^{(0)} \, \mathbf{C}_{\vec{\tau}}^{(0)T} = \mathbf{L}_{\vec{\tau}}^{(0)} \tag{3.23}$$

(where for notational convenience we have also dropped the subscripts denoting the quantum numbers of interest with $\vec{\tau}$). By construction this matrix is symmetric. Moreover, it is diagonal, because only correlators on the diagonal have disconnected diagrams. Very explicitly [8][10]

$$\mathbf{L}_{\vec{\tau}}^{(0)} = \text{diag}\left[\frac{(a+1)(a+2+2r+q-p)(l+1)(l+2+\tau)(1+r+q-p)_{r+1}(2+a+r+q-p)_{2+a+r}}{(1-\delta_{p,q}/2)(p-1-r)!(p-2-r-a)!(q+r)!(q+r+1+a)!r!(r+1+a)!}\Pi_{\frac{\tau}{2}}\Pi_{l+1+\frac{\tau}{2}}\right]_{\substack{p=2+a+i+r \\ q=p+b-2r}} \tag{3.24}$$

where (i,r) label the pairs $(pq) \in R_{\tau,l,[aba]}$, as explained (3.10), and the function Π_s can be given in the form,

$$\Pi_s = \frac{\left(s-\frac{q-p}{2}\right)!\left(s+\frac{q+p}{2}\right)!}{(2s)!} \times \frac{\prod_{k=0}^{b+2a+2+i}\left(s+\frac{p+q}{2}-k\right)}{(s+\frac{4+2a+b}{2})(s+\frac{2+b}{2})(s-\frac{b}{2})} \times \prod_{m=1}^{i}\left(s-\frac{p+q}{2}+m\right). \tag{3.25}$$

The three factors $(s+\frac{4+2a+b}{2})(s+\frac{2+b}{2})(s-\frac{b}{2})$ cancel against the numerator for the values of $k=i, i+a+1, i+a+b+2$ respectively. Then, for fixed $\vec{\tau}$, the spin dependent factors of the elements of the disconnected free theory matrix $\mathbf{L}_{\vec{\tau}}^{(0)}$ are the following

$$L_{pqpq;\vec{\tau}}^{(0)} \sim \left[\frac{\left(\frac{2+2l+\tau+q-p}{2}\right)!\left(\frac{2+2l+\tau-q+p}{2}\right)!}{(2+2l+\tau)!}\right] \times (l+1)(l+\tau+2)\prod_{k=0}^{a-1}\frac{2l+2+\tau-b-2a+2k}{2}$$

$$\times \prod_{k=1}^{b}\frac{2l+2+\tau-b+2k}{2}\prod_{k=2}^{a+1}\frac{2l+2+\tau+b+2k}{2}$$

$$\times \prod_{k=1}^{\frac{1}{2}(p+q-(b+2a+4))}\left(\frac{2l+2+\tau-b-2a-2-2k}{2}\right)\left(\frac{2l+2+\tau+b+2a+4+2k}{2}\right) \tag{3.26}$$

Aside from the factorials in the square brackets, the dependence is a polynomial in spin of degree $(p+q-2)$ which is fully factorised into linear factors. It follows that for fixed

---

[10]A similar formula was given in [9].

$\vec{\tau}$, the highest degree polynomial factor in $\mathbf{L}^{(0)}_{\vec{\tau}}$ occurs for correlators with $p + q = \tau$. For fixed $\vec{\tau}$ the polynomial depends only on the combination $p + q$. The factorials in $L^{(0)}_{pqpq;\vec{\tau}}$ instead depend on both p and q.

Define also the matrix $\mathbf{M}^{(1)}_{\vec{\tau}}$ of leading log tree-level CPW coefficients of operators with given quantum numbers $\vec{\tau}$, at order $1/N^2$, and the (diagonal) matrix of their anomalous dimensions $\boldsymbol{\eta}_{\vec{\tau}}$. Then (3.13) becomes

$$\frac{1}{2}\mathbf{C}^{(0)}_{\vec{\tau}}\,\boldsymbol{\eta}_{\vec{\tau}}\,\mathbf{C}^{(0)T}_{\vec{\tau}} = \mathbf{M}^{(1)}_{\vec{\tau}}\,. \tag{3.27}$$

In matrix form it is straightforward to see that the CPW coefficients contributing to the $\log^2 u$ discontinuity at one-loop are given by

$$\mathbf{N}^{(2)}_{\vec{\tau}} = \frac{1}{8}\mathbf{C}^{(0)}_{\vec{\tau}}\,\boldsymbol{\eta}^2_{\vec{\tau}}\,\mathbf{C}^{(0)T}_{\vec{\tau}} = \frac{1}{2}\mathbf{M}^{(1)}_{\vec{\tau}}\left(\mathbf{L}^{(0)}_{\vec{\tau}}\right)^{-1}\mathbf{M}^{(1)}_{\vec{\tau}}\,, \tag{3.28}$$

i.e. purely in terms of tree-level CPW coefficients. The second equality is obtained straightforwardly from (3.23) and (3.27).

In order to have $\mathbf{M}^{(1)}_{\vec{\tau}}$ explicitly, we first need an expression for the dynamical function $\mathcal{D}^{(1)}_{\vec{p}}$ at $O(1/N^2)$. We obtain $\mathcal{D}^{(1)}_{\vec{p}}$ by using the Mellin Amplitude of Rastelli and Zhou [1], together with the normalisation derived in [8]. Then, we use this to read off its partial wave decomposition and construct $\mathbf{M}^{(1)}_{\vec{\tau}}$. The details of this procedure are discussed in appendix B.2.

Finally we use (3.28) to obtain the full double log discontinuity at $O(1/N^4)$,

$$\mathcal{D}^{(2)}_{\vec{p}}\Big|_{\log^2 u} = \sum_{\tau \geq \tau^{\max},l,a,b}\left(\mathbf{N}^{(2)}_{\vec{\tau}}\right)_{(p_1 p_2),(p_3 p_4)}\widetilde{\mathbb{L}}_{\vec{p};\vec{\tau}}\,, \tag{3.29}$$

where we recall that the long blocks take the form $\mathbb{L} = \mathcal{P} \times \mathcal{I} \times \widetilde{\mathbb{L}}$ as in (2.19).

Note that the above method does not require us to find the leading anomalous dimensions $\boldsymbol{\eta}$ or 3-point functions $\mathbf{C}^{(0)}$ themselves. The anomalous dimensions $\boldsymbol{\eta}_{\vec{\tau}}$ are just the eigenvalues of

$$\frac{1}{2}\mathbf{C}^{(0)}_{\vec{\tau}}\,\boldsymbol{\eta}_{\vec{\tau}}\,(\mathbf{C}^{(0)}_{\vec{\tau}})^{-1} = \mathbf{M}^{(1)}_{\vec{\tau}}\left(\mathbf{L}^{(0)}_{\vec{\tau}}\right)^{-1} \tag{3.30}$$

and eigenvalues can always be found unambiguously. The eigenvectors however, are ambiguous if there are repeated eigenvalues, as turns out to be the case here [8]. This is the aforementioned degeneracy of anomalous dimensions, which is consequence of a surprising physical statement about tree-level physics: there is a hidden ten-dimensional conformal symmetry which prevents the spectrum from being fully unmixed at tree-level [9].

The general solution for the anomalous dimensions is [8]

$$\eta_{\mathcal{K}_{pq}} = -\frac{1}{N^2}\frac{\delta^{(4)}_t \delta^{(4)}_{t+l+1}}{\left(l + 2p - 2 - a - \frac{1+(-)^{a+l}}{2}\right)_6} \tag{3.31}$$

where $(\ldots)_6$ is the Pochhammer symbol, $t \equiv \frac{(\tau-b)}{2} - a$ was defined in (3.11), and[11]

$$\delta^{(4)}_t \equiv 2(t-1)(t+a)(t+a+b+1)(t+2a+b+2)\,. \tag{3.32}$$

---

[11]Compared to [8] we added a factor of 2 in the definition of $\delta^{(4)}_t$.

Since η_{pq} depends only on p and not q, the anomalous dimensions are in general partially degenerate. States which all lie on the same vertical line in the figure $R_{\vec{\tau}}$ have the same anomalous dimension.

Notice that the r.h.s. of (3.30) is rational in spin because it is obtained from $\mathbf{M}_{\vec{\tau}}^{(1)}$ and $\mathbf{L}_{\vec{\tau}}^{(0)}$ which are both rational in spin. Indeed, we emphasise that the original eigenvalue problem set up in [4] is more sophisticated than (3.30), since it was set up to have a direct correspondence between eigenvectors and three-point couplings.

3.2 Below threshold predictions

A feature of four-point correlators of half-BPS single-particle operators with generic charges is that one can bootstrap one-loop pieces of the $\log u$ and analytic part, below threshold. As well as the double log discontinuity, which is entirely above threshold, there is information from within the window and below the window which further constrains the four-point function. Remarkably using all of this available lower order data always fixes the one-loop four-point function up to certain well understood ambiguities which only have finite spin dependence in the SCPW expansion.

To begin with consider long SCPW coefficients of the analytic part of the tree-level correlator $L^{(1)}$ arising from operators in the window region, $\tau^{\min} \leq \tau < \tau^{\max}$ (see figure 1). For simplicity assume $p_1 + p_2 \geq p_3 + p_4$ (the other case is similar), then (3.15) becomes

$$L_{\vec{p};\vec{\tau}}^{(1)} = \sum_{(pq) \in R_{\vec{\tau}}} C_{p_1 p_2 \mathcal{K}_{pq}}^{(1)} C_{p_3 p_4 \mathcal{K}_{pq}}^{(0)} \qquad \tau^{\min} \leq \tau < \tau^{\max}, \quad p_1 + p_2 \geq p_3 + p_4. \qquad (3.33)$$

The key point here is that there are new, leading three-point functions at $O(1/N^2)$ — $C_{p_1 p_2 \mathcal{K}_{pq}}^{(1)}$ — with below threshold twist $\tau < p_1 + p_2$.

Fixing $(p_1 p_2)$ and $\vec{\tau}$, let us consider all values of $(p_3 p_4) \in R_{\vec{\tau}}$ and rewrite (3.33) as a vector equation

$$\mathbf{L}_{(p_1 p_2);\vec{\tau}}^{(1)} = \mathbf{C}_{(p_1 p_2);\vec{\tau}}^{(1)} \mathbf{C}_{\vec{\tau}}^{(0)T}. \qquad (3.34)$$

Here we have defined the row vector $\mathbf{C}_{(p_1 p_2);\vec{\tau}}^{(1)}$ with entries

$$\left(\mathbf{C}_{(p_1 p_2);\vec{\tau}}^{(1)} \right)_{(pq)} = C_{p_1 p_2 \mathcal{K}_{(pq);\vec{\tau}}}^{(1)} \qquad \forall (pq) \in R_{\vec{\tau}} \qquad (3.35)$$

and the row vector of no-log $O(1/N^2)$ CPW coefficients, $\mathbf{L}_{(p_1 p_2);\vec{\tau}}^{(1)}$, with entries

$$\left(\mathbf{L}_{(p_1 p_2);\vec{\tau}}^{(1)} \right)_{(p_3 p_4)} = L_{\vec{p};\vec{\tau}}^{(1)} \qquad \forall (p_3 p_4) \in R_{\vec{\tau}}. \qquad (3.36)$$

The other ingredient is the matrix of leading three-point couplings $\mathbf{C}_{\vec{\tau}}^{(0)}$ defined in (3.21).

Consider now the $\log u$ part of the one-loop four-point function with SCPW coefficients $M^{(2)}$ (3.16). In direct analogy to $\mathbf{L}_{(p_1 p_2);\vec{\tau}}^{(1)}$ above, define the vector $\mathbf{M}_{(p_1 p_2);\vec{\tau}}^{(2)}$. The OPE in the window (3.16) (for varying $(p_3 p_4) \in R_{\vec{\tau}}$), becomes the vector equation

$$\mathbf{M}_{(p_1 p_2);\vec{\tau}}^{(2)} = \frac{1}{2} \mathbf{C}_{(p_1 p_2);\vec{\tau}}^{(1)} \boldsymbol{\eta}_{\vec{\tau}} \mathbf{C}_{\vec{\tau}}^{(0)T}. \qquad (3.37)$$

If we know $\mathbf{C}_{\vec{\tau}}^{(0)}$ we can explicitly solve for $\mathbf{C}_{(p_1p_2);\vec{\tau}}^{(1)}$ using (3.34) and plug in here to get the one-loop SCPW coefficients $\mathbf{M}_{(p_1p_2);\vec{\tau}}^{(2)}$. However even if we don't, because of the degeneracy of the anomalous dimensions, we see that by using (3.23), (3.27) and (3.34) we obtain $\mathbf{M}_{(p_1p_2);\vec{\tau}}^{(2)}$ purely in terms of tree-level SCPW data as

$$\mathbf{M}_{(p_1p_2);\vec{\tau}}^{(2)} = \mathbf{L}_{(p_1p_2);\vec{\tau}}^{(1)}(\mathbf{L}_{\vec{\tau}}^{(0)})^{-1}\mathbf{M}_{\vec{\tau}}^{(1)} . \tag{3.38}$$

We thus bootstrap a piece of the single log coefficient of the one-loop correlator from tree-level data.

In a very similar way, pieces of the analytic part of the one-loop four-point function, namely the coefficients $L^{(2)}$ for twists below the window, can be determined purely in terms of tree-level SCPW coefficients. From (3.17) we find

$$L_{\vec{p};\vec{\tau}}^{(2)} = \mathbf{C}_{(p_1p_2);\vec{\tau}}^{(1)}\mathbf{C}_{(p_3p_4);\vec{\tau}}^{(1)T} = \mathbf{L}_{(p_1p_2);\vec{\tau}}^{(1)}(\mathbf{L}_{\vec{\tau}}^{(0)})^{-1}\mathbf{L}_{(p_3p_4);\vec{\tau}}^{(1)T} \qquad 4+2a+b \leq \tau < p_3+p_4 . \tag{3.39}$$

Recall that the SCPW coefficients $L^{(1)}$ appearing in (3.39) are determined by summing contributions of $\mathcal{D}_{\vec{p}}^{(1)}$, and connected free theory, as in (3.8). A general formula for connected free theory at order $1/N^2$ can be obtained using results in [8], and was presented already in [9]. In our notation, it is recorded in appendix B.1.

In the above discussion we suppressed the dependence on the spin l of the exchanged operator. Let us now be more concrete and describe how the quantities $\mathbf{M}_{(p_1p_2);\vec{\tau}}^{(2)}$ and $L_{\vec{p};\vec{\tau}}^{(2)}$, obtained from matrix multiplication, depend on spin. In fact, such a spin dependence follows from a) the knowledge of the spin dependence of disconnected free theory, b) the spin dependence of tree-level SCPW coefficients [8], given explicitly in (B.23) appendix B.2.3, and c) reciprocity symmetry $l \leftrightarrow -l-\tau-3$. Proofs of the following formulas are collected in appendix D.

When b is even we can treat even and odd spins separately. In each of these cases we find,

$$M_{\vec{p};\tau,l,[aba]}^{(2)} = \frac{\left(\frac{2+2l+\tau+p_{43}}{2}\right)!\left(\frac{2+2l+\tau-p_{21}}{2}\right)!}{(2+2l+\tau)!} \times \frac{num_{M^{(2)}}(2l+\tau+3)}{den_{M^{(2)}}(2l+\tau+3)} \tag{3.40}$$

$$L_{\vec{p};\tau,l,[aba]}^{(2)} = \frac{\left(\frac{2+2l+\tau+p_{43}}{2}\right)!\left(\frac{2+2l+\tau-p_{21}}{2}\right)!}{(2+2l+\tau)!} \times \frac{num_{L^{(2)}}(2l+\tau+3)}{den_{L^{(2)}}(2l+\tau+3)} \tag{3.41}$$

The polynomials num and den are even in the variable $(2l+\tau+3)$, with coefficients which depend on τ. The denominators are fully predicted by the formula (3.26) for $p+q=\tau$. In particular they have degree $\tau-2$. The numerators have degree,

$$\text{degree } num_{M^{(2)}}(l) = (\tau-4) + 2(\kappa_{\vec{p}}-2) + p_{21} \tag{3.42}$$

$$\text{degree } num_{L^{(2)}}(l) = 2(\tau-4) - (p_{43}-p_{21}) \tag{3.43}$$

When b is odd, the symmetry $l \leftrightarrow -l - \tau - 3$ exchanges even and odd spin. By picking the sector of even spins as representative, we find

$$M^{(2)}_{\vec{p};\tau,\text{even},[aba]} = \frac{\left(\frac{2+2l+\tau+p_{43}}{2}\right)! \left(\frac{2+2l+\tau-p_{21}}{2}\right)!}{(2+2l+\tau)!} \times \frac{(l+\tau+2)^2 num^{\text{even}}_{M^{(2)}}(l)}{den^{\text{even}}_{M^{(2)}}(l)} \quad (3.44)$$

$$L^{(2)}_{\vec{p};\tau,\text{even},[aba]} = \frac{\left(\frac{2+2l+\tau+p_{43}}{2}\right)! \left(\frac{2+2l+\tau-p_{21}}{2}\right)!}{(2+2l+\tau)!} \times \frac{(l+\tau+2)^2 num_{L^{(2)}}(l)}{den_{L^{(2)}}(l)} \quad (3.45)$$

where the denominators are again predicted by the formula (3.26) for $p + q = \tau$, and

$$\text{degree } num_{M^{(2)}}(l) = (\tau - 4) + 2(\kappa_{\vec{p}} - 2) + p_{21} - 2 \quad (3.46)$$

$$\text{degree } num_{L^{(2)}}(l) = 2(\tau - 4) - (p_{43} - p_{21}) - 2 \quad (3.47)$$

Summarizing, from all the results given above we can determine the following pieces of the $O(1/N^2)$ four-point functions.

- $\log^1 u$ stratum obtained from a finite number of twists, $\forall$ spin,

$$\mathcal{D}^{(2)}_{\vec{p}}\Big|_{\log^1 u} = \frac{1}{2} \log^1 u \sum_{l,a,b} \sum_{\tau=\tau^{\min}}^{\tau^{\max}-2} \left(\mathbf{M}^{(2)}_{(p_1 p_2);\vec{\tau}}\right)_{(p_3 p_4)} \widetilde{\mathbb{L}}_{\vec{p};\vec{\tau}} + \ldots \quad (3.48)$$

where $\mathbf{M}^{(2)}$ is given in (3.38) and we are omitting here terms contributing to twist $\tau \geq \tau^{\max}$.

- $\log^0 u$ stratum obtained from a finite number of twist, $\forall$ spin,

$$\mathcal{D}^{(2)}_{\vec{p}}\Big|_{\log^0 u} = \mathcal{D}^{(2)}_{\text{bound}} + \sum_{l,a,b} \sum_{\tau=4+2a+b}^{\tau^{\min}-2} L^{(2)}_{\vec{p};\vec{\tau}} \widetilde{\mathbb{L}}_{\vec{p};\vec{\tau}} + \ldots \quad (3.49)$$

with $L^{(2)}$ given in (3.39) and we are omitting here terms contributing to twist $\tau \geq \tau^{\min}$. There is an extra subtlety which needs to be tackle in order to determine fully the $\log^0 u$ stratum. It enters the contribution called $\mathcal{D}^{(2)}_{\text{bound}}$, and has to do with multiplet recombination at the unitarity bound, $\tau = 2 + 2a + b$, in each channel. We will discuss this in the next section.

Equations (3.38) and (3.39) show how to obtain SCPW coefficients of the one-loop four-point functions directly in terms of SCPW coefficients of tree-level functions. Unmixing of the CFT data is not necessary to achieve this.

There are cases in which we can unmix explicitly the three-point couplings $\mathbf{C}^{(0)}$. Referring these cases to rectangle $R_{\vec{\tau}}$ given in (3.10), they correspond to operators in $su(4)$ representations $[n, 0, n]$ and $[n, 1, n]$, label by $\mu = 1 \forall t$, and operators $\mathcal{K} \in R_{4+2a+b,l,[aba]}$, labeled by $t = 2 \forall \mu$, for which the rectangle $R_{\vec{\tau}}$ collapses into a line, and therefore the degeneracy has no space yet to show up. Given the knowledge of $\mathbf{C}^{(0)}$ we can proceed to obtain the subleading three-point couplings. A number of examples is given in appendix C, where we also comment on the so called derivative relation.

Despite the fact that the explicit three-point functions and anomalous dimensions are not needed to produce the one-loop OPE predictions described above, we emphasise that they do follow a *significantly* simpler pattern, compared to the partial wave coefficients they are obtained from. The beauty of the pattern is manifest in the structure of the anomalous dimension (3.31), but also in the three-point couplings unmixing when possible. As it was found in [4] and [7], the three-point couplings unmixing always reduces to the problem of finding unitarity matrices with predicted spin dependence.

3.3 Semi-short sector predictions

As anticipated in section 2.4, we now come back to the delicate point of multiplet recombination at the unitarity bound. In (3.49) we gave the one-loop $\log^0 u$ predictions which originate from twists *above* the unitary bound, i.e $4 + 2a + b \le \tau < \tau_{\vec{p}}^{\min}$. In addition, we claim that the dynamical one-loop function must contain a contribution *at* the unitarity bound $\tau = 2 + 2a + b$, which we also predicted. Namely,

$$L_{\vec{p};\vec{\tau}}^{(2)} = L_{\vec{p};\vec{\tau}}^{(2)f} + L_{\vec{p};\vec{\tau}}^{(2)\mathcal{D}} = 0, \qquad \tau = 2 + 2a + b \qquad (3.50)$$

$$L_{\vec{p};2+2a+b,l,[aba]}^{(2)f} = \sum_{k=0}^{a}(-1)^k A_{[l+2+k,1^{a-k}]}\Big|_{\frac{1}{N^4}} - \sum_{k=0}^{a}(-1)^k S_{[l+2+k,1^{a-k}]} \qquad (3.51)$$

The coefficient $L_{\tau=2+2a+b}^{(2)f}$ was obtained in (2.44). Its first term is given by the CPW of connected free theory $A_{2+2a+b-2k,[l+2+k,1^{a-k}]}$ restricted at order $1/N^4$. Its second term is given by summing over the new coefficients $S_{2+2a+b-2k,[l+2+k,1^{a-k}]}$, and it follows non trivially from the analysis of the semishort sector, which by construction is of order $1/N^4$. The contributions to the analytic, $\log^0 u$ part, of $\mathcal{D}^{(2)}$, which come from twists at the unitarity bound, combine to give the function $\mathcal{D}_{\text{bound}}^{(2)}$ in (3.49) in the form

$$\mathcal{D}_{\text{bound}}^{(2)} = -\sum_{l,a,b} L_{\vec{p};\vec{\tau}}^{(2)f} \widetilde{\mathbb{L}}_{\vec{p};\vec{\tau}}. \qquad (3.52)$$

The reason for (3.50) is the following: the OPE of $\mathcal{O}_{p_i}\mathcal{O}_{p_j}$ in free theory runs, by definition, over all operators of $\mathcal{N} = 4$ SYM, but supergravity states correspond only to operators built from half-BPS operators, i.e. they are either half-BPS operators themselves or multi-particle operators. Other single-trace operators at the unitarity bound, which are present in free theory, correspond thus to excited string states, and should be absent from the OPE in the supergravity regime.

Simple examples of operators which correspond to excited string states are the Konishi operator $\text{tr}(\phi^2)$ in the [000] representation, and twist 3 superconformal primaries of the form $\text{tr}(\phi^3)$ in the [010] representation. However, these two cases are special because there are no other existing operators with such quantum numbers. In particular, there will be no S-type contribution in (3.51). Beyond twist 3, instead, we have to distinguish carefully between multi-trace semishort operators, which do remain in the spectrum of supergravity, and excited string states, as was done in section 2.4.

It is very instructive to compare the new features of $1/N^4$ physics with corresponding tree-level terms. Let us begin from the analogue of equation (3.50) at tree level. It reduces to

$$L_{\vec{p};2+2a+b,l,[aba]}^{(1)} = \underbrace{\sum_{k=0}^{a}(-1)^k A_{[l+2+k,1^{a-k}]}\Big|_{\frac{1}{N^2}}}_{=L_{\vec{p};2+2a+b,l,[aba]}^{(1)f}} + L_{\vec{p};2+2a+b,l,[aba]}^{(1)\mathcal{D}} = 0 \qquad (3.53)$$

The difference compared to (3.50), is precisely the difference between performing multiplet recombination with CPW coefficients of connected free theory — assuming all below threshold ($\tau = 2 + 2a + b < \tau^{\min}$) semishort operators are absent — and performing multiplet recombination with remaining below threshold semishort operators. This is just because the semishort three-point functions are all of $O(1/N^2)$ and so are only visible in the SCPW decomposition at $O(1/N^4)$.

Indeed, the leading three-point function $C_{p_i p_j K_{pq}}^{(0)} = 0$ whenever $p_i + p_j > \tau$, thus this vanishing condition extends to the non-semishort "below window" sector, $\tau < \tau^{\min}$ at tree level. Thus

$$L_{\vec{p};\vec{\tau}}^{(1)} = L_{\vec{\tau}}^{(1)f} + L_{\vec{p};\vec{\tau}}^{(1)\mathcal{D}} = 0 \qquad \forall \tau < \tau^{\min}, \qquad (3.54)$$

with the free theory part, $L^{(1)f}$ given in (2.30) when τ is above the unitary bound $\tau \geq 4 + 2a + b$ and (3.53) when at the bound $\tau = 2 + 2a + b$.

3.4 Back to the bootstrap

In the previous three sections we have explained how to bootstrap, from tree-level results, predictions about the dynamical one-loop function of $\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle$. Summarizing, we have obtained the leading $\log^2 u$ discontinuity (see discussion around (3.28)–(3.29)). Then, we have obtained pieces of the single $\log^1 u$ from exchanged operators in the window (see discussion around (3.38) and (3.48)), and also pieces of the analytic $\log^0 u$ part of the correlator from below window data (see discussion around (3.49) and (3.39)). Finally, we understood how to deal with the SCPW coefficient of long operators at the unitarity bound in (3.51). We emphasize that even though the leading log discontinuity can be obtained more elegantly by using the hidden symmetry of [9], our approach here allows us to go beyond that, and compute $M^{(2)}$ and $L^{(2)}$, which are very important pieces of our bootstrap program.

The OPE predictions introduced so far were organised according to the $\log u$ stratification of the correlators given in (3.4)–(3.6). We now point out that the structure of the $O(1/N^4)$ dynamical function admits a further refinement.

Consider first the following observation: looking at below threshold physics at tree level we found that the analytic sector of the dynamical function $\mathcal{D}^{(1)}$ is subject to the constraints (3.54), i.e

$$L_{\vec{p};\vec{\tau}}^{(1)\mathcal{D}} = -L_{\vec{\tau}}^{(1)f} \qquad \forall 4 + 2a + b \leq \tau < \tau^{\min}, \qquad (3.55)$$

augmented by a similar constraint at the unitarity bound, given in (3.53). We claim (and we will show in section 5) that $\mathcal{D}^{(1)}$ is entirely fixed by these constraints, together with the requirement that its $\log u$ discontinuity has threshold twist $\tau^{\max}$.

Consider now the analytic sector at one-loop, we find instead

$$L_{\vec{p};\vec{\tau}}^{(2)\mathcal{D}} = -L_{\vec{\tau}}^{(2)f} + L_{\vec{p};\vec{\tau}}^{(2)} \qquad 4 + 2a + b \le \tau < \tau^{\min}, \qquad (3.56)$$

where $L^{(2)}$ is the new $O(1/N^4)$ prediction (3.39) arising from the tree-level data via the OPE. It is clear then that the analytic part of $\mathcal{D}^{(2)}$ has two separate contributions, one cancelling free theory contribution, i.e $-L_{\vec{\tau}}^{(2)f}$, and another one linked to predictions from tree-level data $L_{\vec{p};\vec{\tau}}^{(2)}$. Furthermore, at the unitarity bound we find a similar split into a piece depending directly on free theory SCPW coefficients and a non-trivial prediction arising from correlators of different charges (3.51). Since the $\log^2$ and $\log^1$ strata of $\mathcal{D}^{(2)}$ are determined uniquely by tree-level data via the OPE (see sections 3.1 and 3.2), and have no free theory contribution, it is natural to split the one-loop function into

$$\mathcal{D}^{(2)} = \mathcal{T}^{(2)} + \mathcal{H}^{(2)}, \qquad (3.57)$$

where $\mathcal{T}^{(2)}$ and $\mathcal{H}^{(2)}$ have a different interplay with connected free theory.

The function $\mathcal{T}^{(2)}$ generalises the tree-level function $\mathcal{D}^{(1)}$, and it will be defined by the properties that it has $\log^1 u$ discontinuity with threshold twist $\tau^{\max}$, no $\log^2 u$ double discontinuity, and it fully cancels below window long contributions coming from recombined free theory, hence the name of generalised tree-level function. Indeed, for any $4 + 2a + b \le \tau < \tau^{\min}$, i.e. strictly above the unitary bound, we expect

$$L_{\vec{p};\vec{\tau}}^{(2)\mathcal{T}} = -L_{\vec{p};\vec{\tau}}^{(2)f} \qquad (3.58)$$

with $L^{(2)f}$ the $O(1/N^4)$ part of (2.30), and at the unitarity bound

$$L_{\vec{p};2+2a+b,l,[aba]}^{(2)\mathcal{T}} = -\sum_{k=0}^{a}(-1)^k A_{[l+2+k,1^{a-k}]}\Big|_{\frac{1}{N^4}} \qquad (3.59)$$

It follows that the one-loop OPE predictions (3.39), below the window, will be encoded only in the function $\mathcal{H}^{(2)}$, i.e.

$$L_{\vec{p};\vec{\tau}}^{(2)\mathcal{H}} = L_{\vec{p};\vec{\tau}}^{(2)} \qquad 4 + 2a + b \le \tau < \tau^{\min}, \qquad (3.60)$$

and at the unitarity bound

$$L_{\vec{p};2+2a+b,l,[aba]}^{(2)\mathcal{H}} = +\sum_{k=0}^{a}(-1)^k S_{[l+2+k,1^{a-k}]} \qquad (3.61)$$

Our task now is to construct the one-loop correlator $\mathcal{D}^{(2)}$ consistently with the OPE predictions. We will see that the splitting $\mathcal{D}^{(2)} = \mathcal{H}^{(2)} + \mathcal{T}^{(2)}$ is also strongly motivated by features of the $\log^2 u$ discontinuity. In fact, we will discover that $\mathcal{H}^{(2)}$ is the minimal one-loop function which consistently emanates top-down from the leading $\log^2 u$ discontinuity. Furthermore, we will find that $\mathcal{T}$ can be constructed as an exact function of N. The interplay of $\mathcal{H}^{(2)}$ with the semishort prediction (3.61) is very remarkable. When we think of it as descending from the double logarithmic discontinuity, it is a tangible triumph of supergravity within our $\mathcal{N} = 4$ bootstrap program.

4 One-loop correlators

The discussion throughout section 3 provided us with predictions for the one-loop function that we now make operative by introducing an ansatz which will suit them all. To understand this ansatz and impose as many constraints as possible, we will first consider the consequence of crossing symmetry and those of the OPE on the structure of one-loop correlators, and secondly we will obtain a (two-variable) resummation of the leading double log discontinuity. We then go on to assemble this and the information below threshold to obtain the one-loop correlator.

4.1 Structure of one-loop correlators

From the OPE we expect different parts of the correlator to possess contributions from operators of different twists (see the previous section). The $\log^2 u$ discontinuity has contributions only from operators above threshold $\tau \geq \tau^{\max}$. The $\log^1 u$ part can have contributions from the window, $\tau \geq \tau^{\min}$. Finally, the analytic $\log^0 u$ part, can have contributions starting from the semishort operators with $\tau \geq p_{43} + 2$ (see figure 1).

Because a long operator of twist τ gives a contribution to the correlator which goes like $\mathcal{P} \times \mathcal{I} \times u^{\frac{1}{2}(\tau - p_{43})}$ for small u, the OPE then dictates that

$$\mathcal{D}_{\vec{p}}^{(2)}\big|_{\log^2 u} = O\left(u^{\frac{1}{2}(\tau^{\max} - p_{43})}\right)$$
$$\mathcal{D}_{\vec{p}}^{(2)}\big|_{\log^1 u} = O\left(u^{\frac{1}{2}(\tau^{\min} - p_{43})}\right)$$
$$\mathcal{D}_{\vec{p}}^{(2)}\big|_{\log^0 u} = O(u), \tag{4.1}$$

where

$$\frac{1}{2}(\tau^{\max} - p_{43}) = \max\left(p_3, \frac{p_1 + p_2 + p_3 - p_4}{2}\right)$$
$$\frac{1}{2}(\tau^{\min} - p_{43}) = \min\left(p_3, \frac{p_1 + p_2 + p_3 - p_4}{2}\right) = \kappa_{\vec{p}} \tag{4.2}$$

The latter is precisely the degree of extremality.

Consider now the split $\mathcal{D}^{(2)} = \mathcal{T}^{(2)} + \mathcal{H}^{(2)}$. We claim that only $\mathcal{T}^{(2)}$ has a contribution at $O(u)$ whereas $\mathcal{H}^{(2)} = O(u^2)$. The reason for this follows again from the detailed understanding of the semishort sector: the contributions at $O(u)$ arise from semishort operators with twist $p_{43} + 2$ in the $[a = 0, p_{43}, 0]$ $su(4)$ representation. In this case there is a single A-type contribution in the sum of (2.44), which is to be dealt with by $\mathcal{T}^{(2)}$, and a single S contribution, to be dealt with by $\mathcal{H}^{(2)}$. Recall that we deal with the split $\mathcal{D}^{(2)} = \mathcal{T}^{(2)} + \mathcal{H}^{(2)}$ by using (3.59) and (3.61). Then notice that the S contribution itself is obtained in (2.43) in terms of SCPW coefficients $S_{q_r \tilde{q}_r p_3 p_4}$ where $q_r + \tilde{q}_r = p_{43} + 2$. But these correlators are next to extremal, and they completely vanish when we use the correct definition of single-particle operators — as discussed below (2.12) — so the S contribution vanishes at this twist. Thus

$$\mathcal{T}^{(2)}\big|_{\log^0 u} = O(u) \qquad \mathcal{H}^{(2)}\big|_{\log^0 u} = O(u^2). \tag{4.3}$$

Under crossing $u \leftrightarrow v$, the analysis of the small u expansion in (4.1) translates into predictions for the small v expansion, which is then useful to understand how to constrain the ansatz for the full function.

For the correlator itself crossing symmetry very simply implies that

$$\langle \mathcal{O}_{p_1}(x_1) \mathcal{O}_{p_2}(x_2) \mathcal{O}_{p_3}(x_3) \mathcal{O}_{p_4}(x_4) \rangle = \langle \mathcal{O}_{p_{\sigma_1}}(x_{\sigma_1}) \mathcal{O}_{p_{\sigma_2}}(x_{\sigma_2}) \mathcal{O}_{p_{\sigma_3}}(x_{\sigma_3}) \mathcal{O}_{p_{\sigma_4}}(x_{\sigma_4}) \rangle, \quad (4.4)$$

for any permutation $\sigma \in S_4$. The implications of this taking into account the prefactor $\mathcal{P}$ requires a little care. When defining the prefactor we always made the choice $0 \leq p_{21} \leq p_{43}$ which should therefore be maintained under the permutation, whilst sending $u \leftrightarrow v$. This requires considering a number of different cases for the relative values of the charges p_i. In all cases however there is a unique permutation σ satisfying the above requirements and one finds that for this permutation

$$\mathcal{D}^{(2)}_{p_1 p_2 p_3 p_4}(u, v) = \left(\frac{u\tau}{v} \right)^{\kappa_{\vec{p}}} \mathcal{D}^{(2)}_{p_{\sigma_1} p_{\sigma_2} p_{\sigma_3} p_{\sigma_4}}(v, u). \quad (4.5)$$

The small u behaviour of $D^{(2)}_{p_{\sigma_1} p_{\sigma_2} p_{\sigma_3} p_{\sigma_4}}(u, v)$ given in (4.1) then yields the following small v behaviour of $\mathcal{D}^{(2)}_{\vec{p}}(u, v)$

$$\mathcal{D}^{(2)}_{\vec{p}}\big|_{\log^2 v} = O\left(v^{\frac{1}{2}(p_1 + p_4 - p_2 - p_3)} \right)$$
$$\mathcal{D}^{(2)}_{\vec{p}}\big|_{\log^1 v} = O(v^0)$$
$$\mathcal{D}^{(2)}_{\vec{p}}\big|_{\log^0 v} = O(1/v^{\kappa_{\vec{p}} - 1}). \quad (4.6)$$

Further, the different small u behaviour of $\mathcal{T}^{(2)}$ and $\mathcal{H}^{(2)}$ in (4.3) implies a different small v limit, namely

$$\mathcal{T}^{(2)}_{\vec{p}}\big|_{\log^0 v} = O(1/v^{\kappa_{\vec{p}} - 1}) \qquad \qquad \mathcal{H}^{(2)}_{\vec{p}}\big|_{\log^0 v} = O(1/v^{\kappa_{\vec{p}} - 2}). \quad (4.7)$$

The differences in the v behaviour between $\mathcal{H}$ and $\mathcal{T}$ are crucial in the determination of our ansatz.

4.2 Resummation of leading logs

Only $\mathcal{H}^{(2)}$ carries the $\log^2 u$ discontinuity in the splitting $\mathcal{D}^{(2)} = \mathcal{T}^{(2)} + \mathcal{H}^{(2)}$, by definition. We can then use the arguments of the previous section to infer that in the small u and v expansion we expect

$$\mathcal{H}^{(2)}_{\vec{p}}\bigg|_{\log^2 u} = u^{-\frac{p_{43}}{2} + \frac{\max(p_1 + p_2, p_3 + p_4)}{2}} \left[O\left(v^{\frac{p_{43} - p_{21}}{2}} \right) \log^2 v + O(v^0) \log v + O(1/v^{\kappa_{\vec{p}} - 2}) \right] \quad (4.8)$$

As explained in section 3.1, the leading log discontinuity is defined by the sum

$$\mathcal{D}^{(2)}_{\vec{p}}\bigg|_{\log^2 u} = \sum_{\vec{\tau} : \tau \geq \tau^{\max}} \left(\mathbf{N}^{(2)}_{\vec{\tau}} \right)_{(p_1 p_2), (p_3 p_4)} \widetilde{\mathbb{L}}_{\vec{p}; \vec{\tau}} \quad (4.9)$$

where $\mathbf{N}^{(2)} = \frac{1}{2} \mathbf{M}^{(1)} \left(\mathbf{L}^{(0)} \right)^{-1} \mathbf{M}^{(1)}$ is the matrix assembled from tree level data.

By explicit computation of (4.9) to higher order in τ we obtained the resummation of the leading log discontinuity in a number of cases, and deduced that, as function of the external charges, it always has the structure

$$
\mathcal{D}^{(2)}_{\vec{p}}\Big|_{\log^2 u} = \frac{P_{2,1}(x,\bar{x},\sigma,\tau)}{(x-\bar{x})^{\mathbf{d}_{\vec{p}}+8}} \left[\mathrm{Li}_2(x) - \mathrm{Li}_2(\bar{x}) \right] + \frac{P_{1,2}(x,\bar{x},\sigma,\tau)}{(x-\bar{x})^{\mathbf{d}_{\vec{p}}+8}} \left[\mathrm{Li}_1^2(x) - \mathrm{Li}_1^2(\bar{x}) \right]
$$
$$
+ \frac{P_{1,-}(x,\bar{x},\sigma,\tau)}{(x-\bar{x})^{\mathbf{d}_{\vec{p}}+8}} \left[\mathrm{Li}_1(x) - \mathrm{Li}_1(\bar{x}) \right] + \frac{P_{1,+}(x,\bar{x},\sigma,\tau)}{(x-\bar{x})^{\mathbf{d}_{\vec{p}}+7}} \log(v) + \frac{P_0(x,\bar{x},\sigma,\tau)}{(x-\bar{x})^{\mathbf{d}_{\vec{p}}+7}} \frac{1}{v^{\kappa_{\vec{p}}-2}}
$$
$$(4.10)$$

where

$$
\mathbf{d}_{\vec{p}} = p_1 + p_2 + p_3 + p_4 - 1, \tag{4.11}
$$

for certain polynomials P depending implicitly on the external charges $\vec{p}$. These polynomials are obtained by matching the series expansion in x and $\bar{x}$ of (4.10), with that on the r.h.s. of (4.9). The latter is obtained by considering that each conformal block, of twist τ and spin l, has a series representation of the form $u^\tau (1-v)^l f(x,\bar{x})$ where $u = x\bar{x}$, $v = (1-x)(1-\bar{x})$, and f is an analytic symmetric function in x and $\bar{x}$. We call an expression of the form (4.10) a *two-variable resummation*.

After a case-by-case inspection of (4.10) we indeed verify its consistency with the general structure (4.8). In particular, we always find an overall $u^{(\tau^{\max}-p_{43})/2}$ factor, which gives the leading term in the small u expansion. Then, in the small v expansion, the $\log^2 v$ behaviour is dictated by $P_{1,2}$, and goes like $v^{(p_{43}-p_{21})/2}$, the $\log^1 v$ behaviour is given by the limit of $-P_{1,-} + P_{1,+}$, and goes like v^0, and finally only the $\log^0 v$ contribution has a singularity of the form $1/v^{\kappa_{\vec{p}}-2}$.

In fact a ten-dimensional conformal structure observed in [9] was found to give a direct formula for these leading logs. We checked in many cases that our results agree, and we postpone to section 6 a more detailed description of this ten-dimensional structure.

4.3 Minimal one-loop functions

We now have all the relevant information to write an ansatz for the minimal one-loop function $\mathcal{H}^{(2)}$, which is consistent with crossing symmetry, and matches the two-variable resummation of the leading $\log^2 u$ discontinuity.

We consider single-valued transcendental functions up to weight-4 functions antisymmetric in $x \leftrightarrow \bar{x}$. The weight counting follows from the resummation (4.10) in which we find an overall $\log^2 u$ paired at most with weight-2 anti-symmetric transcendental functions. Therefore we need a basis for weight-4 antisymmetric transcendental functions, and their lower weight completion. We can make it very explicit, by introducing the series of ladder integrals,

$$
\phi^{(\ell)} = \sum_{r=0}^{\ell} \frac{(-)^r (2\ell-r)!}{l!(l-r)!r!} (\log u)^r \left(\mathrm{Li}_{2\ell-r}(x) - \mathrm{Li}_{2\ell-r}(\bar{x}) \right) \tag{4.12}
$$

Then, the basis has the following form

$$
\begin{aligned}
W_{4-} &= h_1\,\phi^{(2)}(x_1', x_2') + h_2\,\phi^{(2)}(x, \bar{x}) + h_3\,\phi^{(2)}(1-x, 1-\bar{x}) \\
W_{3-} &= h_4\,x\partial_x\phi^{(2)}(x, \bar{x}) + h_5\,(x-1)\partial_x\phi^{(2)}(1-x, 1-\bar{x}) - (x \leftrightarrow \bar{x}) \\
W_{3+} &= (x - \bar{x})\Big(h_6\,\partial_v\phi^{(2)}(x, \bar{x}) + h_7\,\partial_u\phi^{(2)}(1-x, 1-\bar{x})\Big) + h_8\,\zeta_3 \\
W_{2+} &= h_9\log(u)\log(v) + h_{10}\log^2 v + h_{11}\log^2 u
\end{aligned}
\tag{4.13}
$$

and

$$
\begin{aligned}
W_{2-} &= h_\square\,\phi^{(1)}(x, \bar{x}) & W_0 &= h_0 \\
W_{1u} &= h_u\log u & W_{1v} &= h_v\log v
\end{aligned}
\tag{4.14}
$$

The weight -4 and -3 basis have been written in terms of the double box function, which is the $\ell = 2$ integral in the ladder series. The weight-2 anti-symmetric element is instead the $\ell = 1$ box function. Each coefficient function $h_{i=1,\dots,11,\square,u,v,0}$ will be polynomial in the variables $x, \bar{x}, \hat{\sigma}, \hat{\tau}$.

From considerations about crossing in (4.7), and the structure of the two-variable resummations (4.10), we conclude that the ansatz for the minimal one-loop function is given by

$$
\mathcal{H}_{\vec{p}}^{(2)} = \frac{W_{4-} + W_{3-}}{(x - \bar{x})^{\mathbf{d}_{\vec{p}}+8}} + \frac{1}{(x - \bar{x})^{\mathbf{d}_{\vec{p}}+7}}\left[W_{3+} + \frac{W_{2+}}{v^{\kappa_{\vec{p}}-2}}\right]
$$
$$
+ \frac{1}{v^{\kappa_{\vec{p}}-2}}\left[\frac{W_{2-}}{(x - \bar{x})^{\mathbf{d}_{\vec{p}}+8}} + \frac{W_{1v} + W_{1u}}{(x - \bar{x})^{\mathbf{d}_{\vec{p}}+7}} + \frac{W_0}{(x - \bar{x})^{\mathbf{d}_{\vec{p}}+5}}\right]
\tag{4.15}
$$

where recall

$$
\mathbf{d}_{\vec{p}} = p_1 + p_2 + p_3 + p_4 - 1, \qquad \kappa_{\vec{p}} = \frac{\min(p_1 + p_2, p_3 + p_4)}{2} - \frac{p_{43}}{2}.
\tag{4.16}
$$

A smaller basis made of just the box function $\phi^{(1)}$, together with its weight one and weight zero completions will be referred to as *tree-like*. For example, any $\overline{D}$ function can be decomposed in such a basis. However, consistently with our splitting of the one-loop function as $\mathcal{D}^{(2)} = \mathcal{T}^{(2)} + \mathcal{H}^{(2)}$, we will point out in which way the tree-like coefficient functions for $W_{2-}, W_{1u}, W_{1v}, W_0$ really encode physics beyond tree level.

In the following we describe our bootstrap algorithm, going through the sequence of steps that need to be performed in order to obtain $\mathcal{H}_{\vec{p}}^{(2)}$.

Crossing symmetry and leading log matching. For any orientation of the external charges $\vec{p}$, we consider the $\log^2 u$ projection of the ansatz and match with the explicit two-variable resummation described in 4.2. This fixes combinations of coefficient functions from $W_{4-}, W_{3\pm}, W_{2+}$. Note that the power $v^{\kappa_{\vec{p}}-2}$ in the denominator of W_{2+} (4.15) is consistent with the weight-0 part of the leading log (4.10). Matching all independent leading log discontinuities actually fixes completely the polynomials $h_{1,2,\dots,7,9,\dots11}$. When $\kappa_{\vec{p}} = 2$, the correlators are next-to-next-to extremal, for example 2222, and 2233, and there is no singular v behavior in the ansatz. In these cases our ansatz (4.15) reduces to the ansatz considered in our previous works [3, 7].

Absence of unphysical poles. Any leading log discontinuity has itself no poles at $x = \bar{x}$. However, that only counts the $\log^2 u$ projection of the function $\mathcal{H}^{(2)}$. In order for the ansatz to produce a well defined function we have to ensure that globally there are no unphysical poles. In this way, lower weight coefficient functions become entangled with those at weights -4, -3 and -2 symmetric. In particular, both the power of $(x - \bar{x})$ in the denominators, and the coefficient functions of W_{2-}, W_{1u}, W_{1v} and W_0, have the right structure such that all $x = \bar{x}$ poles coming from weight -4, -3 and -2 symmetric coefficient functions can be cancelled. For this reason the 'tree-like' coefficients functions of $\mathcal{H}^{(2)}$, $h_{\square}, h_u, h_v, h_0$, have quite different features compared to their counterparts at tree level. In this process we can keep $v^{\kappa_{\vec{p}}-2}$ as the maximum singular power in the denominator.

Matching the OPE prediction in and below window. At this stage of the algorithm we have found a well defined ansatz with the correct $\log^2 u$ discontinuities. It differs from $\mathcal{H}^{(2)}_{\vec{p}}$ because we have not yet imposed the remaining predictions in and below window, which we have to compute explicitly by using the strategy outlined in section 3.2 and 3.3. Such OPE predictions come as SCPW coefficients at fixed twist, and varying spin, i.e. from a sum like

$$\sum_l c_l^{\tau_0} \mathcal{B}^{\tau_0, l} + \dots + \sum_l c_l^{\tau_k} \mathcal{B}^{\tau_k, l} \,, \tag{4.17}$$

where c_l^τ stands for $M_{\tau,l}^{(2)}$ or $L_{\tau,l}^{(2)}$, and $k < \tau^{\max}$ is finite. Given the analytic representation of the conformal blocks, we can series expand the sum (4.17) in the form

$$u^{\tau_0} \sum_{n=0}^{\tau_k - \tau_0} \sum_{m=0}^{\infty} d_{nm} x^n \bar{x}^m \tag{4.18}$$

and then resum it as

$$x^{\tau_0} \sum_{n=0}^{\tau_k - \tau_0} x^n g_n(\bar{x}) \,, \tag{4.19}$$

where the functions g_n contain transcendetal functions of one-variable. Indeed the ansatz for these g_n descends from the two-variable ansatz (4.15), upon performing the same series expansion as in (4.19). We call an expression of the form (4.19) a *one-variable resummation*.

The initial number of free coefficients grows with $p_1 + p_2 + p_3 + p_4$, because of the denominator factors $(x - \bar{x})$ in (4.15), and obviously with the number of $su(4)$ channels. Cancelling $x = \bar{x}$ poles alone still leaves a large number of free coefficients. Imposing OPE predictions in and below window is indeed crucial to finally obtain the minimal loop functions.

Ambiguities. Imposing predictions in and below window fixes the majority of the free coefficients in the ansatz. A sample of this process is illustrated in table 1. The free parameters left are associated to a restricted class of tree-like functions, which we call *ambiguities*. By construction, these ambiguities do not contribute to the $\log^2 u$ discontinuity in any channel, obey the correct crossing transformations by themselves, have no $x = \bar{x}$ poles, and contribute only above window, i.e for twists $\tau \geq \tau_{\vec{p}}^{\max}$. Furthermore, we find the special feature that their SCPW coefficients have finite spin support, $l = 0, 1$.

correlator	initial free coeffs. in $h_\square, h_u, h_v, h_1$	after leading log matching and pole cancellation	after OPE predictions in and below window
2222	1×378	1	1
2233	1×496	16	2
2244	1×579	20	2
3333	3×579	20	2
4444	6×946	68	4

Table 1. Number of tree-like free coefficients across the three steps of our algorithm.

The Mellin amplitude corresponding to the ambiguities is very simple, since it can be at most linear in the Mellin variables (s, t), for two reasons. Firstly, it cannot be rational, as any additional pole would spoil our predictions in and below the window. Therefore it has to be polynomial. Secondly, this polynomial cannot be higher order than linear, as it would generate tree-like terms with a higher degree denominator than allowed by our ansatz (4.15) for the minimal one-loop function $\mathcal{H}^{(2)}_{\vec{p}}$.

For a generic correlator without any crossing symmetries, we can parametrise the full set of ambiguities by

$$\mathcal{H}^{(2)}_{\vec{p}}\Big|_{\text{ambiguity}} = \frac{u^{\frac{p_3-p_4}{2}}}{v^{\frac{p_2+p_3}{2}}} \oint u^{\frac{s}{2}} v^{\frac{t}{2}} \, \Gamma_{\vec{p}} \sum_{i=0}^{\kappa_{\vec{p}}-2} \sum_{j=0}^{\kappa_{\vec{p}}-2-i} \left(\alpha^{(1,ij)}_{\vec{p}} + \alpha^{(s,ij)}_{\vec{p}} s + \alpha^{(t,ij)}_{\vec{p}} t \right) \hat{\sigma}^i \hat{\tau}^j, \quad (4.20)$$

where $\kappa_{\vec{p}}$ is the degree of extremality (2.14), and $\Gamma_{\vec{p}}$ is the combination of Mack's Gamma functions

$$\Gamma_{\vec{p}} = \Gamma\left[\frac{p_1+p_2-s}{2}\right] \Gamma\left[\frac{p_3+p_4-s}{2}\right] \Gamma\left[\frac{p_1+p_4-t}{2}\right] \Gamma\left[\frac{p_2+p_3-t}{2}\right] \Gamma\left[\frac{p_1+p_3-U}{2}\right] \Gamma\left[\frac{p_2+p_4-U}{2}\right]. \tag{4.21}$$

In (4.21) we have introduced an auxiliary Mellin variable U which makes crossing symmetry manifest, and it is defined as

$$s + t + U = p_1 + p_2 + p_3 + p_4 - 4. \tag{4.22}$$

Thus, for a generic correlator, we find $\frac{3(\kappa_{\vec{p}}-1)\kappa_{\vec{p}}}{2}$ undetermined ambiguities. In cases in which the correlator has some crossing symmetry, we have to count only crossing symmetric combinations.

Let us construct explicitly the ambiguities for the correlators we will discuss in the next sections:

- $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle$. The only fully crossing symmetric combination one can build is the constant Mellin amplitude 1, so there can only be a single ambiguity: α^1_{2222}.[12]

---

[12]The corresponding function in position space is $\bar{D}_{4444}$ [3]. The value of $\alpha^{(1)}_{2222} = 60$ (in the conventions of [3]) was found by using a supersymmetric localisation computation [13]. Such a non-zero value breaks analyticity in spin for the twist 4 one-loop anomalous dimension at spin zero, in agreement with the argument from the Lorentzian inversion formula [6].

- $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_p \mathcal{O}_p \rangle$. This family of correlators is not fully crossing symmetric if $p > 2$. The remaining invariance can be understood as invariance under $t \leftrightarrow U$. As a result, we are left with two out of three ambiguities,

$$\alpha_{22pp}^1, \qquad \text{and} \qquad \alpha_{22pp}^2 s. \tag{4.23}$$

- $\langle \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \rangle$. This correlator admits up to linear terms in $\hat{\sigma}$, and $\hat{\tau}$, but crossing symmetry only allows two (fully symmetric) ambiguities,

$$\alpha_{3333}^1(1 + \hat{\sigma} + \hat{\tau}) \qquad \text{and} \qquad \alpha_{3333}^2(s + U\hat{\sigma} + t\hat{\tau}). \tag{4.24}$$

A correlator with $\kappa = 3$ but no crossing symmetries would admit 9 ambiguities.

- $\langle \mathcal{O}_4 \mathcal{O}_4 \mathcal{O}_4 \mathcal{O}_4 \rangle$. The full crossing symmetry of this correlator greatly reduces the number of ambiguities. With at most quadratic terms in $\hat{\sigma}$, and $\hat{\tau}$, one can construct four independent ambiguities: two ambiguities with constant Mellin amplitudes

$$\alpha_{4444}^1(1 + \hat{\sigma}^2 + \hat{\tau}^2) \qquad \text{and} \qquad \alpha_{4444}^2(\hat{\sigma} + \hat{\tau} + \hat{\sigma}\tau) \tag{4.25}$$

and two other ambiguities with linear terms

$$\alpha_{4444}^3(s + U\hat{\sigma}^2 + t\hat{\tau}^2), \qquad \text{and} \qquad \alpha_{4444}^4(t\hat{\sigma} + U\hat{\tau} + s\hat{\sigma}\hat{\tau}). \tag{4.26}$$

A correlator with $\kappa = 4$ but no crossing symmetries would admit 18 ambiguities.

Notice that our analysis here is already in agreement with the observed number of ambiguities shown in table 1.

Since our bootstrap algorithm has a built-in position space implementation, it will be useful to rewrite the Mellin amplitude for the ambiguities in such a way that the comparison with the position space result is simple. This rewriting follows our organisation of the OPE into above threshold, in and below window, and it is explained in appendix B.2.

4.4 $\langle \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \rangle$

We begin illustrating our bootstrap algorithm with $\mathcal{H}_{3333}^{(2)}$. The solution for the polynomial coefficients $h_1, \ldots h_{11}, h_\square, h_u, h_v, h_0$ is listed in a Mathematica notebook attached to the arXiv submission of this article. For simplicity, the ancillary file contains $\mathcal{H}_{3333}^{(2)}$ with a particular value of the ambiguities.

The 3333 correlator has degree of extremality 3 and full crossing symmetry. The long sector decomposes into three representations, [000], [101] and [020], with threshold twist $\tau^{\max} = 6$.

The resummation of the $\log^2 u$ discontinuity can be obtained from $\widehat{\mathcal{D}}_{3333}$ and $\Delta^{(8)}$ as explained in section 6.1. With this data we can then initiate the first step of our algorithm, by matching and imposing crossing symmetry of the ansatz.

In the second step of the algorithm we impose absence of $x = \bar{x}$ poles on the ansatz.

Finally, we have to impose OPE predictions in and below window. Here the window is empty, since all external charges are equals. This implies that upon projecting the ansatz

onto the $\log^1 u$ stratum we have to set to zero the one-variable expansion up to order $O(x^3)$. OPE predictions below window are instead non trivial.

In [000] the unitary bound is $\tau = 2$, and no long supergravity states contribute, since these are all string states. A non trivial prediction comes in at twist 4. Here there is only one double trace operator $\mathcal{K}_{22;4,l,[000]}$. Using (3.39) we thus get a prediction for $L^{(2)\mathcal{H}}_{3333;4,l,[000]}$.[13] We have

$$L^{(2)\mathcal{H}}_{3333;2,l,[000]} = 0 \tag{4.27}$$

$$L^{(2)\mathcal{H}}_{3333;4,l,[000]} = \frac{9 \times 4800}{(l+1)(l+6)} \frac{((l+3)!)^2}{(2l+6)!} \frac{1+(-1)^l}{2}. \tag{4.28}$$

The one-variable resummation of (4.28) input into (3.49) reads

$$\mathcal{H}^{(2)}_{3333}\Big|_{\Upsilon_{[000]}\log^0 u} = 9 \times 6! \times \frac{x^2}{\bar{x}^4}\left[5(\bar{x}-2)\bar{x}\mathrm{Li}_1(\bar{x}) + \frac{5}{3}(6-6\bar{x}+\bar{x}^2)\mathrm{Li}_1^2(\bar{x})\right] + O(x^3). \tag{4.29}$$

In the [101] and [020] sectors, the unitary bound is $\tau = 4$. There are no predictions descending from the long sectors at tree level. Instead, this is the first case in which we need to consider the consequences of protected semishort operators at twist 4, through our formula (3.61) and the results for $S_{4;l+2,[1]}$ and $S_{4;l+2}$ given in (2.46). More precisely, there is an $S_{4;l+2,[1]}$ for [101], which implies

$$L^{(2)\mathcal{H}}_{3333;4,l,[101]} = \frac{9 \times 576}{(l+2)(l+5)} \frac{((l+3)!)^2}{(2l+6)!} \frac{1-(-1)^l}{2} \tag{4.30}$$

with corresponding one-variable resummation,

$$\mathcal{H}^{(2)}_{3333}\Big|_{\Upsilon_{[101]}\log^0 u} = 9 \times 6! \times \frac{x^2}{\bar{x}^4}\left[3(\bar{x}-2)\bar{x} + \left(6-6\bar{x}+\frac{7}{5}\bar{x}^2\right)\mathrm{Li}_1(\bar{x}) + \frac{1}{5}(\bar{x}-2)\bar{x}\mathrm{Li}_1^2(\bar{x})\right] + O(x^3) \tag{4.31}$$

and an $S_{4,l+2,[0]}$ for [020] which gives,

$$L^{(2)\mathcal{H}}_{3333;4,l,[020]} = \frac{9 \times 288}{(l+3)(l+4)} \frac{((l+3)!)^2}{(2l+6)!} \frac{1+(-1)^l}{2} \tag{4.32}$$

with one-variable resummation

$$\mathcal{H}^{(2)}_{3333}\Big|_{\Upsilon_{[020]}\log^0 u} = 9 \times 6! \times \frac{x^2}{\bar{x}^4}\left[\frac{6}{5}\bar{x}^2 + \frac{3}{5}(\bar{x}-2)\bar{x}\mathrm{Li}_1(\bar{x}) + \frac{1}{10}\bar{x}^2\mathrm{Li}_1^2(\bar{x})\right] + O(x^3) \tag{4.33}$$

There is an important logical distinction between $L^{(2)\mathcal{H}}_{3333;4,l,[101]}$ and $L^{(2)\mathcal{H}}_{3333;4,l,[020]}$ we should highlight. The twist 4 of [020] lies at the bottom of multiplet recombination, in the sense that $\tau = 2a + b + 2$ with $b = 2$ and $a = 0$. This means that the corresponding SCPW does not get shifted by multiplet recombination in another $su(4)$ representation. In fact, our formula (3.61) makes explicit that there is no extra summation over a that needs to be taken into account. This is not the case for the twist 4 of [101], where instead the

---

[13]The explicit expression for $C^{(1)}_{33;4,l,[000]}$ is in appendix, given by (C.8).

SCPW coefficient receives a contribution due to multiplet recombination of [000] at twist 2. However, there is no $S_{2;l+2,[0]}$ contribution, therefore $L^{(2)\mathcal{H}}_{3333;4,l,[101]} = S_{4;l+2,[1]}$ holds exactly.

Coming back to our ansatz, we match (4.29), (4.31) and (4.33). Recall that we had 20 free coefficients, i.e. coefficients not fixed by demanding absence of $x = \bar{x}$ poles. However after matching the OPE predictions below window we are left with only 2 free coefficients. The functions they span are the final ambiguities. They come out in the form

$$\mathcal{H}^{(2)}_{3333}\Big|_{\text{ambiguity}} = u^3 \oint u^s v^t \; \Gamma[-s]^2\Gamma[-t]^2\Gamma[s+t+5]^2$$
$$\times \left[(1 + \hat{\sigma} + \hat{\tau})\beta^1_{3333} + \left(s + \hat{\tau}t - \hat{\sigma}(5 + s + t)\right)\beta^2_{3333}\right] \qquad (4.34)$$

By redefining the Mellin variables s, t we obtain a perfect match with our previous discussion in (4.24). Upon inspection, the SCPW of (4.34) only contributes at spin $l = 0$ for twist above threshold.

4.5 $\langle \mathcal{O}_4 \mathcal{O}_4 \mathcal{O}_4 \mathcal{O}_4 \rangle$

The next correlator we study is 4444. The solution of our bootstrap problem, written up in the basis (4.15), is appended in a Mathematica notebook attached to the arXiv submission of this article. For simplicity, the ancillary file contains $\mathcal{H}^{(2)}_{4444}$ with a particular value of the ambiguities.

The 4444 correlator is fully crossing symmetric and has degree of extremality 4. The long sector decomposes into six $su(4)$ channels: [000], [101], [020] and [202], [121], [040]. The threshold twist is $\tau^{\text{max}} = 8$.

The leading $\log^2 u$ resummation is obtained by acting with $\widehat{\mathcal{D}}_{4444}$ and $\Delta^{(8)}$, as explained in section 6.1. We then initiate the algorithm by matching, imposing crossing symmetry, and absence of $x = \bar{x}$ poles.

We finally come to the OPE predictions in and below window. Being the window empty, we project the ansatz onto the $\log^1 u$ stratum and we set to zero the one-variable expansion up to $O(x^4)$. Below window we find instead non trivial physics. For the representations [000], [101], [020], the discussion is similar to that in 3333 for twist 4, and continues at twist 6 by including predictions coming from the long sector at tree level. For [202], [121], [040] we will have to consider non trivial multiplet recombination taking into account the predictions arising from the semishort sector. We proceed in order.

In the singlet channel [000], there is an empty twist 2 sector, then the $1/N^2$ subleading three-point couplings $C^{(1)}_{44;4,l,[000]}$ and $\mathbf{C}^{(1)}_{44;6,l,[000]}$ give non trivial predictions at twist 4 and twist 6.[14] Recall that there are two double trace operators in $R_{6,l,[000]}$, therefore the twist 6 computation yields a vector three-point function. We find, from (3.39),

$$L^{(2)\mathcal{H}}_{4444;2,l,[000]} = 0 \qquad (4.35)$$

$$L^{(2)\mathcal{H}}_{4444;4,l,[000]} = \frac{16 \times 4800}{(l+1)(l+6)} \frac{((l+3)!)^2}{(2l+6)!} \frac{1 + (-1)^l}{2} \qquad (4.36)$$

$$L^{(2)\mathcal{H}}_{4444;6,l,[000]} = \frac{16 \times 360(2l+9)^2(119 + (2l+9)^2)}{(l+1)(l+2)(l+7)(l+8)} \frac{((l+4)!)^2}{(2l+8)!} \frac{1 + (-1)^l}{2} \qquad (4.37)$$

---

[14]Explicit expressions for $C^{(1)}_{44;4,l,[000]}$ and $\mathbf{C}^{(1)}_{44;6,l,[000]}$ are given in (C.8) and (C.12), respectively.

The corresponding one variable resummation is

$$
\mathcal{H}^{(2)}_{4444}\Big|_{\Upsilon_{[000]}\log^0 u} = 16 \times 6!
$$
$$
\times \left(\frac{x^2}{\bar{x}^4}\left[5(\bar{x}-2)\bar{x}\mathrm{Li}_1(\bar{x}) + \frac{5}{3}(6-6\bar{x}+\bar{x}^2)\mathrm{Li}_1^2(\bar{x}) \right] + \right.
$$
$$
\frac{x^3}{\bar{x}^5}\left[\bar{x}^2(\bar{x}-2)\left(\frac{4\bar{x}^2}{\bar{x}-1}+205\right) + (3440-5590\bar{x}+\frac{7484}{3}\bar{x}^2-244\bar{x}^3)\mathrm{Li}_1^2(\bar{x}) \right.
$$
$$
\left.\left. -300\bar{x}(6-6\bar{x}+\bar{x}^2)\mathrm{Li}_2(\bar{x}) - \bar{x}(1230-1210\bar{x}+211\bar{x}^2)\mathrm{Li}_1(\bar{x}) \right] \right) + O(x^4)
\tag{4.38}
$$

In [101] and [020] there are twist 4 predictions coming from semishorts operators at the unitarity bound, similarly to the case of 3333 discussed previously. In particular, there is an $S_{4;l+2,[1]}$ for [101] and an $S_{4;l+2,[0]}$ for [020], computed in (2.48) In addition we have $1/N^2$ three-point couplings $C^{(1)}_{44;6,l,[101]}$ and $\mathbf{C}^{(1)}_{44;6,l,[020]}$ which give predictions at twist 6. Notice that twist 6 is the first twist available for a long rep [020], but the latter has doubling of the operators, i.e. $\mu = 2$.

The list of results in [101] is

$$
L^{(2)\mathcal{H}}_{4444;4,l,[101]} = \frac{16\times1600}{(l+2)(l+5)}\frac{((l+3)!)^2}{(2l+6)!}\frac{1-(-1)^l}{2}
\tag{4.39}
$$
$$
L^{(2)\mathcal{H}}_{4444;6,l,[101]} = \frac{16\times7200(l+1)(l+8)}{(l+3)(l+6)}\frac{((l+4)!)^2}{(2l+8)!}\frac{1-(-1)^l}{2}
\tag{4.40}
$$

with resummation

$$
\mathcal{H}^{(2)}_{4444}\Big|_{\Upsilon_{[101]}\log^0 u} = 16 \times \frac{5^2 6!}{3^2}
$$
$$
\times \left(\frac{x^2}{\bar{x}^4}\left[3(\bar{x}-2)\bar{x} + \left(6-6\bar{x}+\frac{7}{5}\bar{x}^2\right)\mathrm{Li}_1(\bar{x}) + \frac{1}{5}(\bar{x}-2)\bar{x}\mathrm{Li}_1^2(\bar{x}) \right] - \right.
$$
$$
\frac{x^3}{\bar{x}^5}\left[\frac{4}{75}(1527\bar{x}-3014)\bar{x}^2 + \frac{9(\bar{x}-2)\bar{x}^4}{5(\bar{x}-1)} + \frac{16}{75}(746-766\bar{x}+201\bar{x}^2)\bar{x}\mathrm{Li}_1(\bar{x}) \right.
$$
$$
\left.\left. + \frac{1}{10}(4-176\bar{x}+87\bar{x}^2)\bar{x}\mathrm{Li}_1^2(\bar{x}) + \frac{2}{5}(4+4\bar{x}-3\bar{x}^2)\bar{x}\mathrm{Li}_2(\bar{x}) \right] \right) + O(x^4)
\tag{4.41}
$$

The list of results in [020] is

$$
L^{(2)\mathcal{H}}_{4444;4,l,[020]} = \frac{16\times1152}{(l+3)(l+4)}\frac{((l+3)!)^2}{(2l+6)!}\frac{1+(-1)^l}{2}
\tag{4.42}
$$
$$
L^{(2)\mathcal{H}}_{4444;6,l,[020]} = \frac{16\times864(40817+16702(2l+9)^2+81(2l+9)^4)}{80(l+1)(l+4)(l+5)(l+8)}\frac{((l+4)!)^2}{(2l+8)!}\frac{1+(-1)^l}{2}
\tag{4.43}
$$

with resummation

$$\mathcal{H}^{(2)}_{4444}\Big|_{\Upsilon_{[020]}\log^0 u} = 16 \times 2^2 6!$$

$$\times \left(\frac{x^2}{\bar{x}^4}\left[\frac{6}{5}\bar{x}^2 + \frac{3}{5}(\bar{x}-2)\bar{x}\mathrm{Li}_1(\bar{x}) + \frac{1}{10}\bar{x}^2\mathrm{Li}_1^2(\bar{x}) \right] + \right.$$

$$\frac{x^3}{\bar{x}^5}\left[\frac{3}{5}(382-175\bar{x}) + \frac{243(\bar{x}-2)\bar{x}^2}{25(\bar{x}-1)} - \frac{3}{5}(11358-11342\bar{x}+1981\bar{x}^2)\bar{x}\mathrm{Li}_1(\bar{x}) \right.$$

$$\left.\left. \frac{1}{5^2}(164640-246960\bar{x}+98794\bar{x}^2-8667\bar{x}^3)\mathrm{Li}_1^2(\bar{x}) - 72\bar{x}^3\mathrm{Li}_2(\bar{x}) \right] \right) + O(x^4) \tag{4.44}$$

Finally we arrive at the representations [202], [121] and [040]. The unitary bound for all of them is twist 6, and the semishort predictions were computed in (2.49). In the 4444 correlator, the twist 6 of [040] lies at the bottom of multiplet recombination because $\tau = 2a + b + 2$ with $b = 4$ and $a = 0$. The prediction for $L^{(2)\mathcal{H}}_{4444;6,l,[040]}$ is thus straightforward. The predictions for [121] and [202] involve further shifts, which we now describe. From (3.61) we find

$$L^{(2)\mathcal{H}}_{4444;6,l,[040]} = -S_{4444;6,l+2;[0]} \tag{4.45}$$

$$L^{(2)\mathcal{H}}_{4444;6,l,[121]} = +S_{4444;6,l+2;[1]} - S_{4444;4,l+3;[0]} \tag{4.46}$$

$$L^{(2)\mathcal{H}}_{4444;6,l,[202]} = -S_{4444;6,l+2;[2]} + S_{4444;4,l+3;[1]} \tag{4.47}$$

where in the last line we already implemented the absence of $S_{4444;2,l+2;[0]}$. Formulas (4.46) and (4.47) correctly include shifts to due to multiplet recombination at twist 4 in [020] and [101], respectively. Let us give the explicit expressions here below.

For [040] we find

$$L^{(2)\mathcal{H}}_{4444;6,l,[040]} = \frac{16 \times 384(29 + 3(2l+9)^2)}{(l+3)(l+6)} \frac{(l+4)!^2}{(2l+8)!} \frac{1+(-)^l}{2} \tag{4.48}$$

with resummation

$$\mathcal{H}^{(2)}_{4444}\Big|_{\Upsilon_{[040]}\log^0 u} = 16 \times 6! \times \frac{x^3}{\bar{x}^5}$$

$$\times \left(\left(208 - \frac{16\bar{x}^2}{5(\bar{x}-1)} + \frac{112}{15}\mathrm{Li}_1^2(\bar{x}) \right)(2-\bar{x})\bar{x}^2 - \frac{16}{3}(78-78\bar{x}+17\bar{x}^2)\bar{x}\mathrm{Li}_1(\bar{x}) \right) + O(x^4) \tag{4.49}$$

For [202] we find

$$L^{(2)\mathcal{H}}_{4444;6,l,[202]} = 16 \times \left[\frac{2400(l+2)(l+7)}{(l+3)(l+6)} - \left(\frac{1600}{(l+2)(l+5)} \right)_{l\to l+1} \right] \frac{(l+4)!^2}{(2l+8)!} \frac{1+(-)^l}{2}$$

$$= \frac{16 \times 200\left(-83 + 3(2l+9)^2\right)}{(l+3)(l+6)} \frac{(l+4)!^2}{(2l+8)!} \frac{1+(-)^l}{2} \tag{4.50}$$

with resummation

$$
\left.\mathcal{H}_{4444}^{(2)}\right|_{\Upsilon_{[202]}\log^0 u} = 16 \times 6! \times \frac{x^3}{\bar{x}^5}
$$
$$
\times \left(\left(\frac{25}{3} + \frac{5\bar{x}^2}{3(\bar{x}-1)} + \frac{35}{9}\mathrm{Li}_1^2(\bar{x}) \right) \bar{x}^2(\bar{x}-2) + \frac{5}{9}(30-30\bar{x}+13\bar{x}^2)\mathrm{Li}_1(\bar{x}) \right) + O(x^4)
$$

(4.51)

For [121] we find

$$
L_{4444;6,l,[121]}^{(2)\mathcal{H}} = 16 \times \left[\frac{72(401+174(2l+9)^2+(2l+9)^4)}{(l+2)(l+4)(l+5)(l+7)} - \frac{1152}{(l+4)(l+5)} \right] \frac{(l+4)!^2}{(2l+8)!} \frac{1-(-)^l}{2}
$$
$$
= \frac{16 \times 72(3+(2l+9)^2)(167+(2l+9)^2)}{(l+2)(l+4)(l+5)(l+7)} \frac{(l+4)!^2}{(2l+8)!} \frac{1-(-)^l}{2}
$$

(4.52)

with resummation

$$
\left.\mathcal{H}_{4444}^{(2)}\right|_{\Upsilon_{[121]}\log^0 u} = 16 \times 6! \times \frac{x^3}{\bar{x}^5}\left(\left(\frac{4\bar{x}^3(2-2\bar{x}+\bar{x}^2)}{5(\bar{x}-1)} - 2184\bar{x}(\bar{x}-1) + \frac{1874}{5}\bar{x}^3 \right) \right.
$$
$$
\left. - \left(2184 - 3276\bar{x} + \frac{7104}{5}\bar{x}^2 - \frac{822}{5}\bar{x}^3 \right)\mathrm{Li}_1(\bar{x}) + \frac{14}{5}(48-48\bar{x}+7\bar{x}^2)\mathrm{Li}_1^2(\bar{x}) \right) + O(x^4)
$$

Incredibly all these predictions are consistent with the minimal ansatz (4.15) and uniquely fix the remaining coefficients, leaving only 4 ambiguities. These are

$$
\left.\mathcal{H}_{4444}^{(2)}\right|_{\mathrm{ambiguity}} = u^4 \oint u^s v^t \, \Gamma[-s]^2\Gamma[-t]^2\Gamma[s+t+6]^2
$$
$$
\times \Big[(1+\hat{\sigma}^2+\hat{\tau}^2)\beta_{4444}^1 + \big(s+\hat{\tau}^2 t - \hat{\sigma}^2(6+s+t)\big)\beta_{4444}^2
$$
$$
+ (\hat{\tau}+\hat{\sigma}+\hat{\tau}\hat{\sigma})\beta_{4444}^3 + \big(\hat{\tau}(1+s+t) - \hat{\sigma}(5+t) - \hat{\sigma}\hat{\tau}(5+s)\big)\beta_{4444}^4 \Big]
$$

(4.53)

Again, we find perfect match with our previous discussion in (4.25)–(4.26), after the redefinition of Mellin variables s and t. The ambiguities have only spin $l = 0$ support in any $su(4)$ channel, for twist above threshold.

4.6 Next-to-next-to-extremal correlators

In this section we will consider four point correlators $\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4} \rangle$ with external charges $\langle 2244 \rangle$, $\langle 3335 \rangle$ and $\langle 4424 \rangle$. These correlators are all N^{2}E.

For N^{2}E correlators there are no OPE predictions below window. In particular, semishort predictions $S_{p_1p_2p_3p_4}$ vanish because they are determined through (2.43) in terms of SCPW coefficients $S_{p(r)q(r)p_3p_4}$, where $p(r) + q(r) = p_{43} + 2$, and these correlators are next to extremal, thus completely vanishing as a consequence of our definition of external single particles states. Because of the split $\mathcal{T}^{(2)} + \mathcal{H}^{(2)}$, it follows $L_{2+p_{43},[0p_{43}0]}^{(2)\mathcal{H}} = 0$.

An additional peculiarity of $\langle 3335 \rangle$ and $\langle 4424 \rangle$, which generalises to other N^{2}E correlators as discussed later on in section 5.3, is the fact that the tree level functions $\mathcal{D}_{3335}^{(1)}$ and $\mathcal{D}_{4424}^{(1)}$ are proportional. This implies that, after taking into account a normalization, both correlators have the same one-loop $\log^2 u$ discontinuity. Therefore, an ansatz having

the correct crossing symmetries, and constructed by matching the $\log^2 u$ discontinuity and imposing absence of $x = \bar{x}$ poles, cannot distinguish between $\mathcal{H}^{(2)}_{3335}$ and $\mathcal{H}^{(2)}_{4424}$. Very interestingly, this type of degeneracy is actually lifted at one-loop, because of the different OPE predictions in the window. This illustrates another important aspect of the OPE predictions in and below window. In general, we expect the situation to be as follows: pairs of correlators which are degenerate at tree level will have instead different minimal one-loop functions, distinguished by the OPE predictions in and below window.

For $\langle 3335 \rangle$ and $\langle 4424 \rangle$ we have $\log^1 u$ twist 6 prediction (in the [020] representation). Making manifest reciprocity symmetry, we write them in the form (3.40),

$$
(\tau = 6) \qquad
\begin{cases}
\dfrac{Y^{(0)}_{3335} + Y^{(2)}_{3335}(l + \frac{9}{2})^2}{(l+1)(l+4)(l+5)(l+8)} \dfrac{(l+4)!(l+5)!}{(2l+8)!} \\[4mm]
\dfrac{Y^{(0)}_{4424} + Y^{(2)}_{4424}(l + \frac{9}{2})^2}{(l+1)(l+4)(l+5)(l+8)} \dfrac{(l+4)!(l+5)!}{(2l+8)!}
\end{cases}
\tag{4.54}
$$

The values of the free Y coefficient above, obtained from the OPE predictions, are

$$
\begin{cases}
Y^{(0)}_{3335} = -4762800 \\
Y^{(2)}_{3335} = \frac{4}{35} Y^{(0)}_{3335}
\end{cases}
\quad ; \quad
\begin{cases}
Y^{(0)}_{4424} = -4628736 \\
Y^{(2)}_{4424} = \frac{55}{2009} Y^{(0)}_{4424}
\end{cases}
\tag{4.55}
$$

We proceed as in previous sections. We construct an ansatz which matches the leading logs, has the correct symmetries and no $x = \bar{x}$ poles. We then impose the OPE predictions in the window.

The result for $\langle 3335 \rangle$ and $\langle 4424 \rangle$ can be obtained in the following instructive way. We initially normalize both correlators in a way that the leading logs are the same. Thus we construct *one* ansatz. Before imposing OPE predictions, this ansatz has six free coefficients. We now insist that the SCPW coefficients of the ansatz at $\tau = 6$ have the form (4.54), where we do not specify the values of $Y^{(0)}_{\vec{p}}$ and $Y^{(1)}_{\vec{p}}$ yet. This constraint returns a one-parameter ansatz with one additional ambiguity. We go back to the correct normalization for the correlators, and we keep $Y^{(0)}_{\vec{p}}$ as the free parameter, isolating the tree-like function it multiplies. Then, we can write the minimal loop functions in the following form

$$
\mathcal{H}^{(2)}_{\vec{p}} = \mathcal{N}_{\vec{p}} \overline{\mathcal{H}}^{(2)} + \frac{1}{882} Y^{(0)}_{\vec{p}} u^2 \overline{D}_{4444} \qquad \vec{p} = 3335 || 4424
\tag{4.56}
$$

where $\mathcal{N}_{4424} = 128$ and $\mathcal{N}_{3335} = 135$. Because $Y^{(0)}_{3335} \neq Y^{(0)}_{4424}$, we find $\mathcal{H}^{(2)}_{3335} \neq \mathcal{H}^{(2)}_{4424}$. Differently from tree level, the minimal one-loop functions are distinct.

The ambiguity has the same functional form for both $\langle 3335 \rangle$ and $\langle 4424 \rangle$. Written in Mellin space it reads

$$
\mathcal{H}^{(2)}_{3335 || 4424}\Big|_{\text{ambiguity}} = u^3 v \oint u^s v^t \Gamma[-s]\Gamma[-s-1]\Gamma[-t]\Gamma[-t-1]\Gamma[s+t+6]\Gamma[s+t+7]\beta_{\vec{p}}
\tag{4.57}
$$

Notice the combinations of Mack's $\Gamma_{\vec{p}}$ is the same for both $\langle 3335 \rangle$ and $\langle 4424 \rangle$.

The result for $\langle 2244 \rangle$ is more straightforward. In the window, we have both twist 4 and 6 predictions (in the [000] representation). Again we write them in the form (3.40),

namely

$$
(\tau = 4) \qquad \frac{X_{2244}^{(0)}}{(l+1)(l+6)} \frac{((l+3)!)^2}{(2l+6)!} \tag{4.58}
$$

$$
(\tau = 6) \qquad \frac{Y_{2244}^{(0)} + Y_{2244}^{(2)}(l+\frac{9}{2})^2}{(l+1)(l+2)(l+7)(l+8)} \frac{((l+4)!)^2}{(2l+8)!} \tag{4.59}
$$

with predicted values,

$$
X_{2244}^{(0)} = -8 \times 1920, \qquad Y_{2244}^{(0)} = +8 \times 176400, \qquad Y_{2244}^{(2)} = \frac{76}{245} Y_{2244}^{(0)}. \tag{4.60}
$$

In the orientation $\langle 2424 \rangle$ the window is empty.

The bootstrap algorithm returns $\mathcal{H}_{2244}$ leaving only two ambiguities as we discussed in (4.23). In Mellin space we find

$$
\mathcal{H}_{2244}^{(2)}\Big|_{\text{ambiguity}} = u^4 \oint u^s v^t \Gamma[-s]\Gamma[-s-2]\Gamma[-t]^2 \Gamma[s+t+6]^2 \left[\beta_{2244}^1 + \beta_{2244}^2 \, s \right] \tag{4.61}
$$

The minimal one-loop functions corresponding to $\langle 3335 \rangle$ and $\langle 4424 \rangle$, and $\langle 2244 \rangle$, are given in a Mathematica notebook attached to the arXiv submission of this article. For $\langle 3335 \rangle$ and $\langle 4424 \rangle$ we only included $\overline{\mathcal{H}}$. In both cases, we have fixed some value of the ambiguities.

5 Upgraded tree level Mellin amplitudes

In previous sections, we showed that the one-loop function $\mathcal{D}^{(2)}$ admits the split $\mathcal{D}^{(2)} = \mathcal{T}^{(2)} + \mathcal{H}^{(2)}$, where $\mathcal{H}^{(2)}$ encodes all the non trivial OPE predictions at $O(1/N^4)$ whereas $\mathcal{T}^{(2)}$ is a generalised tree-level function having no $\log^2$ contribution. Our final task is to bootstrap $\mathcal{T}$.

The generalised tree level function $\mathcal{T}_{\vec{p}}$ is defined as the unique function, within the ansatz:

$$
\mathcal{T}_{\vec{p}} = \frac{P_\square \, \phi^{(1)}(x,\bar{x})}{(x-\bar{x})^{\mathbf{d}+2}} + \frac{P_v \log(v)}{(x-\bar{x})^{\mathbf{d}+1}} + \frac{1}{v^{\kappa_{\vec{p}}-1}} \left[\frac{P_u \log(u)}{(x-\bar{x})^{\mathbf{d}+1}} + \frac{P_1}{(x-\bar{x})^{\mathbf{d}-1}} \right] \tag{5.1}
$$

$$
\mathbf{d}_{\vec{p}} = p_1 + p_2 + p_3 + p_4 - 1, \qquad \kappa_{\vec{p}} = \min\left(p_3, \frac{p_1 + p_2 + p_3 - p_4}{2} \right) \tag{5.2}
$$

such that:

(a) the threshold twist for the $\log(u)$ discontinuity is $\tau = \tau^{\text{max}}$.

(b) the SCPW expansion below window *completely cancels* free theory contributions as described in (3.58) and (3.59)

(c) there are no unphysical $x = \bar{x}$ poles in (5.1).

The coefficient functions denoted by P are polynomials in $x, \bar{x}$ and σ, τ. As functions of $x, \bar{x}$ variables, these polynomials have a Taylor expansion of the form $x^n \bar{x}^m$ with $m + n \leq p_1 +$

$p_2 + p_3 + p_4$. The function $\mathcal{T}_{\vec{p}}$ is symmetric under $x \leftrightarrow \bar{x}$, therefore a given polynomial P has the same symmetry as the transcendental function it multiplies. The $su(4)$ decomposition of $\mathcal{T}_{\vec{p}}$ is obviously the same as for the full dynamical function $\mathcal{D}_{\vec{p}}$.

Implementing condition (a) implies

$$P_{\square} = O\left(u^{-\frac{p_{43}}{2} + \frac{\max(p_1+p_2, p_3+p_4)}{2}}\right) \tag{5.3}$$

$$P_u = O\left(u^{-\frac{p_{43}}{2} + \frac{\max(p_1+p_2, p_3+p_4)}{2}}\right) \tag{5.4}$$

The above conditions in fact define a generalised tree function $\mathcal{T}_{\vec{p}}$ for *any* free $\mathcal{N} = 4$ theory, i.e. for all N. Indeed we can define it in terms of the coefficients A_γ^k in front of each propagator structure in (2.8), which we can leave completely arbitrary (other than the relations between them arising from imposing crossing symmetry). The polynomials P in (5.1) become function of the free propagator coefficients, $P[\{A_\gamma^k\}]$. The precise value of these A_γ^k does not affect any step of this algorithm. Furthemore, condition (b) is overconstraining, and therefore the solution we find is unique, i.e. $\mathcal{T}_{\vec{p}}$ is unique.

Because of this uniqueness, we expect our function $\mathcal{T}_{\vec{p}}$ to reduce to known results at tree level when the propagator coefficients A_γ^k take on their free theory values. Indeed, when the external charges are equal, our conditions are precisely those imposed in [36], and for arbitrary charges we expect to recover the tree-level correlators of Rastelli and Zhou [1]. Notice that in position space the function of [1] is described by the same ansatz as in (5.1), except for the change $\mathbf{d}_{\vec{p}} \to \mathbf{d}_{\vec{p}} - 2$. (Various examples can be found in [4] by rewriting the corresponding $\overline{D}$ representation.) Quite non trivially, $\mathcal{T}_{\vec{p}}$ does reduce to the function of [1] when the coefficients A_γ^k are truncated at $O(1/N^2)$. In fact, we find that all polynomials $P_{\square,u,v,1}[\{A_\gamma^k\}]$ acquire an extra double zero $(x - \bar{x})^2$ when we restrict the A_γ^k to their tree level value.

At tree level, the free theory coefficients $A_\gamma\big|_{1/N^2}$ are all proportional to each other, and thus satisfy linear relations. Therefore, we can understand the tree-level degeneration as the result of imposing on $P_{\square,u,v,1}[\{A_\gamma^k\}]$ these tree-level linear relations. However, the non-planar values of the A_γ^k are not as simple, and the corresponding relations become non-linear.

Similarly to the function of Rastelli and Zhou [1] the most transparent representation of $\mathcal{T}_{\vec{p}}$ is given in Mellin space. We thus define the corresponding Mellin amplitude $\mathcal{M}[\mathcal{T}_{\vec{p}}](s, t)$ of the generalised tree similarly to that of the tree-level function of [1]. Amazingly all the generalised tree-level functions $\mathcal{T}_{\vec{p}}$ — defined by the above conditions $(a), (b), (c)$ — can be written in this form with a simple rational Mellin amplitude with only simple poles.

The specific form of $\mathcal{M}[\mathcal{T}_{\vec{p}}](s, t)$, i.e. finiteness and rationality, translates into the observation that the entire function $\mathcal{T}_{\vec{p}}$ is determined uniquely in terms of the coefficient $P_{\square}$ in from of the box function. This can be understood from the fact that the box function contains a $\log u \log v$ term, which on the other hand arises only from a double pole in both s and t in the Mellin transform. More details about this statement are given in appendix B.2.

In the next sections we make our discussion concrete by considering $\mathcal{T}_{3333}$ and $\mathcal{T}_{4444}$. As a bonus of our definition of single-particle operators, we will also show that $\mathcal{T}_{\vec{p}}$ for next-to-next-to extremal correlators coincides with the function of Rastelli and Zhou.

5.1 $\langle \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \rangle$

The result for connected free theory was given in (2.6). We rewrite it here below for convenience,

$$
\langle \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \rangle_{\text{conn. free}} = \frac{9(N^2-4)^2(N^2-1)}{N^2}
$$

$$
\times \left[9\left(u\hat{\sigma} + \frac{u\hat{\tau}}{v} + u^2\hat{\sigma}^2 + \frac{u^2\hat{\tau}^2}{v^2} + \frac{u^3\hat{\sigma}^2\hat{\tau}}{v^2} + \frac{u^3\hat{\sigma}\hat{\tau}^2}{v^2} \right) + \frac{18(N^2-12)}{(N^2-4)}\frac{u^2\hat{\sigma}\hat{\tau}}{v} \right]. \tag{5.5}
$$

Crossing invariance of $\langle \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_3 \rangle$ restricts the total number of connected coefficients $\{A_2^k, A_4^k, A_6^k\}$, in the generic sum over propagator structures (2.8), to only two independent ones. We have indeed

$$
A_2^0 = A_2^1 = A_4^0 = A_4^2 = A_6^1 = A_6^2 = \frac{9}{(N^2-1)}A_0^0 \tag{5.6}
$$

$$
A_4^1 = \frac{18(N^2-12)}{(N^2-4)(N^2-1)}A_0^0 \tag{5.7}
$$

where $A_0^0 = \frac{9(N^2-4)^2(N^2-1)^2}{N^2}$.

The generalised tree level function in Mellin space is

$$
\mathcal{T}_{3333} = u^3 \oint u^s v^t \, \Gamma[-s]^2\Gamma[-t]^2\Gamma\big[s+t+5\big]^2 \mathcal{M}[\mathcal{T}_{3333}], \tag{5.8}
$$

with

$$
\mathcal{M}[\mathcal{T}_{3333}] = \frac{1}{(s+2)(t+1)(s+t+4)}\left[-A_2^0 + \frac{1}{4}(s+2)(A_4^1 - 2A_2^0) \right] +
$$

$$
\frac{\hat{\tau}}{(s+1)(t+2)(s+t+4)}\left[-A_2^0 + \frac{1}{4}(t+2)(A_4^1 - 2A_2^0) \right] + \tag{5.9}
$$

$$
\frac{\hat{\sigma}}{(s+1)(t+1)(s+t+3)}\left[-A_2^0 + \frac{1}{4}(s+t+3)(A_4^1 - 2A_2^0) \right].
$$

Notice that $\mathcal{M}[\mathcal{T}_{3333}] = m_{3333}^1(s,t) + \hat{\sigma}m_{3333}^{\hat{\sigma}}(s,t) + \hat{\tau}m_{3333}^{\hat{\tau}}(s,t)$. Exploiting full crossing of $\mathcal{T}_{3333}$ it also useful to write

$$
\mathcal{T}_{3333} = \frac{1}{u^2}\left(\mathcal{F}(u,v) + \hat{\sigma}u^5\mathcal{F}(1/v, u/v) + \frac{\hat{\tau}u^5}{v^5}\mathcal{F}(v,u) \right), \tag{5.10}
$$

with $\mathcal{F}$ such that $\mathcal{F}(u,v) = \mathcal{F}(u/v, 1/v)$. The amplitude $m_{3333}^1(s,t)$ is indeed the Mellin transform of $\mathcal{F}$. Then, crossing invariance of $\mathcal{F}$ under $u \to u/v$ and $v \to 1/v$ corresponds to the identity

$$
m_{3333}^1(s,t) = m_{3333}^1(s, -t-s-5). \tag{5.11}
$$

The other two identities which follow from (5.10) are

$$
m_{3333}^{\hat{\sigma}}(s,t) = m_{3333}^1(-s-t-5, s), \qquad m_{3333}^{\hat{\tau}}(s,t) = m_{3333}^1(t,s). \tag{5.12}
$$

In our notation, the amplitude of Rastelli and Zhou would correspond only to the term multiplied by A_2^0 in (5.9). Indeed, the new contribution, proportional to $A_4^1 - 2A_2^0$, vanishes when we plug in (5.6) and (5.7), and we expand at order $1/N^2$. In the next example, i.e. the generalised tree-level correlator $\mathcal{T}_{4444}$, we will see similar features showing up, and we will comment more in general about what is the pattern of $\mathcal{M}[\mathcal{T}]$.

5.2 $\langle \mathcal{O}_4 \mathcal{O}_4 \mathcal{O}_4 \mathcal{O}_4 \rangle$

There are in total $3 + 12$ propagator structures in free theory. The first three are disconnected and not relevant here. We quote the result for connected free theory,

$$
\langle \mathcal{O}_4 \mathcal{O}_4 \mathcal{O}_4 \mathcal{O}_4 \rangle_{\text{conn. free}} = \frac{16(N^2-9)^2(N^2-4)^2(N^2-1)^2}{(N^2+1)^2}
$$

$$
\times \left[\frac{16}{N^2-1} \left(u\hat{\sigma} + \frac{u\hat{\tau}}{v} + u^3\hat{\sigma}^3 + \frac{u^3\hat{\tau}^3}{v^3} + \frac{u^4\hat{\sigma}^3\hat{\tau}}{v} + \frac{u^4\hat{\sigma}\hat{\tau}^3}{v^3} \right) + \right.
$$

$$
8 \left(\frac{27}{N^2(N^2-9)} + \frac{9}{N^2+1} - \frac{7N^2+4}{(N^2-4)(N^2-1)} \right) \left(u^2\hat{\sigma}^2 + \frac{u^2\hat{\tau}^2}{v^2} + \frac{u^4\hat{\sigma}^2\hat{\tau}^2}{v^2} \right) +
$$

$$
\left. 16 \left(\frac{54}{N^2(N^2-9)} + \frac{18}{N^2+1} - \frac{16N^2+25}{(N^2-4)(N^2-1)} \right) \left(\frac{u^2\hat{\sigma}\hat{\tau}}{v} + \frac{u^3\hat{\sigma}^2\hat{\tau}}{v} + \frac{u^3\hat{\sigma}\hat{\tau}^2}{v^2} \right) \right]. \quad (5.13)
$$

Written as a sum over propagator structures as in (2.8), connected free theory is constrained by crossing symmetry to three independent classes,

$$
A_2^0 = A_2^1 = A_6^0 = A_6^1 = A_8^1 = A_8^3 = \frac{16}{N^2-1}, \quad (5.14)
$$

$$
A_4^0 = A_4^2 = A_8^2 = 8 \left(\frac{27}{N^2(N^2-9)} + \frac{9}{N^2+1} - \frac{7N^2+4}{(N^2-4)(N^2-1)} \right), \quad (5.15)
$$

$$
A_6^1 = A_6^2 = A_4^1 = 16 \left(\frac{54}{N^2(N^2-9)} + \frac{18}{N^2+1} - \frac{16N^2+25}{(N^2-4)(N^2-1)} \right). \quad (5.16)
$$

We will consider $\{A_2^0, A_4^0, A_6^1\}$ as independent.

The generalised tree level function $\mathcal{T}_{4444}$ can be written conveniently in terms of just two independent functions $\widetilde{\mathcal{F}}$ and $\mathcal{F}$, in the following way,

$$
\mathcal{T}_{4444} = \frac{1}{u^2} \left[\mathcal{F}(u,v) + \hat{\sigma}^2 u^6 \mathcal{F}(1/v, u/v) + \frac{\hat{\tau}^2 u^6}{v^6} \mathcal{F}(v,u) \right] \quad (5.17)
$$

$$
+ \frac{1}{u^2} \left[\hat{\sigma}\hat{\tau} \widetilde{\mathcal{F}}(u,v) + \frac{\hat{\sigma} u^6}{v^6} \widetilde{\mathcal{F}}(v,u) + \hat{\tau} u^6 \widetilde{\mathcal{F}}(1/v, u/v) \right], \quad (5.18)
$$

where both $\widetilde{\mathcal{F}}$ and $\mathcal{F}$ are invariant under $u \to u/v$ and $v \to 1/v$. Given the Mellin trasform,

$$
\mathcal{T}_{4444} = u^4 \oint u^s v^t \, \Gamma[-s]^2 \Gamma[-t]^2 \Gamma[s+t+6]^2 \mathcal{M}[\mathcal{T}_{4444}], \quad (5.19)
$$

with

$$
\mathcal{M}[\mathcal{T}_{4444}] = m_{4444}^1 + \hat{\sigma}^2 m_{4444}^{\hat{\sigma}^2} + \hat{\tau}^2 m_{4444}^{\hat{\tau}^2} + \hat{\sigma}\hat{\tau} m_{4444}^{\hat{\sigma}\hat{\tau}} + \hat{\sigma} m_{4444}^{\hat{\sigma}} + \hat{\tau} m_{4444}^{\hat{\tau}} \quad (5.20)
$$

we will specify $m_{4444}^1(s,t)$ and $m_{4444}^{\hat{\sigma}\hat{\tau}}(s,t)$, which are the Mellin transforms of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$, respectively, and reconstruct $\mathcal{M}[\mathcal{T}_{4444}]$ by using symmetries, similarly to (5.17) and (5.18).

The Mellin transforms of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are

$$
m_{4444}^1 = -\frac{A_2^0}{(s+3)(t+1)(s+t+5)} - \frac{\mathscr{L}_{4444}^1}{2(s+2)(t+1)(s+t+5)} - \frac{\mathscr{L}_{4444}^1 + \mathscr{L}_{4444}^2(s+1)}{6(s+1)(t+1)(s+t+5)},
$$

$$
m_{4444}^{\hat{\sigma}\hat{\tau}} = -\frac{A_2^0}{(s+1)(t+2)(s+t+4)} + \frac{\mathscr{L}_{4444}^1}{2(s+1)(t+1)(s+t+4)} + \frac{\mathscr{L}_{4444}^1}{2(s+1)(t+2)(s+t+5)} \quad (5.21)
$$

$$
- \frac{\mathscr{L}_{4444}^1(s+3)}{3(s+1)(t+1)(s+t+5)} - \frac{\mathscr{L}_{4444}^1 - \mathscr{L}_{4444}^2 + (2\mathscr{L}_{4444}^1 - \mathscr{L}_{4444}^2)(s+1)}{3(s+1)(t+2)(s+t+4)},
$$

where we defined

$$\mathscr{L}_{4444}^1 = A_2^0 - A_4^0 \qquad \mathscr{L}_{4444}^2 = \frac{3}{2} A_2^0 - A_4^0 - \frac{1}{4} A_6^1 \tag{5.22}$$

Finally,

$$
\begin{aligned}
m_{4444}^{\hat{\sigma}}(s,t) &= m_{4444}^{\hat{\sigma}\hat{\tau}}(t,s) & m_{4444}^{\hat{\tau}}(s,t) &= m_{4444}^{\hat{\sigma}\hat{\tau}}(-s-t-6,s) \\
m_{4444}^{\hat{\tau}^2}(s,t) &= m_{4444}^1(t,s) & m_{4444}^{\hat{\sigma}^2}(s,t) &= m_{4444}^1(-s-t-6,s)
\end{aligned}
\tag{5.23}
$$

The terms in m_{4444}^1 and $m_{4444}^{\hat{\sigma}\hat{\tau}}$, proportional to A_2^0, give the amplitude of Rastelli and Zhou [1]. The combinations $\mathscr{L}_{4444}^{i=1,2}$ in (5.22) vanish at order $1/N^2$.

Let us highlight some new features of $\mathcal{M}[\mathcal{T}_{4444}]$ beyond tree level. Recall that the result of Rastelli and Zhou [1] can be obtained by considering an ansatz in Mellin space such that each monomial $\hat{\sigma}^i \hat{\tau}^j$ is accompanied by a *single* pole in the plane (s,t). In comparison, the upgraded tree level amplitude has more structure than this. In particular, poles like $(s+2)(t+1)$ and $(s+1)(t+1)$, corresponding to powers of u^2 and u^3 in the small u expasion, and therefore corresponding to allowed twists below window, are also turned on. We see now that their residue is proportional to the linear constraints $\mathscr{L}_{4444}^{i=1,2}$, which indeed vanish at order $1/N^2$. We also notice that by writing each pole in the form $\frac{1}{(s+n_1)(t+n_2)(s+t+n_3)}$ with integers $n_{i=1,2,3}$, the numerator is at most linear in s and t. Therefore, we infer that the limit $s \to \beta s$ and $t \to \beta t$ with large β scales like $O(\beta^{-2})$, i.e. one more power than the $O(\beta^{-3})$ of the tree level function of Rastelli and Zhou.

The case of $\mathcal{M}[\mathcal{T}_{4444}]$ exemplifies well what is the general pattern of $\mathcal{M}[\mathcal{T}_{\vec{p}}]$ in Mellin space. In fact we expect $\mathcal{M}[\mathcal{T}_{\vec{p}}]$ to be a rational function in which all allowed poles in the plane (s,t) are turned on, eventually decorated by a non trivial numerator, which is nevertheless constrained by the large s and t behavior. Similarly to our position space algorithm, the free coefficients in this ansatz will be fixed by demanding that the SCPW expansion below window *completely cancels* free theory contributions as described in (3.58) and (3.59).

5.3 Next-to-next-to-extremal correlators

A next-to-next-to-extremal correlator is defined by $\kappa_{\vec{p}} = 2$, i.e. a vector of external charges such that $p_3 = 2$ or $p_1 + p_2 + p_3 - p_4 = 4$. There are only six propagator structures available, and indeed these correlators only contribute to a single $su(4)$ channel, namely $[0, p_{43}, 0]$. The definition of single particle states has two non trivial consequences. Firstly, it was proven in [8] that the number of connected propagator structures actually reduces to three. Secondly, connected free theory is given by the exact formula,

$$
\mathcal{P} F_{\vec{p}}(N^2) {}_{p_1 p_2 p_3 p_4} \left[\left(1 + \frac{p_{43} + p_{21}}{2} \right) \frac{u\tau}{v} + \left(1 + \frac{p_{13} + p_{42}}{2} \right) u\sigma + \left(1 + |\frac{p_{23} + p_{14}}{2}| \right) \frac{u^2 \sigma \tau}{v} \right]
\tag{5.24}
$$

where $F_{\vec{p}}$ asymptotes $N^{(p_1 + p_2 + p_3 + p_4 - 4)/2}$ in the large N limit. For example,

$$
F_{2244} = F_{3324} = \frac{\prod_{k=1}^3 (N^2 - k^2)}{(N^2 + 1)}, \qquad F_{3335} = F_{3524} = \frac{\prod_{k=1}^4 (N^2 - k^2)}{N(N^2 + 5)},
$$

$$
F_{4424} = \frac{\prod_{k=1}^3 (N^2 - k^2)(N^4 - 20N^2 + 9)}{N(N^2 + 1)^2}.
$$

Thus, for next-to-next-to-extremal correlators, the non-planar result (5.24) is the factorized product of the $1/N^2$ connected free theory, uplifted to all N by the factor $F_{\vec{p}}(N^2)$. It follows that the all N relative coefficients among the three propapator structures, was already captured by the $1/N^2$ result. Notice also that $F(N^2)$ manifestly vanishes when the number of colors N is less than the charge of any of the external operators. Both these statements would be false if we replace our single particle operator $\mathcal{O}_p$ with the corresponding single trace half-BPS operator, thus dropping the admixture of multi-trace operators.

The particular structure of connected free theory in (5.24) implies the following exact relation on the SCPW coefficients,

$$L^f_{\vec{p},2+p_{43},[0p_{43}0]} = F_{\vec{p}}(N^2) \left[A_{[l+2]}\Big|_{\frac{1}{N^2}} \right], \qquad L^f_{\vec{p},\tau \geq 4+p_{43},[0p_{43}0]} = F_{\vec{p}}(N^2) L^{(1)f}_{\vec{p},\tau,[0p_{43}0]}. \quad (5.25)$$

Therefore, for the purpose of constructing generalised tree-level functions, the defining condition (b) becomes

$$L^{\mathcal{T}}_{\vec{p},2+p_{43},[0p_{43}0]} + F_{\vec{p}}(N^2) \left[A_{[l+2]}\Big|_{\frac{1}{N^2}} \right] = 0 \quad (5.26)$$

and by uniqueness we conclude that for next-to-next-to-extremal correlators the generalised tree level function $\mathcal{T}_{\vec{p}}$ equals the function of Rastelli and Zhou, properly normalized as in [8], multiplied by the factor $F_{\vec{p}}(N^2)$.

Building on the property that $\mathcal{T}_{\vec{p}}$ is uniquely defined by the conditions (a), (b), (c) it is simple to see that next-to-next-to-extremal correlators sometimes have an additional feature. In fact, there are sets of external charges $q_{i=1,\ldots 4}$ and $q'_{i=1,\ldots 4}$ such that the corresponding $1/N^2$ free theories will be proportional. For example, the following two families

$$
\begin{aligned}
q_1 &= p + 2 & q'_1 &= q + 2 \\
q_2 &= p + 2 & q'_2 &= q + 2 \\
q_3 &= 2 & q'_3 &= q'_4 - 2q \\
q_4 &= 2(q+1) & q'_4 &= q + (p+2)
\end{aligned}
\quad (5.27)
$$

Notice that both have the same value of the exponent $\mathbf{d} = p_1 + p_2 + p_3 + p_4 - 1$, and both have the same threshold twist. This happens because the next-to-next-to extremality condition, which we can rewrite as $p_3 + \min(0, \frac{p_1+p_2-p_3-p_4}{2}) = 2$, is achieved by the two different conditions on the minimum. Therefore,

$$q'_3 + \max\left(0, \frac{q'_1 + q'_2 - q'_3 - q'_4}{2}\right) = q_3 + \max\left(0, \frac{q_1 + q_2 - q_3 - q_4}{2}\right). \quad (5.28)$$

In the case $p = 2$ and $q = 1$, we obtain the correlators 4424 and 3335. Indeed, it is simple to verify that the corresponding tree level function from [1] are proportional to each other. This 'degeneracy' is lifted at one-loop. As we explicitly showed in section 4.6, the minimal one-loop functions $\mathcal{H}^{(2)}_{3335}$ and $\mathcal{H}^{(2)}_{4424}$ are genuinely distinguished by OPE predictions in the window.

The constructions of $\mathcal{T}$ in this section, and the one of the minimal one-loop function in the previous section, conclude our journey through the determination of the dynamical one-loop function $\mathcal{D}^{(2)} = \mathcal{T}^{(2)} + \mathcal{H}^{(2)}$. The subject of the next section is instead inspired by the existence of the hidden tree-level symmetry found by [9].

6 Explorations of the 10d symmetry at one-loop

A tree level correlator $\mathcal{D}_{pqrs}^{(1)}$ is obtained by acting with a differential operator $\widehat{\mathcal{D}}_{pqrs}$ on the stress-tensor four point correlator $\mathcal{D}_{2222}^{(1)}$. The existence of these operators is a consequence of the hidden tree level 10d conformal symmetry, unveiled by Caron-Huot and Trinh in [9]. The structure of the anomalous dimensions (3.31), both numerator and denominator can also be understood in terms of this hidden symmetry. In particular, the numerator is the eigenvalue of an 8-th order operator which annihilates protected multiplets, $\Delta^{(8)}$, and the structure of the denominator is in correspondence with that of the partial-wave decomposition of the $2 \to 2$ flat space S-matrix of the type IIB axio-dilaton. Then, the hidden symmetry explains the residual degeneracies of the anomalous dimensions found in [8], and the proportionality of some the next-to-next-to-extremal tree level correlators, as we observed in section 5.3.

An interesting question to ask is whether an organising 10d principle persists at one-loop. We begin investigating this problem by showing that we can recast the expression of our minimal one-loop functions by using the operators $\Delta^{(8)}$ and $\widehat{\mathcal{D}}_{pqrs}$. We shall see that even though it is possible to achieve such a result, the way it happens departs slightly from the way we understood the physical properties of $\mathcal{H}_{\vec{p}}^{(2)}$ in section 4.1 and 4.3.

6.1 Leading logs from $\Delta^{(8)}$ and $\widehat{\mathcal{D}}_{pqrs}$

We introduce the operators $\Delta^{(8)}$ and $\widehat{\mathcal{D}}_{pqrs}$ by recalling few important facts.

It was noticed in [8] that the computation of the one-loop leading $\log^2 u$, i.e (4.9), could be reorganised and simplified drastically by introducing an 8-th order differential Casimir operator. It is indeed possible to rewrite the $\log^2 u$ discontinuity as

$$\mathcal{D}_{\vec{p}}^{(2)}\Big|_{\log^2 u} = \sum_{a,b} \frac{\Upsilon_{[aba]}}{u^2} \left[u^{-\frac{p_{43}}{2}} \, \Delta_{[aba]}^{(8)} \, u^{+\frac{p_{43}}{2}} \right] \mathcal{F}_{\vec{p};[aba]}^{(2)} \tag{6.1}$$

$$\mathcal{F}_{\vec{p};[aba]}^{(2)} = \sum_{\tau \geq \tau^{\max},l} \left(\mathbf{M}_{\vec{\tau}}^{(1)} \frac{\left(\mathbf{L}_{\vec{\tau}}^{(0)}\right)^{-1}}{\delta_{t(\tau)}^{(4)} \delta_{t(\tau)+l+1}^{(4)}} \mathbf{M}_{\vec{\tau}}^{(1)} \right)_{(p_1 p_2),(p_3 p_4)} \frac{\mathcal{B}^{(2+\frac{\tau}{2},l)}}{u^{\frac{p_{43}}{2}}} \tag{6.2}$$

where the Casimir $\Delta_{[aba]}^{(8)}$ is precisely such that its eigenvalue is the numerator of the anomalous dimensions in (3.31), i.e.

$$\delta_t^{(4)} = 2(t-1)(t+a)(t+a+b+1)(t+2a+b+2) \tag{6.3}$$

$$\Delta_{[aba]}^{(8)} \mathcal{B}^{2+\tau,l} = +\delta_t^{(4)} \delta_{t+l+1}^{(4)} \mathcal{B}^{2+\tau,l} \tag{6.4}$$

It turned out that the resummation of $\mathcal{F}_{\vec{p};[aba]}^{(2)}$ was remarkably simpler. (Notice that $\mathcal{F}_{\vec{p};[aba]}^{(2)}$ here has a series expansion in integer powers of u.)

A further improvement of (6.1) was achieved in [9].

Firstly, it is possible to repackage the action of the $su(4)$ dependent operators $\Delta^{(8)}_{[aba]}$, into a single compact operator

$$\Delta^{(8)} = \frac{x\bar{x}y\bar{y}}{(x-\bar{x})(y-\bar{y})} \prod_{i,j=1}^{2} \left(\mathbf{C}^{[+\alpha,+\beta,0]}_{x_i} - \mathbf{C}^{[-\alpha,-\beta,0]}_{y_j} \right) \frac{(x-\bar{x})(y-\bar{y})}{x\bar{x}y\bar{y}} \tag{6.5}$$

where $\alpha = p_{21}/2$, $\beta = p_{34}/2$ and $\mathbf{C}^{[\alpha,\beta,\gamma]}_x$ is the elementary $2d$ casimir

$$\mathbf{C}^{[\alpha,\beta,\gamma]}_x = x^2(1-x)\partial^2_x + x(\gamma - (1+\alpha+\beta)x)\partial_x - \alpha\beta x. \tag{6.6}$$

Secondly, this $\Delta^{(8)}$ has the property that

$$(u\sigma)^{-\frac{p_{43}}{2}} \Delta^{(8)} (u\sigma)^{\frac{p_{43}}{2}} \left[\mathcal{F}^{(2)}_{\vec{p};[aba]} \Upsilon_{a,b} \right] = \Upsilon_{[aba]} \left[u^{-\frac{p_{43}}{2}} \Delta^{(8)}_{[aba]} u^{+\frac{p_{43}}{2}} \right] \mathcal{F}^{(2)}_{\vec{p};[aba]}, \tag{6.7}$$

where $\mathcal{F}^{(2)}_{\vec{p};[aba]}$ does not depend on the $su(4)$ cross ratios. We can then commute $\Upsilon_{[aba]}$ and obtain the $\log^2 u$ discontinuity from the action of $\Delta^{(8)}$ on a prepotential $\mathcal{F}^{(2)}_{\vec{p}}$, namely

$$u^2 \mathcal{D}^{(2)}_{\vec{p}} \Big|_{\log^2 u} = (u\sigma)^{-\frac{p_{43}}{2}} \Delta^{(8)} (u\sigma)^{\frac{p_{43}}{2}} \left[\mathcal{F}^{(2)}_{\vec{p}} \right], \qquad \mathcal{F}^{(2)}_{\vec{p}} \equiv \sum_{a,b} \mathcal{F}^{(2)}_{\vec{p};[aba]} \Upsilon_{[aba]} \tag{6.8}$$

Notice that the conjugated operator $(u\sigma)^{-\frac{p_{43}}{2}} \Delta^{(8)} (u\sigma)^{+\frac{p_{43}}{2}}$ is invariant under the symmetry, $u \to u/v$ and $v \to 1/v$, which in our conventions holds when $p_{21} = 0$. We remark here that the expression of $\Delta^{(8)}$ depends on the choice of external charges!

The remaining dependence on the external charges $\vec{p}$, can be absorbed into the action of the operators $\widehat{\mathcal{D}}_{\vec{p}}$. These operators are defined at tree level by the relations

$$\mathcal{D}^{(1)}_{\vec{p}} = \widehat{\mathcal{D}}_{\vec{p}} \left(u^2 \mathcal{D}^{(1)}_{2222} \right) \qquad \forall \vec{p} = (p_1 p_2 p_3 p_4) \tag{6.9}$$

and are used at one-loop to compute

$$\mathcal{F}^{(2)}_{\vec{p}} = \widehat{\mathcal{D}}_{\vec{p}} \mathcal{F}^{(2)}_{2222}, \tag{6.10}$$

once $\mathcal{F}^{(2)}_{2222}$ is known [9]. The latter can be written in the form

$$\mathcal{F}^{(2)}_{2222} = 16 \left(\frac{x\bar{x}}{\bar{x}-x} \right)^7 \left[\mathscr{F}_2(x,\bar{x}) - \mathscr{F}_2(\bar{x},x) \right], \tag{6.11}$$

$$\mathscr{F}_2 = \left[1 + \frac{x^2}{2} \left[\sum_{i=0}^{2} \left[\frac{\bar{x}-x}{x\bar{x}} \right]^{i+1} \frac{x^i \partial_1 x^i}{i!(5)_i} \right] \right] \left(\frac{(x-2)(1-x)}{x^3} - \frac{(1-x)^5}{8x^5} \mathrm{Li}_1^2(x) + \right.$$

$$\left. + \frac{(1-(1-x)^5)}{4x^5} \mathrm{Li}_2(x) + \frac{(1-x)(7-7x+2x^2)}{8x^4} \mathrm{Li}_1(x) \right).$$

In writing (6.11) we have highlighted the max power of $(x-\bar{x})$ in the denominator. This has to be the same power of the tree-level function $\mathcal{D}^{(1)}_{2222}$ by construction.

In this paper we considered

$$
\widehat{\mathcal{D}}_{2233} = \frac{3}{4 \times 2}(4 - u\partial_u)
$$

$$
\widehat{\mathcal{D}}_{2323} = \frac{3}{4 \times 2}(u\partial_u + v\partial_v)
$$

$$
\widehat{\mathcal{D}}_{2244} = \frac{1}{4}(5 - u\partial_u)(4 - u\partial_u)
$$

$$
\widehat{\mathcal{D}}_{2424} = \frac{1}{4}(1 + u\partial_u + v\partial_v)(u\partial_u + v\partial_v)
$$

$$
\widehat{\mathcal{D}}_{3335} = \frac{3\sqrt{15}}{4 \times 4}(4 - u\partial_u)v\partial_v(-u\partial_u - v\partial_v)
$$

$$
\widehat{\mathcal{D}}_{4424} = \frac{2\sqrt{2}}{4}(4 - u\partial_u)v\partial_v(-u\partial_u - v\partial_v)
$$

$$
\widehat{\mathcal{D}}_{3333} = \frac{9}{4 \times 4}\left[(4 - u\partial_u)^2 + \frac{u\tau}{v}(v\partial_v)^2 + u\sigma(u\partial_u + v\partial_v)^2\right]
$$

$$
\widehat{\mathcal{D}}_{4444} = \frac{1}{4}\left[(5 - u\partial_u)^2 + 4\left[\frac{u\tau}{v}(v\partial_v)^2 + u\sigma(u\partial_u + v\partial_v)^2\right]\right](4 - u\partial_u)^2
$$

$$
+ \frac{1}{4}\left[\frac{(u\tau)^2}{v^2}(1 - v\partial_v)^2(v\partial_v)^2 + (u\sigma)^2(u\partial_u + v\partial_v)^2(1 + u\partial_u + v\partial_v)^2\right]
$$

$$
+ \frac{u\tau}{v}(u\sigma)(v\partial_v)^2(v\partial_v + u\partial_u)^2 \tag{6.12}
$$

For these correlators we have verified explicitly that the prescription (6.10) agrees with the more standard two variable CPW resummation obtained through (4.9). This amazing computation shows a very non trivial outcome of the ten-dimensional conformal symmetry.

More generally, the operators $\widehat{\mathcal{D}}_{\vec{p}}$ have the unique form

$$
\widehat{\mathcal{D}}_{\vec{p}} = \sum_{\gamma=p_{43}}^{\min(p_1+p_2,p_3+p_4)-4}\left[(u\hat{\sigma})^{\frac{\gamma-p_{43}}{2}}\sum_{k=0}^{\frac{\gamma-p_{43}}{2}}\left(\frac{g_{14}g_{23}}{g_{13}g_{24}}\right)^k \mathrm{d}_{\vec{p}}^{\gamma k}\right] \tag{6.13}
$$

where $\mathrm{d}_{\vec{p}}^{\gamma k}$ is a polynomial up to degree $\frac{1}{2}\sum_i(p_i - 2)$ in the letters $u\partial_u$ and $v\partial_v$. The top degree is fixed by the difference of denominator power, compared to $\mathcal{D}_{2222}^{(1)}$. Each monomial $\hat{\sigma}^i\hat{\tau}^j$ in the amplitude of Rastelli and Zhou corresponds to a propagator structure in (6.13). Finally, the sum over γ contains at most propagator structures with $(\hat{\tau}/v)$ to the power $\kappa_{\vec{p}} - 2$. As we show in appendix B.2.2, the differential operator $\widehat{\mathcal{D}}_{\vec{p}}$ can be found just by considering the rational function multiplying the box function $\phi^{(1)}(x,\bar{x})$ in the position space representation of the tree level correlator $\mathcal{D}_{\vec{p}}^1$.

The upgraded tree level functions we constructed in section 5 can be written by generalising the operators $\widehat{\mathcal{D}}_{\vec{p}}$ given in (6.13). In particular, the polynomials $\mathrm{d}_{\vec{p}}^{\gamma k}$ become function of free theory data $\{A_\gamma^k\}$ and have degree bounded by $1 + \frac{1}{2}\sum_i(p_i - 2)$. We quote

the example of $\mathcal{T}_{3333}$ for illustration,

$$\mathcal{T}_{3333} = \frac{1}{16}(4 - u\partial_u)^2 \left[A_2^0 - \frac{\mathscr{L}_{3333}}{4}(3 - u\partial_u) \right] u^2 \mathcal{D}_{2222}^{(1)}$$
$$+ \frac{1}{16}\frac{u\tau}{v}(v\partial_v)^2 \left[A_2^0 + \frac{\mathscr{L}_{3333}}{4}(1 + v\partial_v) \right] u^2 \mathcal{D}_{2222}^{(1)} \qquad (6.14)$$
$$+ \frac{1}{16}u\sigma(u\partial_u + v\partial_v)^2 \left[A_2^0 - \frac{\mathscr{L}_{3333}}{4}(1 - u\partial_u - v\partial_v) \right] u^2 \mathcal{D}_{2222}^{(1)}.$$

6.2 A pre-amplitude study

In this section we will explore the question: "Can we pull out of our minimal one-loop functions the operators $\widehat{\mathcal{D}}_{pqrs}$ and $\Delta^{(8)}$?"

Consider first the one-loop correlators $\langle 22pp \rangle$ for $p = 2, 3$ and 4. (The cases $p = 2, 3$ have been determined in [3] and [7], respectively.) We have managed to rewrite these in terms of certain pre-amplitudes $\mathcal{L}_{22pp}^{(2)}$ such that the following equations are satisfied

$$\mathcal{H}_{2222}^{(2)} = \frac{1}{u^2}\Delta^{(8)}\mathcal{L}_{2222}^{(2)} + 4u^2\overline{D}_{2422} \qquad (6.15)$$

$$\mathcal{H}_{2233}^{(2)} = \frac{1}{u^2}\Delta^{(8)}\mathcal{L}_{2233}^{(2)} - \frac{2u^2}{v^2} + 4u^2(\overline{D}_{1423} + \overline{D}_{1432}) + 24u^2\overline{D}_{2422} - 30u^3\overline{D}_{3522} \qquad (6.16)$$

$$\mathcal{H}_{2244}^{(2)} = \frac{1}{u^2}\Delta^{(8)}\mathcal{L}_{2244}^{(2)} - \frac{8u^2}{v^2} + 24u^3(\overline{D}_{2523} + \overline{D}_{2532}) + 40u^2\overline{D}_{2422} + 48u^3(\overline{D}_{3522} - u\overline{D}_{4622}) \qquad (6.17)$$

The r.h.s. of (6.15)–(6.17) consists of functions $\mathcal{L}_{22pp}^{(2)}$ such that $\Delta^{(8)}\mathcal{L}_{22pp}^{(2)}$ reproduces the minimal one-loop function on the l.h.s. up to some tree level remainders. There is a single remainder for $p = 2$, which happens to have the same structure of $\mathcal{D}_{2222}^{(1)}$. For $p = 3, 4$, there is more structure in the remainders than the corresponding tree level correlators. The origin of these remainders goes together with $\Delta^{(8)}$, as we now explain.

The ansatz for the functions $\mathcal{L}_{22pp}^{(2)}$ has the same form the one for $\mathcal{H}_{\vec{p}}$ in (4.15), with the substitution $\mathbf{d}_{\vec{p}} \to \mathbf{d}_{\vec{p}} - 8$, i.e.

$$\mathcal{L}_{22pp}^{(2)} = \frac{W_{4-} + W_{3-}}{(x - \bar{x})^{\mathbf{d}_{\vec{p}}}} + \frac{W_{3+} + W_{2+}}{(x - \bar{x})^{\mathbf{d}_{\vec{p}}-1}} + \left[\frac{W_{2-}}{(x - \bar{x})^{\mathbf{d}_{\vec{p}}}} + \frac{W_{1v} + W_{1u}}{(x - \bar{x})^{\mathbf{d}_{\vec{p}}-1}} + \frac{W_0}{(x - \bar{x})^{\mathbf{d}_{\vec{p}}-3}} \right] \qquad (6.18)$$

The basis of transcendental functions is unchanged

$$W_{4-} = h_1\,\phi^{(2)}(x_1', x_2') + h_2\,\phi^{(2)}(x, \bar{x}) + h_3\,\phi^{(2)}(1 - x, 1 - \bar{x})$$
$$W_{3-} = h_4\,x\partial_x\phi^{(2)}(x, \bar{x}) + h_5\,(x - 1)\partial_x\phi^{(2)}(1 - x, 1 - \bar{x}) - (x \leftrightarrow \bar{x})$$
$$W_{3+} = (x - \bar{x})\Big(h_6\,\partial_v\phi^{(2)}(x, \bar{x}) + h_7\,\partial_u\phi^{(2)}(1 - x, 1 - \bar{x}) \Big) + h_8\,\zeta_3$$
$$W_{2+} = h_9\log(u)\log(v) + h_{10}\log^2 v + h_{11}\log^2 u \qquad (6.19)$$

and

$$W_{2-} = h_\square\,\phi^{(1)}(x, \bar{x}) \qquad\qquad W_0 = h_0$$
$$W_{1u} = h_u\log u \qquad\qquad\qquad W_{1v} = h_v\log v \qquad (6.20)$$

Each coefficient function $h_{i=1,\dots,11,\square,u,v,0}$ will be polynomial in the variables $x, \bar{x}$, since we are studying next-to-next-to-extremal correlators.

By construction, we make manifest the $\log^2 u$ discontinuity

$$\mathcal{L}^{(2)}_{22pp}\Big|_{\log^2 u} = \widehat{\mathcal{D}}_{22pp} \mathcal{F}^{(2)}_{2222} \tag{6.21}$$

and impose $x_i \to x_i/(x_i - 1)$ crossing symmetry, because this is a symmetry of the $\langle 22pp \rangle$ correlators, and it is a symmetry of $\Delta^{(8)}$. Then, we impose absence of $x = \bar{x}$ poles.

In our algorithm for the minimal one-loop function $\mathcal{H}_{22pp}$ we would cross the ansatz to the orientation $2p2p$ and match the corresponding $\log^2 u$ discontinuity. But $\Delta^{(8)}$ depends on the orientation of the external charges, and we cannot proceed this way. Instead, we stay within the orientation $\vec{p} = 22pp$, and apply $\Delta^{(8)}$ on $\mathcal{L}^{(2)}_{22pp}$. The resulting ansatz can now be understood as the starting point for the construction of our one-loop function, In particular, after crossing to $\vec{p} = 2p2p$, and matching the corresponding leading log, the weight four, three and two-symmetric coefficient functions are fixed.

Moving to the tree-like part, we encounter a major difference compared to our algorithm of section 4.3: the action of $\Delta^{(8)}$ brings $\mathcal{L}^{(2)}_{22pp}$ outside the minimality of $\mathcal{H}^{(2)}_{22pp}$! We thus expect the presence of extra tree-level contributions with additional singular terms in v.[15] Indeed, looking at the relations (6.15)–(6.17), both $\Delta^{(8)}\mathcal{L}^{(2)}_{22pp}$ and the remainders have additional singular terms compared to our minimal one-loop function (4.15), but these cancel in the sum, in such a way that the r.h.s. is indeed our minimal one-loop function.

It is important to realise that the free coefficients we can play with, in order to obtain the remainders on the r.h.s. of (6.15)–(6.17), are the free coefficients in $\Delta^{(8)}\mathcal{L}^{(2)}_{22pp}$, and the $\alpha_{i=1,\dots}$ labeling the ambiguities of $\mathcal{H}_{22pp}$. Indeed, for the purpose of this section it is useful to think of the ambiguities as a sort of 'gauge' parameters which we eventually fix to some particular value. (Of course, the most general form of the ambiguities on the r.h.s. of (6.15)–(6.17) can be added in afterwards.) Operationally, the idea is to fix free parameters in such a way that the resulting tree-like part in the difference $\mathcal{H}^{(2)}_{22pp} - u^{-2}\Delta^{(8)}\mathcal{L}^{(2)}_{22pp}$ has a lower power of $(x - \bar{x})$ in the denominator, compared to $\mathbf{d}_{\vec{p}} + 8$.[16]

Being a differential operator, $\Delta^{(8)}$ has a kernel, therefore any construction of $\mathcal{L}_{22pp}$ is only unique up to such a kernel. The kernel does not show up in the $\log^2 u$ discontinuity, because in that case the use of $\Delta^{(8)}$ is specified by the OPE, and in particular by the form of the anomalous dimensions [8].

So far, the determination of $\mathcal{L}_{22pp}$ did not follow strictly the rules of our algorithm, especially regarding the minimality of our ansatz for $\mathcal{H}_{\vec{p}}$. However, it will be very surprising how the complexity of $\mathcal{L}_{22pp}$ is reduced in comparison to $\mathcal{H}_{22pp}$.

---

[15]This is because weight four and three contributions in the preamplitude $\mathcal{L}_{\vec{p}}$, upon the action of $\Delta^{(8)}$, produce a cascade of tree level contributions with non minimal denominator. Some of these contributions in $\mathcal{L}_{\vec{p}}$ are fixed by matching the pre-amplitude $\log^2 u$ discontinuity.

[16]In order to achieve this result it is useful to impose as many $x = \bar{x}$ zeros as possible in the difference between $\mathcal{H}^{(2)}_{22pp} - u^{-2}\Delta^{(8)}\mathcal{L}^{(2)}_{22pp}$. Similarly, we impose as many $x = 0$ zeros as possible.

The coefficient functions for $\mathcal{L}_{2222}$, listed here below in the basis (6.19)–(6.20), are ($Y_\pm \equiv 1 \pm v$):

$$h_1 = +4u^4 v^2(u-1+v),$$

$$h_2 = -4u^4(u+1-v),$$

$$h_3 = -4v(u^5 - 5u^4 Y_+ - Y_-^4 Y_+ + 5u^3(2Y_+^2 - v) + uY_-^2(5Y_+^2 - 2v) - u^2 Y_+(10Y_+^2 - 19v)),$$

$$h_4 = -\frac{1}{3}u^3(u^4 + Y_-^4 - 4u^3 Y_+ + 8uY_-^2 Y_+ - 2u^2(3Y_-^2 - 2v)),$$

$$h_5 = +\frac{1}{2}h_4 + \frac{1}{6}Y_-(5u^6 - 39u^5 Y_+ + 3Y_-^4(Y_+^2 + 4v) + u^4(73Y_+^2 - 4v)$$
$$- uY_-^2 Y_+(19Y_+^2 + 32v) - 2u^3 Y_+(37Y_+^2 - 8v) + u^2(51Y_+^4 - 56vY_+^2 - 88v^2)),$$

$$h_6 = -\frac{1}{3}u^3 Y_-(u^2 + Y_-^2 + 10uY_+),$$

$$h_7 = -\frac{5}{12}h_8 + \frac{1}{6}(-4u^5 Y_-^2 + 3Y_-^4(Y_+^2 + 4v) + u^4(19 + 48v + 99v^2) - 4uY_-^2 Y_+(4Y_+^2 + 11v) +$$
$$- 4u^3(9 + 39v + 45v^2 + 7v^3) + 2u^2(17Y_+^4 + 11vY_+^2 - 64v^2)),$$

$$h_8 = -\frac{2}{5}u^2(u^4 + Y_-^4 - 2u(2+7v)(u^2 + Y_-^2) + u^2(6 + 24v - 94v^2)),$$

$$h_9 = -\frac{1}{3}(14u^5 - 2u^6 - 3Y_-^5 Y_+ - 14u^4(2 + v - 3v^2) + 2uY_-^3(8Y_+^2 - 5v) +$$
$$+ u^3 Y_-(38Y_-^2 + 135v) - 7u^2(5 - v + v^3 - 5v^4)),$$

$$h_{10} = -\frac{1}{3}v(7u^5 - 3Y_-^4(3+v) - u^4(37 + 35v) + uY_-^2(43 + 49v + 16v^2) +$$
$$+ u^3(78 + 99v + 38v^2) - u^2(82 + 60v + 75v^2 + 35v^3)),$$

$$h_{11} = -\frac{1}{3}u^4(2u^2 - 7Y_-^2 - 7uY_+), \tag{6.22}$$

for weight four, three, and two symmetric, and

$$\widetilde{h}_\square = +\frac{1}{180}(572u^7 - 2939u^6 Y_+ + 234Y_-^6 Y_+ - 3uY_-^4(379Y_+^2 - 4v) + u^5(5295Y_+^2 - 2320v)$$
$$+ 9u^2 Y_-^2 Y_+(193Y_+^2 - 64v) - 4u^4 Y_+(974Y_+^2 - 2087v) + 2u^3(67Y_+^4 - 1322vY_+^2 + 400v^2))$$

$$\widetilde{h}_u = -\frac{1}{5}u(29u^5 - 114u^4 Y_+ - 10Y_-^4 Y_+ + 2uY_-^2(31Y_+^2 - 34v)$$
$$+ u^3(193Y_+^2 - 174v) - 10u^2 Y_+(16Y_+^2 - 43v))$$

$$\widetilde{h}_v = -\frac{1}{2}\widetilde{h}_u - \frac{1}{90}Y_-(317u^5 - 1115u^4 Y_+ - 90Y_-^4 Y_+ + uY_-^2(569Y_+^2 - 656v)$$
$$+ 2u^3(879Y_+^2 - 917v) - u^2(1439 + 451v + 451v^2 + 1439v^3))$$

$$\widetilde{h}_0 = \frac{1}{45}u(47u^3 - 79u^2 Y_+ - 15Y_-^2 Y_+ + u(77 - 4v + 77v^2)) \tag{6.23}$$

for the weight two anti-symmetric, weight one and weight zero parts. The full tree-like function is given by combining $\widetilde{h}_{i=\square,u,v,0}$, and the following function

$$\mathcal{A} = \beta_{2222}\, u(2 - u\partial_u)^2 \overline{D}_{1111} + \mathcal{K}_{\Delta^{(8)}} \tag{6.24}$$

The first term proportional to β_{2222} will span the ambiguity of $\mathcal{H}_{2222}^{(2)}$. The function $\mathcal{K}_{\Delta^{(8)}}$ is the kernel of $\Delta^{(8)}$. A restricted choice of $\mathcal{K}_{\Delta^{(8)}}$ which is compatible with the symmetry $u \to u/v, v \to 1/v$, and the way we construct $\mathcal{L}_{2222}^{(2)}$, is

$$\mathcal{K}_{2222} = k_1\, u\overline{D}_{1111} + k_2\,(1+u+v)\overline{D}_{1111} + k_3\zeta_3 + k_4 \log^2 v + k_5(2\log u - \log v) + k_6 \quad (6.25)$$

We will now present the results for $\mathcal{L}_{2233}$ and $\mathcal{L}_{2244}$, and to do so we will make use of the operators $\widehat{\mathcal{D}}_{pqrs}$. The result for $\mathcal{L}_{2233}^{(2)}$ is

$$\mathcal{L}_{2233}^{(2)} - \widehat{\mathcal{D}}_{2233}\mathcal{L}_{2222}^{(2)} = \left[\frac{1}{2}R_3 + R_5\,(x-1)\partial_x\right]\phi^{(2)}(1-x,1-\bar{x}) - (x \leftrightarrow \bar{x}) +$$
$$+ (x-\bar{x})R_7\,\partial_u\phi^{(2)}(1-x,1-\bar{x}) - 32\zeta_3 + R_9\,\log(u)\log(v) + R_{10}\,\log^2 v\,, \quad (6.26)$$

where

$$\begin{aligned}
R_3 &= -\frac{2}{(x-\bar{x})^3}(u^3 + u(7Y_+^2 + 2v) - 5u^2Y_+ - 3Y_+^3) \\
R_5 &= +\frac{1}{(x-\bar{x})^3}Y_-(7u^2 - 15uY_+ + 8(Y_+^2 - v)) \\
R_7 &= +\frac{1}{(x-\bar{x})^2}(5u^2 - 13uY_+ + 8(Y_+^2 - v)) \\
R_9 &= +\frac{1}{(x-\bar{x})^2}Y_-(6Y_- - 5u) \\
R_{10} &= +\frac{1}{2(x-\bar{x})^2}(5u(5+3v) + 60v - 13u^2 - 12)
\end{aligned} \quad (6.27)$$

The $\log^2 u$ discontinuity of $\mathcal{L}_{2233}^{(2)}$ is captured by $\widehat{\mathcal{D}}_{2233}\mathcal{L}_{2222}^{(2)}$, as it should. Indeed the r.h.s. of (6.26) has no overlap with the $\log^2 u$ projection. Surprisingly, $\widehat{\mathcal{D}}_{2233}\mathcal{L}_{2222}^{(2)}$ also captures relevant parts of the full $\mathcal{L}_{2233}^{(2)}$, but for the r.h.s. of (6.26). The latter has no spurious poles by construction, and it can be shifted by the kernel of $\Delta^{(8)}$ without changing $\mathcal{H}_{2233}^{(2)}$, or (6.16). Notice that in $\mathcal{L}_{2222}^{(2)}$ we have included $\mathcal{K}_{2222}$ as given in (6.25), and $\widehat{\mathcal{D}}_{2233}\mathcal{K}_{2222}$ now produces non-kernel functions proportional to k_1 and k_2. In particular, the r.h.s. of (6.26) holds for specific values of $k_{i=1,\dots 6}$.

The result for $\mathcal{L}_{2244}^{(2)}$ has the same level of complexity, once we use the property that $\widehat{\mathcal{D}}_{2244}$ and $\widehat{\mathcal{D}}_{2233}$ concatenate, i.e $\widehat{\mathcal{D}}_{2244} = \frac{2}{3}(5 - u\partial_u)\widehat{\mathcal{D}}_{2233}$, and therefore we use the whole $\mathcal{L}_{2233}^{(2)}$ as starting point, rather than $\mathcal{L}_{2222}^{(2)}$. Then,

$$\mathcal{L}_{2244}^{(2)} - \frac{2}{3}(5 - u\partial_u)\mathcal{L}_{2233}^{(2)} = \left[\frac{1}{2}R_3 + R_5\,(x-1)\partial_x\right]\phi^{(2)}(1-x,1-\bar{x}) - (x \leftrightarrow \bar{x}) +$$
$$+ (x-\bar{x})R_7\,\partial_u\phi^{(2)}(1-x,1-\bar{x}) - 88\zeta_3 + R_9\,\log(u)\log(v) + R_{10}\,\log^2 v +$$
$$+ R_\square\,\phi^{(1)}(x,\bar{x}) + R_u \log u + R_v \log v\,, \quad (6.28)$$

where

$$R_3 = -\frac{4}{3(x-\bar{x})^3}(u^3 + 17u(Y_+^2 + 2v) - 9u^2 Y_+ - 9Y_+(Y_+^2 + 6v))$$

$$R_5 = +\frac{2}{(x-\bar{x})^3}Y_-(6u^2 - 17uY_+ + (11Y_+^2 + 16v))$$

$$R_7 = +\frac{2}{3(x-\bar{x})^2}(7u^2 - 36uY_+ + (33Y_+^2 + 48v)$$

$$R_9 = +\frac{10}{(x-\bar{x})^2}Y_-(3Y_- - 2u)$$

$$R_{10} = +\frac{5}{(x-\bar{x})^2}(u(13 + 9v) + 30v - 7u^2 - 6) \tag{6.29}$$

$$R_\square = +\frac{72}{(x-\bar{x})^3}uv$$

$$R_u = +\frac{36}{(x-\bar{x})^2}(u(Y_+ - u))$$

$$R_v = +\frac{18}{(x-\bar{x})^2}(Y_+ - u)(Y_- - u))$$

The use of $\mathcal{L}_{2233}^{(2)}$ as starting point, instead of $\mathcal{L}_{2222}^{(2)}$, has the effect of keeping $(x-\bar{x})^3$ the maximal power of the denominator in the functions R given above. Again, the r.h.s. of (6.28) holds for specific values of $k_{i=1,\ldots 6}$.

The construction of $\mathcal{L}_{22pp}^{(2)}$ depends on the initial orientation of the external charges we choose, since the latter goes along with $\Delta^{(8)}$ through the values of p_{21} and p_{34}. In order to obtain $\mathcal{L}_{2p2p}^{(2)}$ we simply repeat the previous construction with minor modifications.

Consider the case of $\mathcal{L}_{2323}^{(2)}$ for illustration. We define $\mathcal{L}_{2323}$ by the equation which extracts $\Delta^{(8)}$ out of $\mathcal{H}_{2323}$,

$$u^2 \mathcal{H}_{2323}^{(2)} = (u\sigma)^{-1/2}\Delta^{(8)}(u\sigma)^{+1/2}\mathcal{L}_{2323}^{(2)} - \frac{u^2(1+v)(3+u-v)}{2v^2}$$
$$+ 4u^4(\overline{D}_{2431} + \overline{D}_{3421}) + 24u^4\overline{D}_{2422} - 30u^4\overline{D}_{3421} \tag{6.30}$$

Then we write $\mathcal{L}_{2323}^{(2)}$ as follows

$$u\mathcal{L}_{2323}^{(2)} - u\widehat{\mathcal{D}}_{2323}\mathcal{L}_{2222}^{(2)} = \left[\frac{1}{2}R_3 + R_5(x-1)\partial_x\right]\phi^{(2)}(1-x, 1-\bar{x}) - (x \leftrightarrow \bar{x}) +$$
$$+ (x-\bar{x})R_7\, \partial_u \phi^{(2)}(1-x, 1-\bar{x}) + R_9 \log(u)\log(v) + R_{10}\log^2 v +$$
$$+ R_\square\, \phi^{(1)}(x, \bar{x})\,. \tag{6.31}$$

Notice a feature of $\mathcal{L}_{2323}^{(2)}$, which did not show up in $\mathcal{L}_{22pp}$: this is the presence of the relative

$1/u$ factor between the r.h.s. and the l.h.s. of (6.31). Indeed the l.h.s. is

$$R_3 = +\frac{1}{2(x-\bar{x})^3}(u^3 - u^2(5v+8) + u(13+28v-5v^2) + 3(3v^3+8v^2-9v-2))$$

$$R_5 = +\frac{1}{(x-\bar{x})}(1-v)R_7 + \frac{1}{(x-\bar{x})^3}uY_-(17-18v+v^2+u^2-2u(v+9))$$

$$R_7 = -\frac{1}{8(x-\bar{x})^2}(41+15v-57v^2+v^3+u^2(v+17)-2u(29+6v+v^2))$$

$$R_9 = -\frac{1}{8(x-\bar{x})^2}(2u^3 - u^2(51+5v) + 12u(5+11v) - 3Y_-^3)$$

$$R_{10} = -\frac{1}{16(x-\bar{x})^2}(Y_-(209+17v(v-2)) + 2u^3 + u^2(197-13v) - u(408+388v-28v^2))$$

$$R_\square = +\frac{1}{8(x-\bar{x})}(3u^2 - 3Y_-^2 - u(31+5v)) \tag{6.32}$$

and is not possible to use the freedom in the construction of $\mathcal{L}_{\vec{p}}$ to reabsorb the extra $1/u$.

Determining $\mathcal{L}_{pqrs}^{(2)}$ for other N²E correlators can be done in a similar way. For such correlators, we can also reverse the procedure and bootstrap directly $\mathcal{H}_{pqrs}$, by making a simpler ansatz for $\mathcal{L}_{pqrs}^{(2)}$, apply $\Delta^{(8)}$, and complete it with a tree like function, Indeed, for a large value of $p_1 + p_2 + p_3 + p_4$, i.e. a large denominator power $(x-\bar{x})^\#$, therefore a large number of initial free coefficients in the polynomial ansatz for $h_{i=1,\ldots}$ in (4.15), the use of $\Delta^{(8)}$ in combination with $\widehat{\mathcal{D}}_{pqrs}$ reduces considerably the complexity of the computation.

As far as we investigated, pulling $\Delta^{(8)}$ out of $\mathcal{H}_{pqrs}$ is possible even for multi-channel correlators. In a first instance, this problem reduces to a collection of single channel computations, because $\Delta^{(8)}$ acts diagonally on the $su(4)$ harmonics $\Upsilon_{[aba]}$. However, some extra care is needed in defining the ansatz for $\mathcal{L}_{pqrs}^{(2)}$, since the latter does not manifestly obey the rules of $\mathcal{H}_{pqrs}$. For example, already in the case of $\mathcal{L}_{2323}^{(2)}$ we have found the need of $1/u$ terms in the coefficient functions. This behavior is generic in multi-channel correlators.

Despite discrepancies, it would be fascinating to take full advantage of $\Delta^{(8)}$ and $\widehat{\mathcal{D}}_{pqrs}$ at one-loop. Perhaps, understanding the fate of the hidden 10d symmetry at one-loop for generic correlators, would provide a major insight on our construction of the preamplitudes $\mathcal{L}_{pqrs}^{(2)}$, thus on our bootstrap program. We leave this for a future work.

Before concluding, we mention a special property of $\langle 2222 \rangle$, or "How to bootstrap $\mathcal{H}_{2222}^{(2)}$ without really trying!" In fact, only in this case it is possible to carry out the following procedure on the ansatz (6.19) for $\mathcal{L}_{2222}^{(2)}$: impose $x \to x/(x-1)$ crossing symmetry, impose absence of $x = \bar{x}$ poles, apply $\Delta^{(8)}$, and impose the remaining crossing invariance under $x \to 1/x$. Without any reference to the $\log^2 u$ discontinuity, the above procedure returns a function with only three independent coefficients. (A fourth one is proportional to u^2/v^2 and can be set to zero.) If we furthermore impose analiticity in spin of the $\log^2 u$ discontinuity, we end up with two independent coefficients. One is multiplying a weight four anti-symmetric function, and the other one is multipling a tree-like function. By looking at the minimality of the weight four anti-symmetric function, we immediately conclude that the result is a linear combination of $\mathcal{H}_{2222}^{(2)} - 4u^2\overline{D}_{2422}$ and $u^2\overline{D}_{4444}$.

7 Conclusions

In this paper we have given a general algorithm for computing all one-loop quantum gravity four-point amplitudes in IIB supergravity on AdS$_5\times$S^5. It works for arbitrary external states, i.e. arbitrary KK modes on the five-sphere, and has been tested explicitly for $\langle 2244\rangle$, $\langle 3333\rangle$,$\langle 4444\rangle$,$\langle 3335\rangle$ and $\langle 4424\rangle$. These results are available in a Mathematica notebook attached to the arXiv submission of this article.

The amplitudes we studied are dual to four-point correlators of single-particle half-BPS operators, which we have properly identified in $\mathcal{N}=4$ SYM, in the regime of strong 't Hooft coupling, at order $1/N^4$ in the large N limit of the gauge group SU(N), Our bootstrap program has its foundations in the detailed understanding of the spectrum of two-particle operators, and OPE arguments on the CFT side. We first determine well-defined pieces of the one-loop correlator by extracting all relevant data from many four-point tree-level correlators. Then we rearrange this data to determine the combination which appear at one-loop, and sum up the result. These pieces of the result are fed into an ansatz for the full function, which yields the final result upon demanding no unphysical (euclidean) singularities, i.e. no poles when $x\to\bar{x}$.

Our algorithm generalises in a non-trivial way the one developed in our previous works [3, 7], where the explicit results for $\langle 2222\rangle$ and $\langle 2233\rangle$ were obtained. There are indeed a substantial number of new features which emerge. In fact, the one-loop amplitude for a correlator of KK modes with arbitrary weights, cannot be fixed just by the knowledge of the $\log^2 u$ discontinuity, as was the case for $\langle 2222\rangle$ and $\langle 2233\rangle$. Fortunately there are more pieces of the one-loop amplitude which are determined via tree-level data. These come from window and below window OPE analysis, as illustrated in figure 1, and explained in section 3. They contribute to the single-log and no-log pieces of the one-loop amplitude, respectively. With the addition of this information the one-loop correlators are then completely determined up to well-understood ambiguities with finite spin support.

Two novel and note-worthy features of generic one-loop amplitudes, which originate from the analysis of CFT data in and below the window, are: 1) the natural splitting of the one-loop dynamical function into two independent pieces, $\mathcal{D}^{(2)}=\mathcal{T}^{(2)}+\mathcal{H}^{(2)}$, and 2) the need of a proper understanding of semishort operators contributions at order $1/N^4$, and multiplet recombination.

The splitting of $\mathcal{D}^{(2)}=\mathcal{T}^{(2)}+\mathcal{H}^{(2)}$ introduces what we call the "generalised tree-level amplitude" $\mathcal{T}$. It is uniquely defined in terms of free theory.[17] It is tree-like, since it is built out of polylogarithms with maximal weight 2, and does not contribute to the $\log^2 u$ discontinuity of $\mathcal{H}^{(2)}$. It has no $\log u$ contributions below threshold, and crucially it completely cancels the contribution of recombined free theory below window. Within the ansatz (5.1) this generalised tree-level amplitude has a unique solution.

All the novel and interesting OPE dynamics is neatly combined into $\mathcal{H}^{(2)}$, which we call the "minimal one-loop function". This function has to match the double logarithmic discontinuity and below threshold predictions (excluding those arising from free theory), as

---

[17]Indeed it can be defined in terms of any generalised free theory with arbitrary coefficients in front of the propagator structure.

explained in section 4. In a very non-trivial way this is always possible within the minimal ansatz (4.15). A much wider ansatz would be needed if $\mathcal{T}$ was not included in the first place.

The semi-short sector on its own has a completely independent story, which we determine at order $1/N^4$ purely from free theory correlators. This is done in section 2, where we also clarify some important issues about the way the protected sector actually contributes in the $1/N$ expansion. Then, through multiplet recombination, we obtain predictions for $\mathcal{H}^{(2)}$ below window at the unitarity bound. What is truly remarkable is the fact that this independent input is consistent with our construction of $\mathcal{H}^{(2)}$, which from the very beginning descends from the double logarithmic discontinuity. In this sense, it would be interesting to have a fuller understanding of the meaning of the generalised tree-level amplitude and the split $\mathcal{D}^{(2)} = \mathcal{T}^{(2)} + \mathcal{H}^{(2)}$.

The remaining ambiguities have a clear description in Mellin space, and simply span the space of arbitrary linear functions in the Mellin variables s, t, consistent with crossing symmetry, as described around (4.20). These ambiguities are manifest in our algorithm, because we have chosen the minimal ansatz. Widening our ansatz would allow in principle more freedom. However, we expect true stringy ambiguities to be in one-to-one correspondence with polynomial Mellin amplitudes of some degree [48, 49]. Indeed, a polynomial Mellin amplitude of order r has at most a spin r contribution in its SCPW expansion, and counts as a contribution in a bulk 10d effective action which involves $\partial^{2r} R^4$. The case $r = 0$, i.e. the R^4 term, appears at one-loop order with $(\alpha')^{-1}$. Dimensional analysis then implies that the $\partial^{2r} R^4$ term comes with coefficient $(\alpha')^{-1+r}$. In summary

$$\mathcal{M}(s,t) = O(s^{r-a} t^a) \quad \leftrightarrow \quad \text{max spin: } r \quad \leftrightarrow \quad \partial^{2r} R^4 \quad \leftrightarrow \quad (\alpha')^{-1+r}. \tag{7.1}$$

The minimal ansatz naturally gives ambiguities that contribute up to order $(\alpha')^0$ only. Widening it would include ambiguities corresponding to higher order α' corrections (7.1).[18] Although within our bootstrap program we cannot fix the value of the ambiguities, it may be possible that a combination of different techniques will. For example, by using localisation techniques [13] obtained the value of the $\langle 2222 \rangle$ ambiguity.

Finally we come to the question of ten-dimensional conformal symmetry [9]. This symmetry implies a beautiful structure for the leading logs at any order. At one-loop the double logarithmic discontinuity for any correlator can be written as $\Delta^{(8)}$ acting on a much simpler object. Furthermore this object can be simplified further by pulling out the differential operators $\widehat{\mathcal{D}}_{pqrs}$. In section 6 we have examined the extent to which this hidden structure transfers to the full one-loop function itself. We have found that the one-loop amplitudes can be written as $\Delta^{(8)}$ acting on a pre-amplitude $\mathcal{L}_{pqrs}^{(2)}$ up to a tree-like remainder. Although the resulting pre-amplitude cannot be written in the way the double logarithmic discontinuity can, i.e. directly as $\widehat{\mathcal{D}}_{pqrs}\mathcal{L}_{2222}^{(2)}$, we have found in a number of examples that the difference $\mathcal{L}_{pqrs}^{(2)} - \widehat{\mathcal{D}}_{pqrs}\mathcal{L}_{2222}^{(2)}$ can be put in a very simple form, i.e. (6.26), (6.28) and (6.31). This pattern may hint at more structure in the result than is currently apparent, and we hope to investigate this possibility in the future. In a similar vein there are hints

---

[18]At one-loop there are also stringy corrections which can be obtained from tree-level string corrections. See [11] for initial work in this direction.

that the ten-dimensional symmetry also controls the leading order corrections in $\lambda^{-\frac{1}{2}}$ at tree level [33]. Perhaps it can be used to study these four-point functions much more generally.

Future directions include a more detailed investigation of the Mellin space representation of our one-loop functions, which would extend the analysis of $\langle 2222 \rangle$ in [11], as well as the possibility of pushing our bootstrap program to two loops. It would be fascinating if the results we obtain in the large N expansion could be compared (possibly taking into account also the α' corrections) to the results based on integrable methods [53–57]. We also emphasise that our fresh new look at free $\mathcal{N} = 4$ SYM, especially our understanding of single particle operators and generalised tree-level functions, suggests a different way to approach the mysterious six-dimensional $(2,0)$ theory, which has been recently studied from a holographic perspective in several papers [58–61].

Acknowledgments

A Superblocks

Here we give the explicit definition of the superblocks $\mathbb{S}^{p_1 p_2 p_3 p_4}_{\gamma, \underline{\lambda}}$ following [34]. They are defined by a determinantal formula. Let us introduce first the function

$$F^{\alpha\beta\gamma\underline{\lambda}} = (-1)^{p+1} \frac{(x-y)(x-\bar{y})(\bar{x}-y)(\bar{x}-\bar{y})}{(x-\bar{x})(y-\bar{y})} \det \begin{pmatrix} F^X_{\underline{\lambda}} & R \\ K_{\underline{\lambda}} & F^Y \end{pmatrix}, \qquad (A.1)$$

where the determinant is taken on the $(p+2) \times (p+2)$ matrix (where $p = \min(\alpha, \beta)$)

$$(F^X_{\underline{\lambda}})_{ij} = \left([x_i^{\lambda_j - j} {}_2F_1(\lambda_j + 1 - j + \alpha, \lambda_j + 1 - j + \beta; 2\lambda_j + 2 - 2j + \gamma; x_i)] \right)_{1 \leq i \leq 2, 1 \leq j \leq p}$$

$$(F^Y)_{ij} = \left((y_j)^{i-1} {}_2F_1(i - \alpha, i - \beta; 2i - \gamma; y_j) \right)_{1 \leq i \leq p, 1 \leq j \leq 2}$$

$$(K_{\underline{\lambda}})_{ij} = \left(- \delta_{i; j - \lambda_j} \right)_{1 \leq i \leq p, 1 \leq j \leq p}$$

$$(R)_{ij} = \left(\frac{1}{x_i - y_j} \right)_{1 \leq i \leq 2, 1 \leq j \leq 2} . \qquad (A.2)$$

(The brackets in the definition of $F^X_{\underline{\lambda}}$ mean deletion of the singular terms in the Taylor expansion in x_i around $x_i = 0$ when $\lambda_j < j$ and we have defined here $x_i = (x, \bar{x})$ $y_j = (y, \bar{y})$

in the matrix.) Then

$$\mathbb{S}^{p_1p_2p_3p_4}_{\gamma,\underline{\lambda}} = \mathcal{P} \left(\frac{x\bar{x}}{y\bar{y}}\right)^{\frac{1}{2}(\gamma-p_4+p_3)} F^{\alpha\beta\gamma\underline{\lambda}}, \qquad \begin{cases} \alpha = \frac{1}{2}(\gamma - p_1 + p_2), \\ \beta = \frac{1}{2}(\gamma + p_3 - p_4). \end{cases} \tag{A.3}$$

The prefactor $\mathcal{P}$ is that of (2.9).

B   Trees and amplitudes

B.1 $1/N^2$ connected free theory

We quote a formula for connected free theory at order $1/N^2$, of a generic four-point function $\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle$. The same formula is described in a different notation in [9].

Each propagator structure in free theory is labelled by monomials of the form $\mathcal{P}\hat{\sigma}^{i-j}\hat{\tau}^j$ where i,j belong to $T = \{(i,j) \mid 0 \le i \le \kappa_{\vec{p}},\ 0 \le j \le i\}$, and the bound $\kappa_{\vec{p}}$ is precisely the degree of extremality. The lattice of points described by T is schematically depicted here below. We shall distinguish the three edges from the interior.

(B.1)
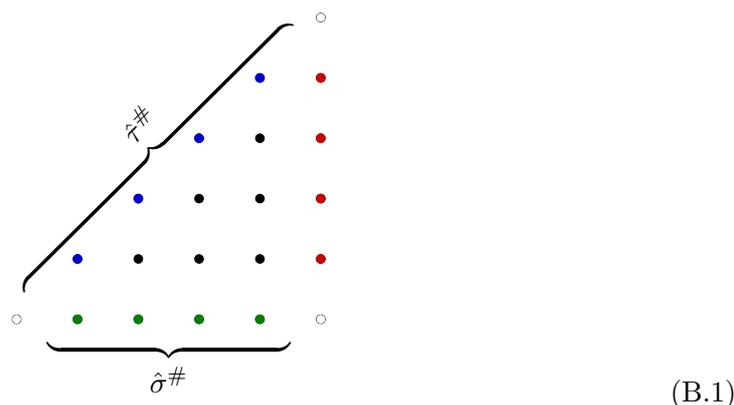
Vertices at the intersection of the edges correspond to disconnected diagrams, when they do not vanish according to our definition of single particle states. In [8] we determined the value of the following connected propagator structure

$$\frac{\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle}{\mathcal{P}} \supset \frac{\sqrt{p_1p_2p_3p_4}}{N^2} \left[1 + \frac{p_{43}+p_{21}}{2}\right]\frac{u\hat{\tau}}{v} \tag{B.2}$$

Looking at the diagram of T, we have determined the coefficient associated to the point $(1,1)$ on the diagonal edge of the triangle. From crossing on the other edges we find

$$\frac{\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle}{\mathcal{P}} \supset \frac{\sqrt{p_1p_2p_3p_4}}{N^2}\Bigg[\left(1+\frac{p_{43}+p_{21}}{2}\right)\sum_{k=1}^{t-1}\left(\frac{u\hat{\tau}}{v}\right)^k + \left(1+\frac{p_{13}+p_{42}}{2}\right)\sum_{k=1}^{t-1}(u\hat{\sigma})^k +$$
$$\left(1+|\frac{p_{23}+p_{14}}{2}|\right)\sum_{k=1}^{t-1}(u\hat{\sigma})^k\left(\frac{u\hat{\tau}}{v}\right)^{t+1-k}\Bigg] \tag{B.3}$$

By including propagator structure in the interior of T we finally obtain the general formula

$$\frac{\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle}{\mathcal{P}} = \frac{\sqrt{p_1 p_2 p_3 p_4}}{N^2} \Bigg[\left(1 + \frac{p_{43} + p_{21}}{2} \right) \sum_{k=1}^{t-1} \left(\frac{u\hat{\tau}}{v} \right)^k + \left(1 + \frac{p_{43} - p_{21}}{2} \right) \sum_{k=1}^{t-1} (u\hat{\sigma})^k +$$

$$\left(1 + |\frac{p_1 + p_2 - p_3 - p_4}{2}| \right) \sum_{k=1}^{t-1} (u\hat{\sigma})^k \left(\frac{u\hat{\tau}}{v} \right)^{t+1-k} + 2 \sum_{T \backslash \text{edges}} (u\hat{\sigma})^{\#_1} \left(\frac{u\hat{\tau}}{v} \right)^{\#_2} \Bigg] \tag{B.4}$$

B.2 Generalised tree level amplitudes

The Mellin transform of generalised tree level functions has the same form of the Rastelli and Zhou integral. By using the conventions of [8] we obtain,

$$\mathcal{T} = \frac{u^{\frac{p_3 - p_4}{2}}}{v^{\frac{p_2 + p_3}{2}}} \oint u^{\frac{s}{2}} v^{\frac{t}{2}} \Gamma_{\vec{p}} \, \mathcal{M}[\mathcal{T}](s, t), \qquad p_{43} \geq p_{21} \geq 0. \tag{B.5}$$

In order to make manifest how the pole structures of $\Gamma_{\vec{p}}$ relates to the OPE expansion in twists, and the log stratification of the tree level dynamical function, it is convenient to make manifest the location of the double poles in s and t and extract

$$\Gamma \left[\max \left(\frac{p_1 + p_2}{2}, \frac{p_3 + p_4}{2} \right) - \frac{s}{2} \right]^2 \Gamma \left[\max \left(\frac{p_1 + p_4}{2}, \frac{p_2 + p_3}{2} \right) - \frac{t}{2} \right]^2 \tag{B.6}$$

In our conventions $p_1 + p_4 \geq p_2 + p_3$, therefore the max of the second Γ in (B.6) is fixed. By changing variables to

$$s \leftrightarrow \frac{s}{2} - \max \left(\frac{p_1 + p_2}{2}, \frac{p_3 + p_4}{2} \right), \qquad t \leftrightarrow \frac{t}{2} - \frac{p_1 + p_4}{2}, \tag{B.7}$$

and introducing

$$P \equiv \left| \frac{p_1 + p_2}{2} - \frac{p_3 + p_4}{2} \right| \qquad Q \equiv \frac{p_4 - p_3 + p_1 - p_2}{2}, \tag{B.8}$$

we rearrange $\mathcal{T}$ into the form

$$\mathcal{T} = u^{\max(\frac{p_1 + p_2 - p_{43}}{2}, p_3)} v^{\frac{p_{43} - p_{21}}{2}} \oint u^s v^t \, \Gamma[-s]^2 \Gamma[-t]^2 \left[\frac{(-)^P}{(s+1)_P} \frac{(-)^Q}{(t+1)_Q} \mathcal{M}[\mathcal{T}](s, t) \right] \tag{B.9}$$

$$\times \Gamma \big[2 + s + t + \max(p_1, p_3 + Q) \big] \Gamma \big[2 + s + t + \max(p_2 + Q, p_4) \big]$$

The object highlighted in brackets [. . .] in the first line, which includes the amplitude and two Pochhammer symbols in the denominator, has only simple poles. The sequence of

double and simple poles splits in three sectors as follows

pole in s	power of $u^{\#}$	pole in t	power of $v^{\#}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	$p_3 + \max(0, \frac{p_1+p_2-p_3-p_4}{2})$	0	$\frac{p_{43}-p_{21}}{2}$
-1	$\vdots$	-1	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$-P$	$p_3 + \min(0, \frac{p_1+p_2-p_3-p_4}{2})$	$-Q$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$-P^*$	1	$-Q^*$	$-p_3 + 1 - \min(0, \frac{p_1+p_2-p_3-p_4}{2})$

$$\tag{B.10}$$

Notice the symmetric relation

$$P^* - P = Q^* - Q = p_3 - 1 + \min\left(0, \frac{p_1 + p_2 - p_3 - p_4}{2}\right). \tag{B.11}$$

The first two sectors of the table above contain information only from $\Gamma_{\vec{p}}$. We have double poles of the form $\Gamma[-s]^2$ and $\Gamma[-t]^2$, which originates from $\Gamma_{\vec{p}}$ in the region where simple poles of the individual Gamma functions in (4.21) overlap. Then, we have remaining simple poles of the form $(s+1)_P(t+1)_Q$, in our notation. The third sector instead arises only from $\mathcal{M}[\mathcal{T}]$. A general subtlety in defining the (rational) Mellin amplitude is due to the choice of contour of integration. This contour should separate poles in s and t from those in $s+t$, in order to have a well defined residue integration a la' Mellin-Barnes. This is achieved by rewriting $\mathcal{M}[\mathcal{T}]$ in the form,

$$\mathcal{M}[\mathcal{T}] = \sum_{\text{poles}} \frac{m^{p,q,r}(s,t)}{(s+p)(t+q)(s+t+r)} \qquad r > -p - q, \tag{B.12}$$

and paying attention that when $\mathcal{M}[\mathcal{T}]$ is restricted to a single complex variable, let's say $(-n, t)$ for example, the only poles in t that count, are those of the form $(t+m)$ in (B.12), and those of the form $(s+t+r)\big|_{s=-n}$ are discarded.

In the special case of equal charges, $p_i = p$, the Mellin integral simplifies to

$$\mathcal{T}_{pppp} = u^p \oint u^s v^t \; \Gamma[-s]^2 \Gamma[-t]^2 \Gamma[s+t+p+2]^2 \times \mathcal{M}[\mathcal{T}_{pppp}] \tag{B.13}$$

Our new Mellin amplitudes (5.9) and (5.21) are presented in this way.

B.2.1 Mellin-Barnes integration

We now perform the residue integration of (B.9) in detail.

The computation will be organised as follows: firstly, we pick double poles in s and double poles in t. Then, we pick two sequences of poles: we pick double poles in s and

simple poles in t, and we pick double poles in t and simple poles in s. Finally we pick only simples poles in both s and t. We will make the symmetry $s \leftrightarrow t$ visible.

The main input is a well known formula for Gamma function

$$\Gamma[-z]^2\Big|_{z \to k} \;\to\; \frac{1}{(k!)^2}\partial_z\left(\frac{1}{k-z}\right) + \frac{1}{(k!)^2}\frac{\psi_{k+1}}{(k-z)} \tag{B.14}$$

Proceeding in the order described above:

Double poles and double poles. The result of contour integration is

$$(-)^{P+Q}\sum_{n,m \geq 0} \frac{u^n}{(n!)^2}\frac{v^m}{(m!)^2}\frac{\Gamma[2+n+m+A]\Gamma[2+n+m+B]}{(n+1)_P(m+1)_Q}\mathcal{M}[n,m]$$
$$\times \Bigg(\left[\log u + \Psi_n[m] + (\psi_{n+1}-\psi_{n+1+P}) + \frac{\partial_s\mathcal{M}}{\mathcal{M}}\right]\left[\log v + \Psi_m[n] + (\psi_{m+1}-\psi_{m+1+Q}) + \frac{\partial_t\mathcal{M}}{\mathcal{M}}\right]$$
$$+ \left[\Psi' + \frac{\partial_t\partial_s\mathcal{M}}{\mathcal{M}} - \frac{\partial_s\mathcal{M}}{\mathcal{M}}\frac{\partial_t\mathcal{M}}{\mathcal{M}}\right]\Bigg) \tag{B.15}$$

For convenience we have defined $A = \max(p_1, p_3 + Q)$, $B = \max(p_2 + Q, p_4)$ and the following two polygamma quantities,

$$\Psi_n[t] = \psi_{2+n+t+A} + \psi_{2+n+t+B} - 2\psi_{n+1} \tag{B.16}$$
$$\Psi_m[s] = \psi_{2+s+m+A} + \psi_{2+s+m+B} - 2\psi_{m+1} \tag{B.17}$$

Notice that $\Psi' = \partial_t\Psi_n[t] = \partial_s\Psi_m[s]$ and the terms $\partial_s\mathcal{M}\partial_t\mathcal{M}$ cancel out.

Simple poles and double poles. The result of contour integration is

$$(-)^{P+Q}\sum_{n \geq 0}\sum_{m=-Q^*}^{-1}\frac{u^n}{(n!)^2}\frac{v^m}{\Gamma[-m]^{-2}}\frac{\Gamma[2+n+m+A]\Gamma[2+n+m+B]}{(n+1)_P}\times$$
$$\left[\log u + \Psi_n[m] + (\psi_{n+1}-\psi_{n+1+P}) + \frac{\partial_s\mathcal{M}}{\mathcal{M}}\right]\mathrm{Res}\left[\frac{\mathcal{M}[n,t]}{(t+1)_Q}\right]_{t=m} \tag{B.18}$$

and

$$(-)^{P+Q}\sum_{m \geq 0}\sum_{n=-P^*}^{-1}\frac{u^n}{\Gamma[-n]^{-2}}\frac{v^m}{(m!)^2}\frac{\Gamma[2+n+m+A]\Gamma[2+n+m+B]}{(m+1)_Q}\times$$
$$\left[\log v + \Psi_m[n] + (\psi_{m+1}-\psi_{m+1+Q}) + \frac{\partial_t\mathcal{M}}{\mathcal{M}}\right]\mathrm{Res}\left[\frac{\mathcal{M}[s,m]}{(s+1)_P}\right]_{s=n} \tag{B.19}$$

Notice that the leading $\log(u)$ discontinuity is obtained by picking, for each double poles in s, both double and simple poles in t.

Simple poles and simple poles. The result of contour integration is

$$(-)^{P+Q}\sum_{n=-P^*}^{-1}\sum_{m=-Q^*}^{-1}\frac{u^n}{\Gamma[-n]^{-2}}\frac{v^m}{\Gamma[-m]^{-2}}\Gamma[2+n+m+A]\Gamma[2+n+m+B]\times$$
$$\mathrm{Res}\left[\frac{\mathcal{M}[n,t]}{(s+1)_P(t+1)_Q}\right]_{\substack{s=n\\t=m}} \tag{B.20}$$

B.2.2 Properties of the tree level Mellin amplitude

We record here an intriguing mathematical observation: the location of the double poles is encoded in the form of Mack's $\Gamma_{\vec{p}}$, and the residue on these poles depends on the Mellin amplitude $\mathcal{M}[\mathcal{T}_{\vec{p}}]$. Thus double poles are found to be in one-to-one correspondence with the polynomial $P_\square$, in the position space representation of the function (5.1), because by residue integration, $P_\square$ is the numerator of the term $\log(u)\log(v) \subset \phi^{(1)}$,

$$\frac{P_\square(x,\bar{x},\hat{\sigma},\hat{\tau})}{(x-\bar{x})^{\mathbf{d}+1}} = u^{\max(\frac{p_1+p_2-p_{43}}{2},\frac{p_3+p_4-p_{43}}{2})}v^{\frac{p_{43}-p_{21}}{2}}\left[\oint u^s v^t \frac{\Gamma[-s]\Gamma[-t]}{\Gamma[s+1]\Gamma[t+1]} \times \right.$$
$$\left. \frac{\Gamma[2+s+t+A]\Gamma[2+s+t+B]}{(-)^{P+s+Q+t}(s+1)_P(t+1)_Q}\mathcal{M}[\mathcal{T}_{\vec{p}}][s,t]\right] \tag{B.21}$$

with P, Q, A and B as in the previous section. Equation (B.21) implies that the Mellin amplitude of $\mathcal{T}$ can be obtained from $P_\square$ upon assuming $\Gamma_{\vec{p}}$, and viceversa. In particular, the conversion makes use of the formula (valid for any $\mathbf{d}$),

$$\frac{u^a v^b}{(x-\bar{x})^{\mathbf{d}+1}} = \frac{\mathbf{d}!}{(2\mathbf{d})!}\oint (-u)^s(-v)^t \frac{\Gamma[-s]\Gamma[-t]}{\Gamma[s+1]\Gamma[t+1]}\Gamma[s+t+X]^2 \times$$
$$\left[(-)^{a+b}\frac{(-s)_a(-t)_b}{(s+1)_{\mathbf{d}-a}(t+1)_{\mathbf{d}-b}}(s+t+X)_{\mathbf{d}+1-a-b-X}(s+t+X)_{\mathbf{d}/2+1-a-b-X}\right] \tag{B.22}$$

applied to each monomial in $P_\square$. The value of X can be tuned afterwards by putting the final result in a canonical form.

Summarizing, there is a bijection $\mathcal{M}[\mathcal{T}] \leftrightarrow P_\square/(x-\bar{x})^{\mathbf{d}_1-1}$, which assumes $\Gamma_{\vec{p}}$. This bijection also implies that the operators $\widehat{\mathcal{D}}_{pqrs}$ introduced in [9] and discussed around (6.13), can be simply obtained from $P_\square$, thus reducing a computation about tree-level functions, to another one involving only rational functions. It would be fascinating to know what is the uplift of $\Gamma_{\vec{p}}$ at one-loop.

B.2.3 On tree level SCPW

Having obtained the explicit tree-level four-point functions in position space as detailed in the previous section we then perform a SCPW decomposition to obtain the corresponding SCPW coefficients $M^{(1)}$.

As a function of twist and spin, $M^{(1)}_{\vec{p};\vec{\tau}}$ fits the ansatz

$$M^{(1)}_{\vec{p};\vec{\tau}} = \frac{\left(\frac{2+2l+\tau+p_{43}}{2}\right)!\left(\frac{2+2l+\tau-p_{21}}{2}\right)!}{(2+2l+\tau)!}\frac{\left(\frac{\tau+\max(p_1+p_2,p_3+p_4)}{2}\right)!\left(\frac{\tau+\min(p_1+p_2,p_3+p_4)}{2}\right)!}{\tau!} \times$$
$$\times \prod_{i=1}^{\max(p_1+p_2,p_3+p_4)/2-\frac{p_{21}}{2}}\left(\frac{\tau-p_{21}}{2}-i+1\right)\prod_{j=1}^{\min(p_1+p_2,p_3+p_4)/2-\frac{p_{43}}{2}}\left(\frac{\tau+p_{43}}{2}+j\right)^{-1}$$
$$\times \sum_{m=0}^{deg}\sum_{n=m}^{deg}\chi_{m,n}\,l^m\tau^{deg-n} \tag{B.23}$$

where $\chi_{m,n}$ are constants, depending only on a, b and $\vec{p}$. The same ansatz works for $L_{\vec{p};\vec{\tau}}^{(1)}$

The degree of the polynomial is

$$deg = -4 + \min(p_1 + p_2, p_3 + p_4) - p_{43} + p_{21} = 2(\kappa_{\vec{p}} - 2) + p_{21}. \qquad \text{(B.24)}$$

When b is even, reciprocity symmetry implies that the polynomial in spin, in the last line (B.23), is even in the variable $2l + \tau + 3$. When b is odd we find the following reduction. For $l \in 2\mathbb{Z}$, the polynomial in spin is of the form

$$\sum_{m=0}^{deg} \sum_{n=m}^{deg} \chi_{m,n} \, l^m \tau^{deg-n} = (l + \tau + 2) \times \sum_{m=0}^{deg-1} \sum_{n=m}^{deg-1} \chi'_{m,n} \, l^m \tau^{deg-1-n} \qquad \text{(B.25)}$$

Consequently, the polynomial for $l \in 2\mathbb{Z}+1$ is obtained from the latter upon $\ell \to -\ell - \tau - 3$.

Notice that the first factor in the second line of (B.23) vanishes for long superblock of twist $\tau = p_{21}, \ldots, p_{43}, \ldots \tau_{\vec{p}}^{\max} - 1$, where $p_{43} \geq p_{21}$.

As function of spin, the analitic part of the tree level correlator fits into

$$\mathcal{B}^{(2+\tau,l)} \left[u^{2+p_{43}} \mathcal{T}_{\vec{p}, [aba]} \Big|_{\frac{1}{N^2}} \right] = \frac{\left(\frac{2+2l+\tau-p_{21}}{2} \right)! \left(\frac{2+2l+\tau+p_{43}}{2} \right)!}{(2+2l+\tau)!} \sum_{m=0}^{deg} \xi'_m(\tau, \vec{p}, a, b) \, l^m \qquad \text{(B.26)}$$

$$+ \theta(\tau - \tau^{\max}) \frac{1}{2} \left[\psi \left(\frac{4+2l+\tau-p_{21}}{2} \right) + \psi \left(\frac{4+2l+\tau+p_{43}}{2} \right) - 2\psi(3+2l+\tau) \right] M_{\vec{p},\tau,l,a,b}^{(1)}$$

for some coefficients ξ'_m. The notation $\mathcal{B}^{(2+\tau,l)}[f]$ means the CPW coefficient of $\mathcal{B}^{(2+\tau,l)}$ in the expansion of f. If $\tau < \tau^{\max}$ we are in the window and we are computing $L_{\vec{p}}^{(1)}$. The derivative relation holds only for N^2E correlators, and it is false otherwise.

C Some subleading three-point couplings

Let us begin from $\mathbf{C}_{(pp),4,l,[000]}^{(1)}$ and $\mathbf{C}_{(pp+1),5,l,[010]}^{(1)}$ with $p = 3, 4 \ldots$, In both cases, we are studying (3.34) in the simplified form,

$$C_{pp\mathcal{K}_4}^{(1)} = \left(C_{22\mathcal{K}_4}^{(0)} \right)^{-1} L_{22pp,\tau=4,l,[000]}^{(1)} \qquad \text{(C.1)}$$

$$C_{pp+1\mathcal{K}_5}^{(1)} = \left(C_{23\mathcal{K}_5}^{(0)} \right)^{-1} L_{23pp+1,\tau=5,l,[010]}^{(1)} \qquad \text{(C.2)}$$

because only one long operator is exchanged. In [010] we should also distinguish between even and odd spins. However, the knowledge of the even spin sector determines the odd sector through the reciprocity symmetry.

Proceeding in the order, we can use results from [4] and [7], to obtain

$$\frac{L_{22pp,\tau=4,l,[000]}^{(1)}}{2p} = \left[\underbrace{(p-1)}_{\text{conn. free}} + \frac{6p}{p-2} \right] \frac{((l+3)!)^2}{3(2l+6)!} = \frac{(p+1)(p+2)}{3(p-2)} \frac{((l+3)!)^2}{(2l+6)!} \qquad \text{(C.3)}$$

$$\frac{L_{23pp+1,\tau=5,l,[010]}^{(1)}}{\sqrt{6p(p+1)}} = \left[\underbrace{(p-1)}_{\text{conn. free}} + \frac{4(2p+1)}{p-2} \right] (l+7) \frac{((l+4)!)^2}{5(2l+8)!}$$

$$= \frac{(p+2)(p+3)}{5(p-2)} (l+7) \frac{((l+4)!)^2}{(2l+8)!} \qquad \text{(C.4)}$$

These same expressions follow from the derivative relation, once analitically continued to the relevant twist. It is interesting to see how this happens: consider first

$$M^{(1)}_{22pp,t=\tau/2,\text{even},[000]} = -\frac{2p}{(p-2)!}\left[\frac{2}{p!}(-)^p(t-p+1)_{2p}\frac{(t!)^2((t+l+1)!)^2}{(2t)!(2t+2l+2)!}\right] \quad (C.5)$$

We should now take a ∂_t of (C.5), and evaluate the result at $t = 2$, which is above the regime of validity of the formula, i.e. $\tau^{\max}_{22pp} = 2p$. Quite non trivially, the only term that will contribute comes from $(t - p + 1)_{2p} = (t - p + 1)_{p-3}(t - 2)(t - 1)_{p+2}$. Indeed one effect of the derivative is to drop the factor $(t - 2)$, which otherwise would make the result vanishing. Then, we find precisely (C.3),

$$\partial_t M^{(1)}_{22pp,t=\tau/2,\text{even},[000]}\Big|_{t=2} = \frac{2p(p+1)(p+2)}{3(p-2)}\frac{((l+3)!)^2}{(2l+6)!} \quad (C.6)$$

Similarly,

$$M^{(1)}_{23pp+1,t=\frac{\tau-1}{2},\text{even},0,1} = \quad (C.7)$$
$$-\frac{\sqrt{6p(p+1)}}{(p-2)!}\left[\frac{2(-)^p}{(p+1)!}(t-p+1)_{2p+1}(l+2t+3)\frac{t!(t+1)!((l+t+2)!)^2}{(2t+1)!(2l+2t+4)!}\right]$$

will have a contribution from $(t - p + 1)_{2p+1}$ after taking the derivative at $t = 2$, and the result will match non trivially (C.4).

The subleading three-point functions we looked for are

$$C^{(1)}_{pp\mathcal{K}_4} = \frac{p(p+1)_2}{\sqrt{3}(p-2)}\frac{(l+3)!}{\sqrt{(2l+6)!}}\frac{1}{\sqrt{(l+1)(l+6)}}\left[\frac{1+(-)^l}{2}\right] \quad (C.8)$$

$$C^{(1)}_{pp+1\mathcal{K}_5} = \frac{\sqrt{2p(p+1)}(p+2)_2}{\sqrt{5}(p-2)}\frac{(l+3)!}{\sqrt{(2l+7)!}}\left[\sqrt{\frac{(l+7)}{(l+1)}}\frac{1+(-)^l}{2} + \sqrt{\frac{(l+1)}{(l+7)}}\frac{1-(-)^l}{2}\right] \quad (C.9)$$

A first example with more than one operator is given by $\mathbf{C}^{(1)}_{(44),6,l,[000]}$. In the following, we refer more directly to the two operators labelled by $R_{6,l,[000]}$, with $\widetilde{\mathcal{K}}_{6,1}$ and $\widetilde{\mathcal{K}}_{6,2}$. Then $\mathbf{C}^{(1)}_{(44),6,l,[000]}$ is the vector made by $C^{(1)}_{44\widetilde{\mathcal{K}}_{6,i=1,2}}$. The inputs we need from [4] are

$$\mathbf{c}^{(0)}_{6,\text{even},[000]} = \begin{bmatrix} +\sqrt{\frac{l+2}{2l+9}} & +\sqrt{\frac{l+7}{2l+9}} \\ -\sqrt{\frac{l+7}{2l+9}} & +\sqrt{\frac{l+2}{2l+9}} \end{bmatrix} \quad (C.10)$$

$$\mathbf{L}^{(0)}_{6,\text{even},[000]} = \frac{(l+1)(l+2)(l+7)(l+8)}{40}\begin{bmatrix} \frac{4}{(l+2)(l+7)} & 0 \\ 0 & 1 \end{bmatrix}\frac{((l+4)!)^2}{(2l+8)!} \quad (C.11)$$

Obtaining $L^{(1)}_{pp44,\tau=6,l,[000]}$ for $p = 2,3$, at twist 6, we can finally solve (3.34) in terms of the 3-pt functions and get the result

$$C^{(1)}_{44\widetilde{\mathcal{K}}_{6,1}} = -6\sqrt{10}\frac{(2l+9)}{\sqrt{(l+2)}}\sqrt{\frac{(l+8)}{(l+1)}}\frac{(l+4)!}{\sqrt{(2l+9)!}}\left[\frac{(1+(-)^l)}{2}\right] \quad (C.12)$$

$$C^{(1)}_{44\widetilde{\mathcal{K}}_{6,2}} = +6\sqrt{10}\frac{(2l+9)}{\sqrt{(l+7)}}\sqrt{\frac{(l+1)}{(l+8)}}\frac{(l+4)!}{\sqrt{(2l+9)!}}\left[\frac{(1+(-)^l)}{2}\right] \quad (C.13)$$

We finish with our list of examples by computing $\mathbf{C}^{(1)}_{(44),6,l,[020]}$ and $\mathbf{C}^{(1)}_{(35),6,l,[020]}$ in $[020]$, in the even spin sector. We need the results

$$\mathbf{c}^{(0)}_{6,\text{even},[020]} = \begin{bmatrix} +\sqrt{\frac{l+5}{2l+9}} & +\sqrt{\frac{l+4}{2l+9}} \\ +\sqrt{\frac{l+4}{2l+9}} & -\sqrt{\frac{l+5}{2l+9}} \end{bmatrix} \tag{C.14}$$

$$\mathbf{L}^{(0)}_{6,\text{even},[020]} = \frac{(l+1)(l+4)(l+5)(l+8)}{30} \begin{bmatrix} \frac{2}{(l+4)} & 0 \\ 0 & \frac{9}{(l+5)} \end{bmatrix} \frac{(l+3)!(l+4)!}{(2l+8)!} \tag{C.15}$$

This case has $\mu = 2$. (The odd spin sector in $[0,2,0]$ has $\mu = 1$ and can be studied as in our first example.) The CPW data entering the r.h.s. of (3.34) for $\mathbf{V}_{44,8,l,[020]}$ and $\mathbf{V}_{35,8,l,[020]}$, is extracted from different correlators. In the first case we consider the CPW data

$$\frac{L^{(1)}_{4424,6,l,[020]}}{\sqrt{128}} = \frac{84}{5} \frac{(l+4)!(l+5)!}{(2l+8)!}, \qquad \frac{L^{(1)}_{4433,6,l,[020]}}{\sqrt{144}} = \frac{9(80+3l(l+9))}{5} \frac{((l+4)!)^2}{(2l+8)!}. \tag{C.16}$$

In the second case,

$$\frac{L^{(1)}_{2435,6,l,[020]}}{\sqrt{120}} = \frac{7(32+l(l+9))}{5} \frac{(l+3)!(l+5)!}{(2l+8)!}, \qquad \frac{L^{(1)}_{3335,6,l,[020]}}{\sqrt{135}} = \frac{52}{15} \frac{(l+4)!(l+5)!}{(2l+8)!}. \tag{C.17}$$

For convenience we again refer to the operators labelled by $R_{6,\text{even spin},[020]}$ with $\mathcal{K}_{6,1}$ and $\mathcal{K}_{6,2}$. Then, by using (3.34) we obtain the results

$$C^{(1)}_{44\mathcal{K}_{6,1}} = -6\sqrt{\frac{6}{5}} \frac{(9l+16)}{\sqrt{(l+5)}} \sqrt{\frac{(l+8)}{(l+1)}} \frac{(l+4)!}{\sqrt{(2l+9)!}} \left[\frac{(1+(-)^l)}{2}\right] \tag{C.18}$$

$$C^{(1)}_{44\mathcal{K}_{6,2}} = +6\sqrt{\frac{6}{5}} \frac{(9l+65)}{\sqrt{(l+4)}} \sqrt{\frac{(l+1)}{(l+8)}} \frac{(l+4)!}{\sqrt{(2l+9)!}} \left[\frac{(1+(-)^l)}{2}\right] \tag{C.19}$$

and

$$C^{(1)}_{35\mathcal{K}_{6,1}} = 21\sqrt{2} \frac{(l+7)}{\sqrt{(l+4)}} \sqrt{\frac{(l+8)}{(l+1)}} \sqrt{\frac{(l+3)!(l+5)!}{(2l+9)!}} \left[\frac{(1+(-)^l)}{2}\right] \tag{C.20}$$

$$C^{(1)}_{35\mathcal{K}_{6,2}} = 21\sqrt{2} \frac{(l+2)}{\sqrt{(l+5)}} \sqrt{\frac{(l+1)}{(l+8)}} \sqrt{\frac{(l+3)!(l+5)!}{(2l+9)!}} \left[\frac{(1+(-)^l)}{2}\right] \tag{C.21}$$

D Spin structure of $M^{(2)}$ and $L^{(2)}$

We discuss the spin structure of our predictions in and below window, focussing directly on the SCPW coefficients. This will provide information on how reciprocity symmetry (see [50–52]) is realised.

Given a pair of external charges $(p_1 p_2)(p_3 p_4)$ with $p_{43} \geq p_{21}$, consider a twist τ such that τ belongs to the window or below the window. Then, the SCPW coefficients we extract from $M^{(2)}$ or $L^{(2)}$ have the form

$$\sum_{(pq) \in R_{\tau,[aba]}} \frac{\langle p_1 p_2 pq \rangle_{\text{conn.}} \langle pq p_3 p_4 \rangle_{conn}}{\langle pqpq \rangle_{\text{disc.}}} \tag{D.1}$$

where $\langle\dots\rangle_{\text{conn.}}$ refers to $M^{(1)}$ or $L^{(1)}$ depending on the situation. The form of (D.1) is actually too schematic, because it assumes $q - p \geq p_{21}$ and $p_{43} \geq q - p$, in the summation over $pq \in R_{\tau,[aba]}$, and this is not always the case.

In order to have better control over the summation, it is useful to visualize it geometrically. Consider first $M^{(2)}$. In the plane (n, m), draw the two lines $p_1 + p_2 = n + m$ and $p_3 + p_4 = n + m$, and the rectangle $R_{\tau,[aba]}$. Since we are considering a τ in the window, a pair of external charges sits inside $R_{\tau,[aba]}$.

$$n + m = \max(p_1 + p_2, p_3 + p_4)$$
$$n + m = \min(p_1 + p_2, p_3 + p_4)$$

(D.2)
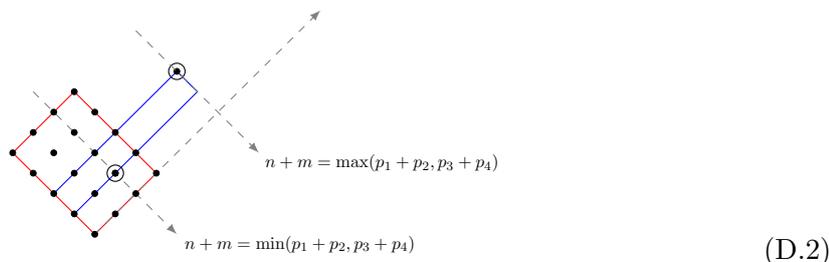
In the figure, $R_{\tau,[aba]}$ is given in red, and each pair $(pq) \in R_{\tau,[aba]}$ is represented by a black dot. The rightmost edge of $R_{\tau,[aba]}$ lies on the line $n + m = \tau$. In fact we can foliate $R_{\tau,[aba]}$ by the lines $n + m = \tau'$ for $\tau' = 4 + 2a + b, \dots \tau$. Running on any such line, the difference $m - n$ increases in the direction $+3\pi/4$ and decreases in the direction $-\pi/4$. The two pairs of external charges $p_1 p_2$ and $p_3 p_4$ are represented by a dot encircled.

There are at most three cases to be taken into account. For a pair $(pq) \in R_{\tau,[aba]}$ belonging to the line $p + q = \tau'$, we can have

(I) $p_{21} \leq p_{43} \leq q - p$ (II) $p_{21} \leq q - p \leq p_{43}$ (III) $q - p \leq p_{21} \leq p_{43}$ (D.3)

These are the three regions in which the blue rectangle divides $R_{\tau,[aba]}$.

We shall now analyze the spin structure case by case, given that for any correlator $\langle q_1 q_2 q_3 q_4 \rangle$ we know the common factor,

$$\frac{\left(\frac{2+2l+\tau+q_{43}}{2}\right)! \left(\frac{2+2l+\tau-q_{21}}{2}\right)!}{(2+2l+\tau)!}$$

(D.4)

and the degrees in spin of all the SCPW involved.

Let's assume without loss of generality that $p_1 + p_2 \leq p_3 + p_4$.

Configuration (II) is the simplest, giving a contribution of the form $\frac{\langle p_1 p_2 pq \rangle_{\text{conn.}} \langle pq p_3 p_4 \rangle_{conn}}{\langle pqpq \rangle_{\text{disc.}}}$, in which the factorials in disconnected free theory cancel with those in the numerator. The total degree of the numerator is

$$\overbrace{\min(p_1 + p_2, p + q) - 4 - (q - p - p_{21})}^{\langle p_1 p_2 pq \rangle} + \overbrace{\min(p + q, p_3 + p_4) - 4 - (p_{43} - q + p)}^{\langle pq p_3 p_4 \rangle}$$
$$= \min(p_1 + p_2, p + q) - 4 - (p_{43} - p_{21}) + (p + q - 4) \equiv e^\star$$

(D.5)

We deduce that any configuration of type (II) contributes in the large spin limit as

$$\frac{\langle p_1 p_2 pq \rangle_{\text{conn.}} \langle pq p_3 p_4 \rangle_{conn}}{\langle pqpq \rangle_{\text{disc.}}} \to \frac{\left(\frac{2+2l+\tau+p_{43}}{2}\right)! \left(\frac{2+2l+\tau-p_{21}}{2}\right)!}{(2+2l+\tau)!} l^{e^\star - (p+q-2)} (1 + O(1/l))$$

(D.6)

Let's consider now a contribution from configurations of type (I), i.e.
$\frac{\langle p_1 p_2 pq\rangle_{\text{conn.}} \langle p_3 p_4 pq\rangle_{conn}}{\langle pqpq\rangle_{\text{disc.}}}$. In this case the factorials come out misaligned. In particular we find

$$\frac{\left(\frac{2+2l+\tau-p_{21}}{2}\right)! \left(\frac{2+2l+\tau+q-p}{2}\right)! \left(\frac{2+2l+\tau-p_{43}}{2}\right)!}{\left(\frac{2+2l+\tau-p+q}{2}\right)!(2+2l+\tau)!}$$

$$= \frac{\left(\frac{2+2l+\tau+p_{43}}{2}\right)! \left(\frac{2+2l+\tau-p_{21}}{2}\right)!}{(2+2l+\tau)!} \left[\frac{\left(\frac{2+2l+\tau+q-p}{2}\right)! \left(\frac{2+2l+\tau-p_{43}}{2}\right)!}{\left(\frac{2+2l+\tau-p+q}{2}\right)! \left(\frac{2+2l+\tau+p_{43}}{2}\right)!}\right] \quad \text{(D.7)}$$

Notice that the term between square brackets is polynomial of degree $q-p-p_{43} \geq 0$, which is positive for type (I) configurations. There are small changes also in the computation of the degree of the numerator, i.e.

$$\overbrace{\min(p_1+p_2, p+q) - 4 - (q-p-p_{21})}^{\langle p_1 p_2 pq\rangle} + \overbrace{\min(p+q, p_3+p_4) - 4 - (q-p-p_{43})}^{\langle p_3 p_4 pq\rangle}$$
$$= e^\star + 2(p_{43} - q + p) \quad \text{(D.8)}$$

The result is that a configuration (I) in the large spin limit contributes as

$$\frac{\langle p_1 p_2 pq\rangle_{\text{conn.}} \langle p_3 p_4 pq\rangle_{conn}}{\langle pqpq\rangle_{\text{disc.}}} \to \frac{\left(\frac{2+2l+\tau+p_{43}}{2}\right)! \left(\frac{2+2l+\tau-p_{21}}{2}\right)!}{(2+2l+\tau)!} \frac{l^{e^\star-(p+q-2)}}{l^{q+p-p_{43}}}(1+O(1/l)) \quad \text{(D.9)}$$

The case of configurations of type (III) mirrors the case of configurations of type (I). Repeating the previous derivation with minor modifications we obtain that in the large spin limit a configurations of type (III) contributes as,

$$\frac{\langle pqp_1 p_2\rangle_{\text{conn.}} \langle pqp_3 p_4\rangle_{conn}}{\langle pqpq\rangle_{\text{disc.}}} \to \frac{\left(\frac{2+2l+\tau+p_{43}}{2}\right)! \left(\frac{2+2l+\tau-p_{21}}{2}\right)!}{(2+2l+\tau)!} \frac{l^{e^\star-(p+q-2)}}{l^{p_{21}-q+p}}(1+O(1/l)) \quad \text{(D.10)}$$

The first conclusion we draw is that when summing over pairs in $(pq) \in R_{\tau,[aba]}$ such that $p+q = \tau'$, the large spin limit is dominated by configurations of type (II), because compared to $l^{e^\star-(p+q-2)}$ both (I) and (III) are further suppressed.

In order to decide which line $p+q = \tau'$, in the range $b+2a+4 \leq \tau' \leq \tau$, gives the leading power law contribution, we further distinguish the two cases: $p_1+p_2 \leq \tau'$ and $4+2a+b \leq \tau' \leq p_1+p_2$. We find

$$\begin{array}{llll} \text{if} & p_1+p_2 \leq \tau' & \text{then} & e^\star - (\tau'-2) = (p_1+p_2) - (p_{43}-p_{21}) - 6 \\ \text{if} & \tau' \leq p_1+p_2 & \text{then} & e^\star - (\tau'-2) = \tau' - (p_{43}-p_{21}) - 6 \end{array} \quad \text{(D.11)}$$

Thus, in the expression

$$M^{(2)}_{\vec{p};\tau,l,[aba]} = \frac{\left(\frac{2+2l+\tau+p_{43}}{2}\right)! \left(\frac{2+2l+\tau-p_{21}}{2}\right)!}{(2+2l+\tau)!} \times \frac{num_{M^{(2)}}(l)}{den_{M^{(2)}}(l)} \quad \text{(D.12)}$$

the power law asymptotics of $num_{M^{(2)}}/den_{M^{(2)}}$, is $l^{-(p_{43}-p_{21})-6}\left[l^{(p_1+p_2)} + \ldots + l^{\tau'} + \ldots\right]$ and $l^{p_1+p_2}$ gives the leading contribution. Since we know independently that the greatest

denominator factor in the summation over $(pq) \in R_{\tau,[aba]}$ is given by disconnected free theory for $p + q = \tau$, we conclude that the degree in spin of $num_{M^{(2)}}$ has to be that of the greatest denominator, i.e $(\tau - 2)$, plus $(p_1 + p_2) - (p_{43} - p_{21}) - 6$. This implies the general result

$$
\begin{aligned}
\text{degree } num_{M^{(2)}}(l) &= (\tau - 4) + (\min(p_1 + p_2, p_3 + p_4) - 4 - (p_{43} + p_{21})) \\
&= (\tau - 4) + 2(\kappa_{\vec{p}} - 2) + p_{21}
\end{aligned}
\tag{D.13}
$$

after using the relation $\kappa_{\vec{p}} = \min\left(\frac{1}{2}(p_1 + p_2 + p_3 - p_4), p_3\right)$.

The details of the computation of $L^{(2)}_{\tau,(p_1 p_2);(p_3 p_4)}$ are very similar to those of $M^{(2)}$, but for an important detail. Since the twist τ belongs to the below window region, the rectangle $R_{\tau,[aba]}$ over which we are summing, is foliated by lines $n + m = \tau'$ with $\tau' = 4 + 2a + b, \ldots \tau < \min(p_1 + p_2, p_3 + p_4)$. Pictorially, the situation is as follows,

$$
\tag{D.14}
$$
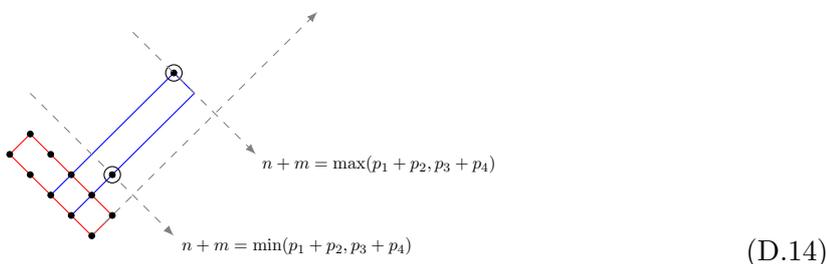
Therefore, comparing with (D.11), we now get for

$$
L^{(2)}_{\vec{p};\tau,l,[aba]} = \frac{\left(\frac{2+2l+\tau+p_{43}}{2}\right)! \left(\frac{2+2l+\tau-p_{21}}{2}\right)!}{(2+2l+\tau)!} \times \frac{num_{L^{(2)}}(l)}{den_{L^{(2)}}(l)}
\tag{D.15}
$$

that the leading asymptotic of $num_{L^{(2)}}/den_{L^{(2)}}$ only counts twists in the region $\tau' < p_1 + p_2$, as it should, thus it is has the form $l^{-(p_{43}-p_{21})-6}\left[l^\tau + \ldots\right]$. We obtain the general result,

$$
\text{degree } num_{L^{(2)}}(l) = 2(\tau - 4) - (p_{43} - p_{21}), \qquad \text{degree } den_{L^{(2)}}(l) = (\tau - 2).
\tag{D.16}
$$

Formulas (D.13) and (D.16) summarize our proof of the spin structure of the SCPW coefficients $M^{(2)}$ and $L^{(2)}$.

Finally, we distinguish between even and odd b.

In the even b case, even and odd spin cases go separately. Reciprocity implies that in both cases num is an even polynomial of the variable $2l + \tau + 3$,

$$
M^{(2)}_{\vec{p};\tau,l,[aba]} = \frac{\left(\frac{2+2l+\tau+p_{43}}{2}\right)! \left(\frac{2+2l+\tau-p_{21}}{2}\right)!}{(2+2l+\tau)!} \times \frac{num_{M^{(2)}}(2l+\tau+3)}{den_{M^{(2)}}(2l+\tau+3)}
\tag{D.17}
$$

$$
L^{(2)}_{\vec{p};\tau,l,[aba]} = \frac{\left(\frac{2+2l+\tau+p_{43}}{2}\right)! \left(\frac{2+2l+\tau-p_{21}}{2}\right)!}{(2+2l+\tau)!} \times \frac{num_{L^{(2)}}(2l+\tau+3)}{den_{L^{(2)}}(2l+\tau+3)}
\tag{D.18}
$$

In this odd b case, the polynomials num and den, for both $M^{(2)}$ and $L^{(2)}$ depend on the spin, whether it is even or odd. Picking

$$M^{(2)}_{\vec{p};\tau,\text{even},[aba]} = \frac{\left(\frac{2+2l+\tau+p_{43}}{2}\right)!\left(\frac{2+2l+\tau-p_{21}}{2}\right)!}{(2+2l+\tau)!} \times \frac{num^{\text{even}}_{M^{(2)}}(l)}{den^{\text{even}}_{M^{(2)}}(l)} \tag{D.19}$$

$$L^{(2)}_{\vec{p};\tau,\text{even},[aba]} = \frac{\left(\frac{2+2l+\tau+p_{43}}{2}\right)!\left(\frac{2+2l+\tau-p_{21}}{2}\right)!}{(2+2l+\tau)!} \times \frac{num_{L^{(2)}}(l)}{den_{L^{(2)}}(l)} \tag{D.20}$$

the SCPW coefficients corresponding to odd spins is then obtained by considering

$$num^{\text{odd}}_{M^{(2)}}(l) = num^{\text{even}}_{M^{(2)}}(-l-\tau-3) \qquad den^{\text{odd}}_{M^{(2)}}(l) = den^{\text{even}}_{M^{(2)}}(-l-\tau-3),$$
$$num^{\text{odd}}_{L^{(2)}}(l) = num^{\text{even}}_{L^{(2)}}(-l-\tau-3) \qquad den^{\text{odd}}_{L^{(2)}}(l) = den^{\text{even}}_{L^{(2)}}(-l-\tau-3).$$

The factorials do not transform.

Because of (B.25) and (3.26) we can say that the ratio $num^{\text{even}}(l)/den^{\text{even}}(l)$ for both $M^{(2)}$ and $L^{(2)}$ will always have a factor of $(l+\tau+2)$, thus reducing the degree of numerator and denominator. In particular, once $(l+\tau+2)$ is factored out, the degree of the auxiliary numerator is down by -2 and that of the denominator is down by -1.

D.1 Refining the one-loop ansatz with reciprocity

We conclude by illustrating the use of reciprocity symmetry in our bootstrap algorithm.

The starting point is the ansatz at the stage in which the leading logs have been matched, and there are no $x = \bar{x}$ poles. The idea is that whenever an OPE predictions in and below window is non trivial, rather than immediately input the prediction, we first impose the correct spin structure of SCPW coefficients on the ansatz, by using (D.17)–(D.20). Recall that in (D.17)–(D.20) we know the denominators. The corresponding numerators instead will be parametrized by a polynomial in l, according to the degree and the parity under $l \leftrightarrow -l-\tau-3$, as we understood in the previous section. We will leave the parameters in these polynomials free. We expect that imposing the spin structure solves a number of free parameters in the ansatz, and trade some of them for the new ones in the various numerators. We shall see in this way how much constraining is the spin structure of the SCPW alone.

3333. As we saw in section 4.4, there are non trivial OPE predictions below window at twist 4 in all $su(4)$ channels. We impose twist 4 SCPW of the form (3.41) of the form

$$\frac{X_{[000]}}{(l+1)(l+6)}, \qquad \frac{X_{[101]}}{(l+2)(l+5)}, \qquad \frac{X_{[020]}}{(l+3)(l+4)}. \tag{D.21}$$

In the table below we report the results of imposing (D.21) on the ansatz,

rep	twist	# free coef. autofit in the ansazt	# free coeff. left over in numerator
[000]	4	3	0
[020]	4	3	0
[101]	4	1	0

$$\tag{D.22}$$

What happens here is that the constraint from reciprocity is so strong that at the same time $X_{[000]}$, $X_{[101]}$, $X_{[020]}$ are fixed to their predicted values, and furthermore, the ansatz is left with no more free coefficients than the ambiguities, i.e. reciprocity fixes $\mathcal{H}^{(2)}_{3333}$ completely. However, we should highlight that the case of twist 4, and so 3333, is actually very special because all numerators entering (D.21) have just zero degree in spin.

We shall see in the next example that reciprocity is still powerful but the ansatz will not be completely fixed.

4444. This correlator exemplifies well the general story about reciprocity symmetry. For given rep $[aba]$ there are non trivial OPE predictions below window, either $\tau = 4$ or $\tau = 6$. We will impose that the ansatz has SCPW coefficients of the form

τ	$[000]$	$[101]$	$[020]$
4	$\dfrac{X_{[000]}}{(l+1)(l+6)}$	$\dfrac{X_{[101]}}{(l+2)(l+5)}$	$\dfrac{X_{[020]}}{(l+3)(l+4)}$
6	$\dfrac{Y^{(0)}_{[000]}+Y^{(2)}_{[000]}(l+\frac{9}{2})^2+Y^{(4)}_{[000]}(l+\frac{9}{2})^4}{(l+1)(l+2)(l+7)(l+8)}$	$\dfrac{Y^{(0)}_{[101]}+Y^{(2)}_{[101]}(l+\frac{9}{2})^2+Y^{(4)}_{[101]}(l+\frac{9}{2})^4}{(l+1)(l+3)(l+6)(l+8)}$	$\dfrac{Y^{(0)}_{[020]}+Y^{(2)}_{[020]}(l+\frac{9}{2})^2+Y^{(4)}_{[020]}(l+\frac{9}{2})^4}{(l+1)(l+4)(l+5)(l+8)}$

$$(\text{D.23})$$

and

τ	$[040]$	$[121]$	$[202]$
6	$\dfrac{Y^{(0)}_{[040]}+Y^{(2)}_{[040]}(l+\frac{9}{2})^2+Y^{(4)}_{[040]}(l+\frac{9}{2})^4}{(l+3)(l+4)(l+5)(l+6)}$	$\dfrac{Y^{(0)}_{[121]}+Y^{(2)}_{[121]}(l+\frac{9}{2})^2+Y^{(4)}_{[121]}(l+\frac{9}{2})^4}{(l+2)(l+4)(l+5)(l+7)}$	$\dfrac{Y^{(0)}_{[202]}+Y^{(2)}_{[202]}(l+\frac{9}{2})^2+Y^{(4)}_{[202]}(l+\frac{9}{2})^4}{(l+2)(l+3)(l+6)(l+7)}$

$$(\text{D.24})$$

with unknown coefficients in the numerators.

The actual OPE predictions give particular polynomials in the various entries of the tables (D.23) and (D.24). Comparing with the results in section 4.5, we see that in some cases the rational functions simplify further. However, according to our discussion about the spin structure, (D.23) and (D.24) are the most general.

The way the ansatz is refined by imposing reciprocity is reported in the table below. What happens is quite remarkable.

rep	twist	# free coef. autofitted in the ansazt	# free coeff. left over in the numerator
$[000]$	4	4	0
	6	4	1
$[101]$	4	4	0
	6	3	1
$[020]$	4	4	0
	6	3	1
$[040]$	6	3	1
$[121]$	6	1	0
$[202]$	6	2	0

$$(\text{D.25})$$

Firstly, at twist 4 reciprocity fixes completely the one variable resummation to its OPE prediction. Secondly, at twist 6, almost all free coefficients in the ansatz are fixed just by the symmetry. We can trade the 4 free coefficients left in the ansatz for those in (D.23) and (D.24). For example, $Y^{(0)}_{[000]}$, $Y^{(0)}_{[101]}$, $Y^{(0)}_{[020]}$, $Y^{(0)}_{[040]}$. Summarizing, there is (only!) a 4 free-coefficients ansatz (with ambiguities) which can potentially describe $\mathcal{H}^{(2)}_{4444}$. By matching the value of these remaining free coefficients in the ansatz to the OPE predictions below window, we obtain $\mathcal{H}^{(2)}_{4444}$.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

[1] L. Rastelli and X. Zhou, *Mellin amplitudes for $AdS_5 \times S^5$*, *Phys. Rev. Lett.* **118** (2017) 091602 [arXiv:1608.06624] [INSPIRE].

[2] L.F. Alday and A. Bissi, *Loop Corrections to Supergravity on $AdS_5 \times S^5$*, *Phys. Rev. Lett.* **119** (2017) 171601 [arXiv:1706.02388] [INSPIRE].

[3] F. Aprile, J.M. Drummond, P. Heslop and H. Paul, *Quantum Gravity from Conformal Field Theory*, *JHEP* **01** (2018) 035 [arXiv:1706.02822] [INSPIRE].

[4] F. Aprile, J.M. Drummond, P. Heslop and H. Paul, *Unmixing Supergravity*, *JHEP* **02** (2018) 133 [arXiv:1706.08456] [INSPIRE].

[5] L. Rastelli and X. Zhou, *How to Succeed at Holographic Correlators Without Really Trying*, *JHEP* **04** (2018) 014 [arXiv:1710.05923] [INSPIRE].

[6] L.F. Alday and S. Caron-Huot, *Gravitational S-matrix from CFT dispersion relations*, *JHEP* **12** (2018) 017 [arXiv:1711.02031] [INSPIRE].

[7] F. Aprile, J.M. Drummond, P. Heslop and H. Paul, *Loop corrections for Kaluza-Klein AdS amplitudes*, *JHEP* **05** (2018) 056 [arXiv:1711.03903] [INSPIRE].

[8] F. Aprile, J. Drummond, P. Heslop and H. Paul, *Double-trace spectrum of $N = 4$ supersymmetric Yang-Mills theory at strong coupling*, *Phys. Rev.* **D 98** (2018) 126008 [arXiv:1802.06889] [INSPIRE].

[9] S. Caron-Huot and A.-K. Trinh, *All tree-level correlators in $AdS_5 \times S_5$ supergravity: hidden ten-dimensional conformal symmetry*, *JHEP* **01** (2019) 196 [arXiv:1809.09173] [INSPIRE].

[10] L.F. Alday, A. Bissi and E. Perlmutter, *Genus-One String Amplitudes from Conformal Field Theory*, *JHEP* **06** (2019) 010 [arXiv:1809.10670] [INSPIRE].

[11] L.F. Alday, *On Genus-one String Amplitudes on $AdS_5 \times S^5$*, arXiv:1812.11783 [INSPIRE].

[12] V. Gonçalves, R. Pereira and X. Zhou, *20′ Five-Point Function from $AdS_5 \times S^5$ Supergravity*, *JHEP* **10** (2019) 247 [arXiv:1906.05305] [INSPIRE].

[13] S.M. Chester, *Genus-2 Holographic Correlator on $AdS_5 \times S^5$ from Localization*, arXiv:1908.05247 [INSPIRE].

[14] O. Aharony, L.F. Alday, A. Bissi and E. Perlmutter, *Loops in AdS from Conformal Field Theory*, *JHEP* **07** (2017) 036 [arXiv:1612.03891] [INSPIRE].

[15] E.Y. Yuan, *Simplicity in AdS Perturbative Dynamics*, arXiv:1801.07283 [INSPIRE].

[16] I. Bertan, I. Sachs and E.D. Skvortsov, *Quantum ϕ^4 Theory in AdS_4 and its CFT Dual*, *JHEP* **02** (2019) 099 [arXiv:1810.00907] [INSPIRE].

[17] K. Ghosh, *Polyakov-Mellin Bootstrap for AdS loops*, *JHEP* **02** (2020) 006 [arXiv:1811.00504] [INSPIRE].

[18] D. Ponomarev, *From bulk loops to boundary large-N expansion*, *JHEP* **01** (2020) 154 [arXiv:1908.03974] [INSPIRE].

[19] D. Carmi, *Loops in AdS: From the Spectral Representation to Position Space*, arXiv:1910.14340 [INSPIRE].

[20] L.J. Dixon, J.M. Drummond, C. Duhr and J. Pennington, *The four-loop remainder function and multi-Regge behavior at NNLLA in planar $N = 4$ super-Yang-Mills theory*, *JHEP* **06** (2014) 116 [arXiv:1402.3300] [INSPIRE].

[21] J.M. Drummond, G. Papathanasiou and M. Spradlin, *A Symbol of Uniqueness: The Cluster Bootstrap for the 3-Loop MHV Heptagon*, *JHEP* **03** (2015) 072 [arXiv:1412.3763] [INSPIRE].

[22] S. Caron-Huot, L.J. Dixon, A. McLeod and M. von Hippel, *Bootstrapping a Five-Loop Amplitude Using Steinmann Relations*, *Phys. Rev. Lett.* **117** (2016) 241601 [arXiv:1609.00669] [INSPIRE].

[23] L.J. Dixon, J. Drummond, T. Harrington, A.J. McLeod, G. Papathanasiou and M. Spradlin, *Heptagons from the Steinmann Cluster Bootstrap*, *JHEP* **02** (2017) 137 [arXiv:1612.08976] [INSPIRE].

[24] J.L. Bourjaily, A. DiRe, A. Shaikh, M. Spradlin and A. Volovich, *The Soft-Collinear Bootstrap: $N = 4$ Yang-Mills Amplitudes at Six and Seven Loops*, *JHEP* **03** (2012) 032 [arXiv:1112.6432] [INSPIRE].

[25] B. Eden, P. Heslop, G.P. Korchemsky and E. Sokatchev, *Constructing the correlation function of four stress-tensor multiplets and the four-particle amplitude in $N = 4$ SYM*, *Nucl. Phys.* **B 862** (2012) 450 [arXiv:1201.5329] [INSPIRE].

[26] D. Chicherin, R. Doobary, B. Eden, P. Heslop, G.P. Korchemsky and E. Sokatchev, *Bootstrapping correlation functions in $N = 4$ SYM*, *JHEP* **03** (2016) 031 [arXiv:1506.04983] [INSPIRE].

[27] D. Chicherin, J. Drummond, P. Heslop and E. Sokatchev, *All three-loop four-point correlators of half-BPS operators in planar $\mathcal{N} = 4$ SYM*, *JHEP* **08** (2016) 053 [arXiv:1512.02926] [INSPIRE].

[28] J.L. Bourjaily, P. Heslop and V.-V. Tran, *Amplitudes and Correlators to Ten Loops Using Simple, Graphical Bootstraps*, *JHEP* **11** (2016) 125 [arXiv:1609.00007] [INSPIRE].

[29] D. Chicherin, A. Georgoudis, V. Gonçalves and R. Pereira, *All five-loop planar four-point functions of half-BPS operators in $\mathcal{N} = 4$ SYM*, *JHEP* **11** (2018) 069 [arXiv:1809.00551] [INSPIRE].

[30] J. Drummond, C. Duhr, B. Eden, P. Heslop, J. Pennington and V.A. Smirnov, *Leading singularities and off-shell conformal integrals*, *JHEP* **08** (2013) 133 [arXiv:1303.6909] [INSPIRE].

[31] V. Gonçalves, *Four point function of $\mathcal{N} = 4$ stress-tensor multiplet at strong coupling*, *JHEP* **04** (2015) 150 [arXiv:1411.1675] [INSPIRE].

[32] D.J. Binder, S.M. Chester, S.S. Pufu and Y. Wang, $\mathcal{N} = 4$ *Super-Yang-Mills correlators at strong coupling from string theory and localization*, *JHEP* **12** (2019) 119 [arXiv:1902.06263] [INSPIRE].

[33] J.M. Drummond, D. Nandan, H. Paul and K.S. Rigatos, *String corrections to AdS amplitudes and the double-trace spectrum of* $\mathcal{N} = 4$ *SYM*, *JHEP* **12** (2019) 173 [arXiv:1907.00992] [INSPIRE].

[34] R. Doobary and P. Heslop, *Superconformal partial waves in Grassmannian field theories*, *JHEP* **12** (2015) 159 [arXiv:1508.03611] [INSPIRE].

[35] B. Eden, A.C. Petkou, C. Schubert and E. Sokatchev, *Partial nonrenormalization of the stress tensor four point function in* $N = 4$ *SYM and AdS/CFT*, *Nucl. Phys.* **B 607** (2001) 191 [hep-th/0009106] [INSPIRE].

[36] F.A. Dolan, M. Nirschl and H. Osborn, *Conjectures for large N superconformal* $N = 4$ *chiral primary four point functions*, *Nucl. Phys.* **B 749** (2006) 109 [hep-th/0601148] [INSPIRE].

[37] G. Arutyunov and S. Frolov, *Some cubic couplings in type IIB supergravity on* $AdS_5 \times S^5$ *and three point functions in* $SYM(4)$ *at large N*, *Phys. Rev.* **D 61** (2000) 064009 [hep-th/9907085] [INSPIRE].

[38] E. D'Hoker, D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, *Extremal correlators in the AdS/CFT correspondence*, hep-th/9908160 [INSPIRE].

[39] G. Arutyunov, R. Klabbers and S. Savin, *Four-point functions of 1/2-BPS operators of any weights in the supergravity approximation*, *JHEP* **09** (2018) 118 [arXiv:1808.06788] [INSPIRE].

[40] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, *Unconstrained* $N = 2$ *Matter, Yang-Mills and Supergravity Theories in Harmonic Superspace*, *Class. Quant. Grav.* **1** (1984) 469 [*Erratum ibid.* **2** (1985) 127] [INSPIRE].

[41] P.S. Howe and G.G. Hartwell, *A Superspace survey*, *Class. Quant. Grav.* **12** (1995) 1823 [INSPIRE].

[42] G.G. Hartwell and P.S. Howe, (N, p, q) *harmonic superspace*, *Int. J. Mod. Phys.* **A 10** (1995) 3901 [hep-th/9412147] [INSPIRE].

[43] P.J. Heslop, *Superfield representations of superconformal groups*, *Class. Quant. Grav.* **19** (2002) 303 [hep-th/0108235] [INSPIRE].

[44] P.J. Heslop and P.S. Howe, *Four point functions in* $N = 4$ *SYM*, *JHEP* **01** (2003) 043 [hep-th/0211252] [INSPIRE].

[45] M. Bianchi, F.A. Dolan, P.J. Heslop and H. Osborn, $N = 4$ *superconformal characters and partition functions*, *Nucl. Phys.* **B 767** (2007) 163 [hep-th/0609179] [INSPIRE].

[46] G. Arutyunov, B. Eden and E. Sokatchev, *On nonrenormalization and OPE in superconformal field theories*, *Nucl. Phys.* **B 619** (2001) 359 [hep-th/0105254] [INSPIRE].

[47] P.J. Heslop and P.S. Howe, *A Note on composite operators in* $N = 4$ *SYM*, *Phys. Lett.* **B 516** (2001) 367 [hep-th/0106238] [INSPIRE].

[48] I. Heemskerk, J. Penedones, J. Polchinski and J. Sully, *Holography from Conformal Field Theory*, *JHEP* **10** (2009) 079 [arXiv:0907.0151] [INSPIRE].

[49] L.F. Alday, A. Bissi and T. Lukowski, *Lessons from crossing symmetry at large N*, *JHEP* **06** (2015) 074 [arXiv:1410.4717] [INSPIRE].

[50] Y.L. Dokshitzer, G. Marchesini and G.P. Salam, *Revisiting parton evolution and the large-x limit*, *Phys. Lett.* **B 634** (2006) 504 [`hep-ph/0511302`] [ɪɴSPIRE].

[51] B. Basso and G.P. Korchemsky, *Anomalous dimensions of high-spin operators beyond the leading order*, *Nucl. Phys.* **B 775** (2007) 1 [`hep-th/0612247`] [ɪɴSPIRE].

[52] L.F. Alday, A. Bissi and T. Lukowski, *Large spin systematics in CFT*, *JHEP* **11** (2015) 101 [`arXiv:1502.07707`] [ɪɴSPIRE].

[53] B. Basso, F. Coronado, S. Komatsu, H.T. Lam, P. Vieira and D.-l. Zhong, *Asymptotic Four Point Functions*, *JHEP* **07** (2019) 082 [`arXiv:1701.04462`] [ɪɴSPIRE].

[54] F. Coronado, *Bootstrapping the simplest correlator in planar $\mathcal{N} = 4$ SYM at all loops*, `arXiv:1811.03282` [ɪɴSPIRE].

[55] T. Bargheer, F. Coronado and P. Vieira, *Octagons I: Combinatorics and Non-Planar Resummations*, *JHEP* **08** (2019) 162 [`arXiv:1904.00965`] [ɪɴSPIRE].

[56] B. Basso and D.-L. Zhong, *Three-point functions at strong coupling in the BMN limit*, `arXiv:1907.01534` [ɪɴSPIRE].

[57] T. Bargheer, F. Coronado and P. Vieira, *Octagons II: Strong Coupling*, `arXiv:1909.04077` [ɪɴSPIRE].

[58] C. Beem, M. Lemos, L. Rastelli and B.C. van Rees, *The $(2,0)$ superconformal bootstrap*, *Phys. Rev.* **D 93** (2016) 025016 [`arXiv:1507.05637`] [ɪɴSPIRE].

[59] P. Heslop and A.E. Lipstein, *M-theory Beyond The Supergravity Approximation*, *JHEP* **02** (2018) 004 [`arXiv:1712.08570`] [ɪɴSPIRE].

[60] S.M. Chester and E. Perlmutter, *M-Theory Reconstruction from $(2,0)$ CFT and the Chiral Algebra Conjecture*, *JHEP* **08** (2018) 116 [`arXiv:1805.00892`] [ɪɴSPIRE].

[61] T. Abl, P. Heslop and A.E. Lipstein, *Recursion relations for anomalous dimensions in the 6d $(2,0)$ theory*, *JHEP* **04** (2019) 038 [`arXiv:1902.00463`] [ɪɴSPIRE].