

Splitting of two-component solitary waves from collisions with narrow potential barriersCallum L. Grimshaw , Simon A. Gardiner, and Boris A. Malomed *Joint Quantum Centre Durham–Newcastle, Department of Physics, Durham University, Durham, DH1 3LE, United Kingdom and Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering, and Center for Light-Matter Interaction, Tel Aviv University, Tel Aviv 69978, Israel*

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We consider the interaction of two-component bright-bright solitons with a narrow potential barrier (splitter) in the framework of a system of two Gross-Pitaevskii (nonlinear-Schrödinger) equations modeling a binary Bose-Einstein condensate, with self-attraction in each component and cross-attraction between them. The objective is to study splitting of composite solitons, which may be used in the design of two-component soliton-interferometer schemes. We produce approximate analytic results, assuming a weak barrier and applying the perturbation theory in the limit in which the system is integrable and the solitary waves may be considered as exact solitons. We do this in the case of negligible interspecies interactions, and also when the nonlinearities are strongly asymmetric, allowing one to neglect the self-interaction in one of the species. Then, we use systematic simulations to study the transmissions of both components in regions outside these approximations and, in particular, to compare numerical results with their analytical counterparts. We conclude that there is an appreciable parameter range where one component is almost entirely transmitted through the barrier, while the other one is reflected. The excitation of internal vibrations in the passing and rebounding solitons is explored as well, with a conclusion that it is weak in the regime of high-quality splitting.

DOI: [10.1103/PhysRevA.101.043623](https://doi.org/10.1103/PhysRevA.101.043623)**I. INTRODUCTION**

Solitons, or more broadly solitary waves, are manifest in a broad range of physical settings [1,2], including, in particular, nonlinear matter waves in atomic Bose-Einstein condensates (BECs) [3–6]. In the mean-field approximation, the commonly adopted dynamical model of a BEC is based on the Gross-Pitaevskii equation (GPE) for a single-component condensate, and a system of coupled GPEs for binary (two-component) mixtures [7]. One of the potential applications of matter-wave solitons is their use in the design of interferometers, in which an incident soliton splits into two fragments upon hitting a narrow potential barrier, followed by recombination of the fragments after rebounding from the confining potential. An object to be probed by the interferometer is placed as an obstacle through which one fragment will pass, which will affect the outcome of the recombination [8–10]. Soliton interferometers have been elaborated theoretically in various configurations [11–20] (including the case when the splitter is inserted as a localized self-repulsive nonlinearity, or its combination with the usual potential barrier [21]) and realized in experiment [9]. Interactions of matter-wave solitons with local potentials have also been studied in other contexts, such as an analytical treatment of the collisions [22], rebound from potential wells [23,24], dynamics of solitons in a dipolar BEC [11], and probing effects of interparticle interactions on tunneling [25,26]. However, the splitting of a fundamental soliton by a linear and/or nonlinear potential barrier implies, in a sense, the application of a “brute force” to the soliton, as its intrinsic structure does not resonate with the action of the splitter. A more natural option, which was

elaborated recently, is fission of a 2-soliton (breather) into its fundamental-soliton constituents [27] (see also Ref. [28]), with the amplitude ratio close to the natural value, 3 : 1 [29] (see also Ref. [30] for a similarly motivated protocol involving a laser pulse in combination with control of the scattering length). These settings may also be realized in the context of optics, with GPE replaced by the nonlinear Schrödinger equation (NLSE) for the spatial-domain propagation of light in planar waveguides [2].

In this work, we aim to elaborate another natural scheme for the splitting, when an incident two-component soliton, governed by a pair of nonlinearly coupled GPEs, hits a narrow splitting barrier. The situation under consideration is one with equal atomic masses and equal negative scattering lengths in the two components, and attractive interaction between the components, while there is no linear coupling (interconversion) between them (interconversion would make splitting of a composite soliton into single-component ones impossible). We note that by replacing time in the coupled GPEs by the propagation distance z , this model also applies to bimodal light propagation in a Kerr-nonlinear waveguide with transverse coordinate x , where ψ_1 and ψ_2 are amplitudes of two components of the electromagnetic wave, corresponding to different carrier wavelengths [2], and where the potential represents transverse modulation of the refractive index. However, in this case the strength of the cross-interaction can only take a single value [$g = 2$, in terms of the notation adopted below in Eqs. (6a) and (6b)], as there is no straightforward optical counterpart to the Feshbach-resonance technique. Alternatively, if ψ_1 and ψ_2 represent amplitudes of two waves with mutually orthogonal linear polarizations,

the relevant value is $g = 2/3$, provided that rapidly oscillating four-wave-mixing terms may be neglected [2].

As mentioned above, previous works addressed collisions of single-component solitons with potential barriers, represented by an ideal δ function or a narrow Gaussian potential barrier, aiming to identify outcomes of the collisions as functions of the velocity of the incoming soliton and the barrier's height and width [8,22]. The dynamics of two-component solitons was studied with regard to their intrinsic vibrations in free space [31], as well as collisions between two solitons in the presence of a narrow Gaussian barrier added to the Manakov's system with equal coefficients of the self- and cross-attraction [32], and also the scattering of dark-bright solitons by impurities [33]. The main objective of the present work is to identify a parameter region in which the collision of a composite solitary wave with the barrier effectively splits it into single-component constituents. The primary control parameters that we consider are the relative norm of the components, defined as parameter f , the velocity of the incident soliton, the strength of the barrier ε [see Eq. (5)], and the relative strength of the interspecies attraction g [see Eqs. (6a) and (6b)]. We first report approximate analytical results, obtained for the system with a weak barrier, in Sec. II. We then summarize results of systematic numerical simulations of the collisions in Sec. III. We compare analytical results to their numerical counterparts in Sec. III, and conclude the paper by Sec. IV. Some technical details are presented in Appendixes A and B.

II. SYSTEM

We consider a binary BEC system, with two components corresponding to different internal states of the same atomic species, and collisions dominated by the s -wave scattering. We model the system by two coupled GPEs, assuming, as usual, that the mean-field wave functions of the two components are radially confined by a tight trapping potential in the transverse (y, z) plane, and weakly confined in the axial (x) direction, if at all. In addition, we assume that an off-resonant sheet of light, propagating perpendicular to the axial direction, with peak beam strength E_B and axial width x_r (defined at the relative amplitude level $1/e^2$), creates a barrier potential for both components, centered at $x = 0$ [10,34]. We assume that the transverse and barrier potentials are insensitive to the internal atomic state, allowing the coupled GPEs to take the form of

$$i\hbar \frac{\partial}{\partial t} \Psi_1(\mathbf{r}) = \left[-\frac{\hbar^2 \nabla^2}{2m} + V(x) + \frac{1}{2} m \omega_r^2 (y^2 + z^2) \right] \Psi_1(\mathbf{r}) + \frac{4\pi \hbar^2 N}{m} [a_{11} |\Psi_1(\mathbf{r})|^2 + a_{12} |\Psi_2(\mathbf{r})|^2] \Psi_1(\mathbf{r}), \quad (1a)$$

$$i\hbar \frac{\partial}{\partial t} \Psi_2(\mathbf{r}) = \left[-\frac{\hbar^2 \nabla^2}{2m} + V(x) + \frac{1}{2} m \omega_r^2 (y^2 + z^2) \right] \Psi_2(\mathbf{r}) + \frac{4\pi \hbar^2 N}{m} [a_{22} |\Psi_2(\mathbf{r})|^2 + a_{12} |\Psi_1(\mathbf{r})|^2] \Psi_2(\mathbf{r}), \quad (1b)$$

where m is the atomic mass; a_{11} , a_{22} , and a_{12} are the intra and interspecies s -wave scattering lengths; $V(x) = E_B e^{-2x^2/x_r^2} + m\omega_r^2 x^2/2$ is, as said above, the combination of the barrier potentials and a weak axial trapping one; ω_T and ω_r are axial and radial trapping frequencies; and N is the total number of particles. Equations (1a) and (1b) are supplemented by the normalization convention

$$\int d\mathbf{r} |\Psi_1(\mathbf{r})|^2 = f, \quad \int d\mathbf{r} |\Psi_2(\mathbf{r})|^2 = 1 - f, \quad (2)$$

so that

$$\int d\mathbf{r} [|\Psi_1(\mathbf{r})|^2 + |\Psi_2(\mathbf{r})|^2] = 1, \quad (3)$$

hence the numbers of particles in the two components are $N_1 = fN$ and $N_2 = (1 - f)N$.

Strong radial confinement then permits us to assume the usual Gaussian ansatz $\phi(y, z) = (m\omega_r/\pi\hbar)^{1/2} \exp(-m\omega_r[y^2 + z^2]/2\hbar)$ for the radial degrees of freedom of the condensate wave functions $\Psi_1(\mathbf{r})$, $\Psi_2(\mathbf{r})$. We integrate over the transverse plane (with coordinates y and z), define $g_{11} \equiv 2\hbar\omega_r a_{11}$, and express the result in terms of a notation with unit coordinate $\hbar^2/m|g_{11}|N$, unit time $\hbar^3/m(g_{11}N)^2$, and unit energy $m(g_{11}N/\hbar)^2$ [which implies unit velocity $|g_{11}|N/\hbar$, and that, after the integration over the transverse plane, we multiply the condensate wave functions by $\hbar/(m|g_{11}|N)^{1/2}$ to render them dimensionless].¹ As a result, the three-dimensional (3D) system of Eqs. (1a) and (1b) is reduced to the one-dimensional (1D) form:

$$i \frac{\partial}{\partial t} \psi_1(x) = \left[-\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} \omega_x^2 x^2 + \varepsilon \eta(x, \sigma) \right] \psi_1(x) - [|\psi_1(x)|^2 + g |\psi_2(x)|^2] \psi_1(x), \quad (4a)$$

$$i \frac{\partial}{\partial t} \psi_2(x) = \left[-\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} \omega_x^2 x^2 + \varepsilon \eta(x, \sigma) \right] \psi_2(x) - [g' |\psi_2(x)|^2 + g |\psi_1(x)|^2] \psi_2(x), \quad (4b)$$

where $g = a_{12}/a_{11}$, $g' = a_{22}/a_{11}$, $\omega_x = \omega_T \hbar^3/m|g_{11}|^2 N^2$, $\varepsilon = E_B x_r (\pi/2)^{1/2}/m^2 |g_{11}|^3 N^3$, $\sigma = x_r/2\hbar^2/m|g_{11}|N$, and

$$\eta(x, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-x^2/2\sigma^2), \quad (5)$$

such that $\lim_{\sigma \rightarrow 0} \eta(x, \sigma) = \delta(x)$, and we assumed all the scattering lengths to be negative (i.e., that all the interactions are attractive). Note that, as mentioned above, the relative strength of the cross-attraction between the components g can be effectively adjusted by means of the Feshbach-resonance technique [35,36], and $\varepsilon > 0$ is the strength of the splitting barrier. Direct control of properties of binary BECs was demonstrated in Ref. [37] for a heteronuclear BEC, and in Ref. [38] for a BEC composed of different hyperfine states of the same species, where the interspecies interaction was varied to probe the miscibility-immiscibility transition. Although it has become a fairly standard technique to control the scattering length in BEC systems, conventionally using

¹This can be thought of heuristically as notation with $\hbar = m = g_{11}N = 1$.

magnetic Feshbach resonances, there are limitations as to what can be achieved in multicomponent systems. For instance, when exploiting a magnetic Feshbach resonance, in principle all of the scattering lengths depend on the value of the applied magnetic field and therefore cannot be varied independently. Hence the Feshbach resonance technique must be utilized in such a way that the three scattering lengths (in the two-component system studied here) are brought as close as possible to their desired values. For the numerical results presented in Sec. III we fix $g' = 1$ in Eq. (4b) (i.e., $a_{11} = a_{22}$) and vary g . We point out that in the particular case of $g = 0$, in the GPE system considered in this paper, the value of g' has a role equivalent to that of f [see Eq. (2)], in that it defines the relative self-interaction strength of the two condensate components. Furthermore, there may be regimes where g , for example, can be controlled essentially independently, keeping the intraspecies scattering lengths close to their background values over relevant values of the magnetic field and $a_{11} \approx a_{22}$. The use of the optical Feshbach-resonance techniques may also be considered, although in this case one would need to accept the large loss rates that typically accompany the optically induced resonance [39,40]. Still another possibility is the use of laser-assisted magnetic Feshbach resonances [41]. To some extent, one additionally has a choice between different atomic species, with different background values of the scattering lengths, and their different dependencies on the applied fields.

In this paper we always assume $(1-f)g' \geq f$, i.e., $f \leq g'/(1+g')$, which effectively defines component 2 as the one with the largest intraspecies mean-field attraction; put in terms of particle numbers and scattering lengths, this means that $N_1 a_{11} \leq N_2 a_{22}$. In the case of $g' = 1$ (i.e., $a_{22} = a_{11}$), this assumption reduces to $f \leq 1/2$ (i.e., $N_1 \leq N_2$); and in the case of $f = 1/2$ (i.e., $N_1 = N_2$), it reduces to $g' \geq 1$ ($a_{22} \geq a_{11}$). In our analytical treatment we choose, in general, to keep both f and g' as separate parameters that may be individually varied. Then, variation of f corresponds to relative changes in the numbers of particles in the two components, and the variation of g' (and g) correspond to changes in the (negative) scattering lengths. This is an appropriate representation of what may be altered in the experimental work with BEC, although the basic results presented in the next section may be readily understood also in the simpler limit case of $g' = 1$.

III. ANALYTICAL CONSIDERATIONS

A. Simplified system

To address an infinitesimally narrow barrier in free 1D space, we set $\omega_x = 0$ and $\sigma \rightarrow 0$ in Eqs. (4) and (5), which leads to a more analytically tractable form of the GPE system:

$$i \frac{\partial \psi_1}{\partial t} = \left[-\frac{1}{2} \frac{\partial^2}{\partial x^2} + \varepsilon \delta(x) - |\psi_1|^2 - g |\psi_2|^2 \right] \psi_1, \quad (6a)$$

$$i \frac{\partial \psi_2}{\partial t} = \left[-\frac{1}{2} \frac{\partial^2}{\partial x^2} + \varepsilon \delta(x) - g' |\psi_2|^2 - g |\psi_1|^2 \right] \psi_2, \quad (6b)$$

which we use in this section. The system can be derived from the Hamiltonian (conserved energy of the mean-field

theory)

$$H[\psi_1, \psi_1^*, \psi_2, \psi_2^*] = \int_{-\infty}^{+\infty} dx \left[\frac{1}{2} \left(\left| \frac{\partial \psi_1}{\partial x} \right|^2 + \left| \frac{\partial \psi_2}{\partial x} \right|^2 \right) - \frac{1}{2} (|\psi_1|^4 + g' |\psi_2|^4 + 2g |\psi_1|^2 |\psi_2|^2) + \varepsilon \delta(x) (|\psi_1|^2 + |\psi_2|^2) \right]. \quad (7)$$

Setting $\varepsilon = 0$, two integrable cases of the system can be identified: If $g = 0$, Eqs. (6) reduce to a pair of uncoupled integrable NLSEs; and, if $g = g' = 1$, Eqs. (6) constitute the integrable Manakov system [42]. Note that the case of $g = -1$, which corresponds to the repulsive interspecies interaction is also integrable. It should also be noted that, even in the attractive system in which the Manakov condition, $g = g' = 1$, does not hold, two-component bright soliton solutions, with identical mode profiles of both components proportional to $\text{sech}(F[x - (x_0 + vt)]/2)$ [the same form as in Eqs. (9), see below], exist if $f + (1-f)g = (1-f)g' + fg \equiv F$. The later condition is tantamount to setting $f = (g' - g)/(g' + 1 - 2g)$, or, in terms of the numbers of particles and (negative) scattering lengths,

$$\frac{N_1}{N} = \frac{a_{12} - a_{22}}{2a_{12} - a_{11} - a_{22}}. \quad (8)$$

The availability of the integrable forms of the system for $\varepsilon = 0$ provides a natural framework for the application of the perturbation theory in the case of relatively small ε .

B. Limit of negligible interspecies interactions

Here we address Eq. (6) with $g = 0$ (no interspecies interactions), with an input in the form of a composite soliton with coinciding centers of both species, moving as $\xi(t) = x_0 + vt$. Far from the potential barrier, the composite soliton is built as

$$\psi_1(x, t) = \frac{1}{2} f \exp[i(vx - \mu_1 t)] \text{sech}\left(\frac{1}{2} f [x - (x_0 + vt)]\right), \quad (9a)$$

$$\psi_2(x, t) = \frac{1}{2} (1-f) \sqrt{g'} \exp[i(vx - \mu_1 t)] \times \text{sech}\left(\frac{1}{2} (1-f) g' [x - (x_0 + vt)]\right), \quad (9b)$$

with chemical potentials

$$\mu_1 = -f^2/8 + v^2/2, \quad (10a)$$

$$\mu_2 = -(1-f)^2 g'^2/8 + v^2/2. \quad (10b)$$

In the present case of $g = 0$, the perturbation theory is based on a natural conjecture that each component passes the barrier under the condition that its center-of-mass kinetic energy exceeds the respective peak potential energy, as given by Eq. (A4) in Appendix A 1. Utilizing the solutions of Eqs. (9), we calculated the respective energy terms from Eq. (7), as summarized in Appendix A 1, and used them to identify regimes in which the components do or do not pass the barrier. In addition, we used these terms to estimate a validity region of this approach. From Eqs. (A3) and (A4) it

follows that the condition for ε to be sufficiently small is

$$\varepsilon \ll f/3. \quad (11)$$

Combining Eqs. (A1) and (A4) results in the conditions

$$v^2 > \varepsilon f/2, \quad (12a)$$

$$v^2 > \varepsilon(1-f)g'/2, \quad (12b)$$

for components 1 and 2, respectively, to be transmitted. We thus predict the incident composite soliton to be split into a pair of a transmitted soliton in component 1 and a reflected one in component 2 in the following interval of velocities:

$$\sqrt{\varepsilon f/2} < |v| < \sqrt{\varepsilon(1-f)g'/2} \quad (13)$$

[recall that we set $f \leq (1-f)g'$]. We expect that the prediction given by Eq. (13) remains valid in the case of nonzero but weak interspecies attraction $g \ll 1$.

A similar approach can also be used in the case when the splitter is nonlinear (corresponding to a steep increase in the interspecies scattering length over a small spatial region). The analysis for the later case is produced in Appendix A 2.

C. Limit of a strongly asymmetric two-component soliton

One can also carry out a perturbative analysis for small ε in the case when the intraspecies self-interaction of component 1 is much weaker than its attractive interaction with component 2, i.e., $f \ll (1-f)g$, and, accordingly, the intraspecies self-attraction of component 2 is much stronger than its interaction with component 1, i.e., $fg \ll (1-f)g'$. These conditions are summarized as

$$\frac{f}{(1-f)} \ll g \ll \frac{g'(1-f)}{f}, \quad (14)$$

which simplify to $f/(1-f) \ll g \ll (1-f)/f$ when $g' = 1$, and to $1 \ll g \ll g'$ when $f = 1 - f = 1/2$.

In the present case, component 2 of the incident mode is essentially the usual NLSE bright soliton, as given by Eqs. (9b) and (10b), while a solution for component 1 is sought for as

$$\psi_1(x, t) = \exp(i[vx - (\mu_1^{(0)} + v^2/2)t])u_1(x - [x_0 + vt]), \quad (15)$$

with u_1 determined by a stationary linear Schrödinger equation

$$\begin{aligned} \mu_1^{(0)}u_1(X) = & -\frac{1}{2}\frac{d^2}{dX^2}u_1(X) - \frac{1}{4}(1-f)^2gg' \\ & \times \operatorname{sech}^2\left[\frac{1}{2}(1-f)g'X\right]u_1(X), \end{aligned} \quad (16)$$

where $X \equiv x - (x_0 + vt) \equiv X$. Equation (16) describes a 1D quantum particle in a Pöschl-Teller potential, which, generally, can be solved in terms of special functions.² The

exact ground-state solution to Eq. (16) and the corresponding eigenvalue are given by

$$u_1(X) = A_1 \left(\operatorname{sech}\left[\frac{1}{2}(1-f)g'X\right] \right)^\alpha, \quad (17a)$$

$$(\mu_1^{(0)})_{\text{ground}} = -\frac{1}{8}\alpha(1-f)g'^2, \quad (17b)$$

where

$$\alpha = \sqrt{\frac{1}{4} + \frac{2g}{g'}} - \frac{1}{2}. \quad (18)$$

Amplitude A_1 in Eq. (17a) is, by itself, arbitrary. However, as the norm of component 1 is fixed to be f , A_1 is determined by condition

$$f \equiv \int_{-\infty}^{+\infty} dX |u_1(X)|^2 = A_1^2 \frac{2\sqrt{\pi} \Gamma(\alpha)}{(1-f)g' \Gamma(\alpha + 1/2)}, \quad (19)$$

where Γ is the Γ function.

Next, we follow the same approach as in the previous subsection, but with the wave form of component 1 given by Eq. (17a), to identify regimes in which component 1 is transmitted (see Appendix A 3 a), while component 2 is reflected by the barrier. Thus, the condition for the reflection of component 1 is obtained as

$$v^2 > \frac{\varepsilon(1-f)g' \Gamma(\alpha + 1/2)}{\sqrt{\pi} \Gamma(\alpha)}. \quad (20)$$

On the other hand, the condition for the reflection of component 2 remains, in the first approximation, the same as given by Eq. (12b). Hence this condition becomes $v^2 < \varepsilon g'(1-f)/2$, and there are intervals of velocities defined by

$$\sqrt{\frac{\varepsilon(1-f)g' \Gamma(\alpha + 1/2)}{\sqrt{\pi} \Gamma(\alpha)}} < |v| < \sqrt{\frac{\varepsilon(1-f)g'}{2}}, \quad (21)$$

[cf. Eq. (13)] in which the collision of the incident composite soliton with the barrier leads to splitting, with component 1 transmitted and component 2 reflected. Note that when $\alpha = 1$ [as follows from Eq. (18), this happens at $g/g' = 1$] the result is $\sqrt{\pi} \Gamma(\alpha) = 2 \Gamma(\alpha + 1/2)$, and interval (21) shrinks to nil; this, in particular, applies for the Manakov system, when $g = g' = 1$. For $g/g' > 1$ the situation inverts, and there is a velocity interval, defined by transposing the upper and lower bounds in Eq. (21), in which component 1 is reflected and component 2 transmitted.

It is important to note that, in the limit case of $g' = 1$, this regime is accessed just by adjusting the relative particle numbers in the two components, with a much smaller population in component 1. Lastly, it is relevant to mention that, while this consideration is entirely valid in the framework of the mean-field theory, it may become irrelevant in the case of a very small number of atoms in component 1, when the GPE cannot be used.

IV. NUMERICAL RESULTS

A. Details of the numerical approach

In numerical simulations, we replace the ideal δ -functional barrier by a Gaussian one, $V(x) = \varepsilon\eta(x, \sigma)$, as defined in Eq. (5). The intention here is to model an experimentally

²This is made substantially simpler upon implementing the change of variable $Y = (1-f)g'X$.

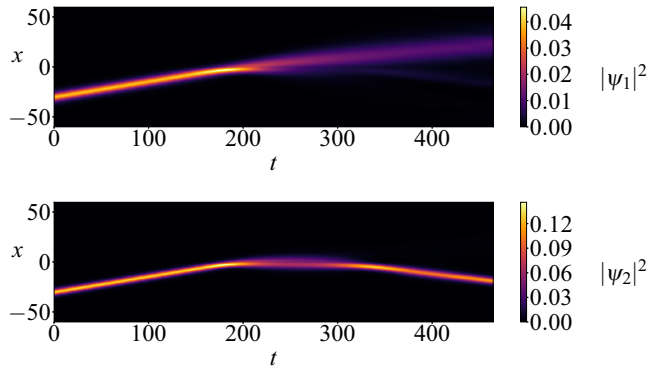


FIG. 1. The evolution of densities of the two components, which demonstrates the splitting of the incident composite soliton into the reflected heavier component and transmitted lighter component close to the splitting-unsplitting boundary, at parameter values $\sigma = 0.4$, $\varepsilon = 0.07$, $f = 0.3$, $g = 0.2$, and $v = 0.155$.

relevant Gaussian-profile off-resonant sheet of light by the regularized version of the δ function, with finite width σ . Except for Figs. 10 and 11, all numerical results presented in this paper were produced by the Gaussian barrier with $\sigma = 0.4$. This value of σ is reasonably small to adequately model the experimentally available barriers [8].

We numerically integrated Eqs. (6a) and (6b), with the potential barrier defined as per Eq. (5), by means of the well-known Fourier-transform split-step method [43,44]. We display typical examples of collisions with the barrier, leading to either the splitting of two-component solitons or their passage through the barrier, in Figs. 1 and 2, respectively. These two examples pertain to slightly different collision velocities and otherwise identical parameters, corresponding to situations close to the boundary between the separation and mutual passage of the components. In Fig. 1 one can see that after the separation of the solitary waves into the smaller population transmitted component and the larger population reflected component the transmitted component has an increased width compared to before the collision. This is due to the diminished post-separation focusing term arising from the nonzero coupling to the larger population component ($g = 0.2$

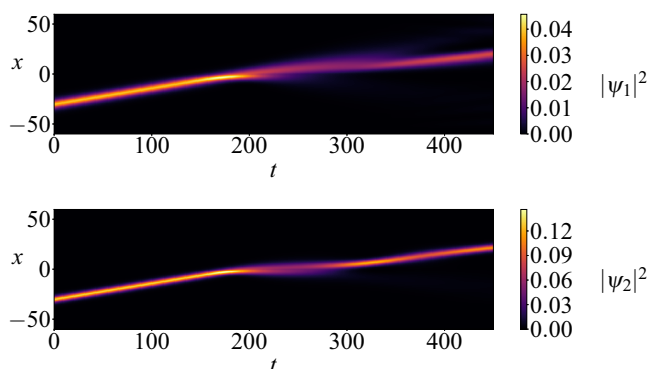


FIG. 2. The same as in Fig. 1, but for the case when the incident composite soliton passes the potential barrier without splitting, close to the splitting threshold. Parameters are the same as in Fig. 1, except for a slightly larger value of the collision velocity $v = 0.16$.

in Figs. 1 and 2). Additionally, there is a small proportion of the smaller population component which is not transmitted in Fig. 1. This would also lead to a larger width of the solitary wave due to decreased focusing by the nonlinearity.

We quantify the transmission of the two components through the barrier by the coefficients

$$T_1 = f^{-1} \int_0^\infty dx |\psi_1(x, t = t_f)|^2, \quad (22)$$

$$T_2 = (1 - f)^{-1} \int_0^\infty dx |\psi_2(x, t = t_f)|^2, \quad (23)$$

which we compute at the “final time” t_f . It is chosen to be $t_f \geq L/2v$ in the cases when the system does not include an axial trapping potential, with v being the velocity of the incident soliton, and L the size of the numerical spatial domain. We set $L = 160$, chosen so that, at the initial location ($x = -L/4$) and final location of any transmitted component ($x = L/4$), the soliton components are far separated from the splitting barrier. Note that the interaction with the barrier decelerates the motion of any transmitted component, meaning that t_f must be increased accordingly. We performed systematic simulations to produce coefficients $T_{1,2}$ as functions of the four control parameters, *viz.*, v , f , ε , and g . Note that the repulsive barrier cannot trap any part of the wave functions, meaning that in the absence (or negligibility) of the axial confinement the reflection coefficients for the two components are $R_{1,2} = 1 - T_{1,2}$.

B. Comparison of numerical results for the transmission with the analytical predictions

1. Weak interspecies interaction

In Figs. 3 and 4 we compare the analytical prediction, given by Eq. (13), with results of the systematic simulations. In general, the agreement is good, provided that g is small (a significant region for the value of $T_1 - T_2 = 1$ is well visible for up to about $g = 0.1$, while the analytical approximation was developed for $g = 0$), and that ε is also relatively small [up to $\varepsilon \approx 0.1$, as expected for $f = 0.3$ from Eq. (11), and seen in Fig. 3].

In Figs. 3 and 4, we mapped out the degree of splitting, as produced by the simulations, in detail by plotting the difference $T_1 - T_2$ as a function of all the control parameters, $\{\varepsilon, v, g, f\}$. The same figures display boundaries (dashed lines) between which the analytical result, given by Eq. (13), predicts splitting to occur. To reiterate, the analytical consideration implies that $T_1 = 1$ and $T_2 = 0$ in the interval of velocities of the incident composite soliton given by Eq. (13), and, on the other hand, $T_1 = T_2$ outside the interval, where the incident soliton does not split. It is clearly seen in Figs. 3 and 4 (as well as in Fig. 11, which is produced below with an effectively exact numerical implementation of a δ -functional barrier) that the analytical prediction gives a good indication of where near-complete splitting occurs for $g \lesssim 1$, gradually deteriorating with increasing g . This is explained, in particular, by the fact that, at relatively large values of g , the attraction between the components naturally tends to suppress the collision-induced splitting. It is also generally

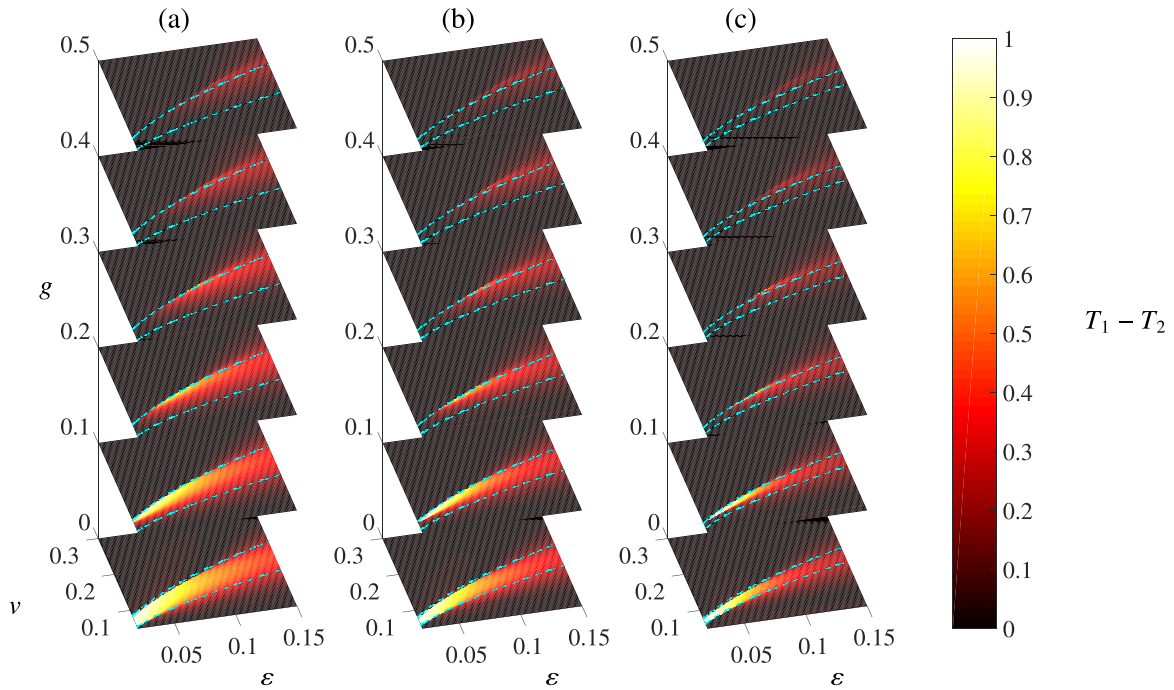


FIG. 3. The transmission difference between the two components $T_1 - T_2$, as produced by simulations of Eqs. (6a) and (6b), in the (v, ε) parameter plane (where v is the collision velocity and ε the strength of the Gaussian potential barrier) at different values of g and f (the cross-attraction strength and scaled population of the first component, respectively). The value of g increases along the vertical axis. (a) $f = 0.3$, (b) $f = 0.35$, and (c) $f = 0.4$. Here and in Fig. 4 the dashed lines display boundaries of the splitting region, as analytically predicted by Eq. (13) in the limit of $g = 0$.

true that, as the barrier area ε increases (which, for fixed width σ , effectively corresponds to increasing the barrier's height), each component is only partly transmitted and partly reflected in the simulations, i.e., T_1 and T_2 take intermediate values

between 0 and 1. The figures also corroborate the prediction of Eq. (13), that the splitting region shrinks markedly as f approaches $1/2$, i.e., the components of the incident composite solitons become nearly equally populated.

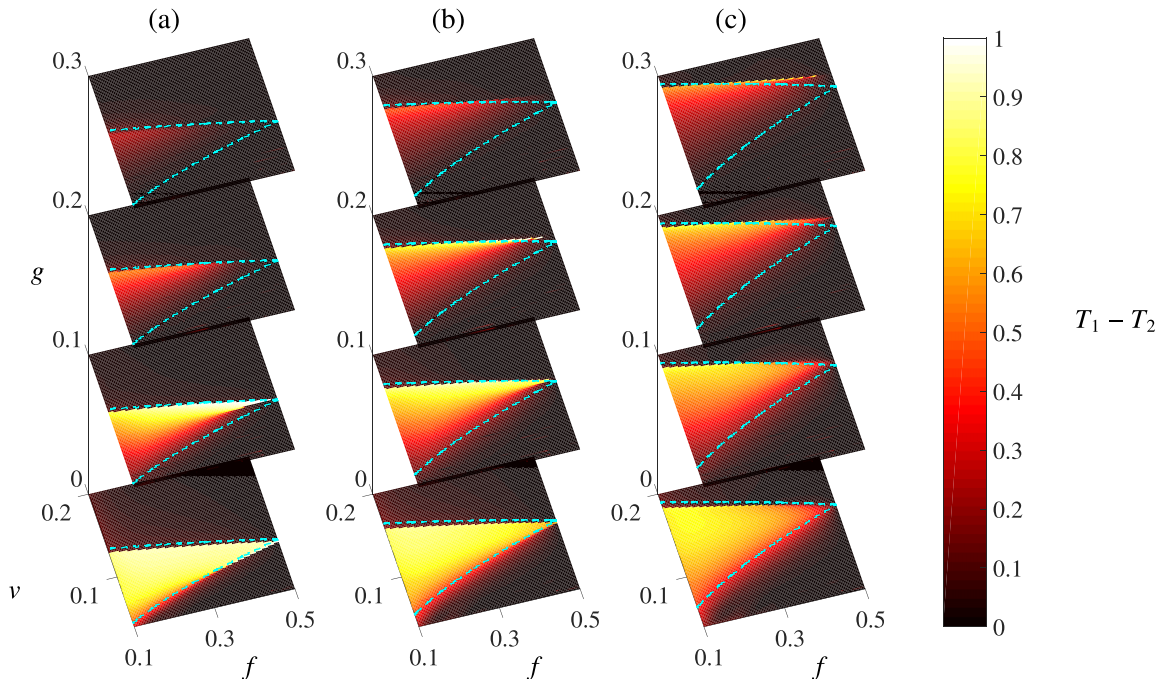


FIG. 4. Similar to Fig. 3, but in the (v, f) parameter plane at different values of g and ε . (a) $\varepsilon = 0.04$, (b) $\varepsilon = 0.06$, and (c) $\varepsilon = 0.08$.

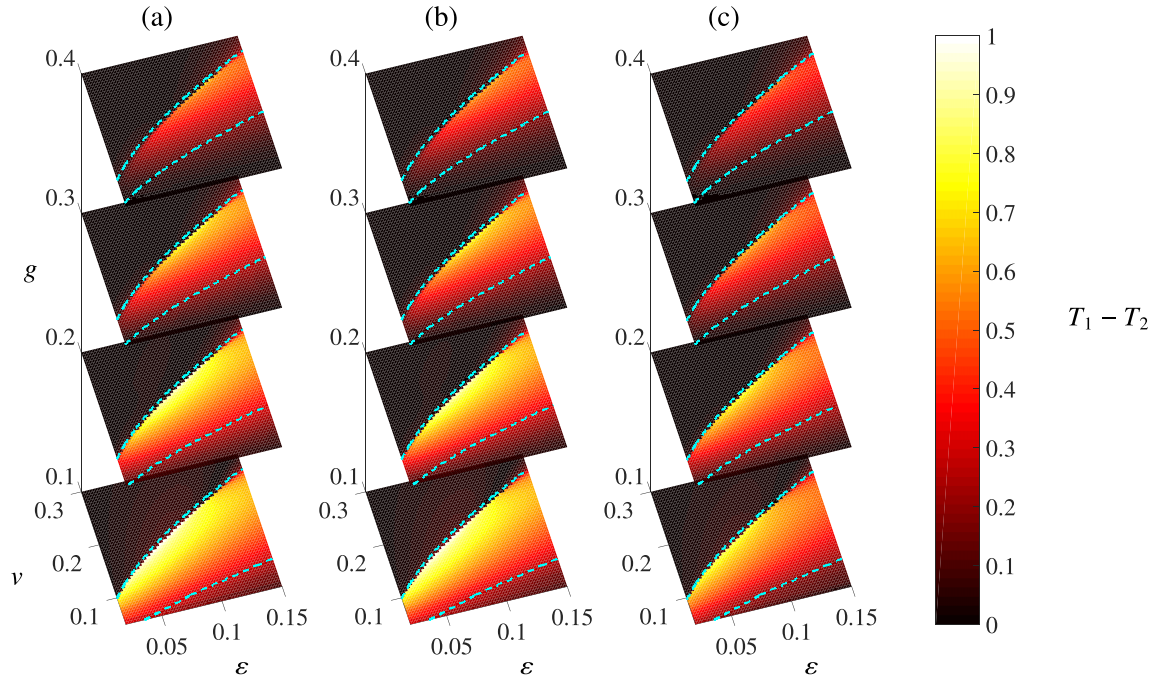


FIG. 5. The transmission difference between the two components $T_1 - T_2$, as produced by simulations of Eqs. (6a) and (6b), in the (v, ε) parameter space (where v is the collision velocity and ε the strength of the Gaussian potential barrier) at different values of g and f (the cross-attraction strength and scaled population of the first component, respectively). The value of g increases along the vertical axis. (a) $f = 0.01$, (b) $f = 0.02$, and (c) $f = 0.05$. The dashed lines display boundaries of the splitting region, as analytically predicted by Eq. (21) in the limit of $f \ll 1$.

2. Strongly asymmetric nonlinearities

We also collected numerical results for the case of strong asymmetry between the two components of the incident composite soliton, i.e., situations satisfying Eq. (14), in which case we confine the consideration to $g' = 1$. These are collected in Fig. 5, which clearly demonstrates that the analytical prediction, elaborated for this case in the form of Eq. (21), is quite accurate, at least up to $f = 0.05$, in broad intervals of values of g and ε . As in the case of small interspecies interactions, we cannot expect perfect splitting (i.e. $T_1 - T_2 \neq 1$) for larger values of g . This is codified in the case of strongly asymmetric nonlinearities in the additional condition for separation given by Eq. (A11), although it produces only a qualitative indication.

Although we found it simpler in our simulations to vary f only, we reiterate that it may not be experimentally practical to work with a very small atomic population in one of the components, for reasons of imaging the density profiles in the weakly populated component, and, generally speaking, validity of the mean-field theory. On the other hand, simulations performed for the alternative regime of $1 \ll g \ll g'$ with $f = 1/2$ (not shown here) yield qualitatively similar results. How best to fulfill Eq. (14) in a particular experimental configuration may depend on what the available values of the scattering lengths are.

C. Continuous variation of the interspecies interaction strength with and without weak axial harmonic confinement

In addition to considering the situation where a soliton moves with a given velocity in free space, the simulations

were also carried out for the soliton beginning its motion from initial position x_0 on one side of an external harmonic-oscillator potential

$$U = \omega_x^2 x^2 / 2. \quad (24)$$

This setting implies that the soliton accelerates to incident velocity

$$v = \omega_x x_0, \quad (25)$$

when it hits the narrow barrier placed at $x = 0$.

Figure 6 shows how the transmission in both components varies in the (v, g) parameter space for $\varepsilon = 0.1$ and $f = 0.3$ in the free-space configuration (no axial trapping). In this case, Eq. (13) predicts that both components of the composite soliton pass the barrier, without splitting, at $v > \sqrt{\varepsilon(1-f)}/2 \approx 0.187$. The numerical findings collected in Fig. 6 support this prediction. This case can be compared to that when the axial harmonic-oscillator confinement is present, as per Eq. (24). Figure 7 shows the results for an equivalent range of parameters with the collision velocity given by Eq. (25). In this figure, the axial trapping produces a harder boundary in the (v, g) parameter space, separating the cases of component 2 being reflected and transmitted.

We display another aspect of the results, collected in Figs. 6 and 7, in Fig. 8 by means of boundaries between parameter regions where the second component is effectively reflected or transmitted for different values of the barrier's strength, ε , while fixing the proportion of the total population in this component at $1 - f = 0.7$. We define this boundary by condition $T_2 = 0.5$. Both sets of Figs. 6 and 8(a), which pertain to the

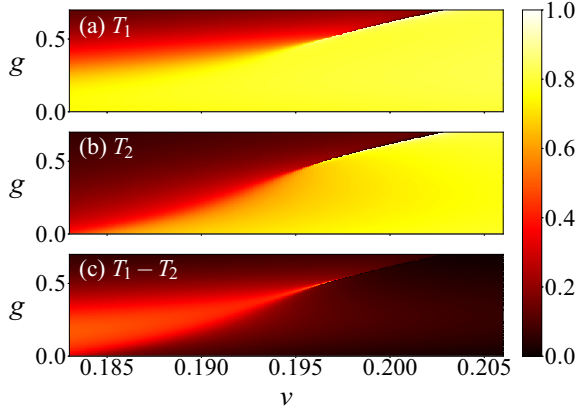


FIG. 6. Transmission coefficients of the two components, produced by the simulations of Eqs. (6a) and (6b) for the composite soliton, with the relative norm of the first component $f = 0.3$, incident in the parameter plane of the collision velocity v and interspecies attraction g . Panels (a)–(c) display the transmission coefficients of the first and second components, and their difference.

soliton-barrier collision in free space, and Figs. 7 and 8(b), which display the numerical findings for the splitter embedded in the external trapping potential [Eq. (24)], demonstrate that making the attraction between the two components stronger, as quantified by increasing the parameter g , while keeping other parameter values fixed (relative population f and barrier area ε), leads to multiple transitions between positive and negative values of the boundary's curvature.

D. Internal excitations in past-collision solitons

An important aspect of the numerical results is the presence of intrinsic excitations in the split and unsplit solitons after the interaction with the barrier. In particular, for the use in interferometers the solitons should keep a nearly fundamental shape, without conspicuous internal vibrations. To this end, we define the measure of the intrinsic excitation in the j -h

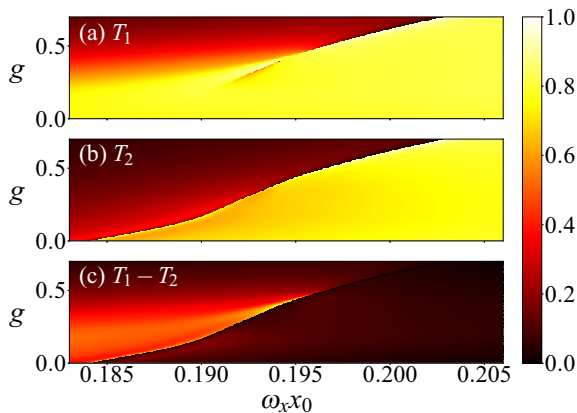


FIG. 7. The same as in Fig. 6, but in the case when the collision velocity is given by Eq. (25) for the incident soliton accelerated from initial position x_0 by the trapping potential (24).

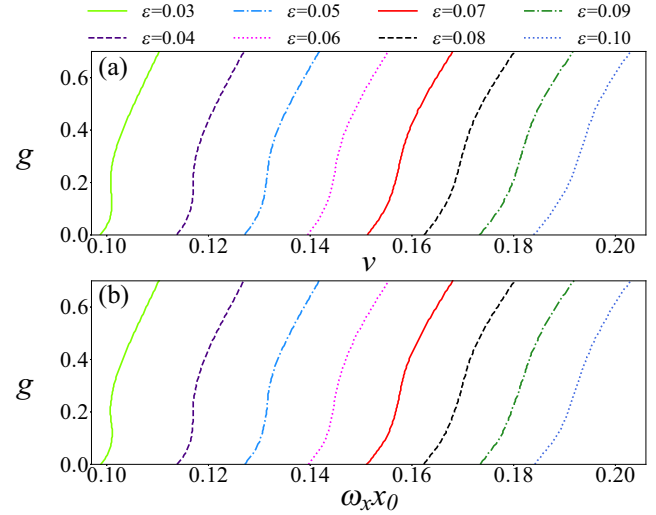


FIG. 8. Boundaries in the (v, g) parameter plane (the collision velocity and relative cross-attraction strength) between regions where the second component of the incident composite soliton, with a fixed relative share of the total norm, $1 - f = 0.7$, bounces (left of the boundary) or passes (right of the boundary), for varying values of the barrier's strength ε . (a) The case of the incident solitons arriving in free space with velocity v ; (b) for the soliton accelerated by the trapping potential (24) as per Eq. (25).

soliton ($j = 1, 2$) as

$$\eta_j = \frac{\max(n_j) - \min(n_j)}{\max(n_j) + \min(n_j)}, \quad (26)$$

where $n_j \equiv |\psi_j(x_c)|^2$ is the density at the soliton's center, with the maximum and minimum taken with respect to the evolution in time. It is shown in Fig. 9 for different values of the interspecies coupling strength, g , and $f = 0.3$, $v = 0.112$, $\varepsilon = 0.04$, which adequately represents a generic situation. In this case, the distance from the barrier $|x_0|$ was increased to 200 to clearly observe the intrinsic oscillations. In accordance with the data displayed in Figs. 3 and 4, the respective transmission coefficient of component 1, T_1 , is close to 1 at $g = 0$, decreasing to 0 at $g = 0.4$, while component 2 bounces back, in agreement with the data collected in Fig. 8. Simultaneously, Fig. 9 shows that the excitation degree in the passing solitons (in component 1) first increases from $\simeq 0.07$ to $\simeq 0.14$, and then decreases. The excitation in the rebounding soliton (in component 2) follows a similar trend, but remaining smaller, roughly, by an order of magnitude. These trends are explained by an effect of increasing attraction between the components on the excitation of the intrinsic vibrations in the solitons with the growth of g , as well as by the effect of the varying shape of the interaction products on the internal excitations in these products. A conclusion is that, in the case of high-quality fission of the incident compound soliton into the passing and rebounding ones, at g sufficiently small, the excitation effect remains weak. This conclusion is quite natural, as the purity of the splitting deteriorates with the increase of g , which leads to deformation of the splitting products, especially the passing soliton, and the deformation excites the intrinsic vibrations.

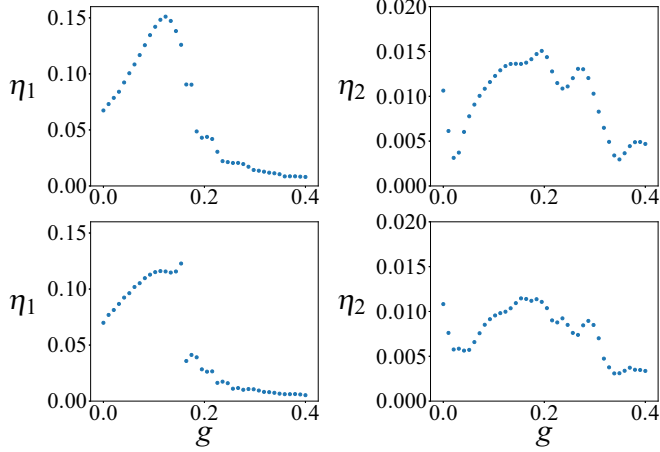


FIG. 9. The amount of the excitation of intrinsic vibrations in the passing and rebounding solitons, in the first and second components, respectively, defined as per Eq. (26), for different values of coupling strength g . The typical situation is displayed for other parameters fixed as $f = 0.3$, $v = 0.112$, $\varepsilon = 0.04$. The top and bottom rows summarize the results of the simulations performed in free space and in the presence of the axial harmonic-oscillator trap. To clearly observe postcollision internal oscillations in the solitons, a symmetric spatial domain is used with the soliton initially placed at $|x_0| = 200$, in both cases.

It is relevant to stress that the excitation degree is much smaller than in the previously studied case of the “brute-force” splitting of a single-component fundamental soliton by a strong barrier [11–20]. To address this point, we note that, in the case of the ideal splitting of an incident fundamental soliton with amplitude $2A$ into two separated fragments, each one, with the center’s coordinate x_0 and phase Φ_0 , is naturally approximated by expression

$$\psi_{\text{frag}}(x) = A \exp(i\Phi_0) \operatorname{sech}[a(x - x_0)], \quad a = 2A, \quad (27)$$

cf. Eq. (9). Further evolution of the “half-soliton,” initiated by this expression, can be produced by the exact Satsuma-Yajima solution [29], which is quite complicated. However, states with the largest and smallest values of the density at the center, which define the excitation measure (26), are ones with zero chirp [45], which makes it possible to approximate them by ansatz (27) with independent values A and a , subject to the conservation of the norm

$$\mathcal{N} \equiv \int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 2A^2 a^{-1} \quad (28)$$

(in the above analysis, normalization $\mathcal{N} = 1$ is adopted). Further, the substitution of the ansatz in the Hamiltonian of the ideal NLSE

$$H_{\text{single}} = \frac{1}{2} \int_{-\infty}^{+\infty} (|\psi_x|^2 - |\psi|^4) dx \quad (29)$$

[cf. Eq. (7)], yields the corresponding value of the energy,

$$E_{\text{single}}(A) = \frac{1}{3}(A^2 a - 2A^4 a^{-1}) \equiv \frac{1}{3}(2\mathcal{N}^{-1}A^4 - \mathcal{N}A^2), \quad (30)$$

where relation (28) is used. Finally, equating energy (30) for the zero-chirp states realizing the states with maximal and minimal densities at the center, the former one given by the

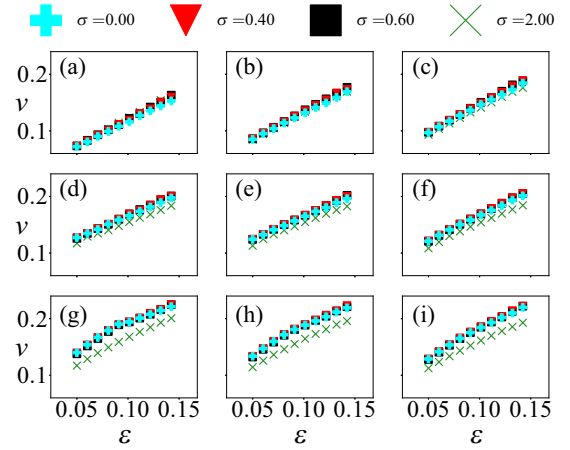


FIG. 10. Lines in parameter space (ε, v) at which the transmission coefficient for the first component is $T_1 = 0.5$, for different values of width σ of the Gaussian barrier (5) ($\sigma = 0$ representing the ideal δ -function). Other parameters are (a) $f = 0.2, g = 0$, (b) $f = 0.3, g = 0$, (c) $f = 0.4, g = 0$, (d) $f = 0.2, g = 0.2$, (e) $f = 0.3, g = 0.2$, (f) $f = 0.4, g = 0.2$, (g) $f = 0.2, g = 0.4$, (h) $f = 0.3, g = 0.4$, and (i) $f = 0.4, g = 0.4$.

split-soliton ansatz (27), it is easy to find the relation between them, $\min(n) = (1/2) \max(n)$ (it does not depend on \mathcal{N}). The respective value of the excitation measure, as given by Eq. (26), is $\eta_{\text{single}} = 1/3$. Thus, the above-mentioned typical values of η_j for the two components of the binary system are smaller than their counterparts in the single-component model, in the case of the ideal splitting, by a factor $\simeq 3-4$. In the real single-component setting, numerical results yield even larger values of the excitation measure $\eta_{\text{single}} \simeq 0.5 - 0.6$ [15]. Thus, the binary system provides, as expected, essential suppression of detrimental effects of the post-collision intrinsic excitation of the fragments.

E. Effect of the finite barrier width

In Fig. 10 we show boundaries corresponding to $T_1 = 0.5$, which separate the effective reflection and transmission of the first component in the parameter space of collision velocity v and barrier area ε , for different fixed values of f, g , and the barrier width σ [see Eq. (5)]. Note that we obtained the results for $\sigma = 0$ by means of the numerical method outlined in Appendix B, in which we represent the “ideal” δ -functional barrier in Fourier space, and incorporate it in the split-step simulation algorithm in the same step as the kinetic energy term [see Eq. (B2)].

We choose parameter ranges in Fig. 10 so as to make them representative for values used in Figs. 1 to 8. From Fig. 10 one can see that the location in the (ε, v) parameter plane where $T_1 = 0.5$ is more sensitive to width σ of the barrier when g is relatively large, and that for the range of values of f and g considered in the analysis, the dynamics are, naturally, more sensitive to the variation of g than the variation of σ . Increasing σ , while keeping other parameter values constant, may cause the value of v at which $T_1 = 0.5$ to become either lower or higher, depending on the other parameters. In particular, a conclusion is that, for $\sigma = 0.4$,

this value of v is consistently larger than that for the ideal δ function ($\sigma = 0$), even if this difference is never greater than 0.01.

V. CONCLUSION

We examined the transmission properties of two-component bright-bright solitary waves colliding with a narrow potential barrier, and considered in detail the effect of varying the barrier strength, incoming soliton velocity, populations of its two components, scattering lengths. We carried this out with the main objective of identifying parameter regions in which the incident composite soliton splits into its components so that one is reflected and the other transmitted, which is an effect of major importance to the design of matter-wave soliton interferometers. For small values of the barrier strength ε , we developed the perturbation theory that effectively predicts the velocity interval in which the splitting takes place for relatively weak interspecies interactions ($g \rightarrow 0$), as well as for the case when the intraspecies interactions are significant for one species only (which implies strongly different populations of the two components, if the two intraspecies scattering lengths are comparable). To obtain analytical estimates for these intervals, we considered a δ -functional barrier. In the numerical part of the work, it was approximated by the corresponding Gaussian, an additional parameter being its width σ . We concluded that values of σ used in the simulations (such as $\sigma = 0.4$, in scaled units), produce results close to those that can be obtained with numerically exact implementation of the δ function (in the Fourier-transform space). By means of the comparison with numerical results, we identified parameter regions in which the perturbation theory accurately predicts the outcomes of the collision of the incident composite soliton with the splitting barrier. The numerical analysis was extended, in the parameter space, beyond those regions, by increasing the strength of the interspecies interaction g and varying the relative population in component 1, f . The excitation of the intrinsic vibrations in the post-collision passing and bounding solitons has also been studied, with a conclusion, that, in cases of high-quality splitting, the excitations remain weak (much smaller than in the previously studied single-component model).

Possible extensions of this work include numerical treatments of the system with unequal intracomponent scattering lengths and different atomic masses in the two components (a heteronuclear binary BEC). Finally, it may also be relevant to consider in detail the splitting by a localized nonlinear potential, which is briefly addressed in Sec. A 2 of the Appendix.

Additional data related to the findings reported in this paper is made available by source [46].

ACKNOWLEDGMENTS

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APPENDIX A: DERIVATION OF THE TRANSMISSION CONDITIONS

1. Limit of the negligible interspecies interactions

The center-of-mass kinetic energies of each component (essentially, half the total mass of the soliton multiplied by the square of the velocity with which it is moving) are written in our notation as [recall the unit energy is $m(g_{11}N/\hbar)^2$ in physical units]

$$(E_{\text{kin}})_1 = f v^2/2, \quad (\text{A1a})$$

$$(E_{\text{kin}})_2 = (1 - f)v^2/2, \quad (\text{A1b})$$

and we determine the intraspecies interaction potential energies from terms in the second line of Eq. (7):

$$(E_{\text{int}})_1 = -f^3/12, \quad (\text{A2a})$$

$$(E_{\text{int}})_2 = -(1 - f)^3(g')^2/12. \quad (\text{A2b})$$

The potential energy of each component associated with the weak potential barrier can be easily found in the framework of the perturbation theory (assuming parameter ε to be sufficiently small), which neglects the deformation of the soliton under the action of the barrier potential [45]:

$$\begin{aligned} U_1(t) &\equiv \varepsilon \int_{-\infty}^{+\infty} dx \delta(x) |\psi_1(x, t)|^2 \\ &= \frac{1}{4} \varepsilon f^2 \text{sech}^2 \left[\frac{1}{2} f(x_0 + vt) \right], \end{aligned} \quad (\text{A3a})$$

$$\begin{aligned} U_2(t) &\equiv \varepsilon \int_{-\infty}^{+\infty} dx \delta(x) |\psi_2(x, t)|^2 \\ &= \frac{1}{4} \varepsilon (1 - f)^2 g'^2 \text{sech}^2 \left[\frac{1}{2} (1 - f)g'(x_0 + vt) \right]. \end{aligned} \quad (\text{A3b})$$

The perturbation theory that we used applies (to each component) provided that the magnitudes of the interaction potential [see Eqs. (A2)] determine the peak values of the barrier-induced energies, as given by Eqs. (A3) when $x_0 + vt = 0$. This yields, eventually, the potential energy associated with the unperturbed component solitons being located exactly on top of the barrier,

$$(U_{\text{max}})_1 = \varepsilon f^2/4, \quad (\text{A4a})$$

$$(U_{\text{max}})_2 = \varepsilon (1 - f)^2 g'^2/4. \quad (\text{A4b})$$

2. Extension for the nonlinear splitter

It is relatively straightforward to extend the theoretical treatment to the case of a nonlinear splitter, as described in Refs. [21,27]. In those works it took the form of a localized self-repulsive nonlinearity (that can be created by a tightly focused laser beam which locally applies by optical Feshbach resonance [47]). The respectively modified system of GPEs (6) is

$$i \frac{\partial \psi_1}{\partial t} = \left[-\frac{1}{2} \frac{\partial^2}{\partial x^2} + \varepsilon_{\text{nonlin}} \delta(x) |\psi_1|^2 - |\psi_1|^2 - g |\psi_2|^2 \right] \psi_1, \quad (\text{A5a})$$

$$i \frac{\partial \psi_2}{\partial t} = \left[-\frac{1}{2} \frac{\partial^2}{\partial x^2} + \varepsilon'_{\text{nonlin}} \delta(x) |\psi_2|^2 - g' |\psi_2|^2 - g |\psi_1|^2 \right] \psi_2, \quad (\text{A5b})$$

where positive $\varepsilon_{\text{nonlin}}$ and $\varepsilon'_{\text{nonlin}}$ quantify the strengths of the nonlinear splitters, which, in principle, may be different for the two atomic species.

Similarly to the linear case, when $g \ll 0$ one can determine velocity intervals in which we predict splitting of an incident composite soliton into a lighter transmitted soliton and a heavier reflected one:

$$\sqrt{\varepsilon_{\text{nonlin}} f^3} / 4 < |v| < g' \sqrt{\varepsilon'_{\text{nonlin}} (1-f)^3} / 4. \quad (\text{A6})$$

Note that the form of this interval implicitly assumes $f^{3/2} \leq (g')^{3/2} \sqrt{\varepsilon'_{\text{nonlin}} / \varepsilon_{\text{nonlin}}}$, which is only fulfilled under the condition that $f \leq (1-f)g'$ when $g' \sqrt{\varepsilon'_{\text{nonlin}} / \varepsilon_{\text{nonlin}}} \leq 1$. For instance, in the case of $g' = 8$ and $f = 13/15$ (with $\varepsilon'_{\text{nonlin}} = \varepsilon_{\text{nonlin}}$), the upper and lower bounds of Eq. (A6) need to be transposed.

One can readily determine nonlinear equivalents of Eq. (A4):

$$(U_{\text{max nonlin}})_1 = \varepsilon_{\text{nonlin}} f^4 / 32, \quad (\text{A7a})$$

$$(U_{\text{max nonlin}})_2 = \varepsilon'_{\text{nonlin}} (1-f)^4 g^2 / 32, \quad (\text{A7b})$$

which, compared to Eq. (A2), reveal that the condition for sufficiently small $\varepsilon_{\text{nonlin}}$ is that it should be significantly smaller than $8/3f$, and, similarly, $\varepsilon'_{\text{nonlin}}$ should be significantly smaller than $8/3(1-f)$. If we consider $\varepsilon_{\text{nonlin}}$ and $\varepsilon'_{\text{nonlin}}$ to be broadly similar in magnitude, this may be simplified to $\{\varepsilon_{\text{nonlin}}, \varepsilon'_{\text{nonlin}}\} \ll 8/3$.

Comparing Eqs. (A6) to (13), we see that the nonlinear splitter manifests much stronger dependence on the norm-distribution parameter $f = N_1/N$, as well as a stronger dependence on the relative magnitude of the intraspecies scattering lengths, via $g' = a_{22}/a_{11}$.

3. Limit of a strongly asymmetric two-component soliton

a. Determining the splitting interval

The kinetic energy $(E_{\text{kin}})_1$ of component 1 is given by Eq. (A1a), while the height of the energy barrier generated by the splitter, in similar fashion to Eq. (A3a), may be determined from Eq. (19) as

$$(U_{\text{max}})_1 = \varepsilon A_1^2 = \frac{\varepsilon f (1-f) g' \Gamma(\alpha + 1/2)}{2\sqrt{\pi} \Gamma(\alpha)}. \quad (\text{A8})$$

Combining these expressions within the energy condition for component 1 to be transmitted through the barrier, $(E_{\text{kin}})_1 > (U_{\text{max}})_1$ [cf. Eq. (12a)], yields the following result:

$$v^2 > \frac{\varepsilon (1-f) g' \Gamma(\alpha + 1/2)}{\sqrt{\pi} \Gamma(\alpha)}. \quad (\text{A9})$$

b. Interaction-energy condition

Strictly speaking, there is an additional condition necessary for the complete collision-induced splitting in free space. The kinetic energy of the transmitted component must exceed its binding energy in the composite soliton, determined by the

cross-attraction $E_{\text{cross}} \equiv -g \int_{-\infty}^{+\infty} dx |\psi_1(x)|^2 |\psi_2(x)|^2$ [in this analysis we assume that the smallness of ε implies that the consideration of the energy described by Eq. (A8) may be neglected altogether], otherwise component 1 will not become a free soliton. Hence, making use of Eqs. (9b), (17a), and (19), we obtain

$$\begin{aligned} E_{\text{cross}} &= -g A_1^2 \frac{(1-f)^2 g'}{4} \int_{-\infty}^{+\infty} dX \left[\text{sech} \left(\frac{[1-f]g'X}{2} \right) \right]^{2(\alpha+1)} \\ &= -g A_1^2 \frac{(1-f) \sqrt{\pi} \Gamma(\alpha+1)}{2 \Gamma(\alpha+3/2)} \\ &= -\frac{\alpha g f (1-f)^2 g'}{2(2\alpha+1)}. \end{aligned} \quad (\text{A10})$$

Substituting expressions (A1) and (A10) in the condition $(E_{\text{kin}})_1 |E_{\text{cross}}|$ yields the final constraint

$$|v| > \sqrt{\frac{\alpha g (1-f)^2 g'}{2(2\alpha+1)}}. \quad (\text{A11})$$

APPENDIX B: NUMERICAL SIMULATIONS WITH REGULARIZED δ FUNCTIONS

The scheme for handling the exact δ -functional barrier in the simulations is adapted from Ref. [48]. This incorporates the Fourier transform of the δ function $\hat{\delta}(k)$ into the part of the split-step method which implements the kinetic-energy term. The relevant expression for the split-step algorithm in the Fourier space is then

$$\mathcal{F}[T + \varepsilon \delta(x)] = \frac{1}{2} k^2 + \varepsilon \hat{\delta}(k). \quad (\text{B1})$$

Due to the fact that one is conflating an analytical expression for the Fourier transform and its discrete computational counterpart, one must be careful while defining the periodic domain for the Fourier transform. To use the discrete Fourier transform, in the numerical computations we choose the domain as $-L/2 \leq x < +L/2$, placing the δ function at the center. The corresponding operator for the kinetic energy, combined with the energy introduced by the δ function, is then written as

$$(M_1)_{mn} = \mathcal{F}[T + \varepsilon \delta(x)]_{mn} = \frac{1}{2} k^2 \delta_{mn} + \frac{\varepsilon}{L}, \quad (\text{B2})$$

where k is defined as a discrete variable running between $-\pi/L$ and $+\pi/L$ with N entries, indexed by integers (m, n) , and δ_{mn} is the Kronecker's delta. When using standard FFT routines in the current context, they must be used in conjunction with two shifting protocols (which shift the location of zero frequency to the center of the array) whenever they are applied, to run them in a way which is consistent with the physically relevant boundary conditions.

Alternatively, one can use only one shifting protocol by accounting for a phase offset in the resulting expression for the δ function in the Fourier space. The expression for the sum of the kinetic energy with the energy of the δ -functional barrier is then

$$\begin{aligned} (M_2)_{mn} &= \mathcal{F}[T + \varepsilon \delta(x)]_{mn} \\ &= \frac{k^2}{2} \delta_{mn} + \frac{\varepsilon}{L} \exp \left(\frac{iL}{2} [k_m - k_n] \right), \end{aligned} \quad (\text{B3})$$

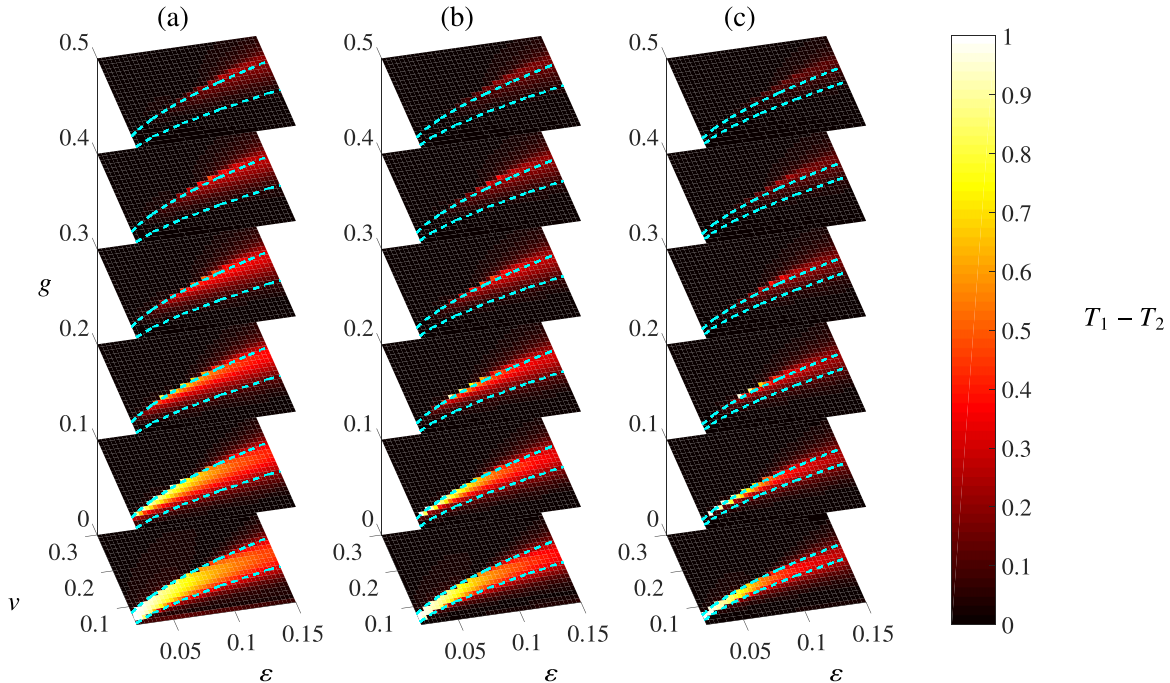


FIG. 11. The same as in Fig. 3, but produced by the numerical algorithm which implements the δ function in the Fourier space, as per Eq. (B2), instead of the Gaussian barrier given by Eq. (5) in the coordinate space.

cf. Eq. (B2). Note that if we were only considering the kinetic energy, this would make no difference, and it is, in fact, common practice to only use one shifting protocol in this case.

To execute this step in the split-step algorithm, one must diagonalize matrix M_1 or M_2 and combine the associated amount of shifts with the Fourier transforms, as mentioned above. Note that the diagonalization need only be done once, as it is constant throughout the simulations (recalculation is required only if ε or L is altered). The need, on a grid with N spatial points, for $(N \times N)$ -dimensional matrix multiplications at each timestep, to implement this method, increases the

computational time. As a result, the resolution of parameter space, as plotted in Fig. 11, is reduced relative to comparable plots presented above when using the Gaussian barrier.

Figure 11 shows a counterpart of Fig. 3, produced by the numerical scheme outlined in this Appendix. Comparison of the plots suggests that, for $g = 0$, the results are quite similar, and the analytical treatment gives a good indication of what to expect. Deviations of the numerical results from the analytical approximation are primarily caused by the nonlinearity and deformation of the solitons' shapes, rather than by the deviation of the numerically approximated potential barrier (as long as it is sufficiently narrow) from the ideal δ function.

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