Efficient construction of the Čech complex

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Abstract

In many applications, the first step into the topological analysis of a discrete point set P sampled from a manifold is the construction of a simplicial complex with vertices on P. In this paper, we present an algorithm for the efficient computation of the Čech complex of P for a given value ε of the radius of the covering balls. Experiments show that the proposed algorithm can generally handle input sets of several thousand points, while for the topologically most interesting small values of ε it can handle inputs with tens of thousands of points. We also present an algorithm for the construction of all possible Čech complexes on P.

Keywords: Čech complex, Vietoris-Rips complex, persistent topology, topological data analysis

1. Introduction

In several computer graphics applications, the modelling and visualisation pipeline starts with the acquisition of the raw data in the form of an unorganised point set, that is, a finite subset $P \subset \mathbf{R}^d$. Typically, $d = 2$ or 3, however, data of higher dimensions also appear in practice. The acquired data is then processed and geometric and topological information is extracted from it. The final goal is to construct and visualize a mathematical model of P, typically a polyhedral mesh capturing as faithfully as possibly the geometry and topology of the original data.

As P is unparametrised, a common first processing step is the computation of a discrete geometric structure defined on it, such as a graph or a

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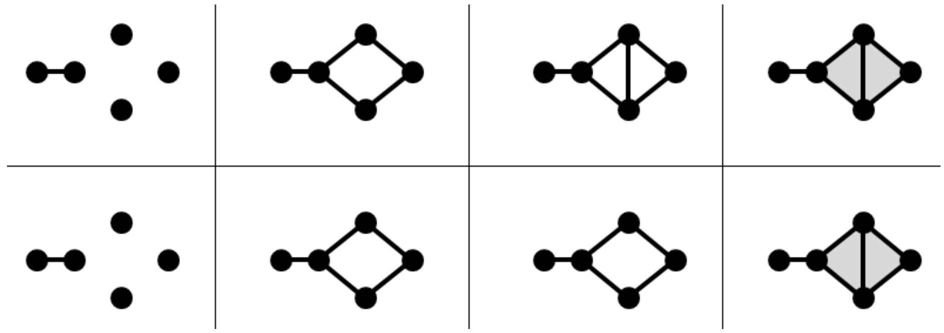

Figure 1: **Left to Right:** Increasing values of ε. **Top:** The Čech complex. **Bottom:** The Rips-Vietoris complex.

simplicial complex connecting points that are close to each other. Regarding graphs, we have the k nearest neighbour graph connecting each point with the k points nearer to it, and the ε neighbour graph, connecting each point with all points at distance less than ε from it. Regarding complexes, the Vietoris-Rips complex [1] connects points in P with a simplex if the distance between any two of them is smaller than ε. The Čech complex [2, 3] connects points in P with a simplex if the radius minimal enclosing sphere of the subset is less than ε. Fig. 1 shows a series of Čech and Vietoris-Rips complices for increasing values of ε.

Recently there has been an increased interest in algorithms for fast computations of neighborhood graphs and complexes [4]. In particular, there is increased interest for algorithms that can efficiently compute a series of neighborhood graphs or complexes for a point set P, corresponding to different values of ε. The computed series of complexes is then further processed to infer the persistent topology of the initial data [5, 6]. In this paper we propose efficient algorithms for the fast computation of the Čech complex.

1.1. Motivation: Vietoris-Rips vs Čech complex

When the input data is voluminous, for example for data sets with several thousand points, the Vietoris-Rips complex rather than the Čech complex is the simplicial model of choice [4]. The main reason is the significantly lower computational costs involved in the construction of the Vietoris-Rips complex. However, this efficiency comes at the cost of some advantages of the Čech complex, which are traded-off.

Firstly, a series of Čech complexes created for several values of ε is expected to be able to capture more topological information than the corre-

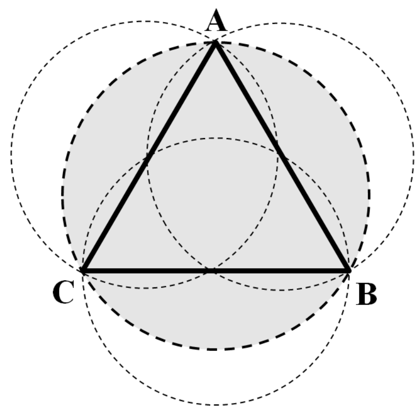

Figure 2: For $1/2 \leq \varepsilon < \sqrt{3}/3$ the Čech and the VR complexes of the unit edge equilateral triangle do not coincide.

sponding series of Vietoris-Rips complexes. Indeed, the Vietoris-Rips complex is the *clique complex* of its skeleton graph, which is the ε-neighbourhood graph of the input P. In other words, for a given ε, each simplex of the Vietoris-Rips complex corresponds to a clique of the ε-neighbourhood graph. That means that, for a given ε, the topological information captured by the Vietoris-Rips complex is equivalent to the topological information captured by the ε-neighbourhood graph. In contrast, a Čech complex conveys more information than its skeleton graph, as it may or may not contain the simplex defined by a clique of its skeleton graph.

To see this with a simple example, let $S = \{A, B, C\}$ be the vertices of an equilateral triangle with edge 1. The minimal enclosing sphere of any of the three edges AB, BC, CA has radius $1/2$, while the minimal enclosing sphere of the triangle ABC, has radius $\sqrt{3}/3$, see Fig. 2. Thus, for $\varepsilon < 1/2$ the Čech complex contains neither the edges of the triangle nor the triangle, for $1/2 \leq \varepsilon < \sqrt{3}/3$ it contains all three edges but not the triangle, while for $\varepsilon \geq \sqrt{3}/3$ it contains both the edges and the triangle. In contrast, the VR complex does not have such discriminatory power as it contains a triangle if and only if it contains all its edges. The same phenomenon can be noticed in the series of complices in Fig. 1.

A second advantage of the Čech complex is the lower asymptotic worst case for the number of maximal simplices. Indeed, the maximal simplices of the Čech complex correspond to maximal enclosing spheres which can be defined by up to $d + 1$ points on them. That is, the number of maximal simplices is bounded by a polynomial of order $d+1$. This argument will become more rigorous in section 2.1 where the details of the computation of the maximal enclosing spheres will be discussed.

In contrast, the maximal simplices of a Vietoris-Rips complex may increase exponentially with the number of points in P. To see this with a simple example, consider a planar regular polygon with $2n$ vertices and diameter 1. The largest distance between two vertices is 1, attained on pairs of antipodal vertices and the second largest distance is $\cos(\pi/2n)$. There are 2^n ways to create a set of n vertices by choosing exactly one vertex from each pair of antipodal vertices and each such set of n vertices gives a maximal simplex of the Vietoris Rips complex for $\cos(\pi/2n) < \varepsilon < 1$. Indeed, the addition of any other vertex in the set will create a pair of antipodal vertices at distance 1 and thus, push the simplex outside the Vietoris Rips complex for that value of ε.

1.2. Related work

Several simplicial constructions have been proposed and are used for the topological analysis of point data. Alpha-shapes [7] is a simplicial construction with well understood properties. The bottleneck in the computation of alpha-complices is the construction of the Voronoi diagram of the input data. Its worst case computational complexity is similar to the the computational complexity of the proposed algorithm for the Čech complex, that is $O(n^d)$ [8]. However, to the best of our knowledge, the construction of the Voronoi diagram for an arbitrary input cannot be localised and thus, it becomes inefficient for large number of input points n and large dimension d.

The beta-complex [9] is a generalisation of the alpha-complex where the input is a set of spheres of various radii rather than a point set. In beta-complices, as in alpha-complices, the computational bottleneck is the construction of the Voronoi diagram of the spheres. The construction of the beta-complex can be accelerated by using quasi-triangulations, that is, the duals of the Voronoi diagram of the spheres [10]. Techniques for the efficient representation and manipulation of quasi-triangulations have been proposed in [11, 12].

Witness complexes [13], which have been successfully used for surface modelling from point clouds [14], can be computed efficiently, however, the understanding of their properties rests on heuristic arguments rather than mathematical proofs.

Unlike the previous examples of simplicial structures which have their roots in computational problems, the Vietoris-Rips and the Čech complex have their roots in mathematical topology. Nevertheless, recently, there is a considerable research effort towards the efficient construction of the Vietoris-Rips complex, which is considered a more tractable problem than the efficient construction of Čech complex. Apart from the academic interest in such constructions, the research is also driven from applications, ranging from coverage problems in sensor networks [15], [16], to the analysis and organisation of large image databases [17].

The current state-of-the-art in computationally efficiency in the construction of the Vietoris-Rips complex is [4]. A memory efficient generation of a stream of simplices for the construction of Vietoris-Rips complexes as required by persistence homology algorithms is proposed in [18]. Publicly available implementations of the construction of the Vietoris-Rips complex can be found in Stanford's Plex family of software for the topological analysis of data. JavaPlex [19], the most recent member of the family, extends Plex [20] and JPlex [21], and incorporates the the construction in [4].

1.3. Contribution

The main contributions of the paper can be summarised as follows:

- A divide and conquer algorithm for the efficient computation of the Čech complex for a given radius ε of the covering balls.
- An algorithm for the efficient computation of all possible Čech complexes on a point set.

1.4. Overview

The rest of the paper is organised as follows. In Section 2, we describe the basic algorithm for the computation of the Čech complex and its divide-and conquer variant we used in the experiments. Section 3 describes an algorithm for the construction of all possible Čech complexes on a point set. In Section 4 we present and discuss experimental results on synthetic and natural point sets and we briefly conclude in Section 5.

2. Construction of the Čech complex

We start with a rigorous definition of the Čech complex and some notation. Consider a finite set P of n points in the d-dimensional Euclidean space $\mathbb{R}^d$ and denote by $\mathcal{B}_\varepsilon(p)$ the open ball of radius ε centered at the point $p \in \mathbb{R}^d$:

$$\mathcal{B}_\varepsilon(p) = \left\{ x \in \mathbb{R}^d \mid \mathrm{d}(p,x) < \varepsilon \right\}.$$

Consider an open cover of P that consists of all such balls centered at the points of P. The Čech complex $\mathcal{N}_\varepsilon(P)$ of the point P at scale ε is a set system that contains precisely those subsets of P for which the covers of all elements of the subset have a common intersection:

$$\mathcal{N}_\varepsilon(P) = \{ S \subseteq P \mid \cap_{p \in S} \mathcal{B}_\varepsilon(p) \neq \emptyset \}.$$

Equivalently, $S \subseteq P$ is a cell of $\mathcal{N}_\varepsilon(P)$ if and only if it is contained within the interior of a sphere of radius ε.

We shall use small letters to name points (i.e. d-dimensional vectors) and capital letters for names of sequences of points. Thus, P will usually denote the sequence $(p_1, \ldots p_n)$, where $n = |P|$ is the length of the sequence P. $P \cup Q$ will denote the merger of the two sequences P and Q (in some unspecified order unless explicitly stated otherwise). Whenever we use terms like “disjoint”, “intersection”, etc applied to sequences, we consider these as sets. Without loss of generality, we assume that there are no repetitions in a sequence of points, i.e. all the points in the sequence are different.

2.1. Minimal Enclosing Sphere

The very basic computation regarding Čech complexes is checking if a given subset of points $S \subseteq P$ is a cell of of the complex $\mathcal{N}_\varepsilon(P)$, i.e.if all the points in P are contained within a sphere of radius ε. This can be done by calculating the minimal enclosing sphere for a set of points, a problem that has been studied from both theoretical (e.g., [22]) and practical point of view [23] (which provides a robust practical implementation). Here, we use our own algorithm, which given a set of n points in $\mathbb{R}^d$ produces the (unique) minimal-radius sphere that contains the points. This is the most basic building block in our computation. We shall denote it by

$$[c, r] := \mathsf{MinSphere}(P)$$

where the input P is a sequence of n d-dimensional points, and the outputs c and r are the centre and the radius of the minimal enclosing sphere, respectively.

A trivial, but very useful, observation is that if the points are in general position (no $d+2$ or more of them are co-spherical), the minimal enclosing sphere has between 2 and $d+1$ points on its boundary (excluding the trivial case of a single point, in which the sphere has radius zero). We call these points the *support* of the sphere. It is now clear that we can easily enumerate all simplices of the complex $\mathcal{N}_\varepsilon(P)$ by simply considering all potential supports. Before we present such an algorithm, we make another trivial observation by noticing that we only need the *maximal cells*, i.e. the simplices of maximal dimension. For a given subset of points $Q \subseteq P$, we can check if it constitutes a maximal cell, denoted by

$$\mathsf{IsMaxCell}(Q)$$

by first checking that the minimal enclosing sphere of the points in Q has a radius smaller than ε and then verifying that for every point p_i not in Q, the minimal enclosing sphere of Q plus p_i has a radius greater than or equal to ε.

2.2. The basic algorithm

We can now describe Algorithm 1, the basic algorithm that enumerates all maximal cells of the Čech complex. It is a simple recursive procedure that generates all possible up-to-$(d+1)$-element subsets of a set of n elements. The set F contains the elements that have already been fixed (to be in the subset generated) while the set R contains the points that are yet to be explored. Thus, the for loop in the else part of the algorithm above, generates all the relevant subsets in lexicographical order. The basis case is handled in the then part, which adds a subset to the output list only if it is a legitimate support of a maximal cell. A call $\mathsf{EnumCells}(\emptyset, P)$ creates a list of all maximal cells of the Čech complex $\mathcal{N}_\varepsilon(P)$.

2.3. Divide-and-conquer construction of Čech complex

We finally present a more efficient divide-and-conquer procedure, which is based on the following simple idea. We first pick a direction, i.e. a d-dimensional vector a of length one, $\|a\| = 1$ and a cutpoint (number) b and partition the set of potential maximal cells C into two subsets C_L and C_R of cells (spheres) whose centres c are to the left or on (i.e. $(a, c) \leq b$) and strictly to the right (i.e. $(a, c) > b$) of the cutpoint. Clearly, C_L depends only on the points $x \in P$ that satisfy $(a, x) \leq b + \varepsilon$, while C_L can be computed only from the points $x \in P$ that satisfy $(a, x) > b - \varepsilon$. This is formalised in the recursive Algorithm 2.

Algorithm 1 $C = \mathsf{EnumCells}\,(F, R)$

Input: F and R are disjoint sequences of points in $\mathbb{R}^d$.

Output: C is the sequence of those maximal cells of the Čech complex $\mathcal{N}_\varepsilon\,(F \cup R)$ that contain all points from F.

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if $|F| = d + 1$ or $|R| = 0$

$[c_1, r_1] := \mathsf{SphereThrough}\,(F)$

$[c_2, r_2] := \mathsf{MinSphere}\,(F)$

$S := \{x \in F \cup R \mid \|x - c_1\| \leq r_1\}$

if $r_1 = r_2$ and $\mathsf{IsMaxCell}\,(S)$

$C = ((c_1, r_1))$

else

if $F \neq \emptyset$, $C := \mathsf{EnumCells}\,(F, \emptyset)$

else $C := \emptyset$

for $i := 1$ to $|R|$

$C := C \cup \mathsf{EnumCells}\left(F \cup (r_i)\,, \left(r_{i+1}, \ldots r_{|R|}\right)\right)$

return C

2.4. Discussion

Here, we discuss some subtle points in and modifications of the lower level primitives which speed up the implementation.

We start with our Minimal Enclosing Sphere algorithm, which uses the following simple heuristic. As explained before, the concept of support is crucial. Given a minimal enclosing sphere for a set of points in dimension d, the support is a set of at most $d + 1$ points that lie on the sphere. The algorithm starts with a trivial support of a single point and gradually extends the radius of the current sphere by performing so-called pivot: pick a point with the largest distance to the centre of the current sphere, and recursively create the minimal enclosing sphere for the current support plus that point. We stop if the current sphere covers all the points. The correctness of this heuristic algorithm, including its termination, is obvious.

Algorithm 2 $C = \mathsf{RecEnum}\,(P)$
Input: P is a sequence of points in $\mathbb{R}^d$.
Output: C is the sequence of all maximal cells of the Čech complex $\mathcal{N}_\varepsilon\,(P)$.

Pick $a \in \mathbb{R}^d$ with $\|a\| = 1$ and $b \in \mathbb{R}$.

$P_L := \{x \in P \mid (a, x) \leq b + \varepsilon\}$, $P_R := \{x \in P \mid (a, x) > b - \varepsilon\}$, $P_C := P_L \cap P_R$.

if P is small or $|P_C|$ is big with respect to $\min\{|P_L|, |P_R|\}$

$C = \mathsf{EnumCells}\,(\emptyset, P)$

else

$C_L := \mathsf{RecEnum}\,(P_L)$, Remove from C_L all cells with centres c s.t. $(a, c) > b$.

$C_R := \mathsf{RecEnum}\,(P_R)$, Remove from C_R all cells with centres c s.t. $(a, c) \leq b$.

$C := C_L \cup C_R$.

Return C

Moreover, in running $\mathsf{IsMaxCell}\,(.)$, we need to check many times if the minimal enclosing sphere radius for a set of points is greater than some given value. This can easily be incorporated into our algorithm, so that it stops with a positive answer as soon as the current sphere radius is large enough (as there is no need to find the final one). We also point out that the recursive call (in the pivot step) involves at most $d+2$ points, and can be implemented by complete enumeration, assuming that the dimension d is very small when compared to the number of points n.

The other algorithm, which deserves attention, is the recursive enumeration of the maximal cells. We have not specified how we partition the point set, and it is clear that this could be crucial in achieving fast running time. We use a very simple heuristic, which is essentially principal components analysis. We pick the direction a to be the largest component and then adjust the threshold b so as to get a “reasonable” split. Moreover, we perform a split only if we the ratio of the number of points in the smaller subset (and recall that the two subsets, in general, overlap and are not necessarily of the same size) to the total number of points is bigger than a prescribed constant, in the experiments 0.6. Otherwise, we simply use the complete enumeration algorithm.

We can make crude estimates of the time complexity of the procedures. It is well known (see, e.g. [22]) that the $\mathsf{MinSphere}(.)$ can be solved by a simple randomised algorithm with linear expected running time $O(d!n)$ where d is the dimension (assumed to be a constant) and n is the number of points. The procedure $\mathsf{IsMaxCell}(.)$ can be implemented in quadratic expected time $O(n^2)$ (we have ignored the constant d here). The main call of the complete enumeration procedure $\mathsf{EnumCells}(\emptyset, .)$ goes through no more than

$$\sum_{i=1}^{d+1} \binom{n}{i}$$

possible bases, which is $O\left(n^{d+1}\right)$, so the overall expected running time is $O\left(n^{d+3}\right)$. The split in the recursive enumeration should in practice improve this, although it could be as bad in theory.

In a final remark, the basic numerical building block of the algorithm is calculating a sphere going through $d+1$ points, and this creates certain numerical instability in practice: whenever we need to decide if a point is inside, outside or on the boundary of such a sphere, we have to do it with certain precision.

3. Construction of a series of Čech complexes

We will finally discuss Algorithm 3, which produces a number of Čech complexes $\mathcal{N}_{\varepsilon}(P)$ of the point P at all "meaningful" scales ε. It is perhaps an interesting point that it is easier to work these out backwards. We start with a single cell that includes all the points, and this is precisely the minimal enclosing sphere of the points. We then iterate a simple rule: from the current set of cells, pick the one with maximal radius ε (assuming without loss of generality that it is unique), and "break" it, i.e. remove each point on the boundary, calculate the minimal enclosing sphere of the others, and add it to the set of cells only if it is a maximal cell at scale ε. The correctness of the iteration is again obvious.

Algorithm 3 $\mathsf{AllComplexes}\,(P)$

Input: P is a set of n points in $\mathbb{R}^d$.

Output: Outputs all different Čech complexes $\mathcal{N}_\varepsilon\,(P)$ starting from the biggest one, where $\varepsilon_{\max}$ is the radius of the minimum enclosing sphere of P and ending at $\varepsilon_{\min} = 0$, where there are $|P|$ single points .

$[c, \varepsilon] := \mathsf{MinSphere}\,(P)$; $C := \{[c, \varepsilon]\}$

repeat

  $\mathsf{Output}\,(\varepsilon, C)$

  $Q := P \cap \partial c$; $R := P \cap c$; $C := C \setminus \{[c, \varepsilon]\}$

  for every $q \in Q$ do

    if $\mathsf{IsMaxCell}\,(R \setminus \{q\}\,, \varepsilon)$ then $C := C \cup \{\mathsf{MinSphere}\,(R \setminus \{q\})\}$

  $i := \mathrm{argmax}_{[c_j, r_j] \in C}\,\{r_j\}$; $\varepsilon := r_i$; $c := c_i$

until $\varepsilon > 0$

4. Experiments

We first tested the computational efficiency of the algorithm on synthetic point sets of various sizes. The test sets were produced through uniform random sampling of the surface of the unit sphere. All algorithms were implemented in Matlab and run on a commodity PC with a core i6 processor and 3Gb RAM. Table 1 (top) shows the timings and Table 1 (bottom) shows the ratio ρ of the number of simplices in the Čech complexes by the number of input points.

As expected, the timings depend not only on the number of input points but also on the value of ε. Indeed, a large values of ε increases the expected number of points in the basic EnumCells call of the algorithm and thus, the localization of the computations is largely lost. The timings show that the algorithm can handle point sets of several thousand points in size, while for values of ε that keep ρ around 2, the algorithm can possibly handle tens of thousand of points. As the triangulations of inputs sampled from low genus surfaces have a value of ρ around 2, complexes with a similar value of ρ are of high interest. Regarding the relationship between ε and ρ, we notice that

Scale	Number of points					
ε	500	1K	2K	5K	10K	50K
0.01	0.6	1.2	2.7	9.6	26	553
0.02	0.7	1.6	4.4	20	84	7941
0.05	1.4	4.9	22	297	2646	-
0.10	5.5	39	307	5920	-	-
0.20	79	700	-	-	-	-

Scale	Number of points					
ε	500	1K	2K	5K	10K	50K
0.01	0.97	0.96	0.93	0.83	0.74	0.90
0.02	0.92	0.86	0.77	0.70	0.82	2.07
0.05	0.74	0.71	0.87	1.41	2.41	-
0.10	0.89	0.21	1.97	4.31	-	-
0.20	2.05	3.43	-	-	-	-

Table 1: **Top:** Timings in seconds. **Bottom:** The ratio ρ of the number of simplices in the complex by the number of input points.

for very small values of ε all the input points are non-connected simplices of dimension 0 of the Čech complex and thus $\rho = 1$. As ε increases, points are connected to form non-connected edges of the complex and the value of ρ decreases and then it increases again together with the complexity of the connectivity.

In a second experiment, we tested the computational efficiency of Algorithm 3 on input point sets sampled from the interior of a planar square region. For each input, we report the running time in seconds and the number of distinct Čech complexes generated by the algorithm. The results are summarised in Table 2. As expected, the algorithm is practical for small point sets only, given that the number of distinct Čech complexes increases rapidly with the number of points in the input.

# input points	10	20	50	100
time	0.35	3.65	120	1493
# complexes	78	453	6169	43292

Table 2: The timings in seconds and the number of distinct complexes generated by Algorithm 3.

Figures 3 and 4 visualise the Čech complexes of various point sets for various values of ε. Each figure shows the minimal enclosing sphere of the maximal simplices of the Čech complex. Blue, green and red colour correspond to minimal enclosing spheres with support of 2 and 3 and 4 points, respectively.

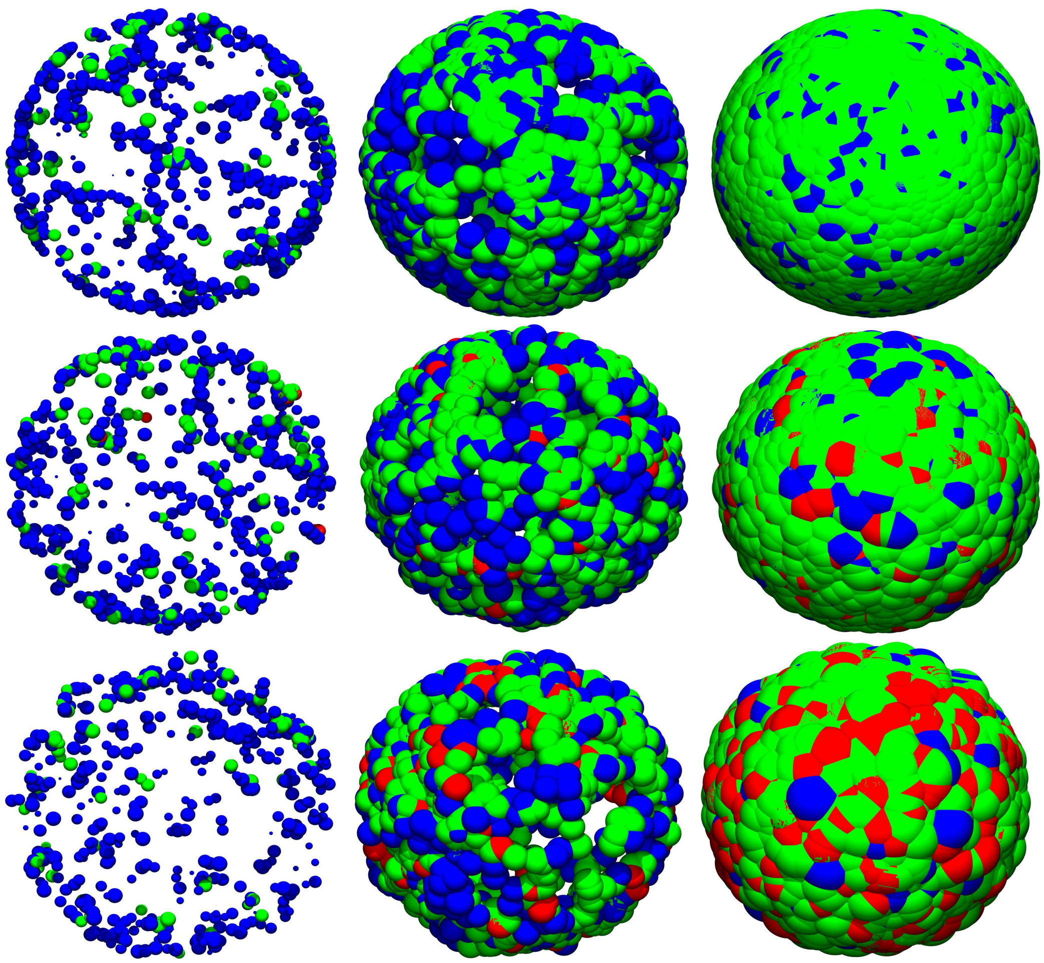

Figure 3: **Top to bottom:** the added noise is equal to 0, 0.05 and 0.1. **Left to right:** $\varepsilon = 0.05, 0.1$ and 0.2. Blue, green and red colour correspond to minimal enclosing spheres with support of 2 and 3 and 4 points, respectively.

In Figure 3, each input set contains 1,000 points sampled from the surface of the unit sphere. Various amounts of noise were added, in the form of a uniform random displacement in the direction of the normal. We notice that

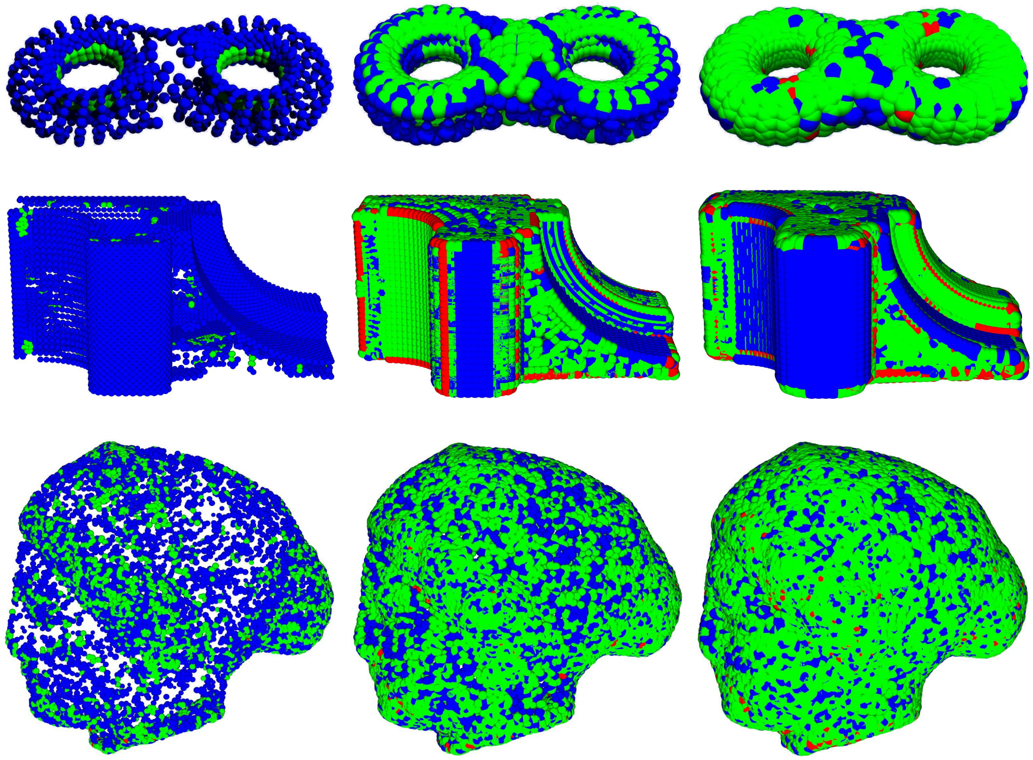

Figure 4: **Top:** $\rho = 1.31$, 1.88 and 2.55. **Middle:** $\rho = 1.48$, 2.36 and 3.89. **Bottom:** $\rho = 1.20$, 1.86 and 2.67. Blue, green and red colour correspond to minimal enclosing spheres with support of 2 and 3 and 4 points, respectively.

when no noise is added and all input points lie on the unit sphere, there is no simplex with minimal enclosing sphere of support 4. Indeed, such a minimal enclosing sphere would have to be the unit sphere itself. We also notice that, as expected, increased noise in the input increases the number of minimal enclosing spheres of support 4. These results illustrate the claim in [24] that, similarly to the Vietoris-Rips complexes, the Čech complexes are also sensitive to spatial noise.

Figure 4 visualises in the same way the Čech complexes of the set of vertices of triangle meshes modelling synthetic or natural objects. In particular, we used the Eight, Fandisk and Igea triangle mesh models with 766, 6475 and 8268 vertices. All cases verify the claim that for values of ρ around 2 the Čech complexes offer good descriptions of the underlying surfaces of the input points. We also note that near surface features, such as sharp edges or corners, the size of the support obtains its maximum value 4.

5. Conclusion

We presented an algorithm for the efficient computation of the Čech complex of a point set for a given value of ε and an algorithm for the efficient computation of all possible Čech complexes on a point set. Experimental results on synthetic and natural 3D point sets sampled from surfaces show that the first algorithm can generally handle sets of several thousand points. The same experiments show that for smaller values of ε, which give Čech complexes with most of their maximal simplices triangles, the algorithm can handle inputs with tens of thousands of points. We conclude that in certain surface modelling applications the computation of the topologically more informative Čech complex can be a viable alternative to the simpler and more efficient computations of Vietoris-Rips complexes.

In the future, we plan to prove theoretical bounds for the number of maximal simplices in the Čech complex of N points sampled from the surface of a sphere. In particular, we conjecture that the number of maximal simplices is $O(N)$. As a second direction of future work, we plan to investigate the differences in the persistence topology of a point set when it is studied through the Čech and the Vietoris-Rips complex, respectively.

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