EXPLICIT SPECTRAL GAPS FOR RANDOM COVERS OF RIEMANN SURFACES

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ABSTRACT. We introduce a permutation model for random degree n covers X_n of a non-elementary convex-cocompact hyperbolic surface $X = \Gamma\backslash\mathbb{H}$. Let δ be the Hausdorff dimension of the limit set of Γ. We say that a resonance of X_n is *new* if it is not a resonance of X, and similarly define new eigenvalues of the Laplacian.

We prove that for any $\epsilon > 0$ and $H > 0$, with probability tending to 1 as $n \to \infty$, there are no new resonances $s = \sigma + it$ of X_n with $\sigma \in [\frac{3}{4}\delta + \epsilon, \delta]$ and $t \in [-H, H]$. This implies in the case of $\delta > \frac{1}{2}$ that there is an explicit interval where there are no new eigenvalues of the Laplacian on X_n. By combining these results with a deterministic 'high frequency' resonance-free strip result, we obtain the corollary that there is an $\eta = \eta(X)$ such that with probability $\to 1$ as $n \to \infty$, there are no new resonances of X_n in the region $\{ s : \mathrm{Re}(s) > \delta - \eta \}$.

CONTENTS

1. Introduction

This paper is about spectral gaps for random Riemann surfaces. More specifically, we are interested in various notions of spectral gap for random covers of a fixed Schottky Riemann surface. This is in close analogy to questions about the spectral gap of a random regular graph, and this analogy informs our model for random coverings, so we begin with a discussion on graphs.

Let G be a k-regular graph on n vertices. Then the adjacency matrix A_G of G has n real eigenvalues in $[-k, k]$ and k appears as an eigenvalue with multiplicity equal to the number of connected components of G. Denoting by

$$k = \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n$$

the eigenvalues of G, the *spectral gap* of G is $\lambda_0 - \lambda_1$. If G is connected, then $\lambda_0 > \lambda_1$ and the spectral gap is related to the exponential rate at which the random walk on G converges to the uniform measure. As such, it is an important quantity in theoretical computer science, and accordingly, there has been a great deal of interest in the spectral gap of a *random* regular graph. Alon's conjecture [Alo86], now a theorem due to Friedman [Fri08], says that for any $\epsilon > 0$, as $n \to \infty$, the probability that $\lambda_1(G_n) > 2\sqrt{k-1} + \epsilon$ tends to zero, when G_n is sampled uniformly at random from k-regular graphs with n vertices. The relevance of the quantity $2\sqrt{k-1}$ is that for any k-regular graph with n vertices, a result of Alon-Boppana [Nil91] says that $\lambda_1(G) \geq 2\sqrt{k-1} - o_n(1)$, so $2\sqrt{k-1}$ is an asymptotically optimal lower bound for $\lambda_1(G)$, often called the *Ramanujan bound* after [LPS88].

The model of a random graph described above chooses random graphs according to a uniform distribution. Another popular model for a random $2k$-regular graph is called the *permutation model* and is the one we wish to focus on in the sequel. Let $\Gamma = \langle \gamma_1, \ldots, \gamma_k, \gamma_1^{-1}, \ldots, \gamma_k^{-1} \rangle$ be a free group on k generators, $k \geq 2$, and let S_n denote the symmetric group on n letters, and ϕ_n be a random homomorphism from Γ to S_n, sampled uniformly from all possible homomorphisms. Since Γ is free, a homomorphism is described simply by choosing the images $\phi_n(\gamma_i)$ of the generators of Γ independently and uniformly from S_n. Then let G_n be the random graph with vertex set $[n] \stackrel{\text{def}}{=} \{1, \ldots, n\}$ and with an edge between i and j if there is a generator γ_a such that $\phi_n(\gamma_a)(i) = j$. We will adapt this model to a model of a random Riemann surface.

Let X be a connected, non-elementary, non-compact, convex co-compact hyperbolic surface. Then $X = \Gamma \backslash \mathbb{H}$ where $\mathbb{H}$ is the hyperbolic upper half plane and Γ is a free subgroup of $\mathrm{SL}_2(\mathbf{R})$. We view X as fixed throughout the paper. We let X_n be the random n-cover of X obtained as a fibered product $X_n \stackrel{\text{def}}{=} \mathbb{H} \times_{\phi_n} [n]$. More precisely, X_n is the quotient of $\mathbb{H} \times [n]$ by the diagonal action of Γ

$$\gamma.(x, i) = (\gamma(x), \phi_n(\gamma)(i)).$$

If $S \subset [n]$ is a set of representatives for the orbits of Γ on $[n]$ via ϕ_n, and $\Gamma_i \stackrel{\text{def}}{=} \mathrm{Stab}_\Gamma(i)$ is the stabilizer of $i \in S$, then X_n is isomorphic to the disjoint union of (connected) covers $\Gamma_i \backslash \mathbb{H}$, i.e.

$$X_n = \bigsqcup_{i \in S} \Gamma_i \backslash \mathbb{H}.$$

Notice that we have

$$\sum_{i \in S} [\Gamma : \Gamma_i] = n.$$

We say that a property $E(\phi_n)$ of the random ϕ_n holds *asymptotically almost surely* (a.a.s.) if as $n \to \infty$, the probability that $E(\phi_n)$ holds tends to 1. It is an elementary calculation[1] that a.a.s. Γ acts transitively on $[n]$ via ϕ_n and hence, a.a.s. X_n is connected. This also follows from the main theorems below. Although we do not assume X_n is connected at any point, it would not hurt to assume this on a first reading.

We now discuss the spectral theory of X and X_n. The group Γ acts properly discontinuously on $\mathbb{H}$, but for any point $o \in \mathbb{H}$, the orbit Γo accumulates on $\partial \mathbb{H} = \mathbf{R} \cup \{\infty\}$ and the accumulation set of this orbit is called the *limit set of* Γ and denoted by $\Lambda(\Gamma)$. This $\Lambda(\Gamma)$ is a perfect nowhere dense fractal and has an associated Hausdorff dimension $\delta \stackrel{\text{def}}{=} \dim_{\mathrm{Haus}}(\Lambda(\Gamma)) \in [0, 1)$. By a result of Lax and Phillips [LP81], the spectrum of the Laplacian Δ_X is discrete in the range $[0, \frac{1}{4})$, and Patterson [Pat76] proved that if $\delta > \frac{1}{2}$, then the lowest eigenvalue of Δ_X is $\delta(1-\delta)$. If $\delta \leq \frac{1}{2}$ then there are no eigenvalues of Δ_X. The same is true for X_n, with the same δ (although $\delta(1-\delta)$ will be simple if and only if X_n is connected). More generally, if λ is any eigenvalue of X, then by lifting eigenfunctions through the covering map, λ is an eigenvalue for X_n with at least as large multiplicity.

The first main theorem of our paper is the following.

Theorem 1.1. *Assume that $\delta > \frac{1}{2}$. Then for any $\sigma_0 \in \left(\frac{3}{4}\delta, \delta\right)$, a.a.s.*

$$\mathrm{spec}(\Delta_{X_n}) \cap [\delta(1-\delta), \sigma_0(1-\sigma_0)] = \mathrm{spec}(\Delta_X) \cap [\delta(1-\delta), \sigma_0(1-\sigma_0)] \tag{1.1}$$

and the multiplicities on both sides are the same.

Remark 1.2. This theorem implies that a.a.s. the X_n have a uniform spectral gap, and this spectral gap only depends on δ and the gap between the first two eigenvalues of X.

Remark 1.3. If $\delta \in (\frac{1}{2}, \frac{2}{3})$ then since X_n has no eigenvalues in $[\frac{1}{4}, \infty)$ by a result of Lax and Phillips [LP81], Theorem 1.1 implies that a.a.s. X_n has no new eigenvalues.

Remark 1.4. Theorem 1.1 can be viewed as a significant sharpening of a result of Brooks and Makover [BM04], albeit in the infinite area setting. See §1.1 for a more detailed discussion of this comparison.

[1] For a concrete reference, this statement follows from [BS87, Thm.13]. One can also prove by elementary combinatorial arguments that the probability that X_n is connected as $n \to +\infty$ is bigger than $1 - \frac{C(k)}{n^{k-1}}$, where $C(k) > 0$ is a constant depending only on k.

Remark 1.5. We point out that it is possible for X_n to not be connected, and in this case, there is no spectral gap. Even further, it is easy to see that X_n can be a connected cyclic cover of X, and by results of [JNS19], these have no uniform spectral gap.

Remark 1.6. In the limit as $\delta \to 1$, the range of forbidden eigenvalues in (1.1) becomes $[0, \frac{3}{16})$. This is interestingly the same range covered by Selberg's $\frac{3}{16}$ Theorem [Sel65] on the spectral gap of congruence covers of the modular surface $\mathrm{SL}_2(\mathbf{Z})\backslash\mathbb{H}$. This should also be compared to the deterministic result of Gamburd [Gam02] for congruence covers of infinite index geometrically finite subgroups of $\mathrm{SL}_2(\mathbf{Z})$: assuming $\delta > \frac{5}{6}$, he shows that the spectrum remains the same in the range $[\delta(1-\delta), \frac{5}{36})$. See also [Mag15] for a generalization of this result to higher dimensions.

We write $\chi(X)$ for the Euler characteristic of X. It has recently been proved by Ballmann, Matthiesen, and Mondal [BMM17] that if $\chi(X) < 0$, Δ_X has at most $-\chi(X)$ eigenvalues. If $\chi(X) = -1$ then this means the only possible eigenvalue of X is at $\delta(1-\delta)$ and thus Theorem 1.1 yields

Corollary 1.7. *Assume that $\delta > \frac{1}{2}$. If X is topologically a pair of pants, or a torus with one hole, then for any $\sigma_0 \in \left(\frac{3}{4}\delta, \delta\right)$, a.a.s.*

$$\mathrm{spec}(\Delta_{X_n}) \cap \left(\delta\left(1-\delta\right), \sigma_0(1-\sigma_0)\right] = \emptyset,$$

and $\delta(1-\delta)$ is a simple eigenvalue of Δ_{X_n}.

We now turn to what we can say about general $\delta \in (0,1)$. In the case $\delta \leq \frac{1}{2}$, Δ_X and Δ_{X_n} will have no discrete L^2 spectrum, so one must consider a more subtle notion of spectral gap.

For any non-elementary convex co-compact hyperbolic Y with $\delta = \delta(Y)$ (e.g. $Y = X$, $Y = X_n$) the *resolvent*

$$R_Y(s) \stackrel{\mathrm{def}}{=} (\Delta_Y - s(1-s))^{-1} : L^2(Y) \to L^2(Y)$$

is, a priori, a meromorphic family of bounded operators in the right half plane $\mathrm{Re}(s) > \frac{1}{2}$ with poles precisely at real s such that $s(1-s)$ is an eigenvalue of Δ_X. By work of Mazzeo-Melrose [MM87], it can be meromorphically continued to a family of bounded operators from $C_0^\infty(Y) \to C^\infty(Y)$ that is meromorphic in $s \in \mathbf{C}$. In the case of hyperbolic surfaces, a simpler proof of the meromorphic continuation is due to Guillopé and Zworski [GZ95], see also the book [Bor16].

The poles of the meromorphically continued resolvent are called *resonances of Y*. In the sequel we write $\mathcal{R}_Y \subset \mathbf{C}$ for the multi-set of resonances, repeated according to multiplicities[2]. Resonances, unlike L^2-eigenvalues, correspond to a non self-adjoint spectral problem and are therefore notoriously difficult to study. There is however a clear analog of the spectral gap in this setting. The 'bass resonance' is located at $s = \delta$ and by a result of Naud [Nau05a] if Y is connected then there exists a constant $\epsilon_\Gamma > 0$ such that

$$\mathcal{R}_Y \cap \{\, s \,:\, \mathrm{Re}(s) \geq \delta - \epsilon_\Gamma \,\} = \{\delta\}.$$

[2]Following [Bor16, Def. 8.2], the multiplicity of a resonance s of Y is given by $\mathrm{rank}\left(\int_\gamma R_Y(s)ds\right)$ where γ is an anticlockwise oriented circle enclosing s and no other resonance of Y.

We call the existence of such a resonance free strip a *spectral gap* for Y. The spectral gap on hyperbolic surfaces has numerous applications, from prime geodesic theorems [Nau05b] to local L^2-asymptotics of waves [GN09]. A recent breakthrough of Bourgain-Dyatlov [BD18] showed that there always exists an *"essential spectral gap"* past the line $\{\mathrm{Re}(s) = \frac{1}{2}\}$, i.e. there exists $\tilde{\epsilon} = \tilde{\epsilon}(Y) > 0$ such that

$$\mathcal{R}_Y \cap \{ s \, : \, \mathrm{Re}(s) \geq \frac{1}{2} - \tilde{\epsilon} \}$$

is a finite set. The proof is based on the general phenomenon of "fractal uncertainty principle", see [Dya19]. We point out that $\widetilde{\epsilon} > 0$ can be made explicit, see Jin-Zhang [JZ17] and also Dyatlov-Jin [DJ18]. For a broader view and a state of the art survey on the mathematical theory of resonances including hyperbolic manifolds and related conjectures, we recommend to read [Zwo17]. Our next main result is the following.

Theorem 1.8. *Fix any $H > 0$ and $\sigma_0 \in \left(\frac{3}{4}\delta, \delta\right)$, and let*

$$\mathrm{Rect}(\sigma_0, H) \stackrel{\mathrm{def}}{=} \{ s = \sigma + it \, : \, \sigma \in [\sigma_0, \delta] \text{ and } |t| \leq H \}.$$

Then a.a.s.

$$\mathcal{R}_{X_n} \bigcap \mathrm{Rect}(\sigma_0, H) = \mathcal{R}_X \bigcap \mathrm{Rect}(\sigma_0, H)$$

where the multiplicities on both sides are the same.

Remark 1.9. Because all eigenvalues $\lambda_\sigma = \sigma(1-\sigma)$ of Δ_{X_n} with $\sigma > \frac{1}{2}$ give a resonance of X_n at σ, with the same multiplicity, and the same is true for X, Theorem 1.8 implies Theorem 1.1 and extends it to resonances in rectangles of explicit width and any bounded height $2H$. We point out that Theorem 1.8 actually yields new information on low frequency resonances past the line $\{\mathrm{Re}(s) = \frac{1}{2}\}$ when $\delta \in (\frac{1}{2}, \frac{2}{3})$.

This leaves the question of how to deal with resonances with large imaginary part. For this we have the following theorem that applies to arbitrary covers. Note that here there is no randomness involved.

Theorem 1.10. *Assume that Γ is a non-elementary convex co-compact group. Then there exist $\epsilon_\Gamma > 0$ and $T_\Gamma > 0$ such that for all finite index subgroups $\widetilde{\Gamma} \subset \Gamma$, we have for $\widetilde{X} = \widetilde{\Gamma} \backslash \mathbb{H}$,*

$$\mathcal{R}_{\widetilde{X}} \cap \{ s \, : \, \mathrm{Re}(s) \geq \delta - \epsilon_\Gamma \text{ and } |\mathrm{Im}(s)| \geq T_\Gamma \} = \emptyset.$$

Remark 1.11. From the work of Bourgain and Dyatlov [BD17], we know that there exists $\varepsilon(\delta) > 0$, depending only on δ and thus uniform on covers such that

$$\mathcal{R}_X \cap \{\mathrm{Re}(s) \geq \delta - \varepsilon(\delta)\}$$

is a **finite** set. However the result of Bourgain and Dyatlov does not provide any information on the finite set of resonances in this uniform strip. Theorem 1.10 shows that new resonances can only appear in a compact region.

Combining Theorem 1.8 with Theorem 1.10 yields the following corollary.

Corollary 1.12. *A.a.s. the random cover $X_n \to X$ has a uniform spectral gap. In particular, above each non elementary surface X, one can produce an infinite family of covers X_n with degree n and having a uniform spectral gap.*

Remark 1.13. When $\delta > \frac{1}{2}$, Corollary 1.12 follows from a mild extension of [BGS11, Thm. 1.2] together with results on random graphs as explained in §1.1. However, when $\delta \leq \frac{1}{2}$, to our knowledge, Corollary 1.12 is completely new: the only result of that type so far is for congruence covers of convex co-compact subgroups of $\mathrm{SL}_2(\mathbf{Z})$, see Oh-Winter [OW16] and the discussion below.

1.1. **Prior work. Brooks and Makover.** Brooks and Makover in [BM04] consider a similar model for random finite area Riemann surfaces. In this model, random surfaces are modeled by random 3-regular oriented graphs sampled according to a refinement of the Bollobás 'bin model' introduced in [Bol88]. Then Brooks and Makover [BM04] construct from a random oriented graph on n vertices a Riemann surface Y_n, tiled by a specific hyperbolic triangle with one vertex at ∞. They then consider a compactification Y_n^c of the cusped surface Y_n. Thus Y_n^c is a random compact Riemann surface; the genus of Y_n^c is however not deterministic[3]. Brooks and Makover proved in *(ibid.)*

Theorem 1.14 (Brooks-Makover)**.** *There is some constant $C > 0$ such that a.a.s. the first non-zero eigenvalue of Y_n^c is $\geq C$.*

Although our main theorems deal instead with infinite area Riemann surfaces, they offer two improvements over Theorem 1.14:

- The range of new forbidden eigenvalues and resonances in Theorems 1.1 and 1.8 are explicit,
- Moreover, we have an entire moduli space of random families (parameterized by the modulus of X) and the range of forbidden eigenvalues and resonances only depends on X in a very mild way, through the Hausdorff dimension of the limit set.

The Brooks-Burger transfer principle. Also relevant to the current work is the following transfer principle for small eigenvalues developed independently by Brooks and Burger in [Bro86, Bur88].

Theorem 1.15 (Brooks-Burger)**.** *Let Y be any compact Riemannian manifold with $\Gamma = \pi_1(Y)$. There is a constant $c(Y) > 0$ and a finite subset $S \subset \Gamma$ such that the following hold. Let Γ' be any finite index subgroup of Γ, with associated Riemannian covering space Y' of Y. Let $\lambda_1(Y')$ be such that $\mathrm{spec}(\Delta_{Y'}) = \{ 0 \leq \lambda_1 \leq \lambda_2 \leq \dots \}$. Let $G = G(\Gamma', S)$ be the Schreier coset graph of S acting on Γ/Γ'. Then*

$$\lambda_1(Y') \geq c(Y) \left(\lambda_0 (G) - \lambda_1(G) \right). \tag{1.2}$$

Theorem 1.15 was extended to Galois covers of non-elementary convex co-compact hyperbolic surfaces by Bourgain, Gamburd and Sarnak in [BGS11, Thm. 1.2] where the left hand side of (1.2)

[3]By a result of Gamburd [Gam06], if $l(Y_n) \overset{\text{def}}{=} \frac{n}{2} + 2 - 2\mathrm{genus}(Y_n^c)$, then as $n \to \infty$, $l(Y_n)$ converges to a Poisson-Dirichlet distribution. The function $l(Y_n)$ coincides with the number of cusps of Y_n.

is replaced by the gap between $\delta(1-\delta)$ and the next eigenvalue of the L^2-Laplacian. This extends to non-Galois covers and therefore applies in the setting of this paper as follows.

Let us assume that $X_n = \Gamma_n\backslash\mathbb{H}$ is connected, for simplicity, although the argument can be adapted to the general case. For fixed $S \subset \Gamma$, the Schreier coset graphs G_n of S acting on $\Gamma/\Gamma_n \cong [n]$ are precisely the random regular graphs of the permutation model, and a.a.s. these have a uniform spectral gap by [BS87, Fri08]. Hence by the extension of [BGS11, Thm. 1.2] the X_n have a uniform spectral gap between $\delta(1-\delta)$ and the next L^2-eigenvalue. Importantly, in all versions of Theorem 1.15, the constant c depends on Y in a complicated way. Because of this, it is unlikely such an argument would lead to e.g. Theorem 1.1. However, this argument does lead to Corollary 1.12 when $\delta > \frac{1}{2}$ (cf. Remark 1.13).

It is also worth mentioning that a variant of Theorem 1.15 has also been developed for resonances in [BGS11, OW16, MOW17], for specific congruence coverings of $Y = \Gamma\backslash\mathbb{H}$ where Γ is an infinite index subgroup of $\mathrm{SL}_2(\mathbf{Z})$. Besides only dealing with Galois covers, the key reason that these methods cannot prove Corollary 1.12 when $\delta \leq \frac{1}{2}$ is the following. The state of the art method [MOW17, Appendix] of dealing with low frequency resonances (a la Theorem 1.8) involves bounds on the dimensions of non-trivial irreducible representations of finite groups $\mathcal{G}$ that are polynomial in $|\mathcal{G}|$. The relevant groups in our setting are S_n, and the issue is that S_n has non-trivial irreducible representations of dimensions that are sub-logarithmic in $|S_n| = n!$.

Finally we point out that the methods of [BGS11, OW16, MOW17] are not well adapted to efficiently tracking constants and hence likely not suitable for producing explicit resonance free regions as in Theorem 1.8.

1.2. **Overview of proofs and paper organization.** All the proofs of the paper rely on a Schottky encoding of the action of Γ on $\mathbf{R}$ that is presented in §2.1. To control resonances (and eigenvalues) we rely on the connection between resonances and zeros of the Selberg zeta function due to Patterson and Perry [PP01]. This connection is explained in §2.2. We then pass to dynamical considerations by the relationship between Selberg zeta functions and dynamical zeta functions explained in §4.1. The relevant dynamical zeta functions are Fredholm determinants of certain transfer operators on vector valued functions, twisted by (random) unitary representations ρ_n^0 of Γ. These are introduced in §2.2. The relevance of these representations is that the zeros of the ρ_n^0-twisted Selberg zeta function of X correspond to new resonances of X_n (see §4.1). These are precisely the objects we wish to control.

Theorems 1.1 and 1.8. Since Theorem 1.8 implies Theorem 1.1 it suffices to discuss the former.

So far we have not been precise about the transfer operators we use. To prove Theorem 1.8 we do not use the 'standard' twisted transfer operators used for example in [BGS11, OW16, MOW17], but rather, we base our twisted operators on the *refined transfer operators* introduced by Bourgain and Dyatlov in [BD17]. The operators are denoted by $\mathcal{L}_{\tau,s,\rho_n^0}$ and defined precisely in §2.2. The parameter s is a frequency parameter, and the parameter τ is a 'discretization parameter' that is taken to be $n^{-\frac{2}{\delta}}$. If we do not use this operator in the definition of the dynamical zeta function, but rather, an iterate of the standard one, without the built in parameter τ, then one can still follow

the strategy of this paper to obtain resonance-free regions. However, these will depend on subtle features of the graph of the pressure functional $P(\sigma)$ defined in §2.1. It is the use of refined transfer operators that allows us to improve on this, and is a key idea in the paper. The functional spaces we use are Bergman spaces, and this gives us crucial access to trace techniques.

To control zeros of the dynamical zeta function in a rectangle, we use Jensen's formula with a circle enclosing the rectangle (cf. Figure 6.1). The strategy is to prove that the expected number of zeros in the region decays as a polynomial in n, so by Markov's inequality, a.a.s. there are none. There are two terms in Jensen's formula we need to control. The first is $\log|\det(1-\mathcal{L}_{\tau,s,\rho_n^0})|$ when s is the center of the circle. As shown in Proposition 4.8, this term decays provided the center of the circle is a sufficiently large real number, which can be arranged. The second term in Jensen's formula is the integral over s in the circle of $\log|\det(1-\mathcal{L}^2_{\tau,s,\rho_n^0})|$. A convenient property of Jensen's formula is that it is an integral formula, and we can take expectations inside the integral. Using Weyl's inequality, and taking expectations, we reduce to bounding the expectation $\mathbb{E}_n\|\mathcal{L}_{\tau,s,\rho_n^0}\|^2_{\text{H.S.}}$ of the squared Hilbert-Schmidt norm of $\mathcal{L}_{\tau,s,\rho_n^0}$ for s on the circle. We need to prove these all decay uniformly and polynomially in n. This estimate is at the core of the proof, is stated precisely in Proposition 5.1, and its proof takes up §5.

We now discuss the proof of Proposition 5.1. The first step is a formula for $\mathbb{E}_n\|\mathcal{L}_{\tau,s,\rho_n^0}\|^2_{\text{H.S.}}$. This uses a deterministic expression for $\|\mathcal{L}_{\tau,s,\rho_n^0}\|^2_{\text{H.S.}}$ involving a Bergman kernel and given in Lemma 4.7. The formula for $\|\mathcal{L}_{\tau,s,\rho_n^0}\|^2_{\text{H.S.}}$ is a complex weighted sum of random variables $\text{Tr}[\rho_n^0(\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1})]$, where $\gamma_{\mathbf{a}'}$ and $\gamma_{\mathbf{b}'}$ are elements of Γ. By linearity of expectations we obtain an expression for $\mathbb{E}_n\|\mathcal{L}_{\tau,s,\rho_n^0}\|^2_{\text{H.S.}}$ as a weighted sum of expectations

$$\mathbb{E}_n[\text{Tr}(\rho_n^0(\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1}))] \tag{1.3}$$

By passing to a majorant, in Lemma 5.4 we reduce our task to estimating a sum of the form

$$\sum_{\mathbf{a},\mathbf{b}\in\overline{Z}(\tau)} |\mathbb{E}_n[\text{Tr}(\rho_n^0(\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1}))]| \tag{1.4}$$

where $\overline{Z}(\tau)$ is a set of words in the generators of Γ, and $\mathbf{a}'$ is $\mathbf{a}$ with the last letter removed.

The strategy is to insert good bounds for (1.3) into (1.4) to obtain the decay we want. This is analogous to the *trace method* used to bound the spectral gap of a random graph, where $\|\mathcal{L}_{\tau,s,\rho_n^0}\|^2_{\text{H.S.}}$ would be replaced by the trace of a power of the adjacency matrix. Indeed, the bounds we use for (1.3) go back to the paper of Broder and Shamir [BS87] who used the trace method to show that the second largest eigenvalue of a $2k$-regular random graph in the permutation model is a.a.s. $\leq 3k^{\frac{3}{4}}$. So the appearance of $\frac{3}{4}$ in Theorems 1.1 and 1.8 is similar to *(ibid.)*.

In *(ibid.)* Broder and Shamir proved, roughly speaking, that $\mathbb{E}_n[\text{Tr}(\rho_n^0(\gamma))]$ has a trivial bound if γ is the identity, a better bound if γ is a proper power of another element in Γ, and an even better bound if γ does not fall in one of the previous two cases. We need a two sided estimate for (1.3) that can be deduced from more recent work of Puder [Pud15] and is stated in Theorem 5.2. According to the three cases above, we partition the range of summation in (1.4) into three different sets.

The hardest of these to deal with in (1.4) is the set PowerPairs(τ) that consists of $\mathbf{a}, \mathbf{b} \in \overline{Z}(\tau)$ such that $\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1}$ is a proper power in Γ. We need to show that the contribution of this set to (1.4) has polynomial decay. We give a precise bound on |PowerPairs(τ)| in Proposition 5.6; this proposition is at the core of the paper so we now explain the ideas of its proof.

Throughout the paper we work with real quantities $\Upsilon_{\mathbf{a}}$, where $\mathbf{a}$ is a word in the generators of Γ. These are defined in §3. Roughly speaking, $\Upsilon_{\mathbf{a}}$ measures the size of the derivative of the associated group element $\gamma_{\mathbf{a}'}$, and the set $\bar{Z}(\tau)$ is the set of words $\mathbf{a}$ such that $\Upsilon_{\mathbf{a}} \approx \tau$. This means that estimating |PowerPairs(τ)| is roughly the same as estimating the sum

$$\sum_{(\mathbf{a},\mathbf{b})\in \mathrm{PowerPairs}(\tau)} \Upsilon_{\mathbf{a}}^{\frac{\delta}{2}} \Upsilon_{\mathbf{b}}^{\frac{\delta}{2}}; \tag{1.5}$$

the choice of the exponent $\frac{\delta}{2}$ optimizes the result we can get from this method. The key combinatorial observation we use to estimate (1.5) is that if $\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1}$ is a proper power, after performing an absolutely bounded finite number of the following operations

- cutting the sequences $\mathbf{a}'$ and $\mathbf{b}'$,
- possibly replacing some cut sequence with its 'mirror',
- and regluing

one can form a long identical pair of sequences. This idea is performed rigorously in §5.4. The result of these operations on the Υ is to introduce a bounded multiplicative constant, since Υ is roughly multiplicative (Lemma 3.4) and behaves well with respect to mirrors (Lemma 3.5). The result of obtaining the long identical pair of sequences is that we get bounds on (1.5) from the relationship between sums of $\Upsilon_{\mathbf{a}}$ and the pressure functional (Lemma 3.10).

Theorem 1.10. The proof of Theorem 1.10 is given in §7. It is based on uniform Dolgopyat estimates for arbitrary unitary representations of Γ. We use the main result of Bourgain and Dyatlov [BD17] on Patterson-Sullivan measures and Fourier decay to provide a short and completely general proof of the uniform Dolgopyat estimates without having to rely on the more difficult technique from [Nau05a], which was also used in [OW16, MOW17].

1.3. **Notation.** If $U \subset \mathbf{C}$ we write $\overline{U}$ for the closure of U. We write $\mathbf{N}$ for the natural numbers and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.

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2. Preliminaries

In this paper we use the notational system for Schottky groups that is used in the papers of Dyatlov and Bourgain [BD17] and Dyatlov and Zworski [DZ17] since it is very convenient for the analysis in the sequel. We follow these papers closely in our development.

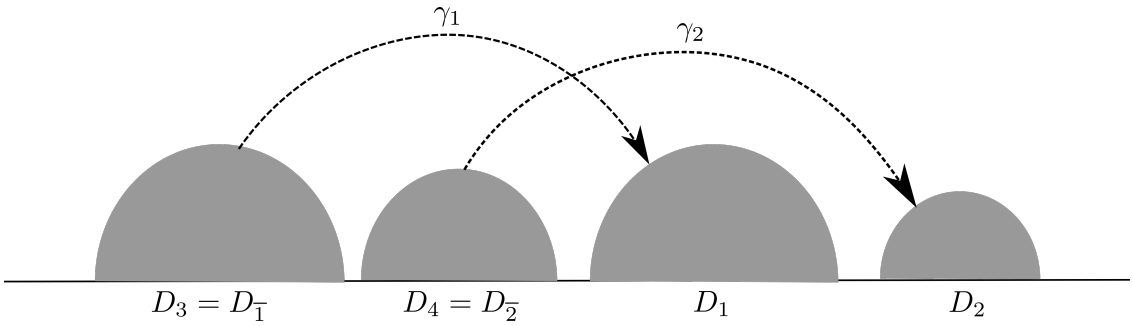

FIGURE 2.1. An example of Schottky pairing with $r = 2$

2.1. **Words, encodings of Schottky groups, and pressure.** Let $r \geq 2$ and $\mathcal{A} = \{1, \ldots, 2r\}$. If $a \in \mathcal{A}$, then we write $\bar{a} = a + r \bmod 2r$. The setup of our paper is that we are given for each $a \in \mathcal{A}$ an *open*[4] disc D_a in $\mathbf{C}$ with center in $\mathbf{R}$. The closures of the discs D_a for $a \in \mathcal{A}$ are assumed to be disjoint from one another. We let $I_a = D_a \cap \mathbf{R}$, an open interval. We write $\mathbf{D} = \cup_{a \in \mathcal{A}} D_a$ for the union of the discs.

We consider the usual action of $\mathrm{SL}_2(\mathbf{R})$ by Möbius transformations on the extended complex plane $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. We are given for each $a \in \mathcal{A}$ a matrix $\gamma_a \in \mathrm{SL}_2(\mathbf{R})$ with the properties

$$\gamma_a \left(\hat{\mathbf{C}} - D_{\bar{a}} \right) = \overline{D_a}, \quad \gamma_{\bar{a}} = \gamma_a^{-1}.$$

We write $\Gamma = \langle \gamma_a \, : \, a \in \mathcal{A} \rangle$ for the group generated by the γ_a. Since the D_a are disjoint, the Ping-Pong Lemma shows that Γ is a free subgroup of $\mathrm{SL}_2(\mathbf{R})$. Any group obtained by this construction is called a *Schottky group*. It is a result of Button [But98] that if $X = \Gamma \backslash \mathbb{H}$ is a connected convex co-compact Riemann surface as in our main theorems, then Γ is a Schottky group; we now fix Γ and assume it arises from the above construction.

The elements of Γ can be encoded by words in the alphabet $\mathcal{A}$ as follows. A *word* is a finite sequence

$$\mathbf{a} = (a_1, \ldots, a_n), \quad n \in \mathbf{N} \cup \{0\}$$

such that $a_i \neq \overline{a_{i+1}}$ for $i = 1, \ldots, n-1$. We say that n is the *length* of $\mathbf{a}$ and denote this by $|\mathbf{a}| = n$. We write $\mathcal{W}$ for the collection of all words, $\mathcal{W}_N$ for the words of length N, and $\mathcal{W}_{\geq N}$ for the words of length $\geq N$. We write $\emptyset$ for the empty word and write $\mathcal{W}^\circ = \mathcal{W} - \{\emptyset\}$. For $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots b_m) \in \mathcal{W}$ we write

- $\mathbf{a}' = (a_1, \ldots, a_{n-1})$ if $\mathbf{a} = (a_1, \ldots, a_n)$ and $n \geq 1$.
- $\mathbf{a} \to \mathbf{b}$ if either of $\mathbf{a}$ or $\mathbf{b}$ is empty, or else $a_n \neq \overline{b_1}$, in which case $(a_1, \ldots, a_n, b_1, \ldots, b_m)$ is in $\mathcal{W}^\circ$ and we write $\mathbf{ab}$ for this concatenation.
- $\mathbf{a} \rightsquigarrow \mathbf{b}$ if $\mathbf{a}, \mathbf{b} \in \mathcal{W}^\circ$ and $a_n = b_1$, which case $\mathbf{a}'\mathbf{b}$ is in $\mathcal{W}^\circ$.

If $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{W}$ then we associate to $\mathbf{a}$ the group element $\gamma_{\mathbf{a}} \stackrel{\text{def}}{=} \gamma_{a_1} \ldots \gamma_{a_n}$; here $\gamma_\emptyset = \mathrm{id}$. The map $\mathbf{a} \in \mathcal{W} \mapsto \gamma_{\mathbf{a}} \in \Gamma$ is a one-to-one encoding of Γ. We write $\overline{\mathbf{a}} \stackrel{\text{def}}{=} (\overline{a_n}, \ldots, \overline{a_1})$ and call this

[4]This is a difference from the notation of [BD17] that we make the reader aware of.

the *mirror* of $\mathbf{a}$. Note that $\gamma_{\overline{\mathbf{a}}} = \gamma_{\mathbf{a}}^{-1}$. If $\mathbf{a} = (a_1, \dots, a_n) \in \mathcal{W}^\circ$ we let

$$D_{\mathbf{a}} = \gamma_{\mathbf{a}'}(D_{a_n}), \quad I_{\mathbf{a}} = \gamma_{\mathbf{a}'}(I_{a_n})$$

and write $|I_{\mathbf{a}}|$ for the length of the open interval $I_{\mathbf{a}}$.

The *Bowen-Series map* $T : \mathbf{D} \to \hat{\mathbf{C}}$ is given by

$$T|_{D_a} = \gamma_a^{-1} = \gamma_{\bar{a}}.$$

The Bowen-Series map is eventually expanding [Bor16, Prop. 15.5]; this will be made explicit below so we do not give the general definition now. The limit set $\Lambda = \Lambda(\Gamma)$ of Γ, defined in the Introduction, coincides with the non-wandering set of T:

$$\Lambda(\Gamma) = \bigcap_{n=1}^{\infty} T^{-n}(\mathbf{D}).$$

The limit set Λ is a compact T-invariant subset of $\mathbf{R}$. Given a Hölder continuous map $\varphi : \Lambda \to \mathbf{R}$, the *topological pressure* $P(\varphi)$ can be defined through the variational formula:

$$P(\varphi) = \sup_{\mu} \left(h_\mu(T) + \int_\Lambda \varphi d\mu \right),$$

where the supremum is taken over all T-invariant probability measures on Λ, and $h_\mu(T)$ stands for the measure-theoretic entropy. A celebrated result of Bowen [Bow79] says that the map

$$\sigma \mapsto P(-\sigma \log |T'|)$$

is convex[5], strictly decreasing and vanishes exactly at $\sigma = \delta(\Gamma)$, the Hausdorff dimension of the limit set Λ. In addition, it is not difficult to see from the variational formula that $P(-\sigma \log |T'|)$ tends to $-\infty$ as $\sigma \to +\infty$. For simplicity, we will use the notation $P(\sigma)$ in place of $P(-\sigma \log |T'|)$. The pressure will play a role in some of the estimates in the sequel.

2.2. **Functional spaces and transfer operators.** Let V be any Hilbert space. If Ω is any open subset of the complex numbers $\mathbf{C}$, we consider the Bergman space $\mathcal{H}(\Omega; V)$ that is the space of V-valued holomorphic functions on Ω with finite norm with respect to the given inner product

$$\langle f, g \rangle \stackrel{\text{def}}{=} \int_\Omega \langle f(x), g(x) \rangle_V dm(x).$$

Here dm is Lebesgue measure on Ω. If V is separable, then $\mathcal{H}(\Omega; V)$ is a separable Hilbert space; in this paper V will always be finite dimensional.

Of particular interest is $\mathcal{H}(\mathbf{D}; V)$. This splits as an orthogonal direct sum

$$\mathcal{H}(\mathbf{D}; V) = \bigoplus_{a \in \mathcal{A}} \mathcal{H}(D_a; V).$$

[5]Convexity follows obviously from the variational formula above.

If $\{\mathbf{e}_k\}_{k=1}^{\infty}$ is any orthonormal basis of $\mathcal{H}(D_a;\mathbf{C})$, and $x_1, x_2 \in D_a$, then the sum

$$\sum_{k=1}^{\infty} \mathbf{e}_k(x_1)\overline{\mathbf{e}_k(x_2)} \stackrel{\text{def}}{=} B_{D_a}(x_1,x_2)$$

converges and the resulting kernel is called the *Bergman kernel of* D_a. It is given by the explicit formula (cf. [Bor16, pg. 378])

$$B_{D_a}(x_1,x_2) = \frac{r_a^2}{\pi\left[r_a^2 - (\overline{x}_2 - c_a)(x_1 - c_a)\right]^2} \tag{2.1}$$

where r_a, c_a are the radius and center of D_a.

Throughout the sequel, $\rho : \Gamma \to \mathcal{U}(V)$ will be a unitary representation of the Schottky group Γ. If $Z \subset \mathcal{W}^\circ$ is any finite subset of words, then we define

$$\mathcal{L}_{Z,s,\rho}[f](x) = \sum_{\substack{\mathbf{a}\in Z\\ \mathbf{a}\rightsquigarrow b}} \gamma'_{\mathbf{a}'}(x)^s \rho(\gamma_{\mathbf{a}'}^{-1}) f(\gamma_{\mathbf{a}'}(x)) \quad x \in D_b, b \in \mathcal{A}.$$

The complex power $\gamma'_{\mathbf{a}'}(x)^s$ is defined by analytic continuation using that $\gamma'_{\mathbf{a}'}(x)$ is positive on I_b and never a negative real on D_b. One has $\mathcal{L}_{Z,s,\rho} : \mathcal{H}(\mathbf{D};V) \to \mathcal{H}(\mathbf{D};V)$. Certain particular choices of Z are made throughout the paper. The basic type of transfer operator that is considered corresponds to the choice $Z = \mathcal{W}_2$. We write $\mathcal{L}_{s,\rho} \stackrel{\text{def}}{=} \mathcal{L}_{\mathcal{W}_2,s,\rho}$. This operator can be written as

$$\mathcal{L}_{s,\rho}[f](x) = \sum_{\substack{a\in\mathcal{A}\\ a\to b}} \gamma'_a(x)^s \rho(\gamma_a^{-1}) f(\gamma_a(x)) \quad x \in D_b, b \in \mathcal{A}.$$

In the following we follow Dyatlov and Zworski [DZ17, §2.4].

Definition 2.1. A subset $Z \subset \mathcal{W}^\circ$ is a *partition* if there is $N \geq 0$ such that for all $\mathbf{a} \in \mathcal{W}$ with $|\mathbf{a}| \geq N$, there is a unique $\mathbf{b} \in Z$ that is a prefix of $\mathbf{a}$.

One particular family of partitions, introduced by Bourgain and Dyatlov [BD17], plays an important role in this paper. For any $\tau > 0$ we define

$$Z(\tau) \stackrel{\text{def}}{=} \{\, \mathbf{a} \in \mathcal{W}^\circ : |I_{\mathbf{a}}| \leq \tau < |I_{\mathbf{a}'}| \,\}.$$

It is shown by Dyatlov and Zworski [DZ17, eqs (2.7), (2.15)] that this is indeed a partition. Not only is the partition $Z(\tau)$ important to us, but so too is its mirror set

$$\overline{Z}(\tau) \stackrel{\text{def}}{=} \{\, \mathbf{a} \in \mathcal{W}^\circ : \overline{\mathbf{a}} \in Z(\tau) \,\}.$$

The reason for introducing this mirror set is to make Lemma 4.5 below work. Note that $\overline{Z}(\tau)$ may not be a partition, although this will not matter. We write $\mathcal{L}_{\tau,s,\rho} \stackrel{\text{def}}{=} \mathcal{L}_{\overline{Z}(\tau),s,\rho}$.

2.3. **The representations appearing in this paper.** In this paper we consider particular types of representations $\rho : \Gamma \to \mathcal{U}(V)$ as follows. We consider $n \in \mathbf{N}$ and the family of symmetric groups S_n on n letters. Let $V_n \stackrel{\text{def}}{=} \ell^2(\{1,\dots,n\})$. The group S_n has a standard representation $\mathrm{std}_n : S_n \to \mathcal{U}(V_n)$ where S_n acts by precomposition on ℓ^2 functions $f : \{1,\dots,n\} \to \mathbf{C}$. This representation is not irreducible, but splits as an orthogonal direct sum $\mathbf{1} \oplus V_n^0$ where V_n^0 is an

irreducible representation of dimension $n-1$. We write $\mathrm{std}_n^0 : S_n \to \mathcal{U}(V_n^0)$ for the corresponding homomorphism of the symmetric group.

We now build a representation from a homomorphism $\phi_n : \Gamma \to S_n$. Since Γ is free, ϕ_n is described simply by choosing the images of a generating set of Γ, which may be taken to be the γ_a with $1 \leq a \leq r$. We consider

$$\rho_n \stackrel{\mathrm{def}}{=} \mathrm{std}_n \circ \phi_n, \quad \rho_n^0 \stackrel{\mathrm{def}}{=} \mathrm{std}_n^0 \circ \phi_n. \tag{2.2}$$

These depend on the choice of ϕ_n. Later in the paper we will view $\phi_n : \Gamma \to S_n$ as a random homomorphism; its law is described by choosing the $\phi_n(\gamma_a)$ with $1 \leq a \leq r$ independently and uniformly at random with respect to the uniform measure on S_n. This gives random representations ρ_n and ρ_n^0. We write $\mathbb{E}_n$ to refer to expectations of random variables with repect to the random representation ρ_n^0. For example, if $\gamma \in \Gamma$, then $\mathrm{Tr}[\rho_n^0(\gamma)]$ is a real random variable and we write $\mathbb{E}_n(\mathrm{Tr}[\rho_n^0(\gamma)])$ for its expectation. At other times we view ϕ_n, ρ_n, ρ_n^0 as fixed and coupled to one another; it will be clear from the context whether we make probabilistic or deterministic statements.

2.4. **Selberg zeta functions.** If X is any convex co-compact hyperbolic surface (not necessarily connected), then the *Selberg zeta function* of X is defined for $\mathrm{Re}(s) > \delta$ by

$$Z_X(s) \stackrel{\mathrm{def}}{=} \prod_{\gamma \in \mathcal{P}(X)} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)l(\gamma)}\right)$$

where $\mathcal{P}(X)$ is the collection of primitive[6] closed geodesics on X, and $l(\gamma)$ is the length of such a geodesic. The function $Z_X(s)$ analytically continues to an entire function [Gui92, GLZ04]. One has the following theorem due to Patterson and Perry [PP01, Theorem 1.5] relating resonances of the Laplacian to the Selberg zeta function.

Theorem 2.2 (Patterson-Perry)**.** *If X is any non-elementary convex co-compact hyperbolic surface, then any resonance of X is a zero of Z_X. Conversely, if s is a zero of Z_X with $\mathrm{Re}(s) > 0$ then s is a resonance of X. In all cases, the order of the zero of Z_X is equal to the multiplicity of the corresponding resonance.*

We will also have a use for twisted Selberg zeta functions. If $\rho : \Gamma \to \mathcal{U}(V)$ is any finite dimensional unitary representation of Γ then we let

$$Z_{X,\rho}(s) \stackrel{\mathrm{def}}{=} \prod_{\gamma \in \mathcal{P}(X)} \prod_{k=0}^{\infty} \det\left(1 - \rho(\gamma) e^{-(s+k)l(\gamma)}\right).$$

This converges to a holomorphic function in $\mathrm{Re}(s) > \delta$ and extends to an entire function by results in [FP17].

[6]Primitive here means it is not an iterate of a shorter closed geodesic.

3. Estimates for derivatives

The following section contains certain technical but either easy or well-known estimates for derivatives of Γ that will be used in the sequel. The fundamental estimates for derivatives of elements of Γ are the following:

Lemma 3.1.

Uniform contraction: *There are $C = C(\Gamma) > 0$ and $0 < \bar{\theta} < \theta < 1$ such that for all $\mathbf{a} \in \mathcal{W}$, $b \in \mathcal{A}$ with $\mathbf{a} \to b$, and $x \in D_b$,*

$$C^{-1}\bar{\theta}^{|\mathbf{a}|} \leq |\gamma'_{\mathbf{a}}(x)| \leq C\theta^{|\mathbf{a}|}. \tag{3.1}$$

Bounded distortion I: *There is $K = K(\Gamma) > 0$ such that for all $b \in \mathcal{A}$, $\mathbf{a} \in \mathcal{W}$ such that $\mathbf{a} \to b$ and all $x_1, x_2 \in D_b$,*

$$e^{-|x_1 - x_2|K} \leq \frac{|\gamma'_{\mathbf{a}}(x_1)|}{|\gamma'_{\mathbf{a}}(x_2)|} \leq e^{|x_1 - x_2|K}. \tag{3.2}$$

Bounded distortion II: *There is a constant $c = c(\Gamma) > 0$ such that for $\mathbf{a} \in \mathcal{W}$, $b_1, b_2 \in \mathcal{A}$ with $\mathbf{a} \to b_1, b_2$ and $x_1 \in D_{b_1}$, $x_2 \in D_{b_2}$,*

$$\frac{|\gamma'_{\mathbf{a}}(x_1)|}{|\gamma'_{\mathbf{a}}(x_2)|} \leq c. \tag{3.3}$$

Proof. The first two properties can be found in [Nau14, §2]. The last part is trivial if $\mathbf{a} = \emptyset$. Otherwise, if $|\mathbf{a}| \geq 1$ we can write $\mathbf{a} = \mathbf{a}'a$ with $\mathbf{a}' \in \mathcal{W}$ and $\mathbf{a}' \to a \to b_1, b_2$. Then for $x_i \in D_{b_i}$ we have

$$|\gamma'_{\mathbf{a}}(x_i)| = \gamma'_{\mathbf{a}'}(\gamma_a(x_i))\gamma'_a(x_i) \quad i = 1, 2.$$

We have $\frac{|\gamma'_a(x_1)|}{|\gamma'_a(x_2)|} \leq C$ by (3.1) and since now $\gamma_a(x_1)$ and $\gamma_a(x_2)$ are in D_a, (3.2) gives

$$\frac{|\gamma'_{\mathbf{a}'}(\gamma_a(x_1))|}{|\gamma'_{\mathbf{a}'}(\gamma_a(x_1))|} \leq \exp(K \sup_{b \in \mathcal{A}} \text{diameter}(D_b)).$$

The equation (3.3) now follows. □

In the rest of the paper, for any $\mathbf{a} \in \mathcal{W}^\circ$, we define

$$\Upsilon_{\mathbf{a}} \stackrel{\text{def}}{=} |I_{\mathbf{a}}|.$$

We set $\Upsilon_\emptyset \stackrel{\text{def}}{=} 1$. For $\mathbf{a} \in \mathcal{W}^\circ$, we have

$$\Upsilon_{\mathbf{a}} \leq \Upsilon_{\mathbf{a}'} \tag{3.4}$$

since $I_{\mathbf{a}} \subset I_{\mathbf{a}'}$. Therefore there is $c = c(\Gamma) > 0$ such that for any $\mathbf{a} \in \mathcal{W}$

$$0 < \Upsilon_{\mathbf{a}} \leq c. \tag{3.5}$$

We next recall some useful results of Bourgain-Dyatlov from [BD17, §2].

Lemma 3.2. *There is a constant $K_0 = K_0(\Gamma) > 1$ such that for any $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{W}^\circ$ and $x \in D_{a_n}$*

$$K_0^{-1}\Upsilon_{\mathbf{a}} \leq |\gamma'_{\mathbf{a}'}(x)| \leq K_0\Upsilon_{\mathbf{a}}.$$

Proof. For $x \in I_{a_n}$ this is [BD17, Lemma 2.5, (20)]. The more general result here follows by combining [BD17, Lemma 2.5] with the bounded distortion estimate (3.3). □

The following lemma is [BD17, Lemma 2.10, (30)].

Lemma 3.3. *There is a constant $K_1 = K_1(\Gamma) > 1$ such that for $\tau \in (0,1)$, for any $\mathbf{a} \in \overline{Z}(\tau)$ we have*

$$K_1^{-1}\tau \leq \Upsilon_{\mathbf{a}} \leq K_1\tau.$$

The next lemma says that Υ is coarsely multiplicative.

Lemma 3.4. *There is a constant $K_2 = K_2(\Gamma) > 1$ such that for all $\mathbf{a}, \mathbf{b} \in \mathcal{W}^\circ$ with $\mathbf{a} \rightsquigarrow \mathbf{b}$*

$$K_2^{-1}\Upsilon_{\mathbf{a}}\Upsilon_{\mathbf{b}} \leq \Upsilon_{\mathbf{a}'\mathbf{b}} \leq K_2\Upsilon_{\mathbf{a}}\Upsilon_{\mathbf{b}},$$

and for $\mathbf{a}, \mathbf{b} \in \mathcal{W}$ with $\mathbf{a} \to \mathbf{b}$

$$K_2^{-1}\Upsilon_{\mathbf{a}}\Upsilon_{\mathbf{b}} \leq \Upsilon_{\mathbf{ab}} \leq K_2\Upsilon_{\mathbf{a}}\Upsilon_{\mathbf{b}}. \tag{3.6}$$

Proof. The first set of inequalities is [BD17, Lemma 2.7]. If either $\mathbf{a}$ or $\mathbf{b}$ is $\emptyset$, then (3.6) is trivially true with $K_2 = 1$. So assume $\mathbf{a}, \mathbf{b} \in \mathcal{W}^\circ$. Then (3.6) follows by combining [BD17, Lemmas 2.6 and 2.7]. □

We also have the following 'mirror' estimate for Υ.

Lemma 3.5 (Mirror estimate, [BD17, Lemma 2.8])**.** *There is a constant $K_3 = K_3(\Gamma) > 1$ such that for any $\mathbf{a} \in \mathcal{W}$*

$$K_3^{-1}\Upsilon_{\overline{\mathbf{a}}} \leq \Upsilon_{\mathbf{a}} \leq K_3\Upsilon_{\overline{\mathbf{a}}}.$$

We now state some lemmas about the set $\overline{Z}(\tau)$.

Lemma 3.6. *There is a constant $C_1 = C_1(\Gamma) > 1$ such that for $\mathbf{a} = (a_1, \ldots, a_n) \in \overline{Z}(\tau)$, for any $x \in D_{a_n}$ we have*

$$C_1^{-1}\tau \leq |\gamma'_{\mathbf{a}'}(x)| \leq C_1\tau.$$

Proof. This follows by combining Lemmas 3.2 and 3.3. □

Given Lemma 3.6, we can make the following estimate on the word lengths of elements $\mathbf{a} \in \overline{Z}(\tau)$.

Lemma 3.7. *There are constants $D = D(\Gamma) > 1$ and $\kappa = \kappa(\Gamma) > 0$ such that if $\mathbf{a} \in \overline{Z}(\tau)$, then*

$$D^{-1}\log\tau^{-1} - \kappa \leq |\mathbf{a}| \leq D\log\tau^{-1} + \kappa.$$

Proof. Write $\mathbf{a} = (a_1, \ldots, a_n)$. Pick $x \in D_{a_n}$. By Lemma 3.6 we have

$$C_1^{-1}\tau \leq |\gamma'_{\mathbf{a}'}(x)| \leq C_1\tau,$$

and combining this with (3.1) gives

$$C_1^{-1}C^{-1}\bar{\theta}^{|\mathbf{a}'|} \leq \tau \leq CC_1\theta^{|\mathbf{a}'|}.$$

Since $|\mathbf{a}| = |\mathbf{a}'| + 1$, this gives the result after taking logarithms and rearranging. □

We now note

Lemma 3.8. *There is $0 < \tau_0 < 1$ such that for $\tau < \tau_0$, $\overline{Z}(\tau) \subset \mathcal{W}_{\geq 2}$.*

Proof. This is a direct consequence of Lemma 3.7. □

Throughout the sequel, τ_0 will always be the parameter given by Lemma 3.8. It will also be useful to know roughly how many elements there are in $\overline{Z}(\tau)$. This is given by [BD17, Lemma 2.13] (noting that $|\overline{Z}(\tau)| = |Z(\tau)|$).

Lemma 3.9. *There is $C_2 = C_2(\Gamma) > 1$ such that for $\tau \in (0, 1]$*

$$C_2^{-1}\tau^{-\delta} \leq |\overline{Z}(\tau)| \leq C_2\tau^{-\delta}.$$

To conclude this section, we record that certain sums of derivatives are related to the pressure functional.

Lemma 3.10. *For all $\sigma_1, Q \in \mathbf{R}$ such that $0 \leq \sigma_1 < Q$ there is a constant $C = C(\sigma_1, Q) > 0$ such that for all $N \in \mathbf{N}_0$ and $\sigma \in [\sigma_1, Q]$ we have*

$$\sum_{a\in\mathcal{A}} \sum_{\substack{\mathbf{a}\in\mathcal{W}_N \\ \mathbf{a}\rightsquigarrow a}} \sup_{I_a} |\gamma_{\mathbf{a}'}|^\sigma \leq C \exp(NP(\sigma_1)), \tag{3.7}$$

and

$$\sum_{\mathbf{a}\in\mathcal{W}_N} \Upsilon_{\mathbf{a}}^\sigma \leq C \exp(NP(\sigma_1)). \tag{3.8}$$

Proof. The estimate (3.7) is a standard estimate that appears in [Nau14, Lemma 3.1]. The estimate (3.8) follows by combining (3.7) with Lemma 3.2 and increasing C. □

4. Transfer operators and zeta functions

4.1. Zeta functions.

Lemma 4.1. *For any $Z \subset \mathcal{W}_{\geq 2}$, and any finite dimensional unitary representation ρ of Γ, the operator $\mathcal{L}_{Z,s,\rho}$ is trace class on $\mathcal{H}(\mathbf{D}; V)$.*

Proof. The proof is an easy adaptation of [Bor16, Lemma 15.7]. The condition $Z \subset \mathcal{W}_{\geq 2}$ rules out $\mathcal{L}_{Z,s,\rho}$ having any summand that acts as the identity on some D_a. □

Corollary 4.2. *Let (ρ, V) be any finite dimensional unitary representation of Γ.*

(1) The operator $\mathcal{L}_{s,\rho}$ is trace class on $\mathcal{H}(\mathbf{D}; V)$.
(2) For $\tau < \tau_0$, the operator $\mathcal{L}_{\tau,s,\rho}$ is trace class on $\mathcal{H}(\mathbf{D}; V)$.

Given Corollary 4.2 we can define *zeta functions*

$$\zeta_\rho(s) \stackrel{\text{def}}{=} \det(1 - \mathcal{L}_{s,\rho}),$$
$$\zeta_{\tau,\rho}(s) \stackrel{\text{def}}{=} \det(1 - \mathcal{L}^2_{\tau,s,\rho}).$$

The determinants that appear here are Fredholm determinants. The reason that we have used $\mathcal{L}^2_{\tau,s,\rho}$ in the definition of $\zeta_{\tau,\rho}(s)$ is that it will later allow us to estimate $\log|\zeta_{\tau,\rho}(s)|$ in terms of the Hilbert-Schmidt norm of $\mathcal{L}_{\tau,s,\rho}$ rather than the trace norm (cf. (6.3)). On the other hand, we do not square $\mathcal{L}_{s,\rho}$ in the definition of $\zeta_\rho(s)$ so that we can access known results about $\zeta_\rho(s)$.

By the general theory of Fredholm determinants we have

Lemma 4.3. *Let (ρ, V) be any finite dimensional unitary representation of Γ.*

(1) The function $\zeta_\rho(s)$ is an entire function of $s \in \mathbf{C}$ and

$$\zeta_\rho(s) = 0 \iff \exists\, u \in \mathcal{H}(\mathbf{D}; V) \,:\, \mathcal{L}_{s,\rho} u = u.$$

(2) If $\tau < \tau_0$ then $\zeta_{\tau,\rho}(s)$ is an entire function of $s \in \mathbf{C}$ and

$$\zeta_{\tau,\rho}(s) = 0 \iff \exists\, u \in \mathcal{H}(\mathbf{D}; V) \,:\, \mathcal{L}^2_{\tau,s,\rho} u = u.$$

The relevance of the zeta functions $\zeta_\rho(s)$ are the following:

Proposition 4.4. *Let $\phi_n : \Gamma \to S_n$ be a fixed homomorphism, and (ρ_n, V_n) the unitary representation corresponding to ϕ_n via (2.2). Let X_n be the n-cover of X corresponding to ϕ_n.*

(1) We have $\zeta_{\rho_n}(s) = Z_{X,\rho_n}(s) = Z_{X_n}(s)$.
(2) We have $Z_{X_n}(s) = Z_X(s)\zeta_{\rho_n^0}(s)$.

Proof. Proof of Part 1. A special case of a result of Jakobson, Naud, and Soares [JNS19, Prop. 2.2] for arbitrary finite-dimensional unitary representations gives

$$\zeta_{\rho_n}(s) = Z_{X,\rho_n}(s)$$

where both sides are entire functions of s.

If X_n is connected, then $X_n = \Gamma_n\backslash\mathbb{H}$ for some $\Gamma_n \leq \Gamma$ and $\rho_n = \mathrm{Ind}_{\Gamma_n}^{\Gamma} 1$, the induction of the trivial representation from Γ_n to Γ. In this case the Venkov-Zograf type induction formula proved by Fedosova and Pohl in [FP17, Thm. 6.1(ii)] (cf. [VZ82]) gives

$$Z_{X,\rho_n}(s) = Z_{X_n}(s).$$

If X_n is not connected, let $X_n^{(1)}, \dots, X_n^{(m)}$ denote its connected components, and let $X_n^{(j)} = \Gamma_n^j\backslash\mathbb{H}$ with $\Gamma_n^j \leq \Gamma$. If we let $\rho_n^j = \mathrm{Ind}_{\Gamma_n^j}^{\Gamma} 1$ then we have $\rho_n = \bigoplus_{j=1}^m \rho_n^j$. Then

$$Z_{X_n}(s) = \prod_{j=1}^m Z_{X_n^{(j)}}(s) = \prod_{j=1}^m Z_{X,\rho_n^j}(s) = Z_{X,\rho_n}(s)$$

where the first equality is by definition of the Selberg zeta functions, the second equality uses the induction formula [FP17, Thm. 6.1(ii)] and the last inequality uses the factorization formula [FP17, Thm. 6.1(i)]. Thus we have proved $\zeta_{\rho_n}(s) = Z_{X,\rho_n}(s) = Z_{X_n}(s)$. *This proves Part 1.*

Proof of Part 2. Using [JNS19, Prop. 2.2] again gives

$$\zeta_{\rho_n^0}(s) = Z_{X,\rho_n^0}(s). \tag{4.1}$$

Since $\rho_n = \mathbf{1} \oplus \rho_n^0$, we have

$$Z_{X_n}(s) = Z_{X,\rho_n}(s) = Z_X(s) Z_{X,\rho_n^0}(s) = Z_X(s)\zeta_{\rho_n^0}(s)$$

where the first equality used Part 1 of the lemma, the second used the factorization formula [FP17, Thm. 6.1(i)], and the third used (4.1). *This proves Part 2.* □

The following lemma adapts (a special case of) [DZ17, Lemma 2.4] to our vector-valued setting. The proof is essentially the same.

Lemma 4.5. *For all sufficiently small $\tau > 0$, if $u \in \mathcal{H}(\mathbf{D}; V)$ is such that $\mathcal{L}_{s,\rho} u = u$, then $\mathcal{L}_{\tau,s,\rho} u = u$.*

Corollary 4.6. *For all sufficiently small $\tau > 0$, if $Z_{X_n}(s) = 0$ and $Z_X(s) \neq 0$, then $\zeta_{\tau,\rho_n^0}(s) = 0$.*

Proof. If $Z_{X_n}(s) = 0$, $Z_X(s) \neq 0$, then by Proposition 4.4, Part 2, $\zeta_{\rho_n^0}(s) = 0$. Then by Lemma 4.3, Part 1, there is $u \in \mathcal{H}(\mathbf{D}; V_n^0)$ such that $\mathcal{L}_{s,\rho_n^0} u = u$. By Lemma 4.5, this implies that $\mathcal{L}_{\tau,s,\rho_n^0} u = u$, and hence $\mathcal{L}^2_{\tau,s,\rho_n^0} u = u$. Then by Lemma 4.3, Part 2, $\zeta_{\tau,\rho_n^0}(s) = 0$. □

4.2. **The Hilbert-Schmidt norm of the transfer operator.** Corollary 4.6 reduces controlling zeros of the Selberg zeta function of X_n that do not come from X to controlling zeros of $\zeta_{\tau,\rho_n^0}(s)$. To do this, we will use Jensen's formula, but before doing so, we collect some estimates. The first will be a pointwise lower bound on $|\zeta_{\tau,\rho}(s)|$ when s is a sufficiently large real number (cf. §4.3). The other will be an estimate for the expectation of the squared Hilbert-Schmidt norm $\|\mathcal{L}_{\tau,s,\rho}\|^2_{\text{H.S.}}$ for $\rho = \rho_n^0$. One input to the latter result is a deterministic (non-random) expression for $\|\mathcal{L}_{\tau,s,\rho}\|^2_{\text{H.S.}}$ that we give now.

Lemma 4.7. *Let (ρ, V) be any finite dimensional unitary representation of Γ. We have for any $s \in \mathbf{C}$ and $\tau \leq \tau_0$*

$$\|\mathcal{L}_{\tau,s,\rho}\|^2_{\text{H.S.}} = \sum_{a,b \in \mathcal{A}} \sum_{\substack{\mathbf{a}_1, \mathbf{a}_2 \in \overline{Z}(\tau) \\ a \rightsquigarrow \mathbf{a}_1, \mathbf{a}_2 \rightsquigarrow b}} \text{Tr}(\rho(\gamma_{\mathbf{a}_1'} \gamma_{\mathbf{a}_2'}^{-1})) \int_{D_b} \gamma'_{\mathbf{a}_1'}(x)^s \overline{\gamma'_{\mathbf{a}_2'}(x)^s} B_{D_a}(\gamma_{\mathbf{a}_1'}(x), \gamma_{\mathbf{a}_2'}(x)) dm(x).$$

Here and henceforth we write $a \rightsquigarrow \mathbf{a}_1, \mathbf{a}_2 \rightsquigarrow b$ to mean that both $a \rightsquigarrow \mathbf{a}_1 \rightsquigarrow b$ and $a \rightsquigarrow \mathbf{a}_2 \rightsquigarrow b$.

Proof. This is similar to arguments given by Jakobson and Naud in [JN16, pgs. 466-467]. For $a \in \mathcal{A}$, let $\{\mathbf{e}_k^a\}_{k=1}^\infty$ be an orthonormal basis for $\mathcal{H}(D_a; \mathbf{C})$ and let $\{v_j\}_{j=1}^{\dim V}$ be an orthonormal basis for V. Then $\{\mathbf{e}_k^a \otimes v_j \ : \ a \in \mathcal{A}, k \in \mathbf{N}, 1 \leq j \leq \dim V\}$ is an orthonormal basis for $\mathcal{H}(\mathbf{D}; V)$. We

have

$$\begin{aligned}
\|\mathcal{L}_{\tau,s,\rho}\|_{\text{H.S.}}^2 &= \operatorname{Tr}(\mathcal{L}_{\tau,s,\rho}^*\mathcal{L}_{s,\tau,\rho}) \\
&= \sum_{a\in\mathcal{A},k\in\mathbf{N},1\le j\le \dim V} \langle \mathcal{L}_{\tau,s,\rho}[\mathbf{e}_k^a\otimes v_j], \mathcal{L}_{\tau,s,\rho}[\mathbf{e}_k^a\otimes v_j]\rangle \\
&= \sum_{a\in\mathcal{A},k\in\mathbf{N},1\le j\le \dim V}\sum_{b\in\mathcal{A}}\int_{D_b} \langle \mathcal{L}_{\tau,s,\rho}[\mathbf{e}_k^a\otimes v_j](x), \mathcal{L}_{\tau,s,\rho}[\mathbf{e}_k^a\otimes v_j](x)\rangle dm(x) \\
&= \sum_{a\in\mathcal{A},k\in\mathbf{N},1\le j\le \dim V}\sum_{b\in\mathcal{A}}\sum_{\substack{\mathbf{a}_1,\mathbf{a}_2\in\overline{Z}(\tau)\\ \mathbf{a}_1,\mathbf{a}_2\rightsquigarrow b}} \\
&\int_{D_b} \gamma'_{\mathbf{a}_1'}(x)^s\overline{\gamma'_{\mathbf{a}_2'}(x)^s}\langle \rho(\gamma_{\mathbf{a}_1'}^{-1})\mathbf{e}_k^a\otimes v_j(\gamma_{\mathbf{a}_1'}(x)), \rho(\gamma_{\mathbf{a}_2'}^{-1})\mathbf{e}_k^a\otimes v_j(\gamma_{\mathbf{a}_2'}(x))\rangle_V dm(x) \\
&= \sum_{a,b\in\mathcal{A},k\in\mathbf{N}}\sum_{\substack{\mathbf{a}_1,\mathbf{a}_2\in\overline{Z}(\tau)\\ a\rightsquigarrow\mathbf{a}_1,\mathbf{a}_2\rightsquigarrow b}} \\
&\operatorname{Tr}(\rho(\gamma_{\mathbf{a}_1'}\gamma_{\mathbf{a}_2'}^{-1}))\int_{D_b}\gamma'_{\mathbf{a}_1'}(x)^s\overline{\gamma'_{\mathbf{a}_2'}(x)^s}\mathbf{e}_k^a(\gamma_{\mathbf{a}_1'}(x))\overline{\mathbf{e}_k^a(\gamma_{\mathbf{a}_2'}(x))}dm(x) \\
&= \sum_{a,b\in\mathcal{A}}\sum_{\substack{\mathbf{a}_1,\mathbf{a}_2\in\overline{Z}(\tau)\\ a\rightsquigarrow\mathbf{a}_1,\mathbf{a}_2\rightsquigarrow b}} \operatorname{Tr}(\rho(\gamma_{\mathbf{a}_1'}\gamma_{\mathbf{a}_2'}^{-1}))\int_{D_b}\gamma'_{\mathbf{a}_1'}(x)^s\overline{\gamma'_{\mathbf{a}_2'}(x)^s}B_{D_a}(\gamma_{\mathbf{a}_1'}(x),\gamma_{\mathbf{a}_2'}(x))dm(x).
\end{aligned}$$

The final application of Fubini's theorem is justified since we assume $\tau\le\tau_0$, so $\overline{Z}(\tau)\subset\mathcal{W}_{\ge 2}$, and each $\gamma_{\mathbf{a}_1'},\gamma_{\mathbf{a}_2'}$ maps D_b into a compact subset of D_a, coupled with the fact that the convergence of $\sum_{k=1}^\infty \mathbf{e}_k^{\mathbf{a}}(x_1)\overline{\mathbf{e}_k^{\mathbf{a}}(x_2)}$ to $B_{D_a}(x_1,x_2)$ is uniform on compact subsets of D_a (see, for example, [Bor16, Proof of Thm. 15.7]). □

4.3. **A pointwise estimate for the modulus of a zeta function.**

Proposition 4.8 (Pointwise bound for $|\zeta_{\tau,\rho}(s)|$)**.** *There is $\tau_1\le\tau_0$ and $B\in\mathbf{R}$ with $B>2\delta$ such that if $\tau\le\tau_1$, if $s\in[B,\infty)$, and (ρ,V) is any finite dimensional unitary representation of Γ, we have*

$$-\log|\zeta_{\tau,\rho}(s)|\le(\dim V)\tau.$$

Remark 4.9. A crucial restriction in Proposition 4.8 is $\operatorname{Re}(s)>2\delta$ that results from the presence of $\mathcal{L}_{\tau,s,\rho}^2$ in the definition of $\zeta_{\tau,\rho}$.

Proof of Proposition 4.8. We can write

$$\zeta_{\tau,\rho}(s)=\det(1-\mathcal{L}_{\tau,s,\rho}^2)=\exp\left(-\sum_{k=1}^\infty\frac{1}{k}\operatorname{Tr}\mathcal{L}_{\tau,s,\rho}^{2k}\right)$$

whenever the series inside the exponential is absolutely convergent. We have if $x\in D_b$

$$\mathcal{L}_{\tau,s,\rho}^{2k}[f](x)=\sum_{\substack{\mathbf{a}_1,\dots,\mathbf{a}_{2k}\in\overline{Z}(\tau)\\ \mathbf{a}_1\rightsquigarrow\mathbf{a}_2\rightsquigarrow\cdots\rightsquigarrow\mathbf{a}_{2k}\rightsquigarrow b}}\gamma'_{\mathbf{a}_1'\mathbf{a}_2'\dots\mathbf{a}_{2k}'}(x)^s\rho(\gamma_{\mathbf{a}_1'\mathbf{a}_2'\dots\mathbf{a}_{2k}'}^{-1})f(\gamma_{\mathbf{a}_1'\mathbf{a}_2'\dots\mathbf{a}_{2k}'}(x)).$$

Carefully applying the Lefschetz fixed point formula [Bor16, Lemma 15.9] now gives

$$\mathrm{Tr}\mathcal{L}_{\tau,s,\rho}^{2k} = \sum_{\substack{\mathbf{a}_1,\dots,\mathbf{a}_{2k}\in\overline{Z}(\tau)\\ \mathbf{a}_{2k}\rightsquigarrow\mathbf{a}_1\rightsquigarrow\mathbf{a}_2\rightsquigarrow\cdots\rightsquigarrow\mathbf{a}_{2k}}} \mathrm{Tr}[\rho(\gamma_{\mathbf{a}_1'\mathbf{a}_2'\dots\mathbf{a}_{2k}'}^{-1})]\frac{\gamma'_{\mathbf{a}_1'\mathbf{a}_2'\dots\mathbf{a}_{2k}'}(x_{\mathbf{a}_1'\mathbf{a}_2'\dots\mathbf{a}_{2k}'})^s}{1-\gamma'_{\mathbf{a}_1'\mathbf{a}_2'\dots\mathbf{a}_{2k}'}(x_{\mathbf{a}_1'\mathbf{a}_2'\dots\mathbf{a}_{2k}'})}$$

where $x_{\mathbf{a}_1'\mathbf{a}_2'\dots\mathbf{a}_{2k}'}\in\mathbf{R}$ is the unique attracting fixed point of $\gamma_{\mathbf{a}_1'\mathbf{a}_2'\dots\mathbf{a}_{2k}'}$. Let b denote the last letter of $\mathbf{a}_{2k}$.

By using Lemmas 3.2 and 3.4 ($2k-1$ times) we obtain

$$\begin{aligned}\gamma'_{\mathbf{a}_1'\mathbf{a}_2'\dots\mathbf{a}_{2k}'}(x_{\mathbf{a}_1'\mathbf{a}_2'\dots\mathbf{a}_{2k}'}) &\le K_0\Upsilon_{\mathbf{a}_1'\mathbf{a}_2'\cdots\mathbf{a}_{2k-1}'\mathbf{a}_{2k}}\\ &\le K_0K_2^{2k-1}\Upsilon_{\mathbf{a}_1}\dots\Upsilon_{\mathbf{a}_{2k}}.\end{aligned}$$

Now using Lemma 3.3 we obtain

$$\gamma'_{\mathbf{a}_1'\mathbf{a}_2'\dots\mathbf{a}_{2k}'}(x_{\mathbf{a}_1'\mathbf{a}_2'\dots\mathbf{a}_{2k}'}) \le K_0K_1^{2k-1}K_2^{2k-1}\tau^{2k} \le K^k\tau^{2k}$$

for some $K>1$. We now assume

$$\tau_1 \le \frac{1}{2}K^{-1}$$

so that given $\tau\le\tau_1$ we have

$$\gamma'_{\mathbf{a}_1'\mathbf{a}_2'\dots\mathbf{a}_{2k}'}(x_{\mathbf{a}_1'\mathbf{a}_2'\dots\mathbf{a}_{2k}'}) \le 2^{-2k}.$$

We may also use the simple estimate $\mathrm{Tr}[\rho(\gamma_{\mathbf{a}_1'\mathbf{a}_2'\dots\mathbf{a}_{2k}'}^{-1})]\le\dim V$. Putting this together gives

$$|\mathrm{Tr}\mathcal{L}_{\tau,s,\rho}^{2k}| \le (\dim V)\,(K\tau)^{2ks}\,|\overline{Z}(\tau)|^{2k}.$$

Hence by Lemma 3.9 we obtain

$$|\mathrm{Tr}\mathcal{L}_{\tau,s,\rho}^{2k}| \le (\dim V)\,(K\tau)^{2ks}\,C_2^{2k}\tau^{-2k\delta} = (\dim V)K^{2ks}C_2^{2k}\tau^{(2s-2\delta)k}.$$

Choose B such that $B>\max(1,2\delta)$ and

$$K^B \ge C_2,$$

with the effect of obtaining $|\mathrm{Tr}\mathcal{L}_{\tau,s,\rho}^{2k}| \le (\dim V)K^{4ks}\tau^{(2s-2\delta)k} = (\dim V)(K^4\tau^{(2-\frac{2\delta}{s})})^{sk}$ when $s\ge B$. Now decrease τ_1, if necessary, to ensure

$$K^4\tau_1^{(1-\frac{2\delta}{B})} \le 2^{-1}.$$

Note that $1-\frac{2\delta}{B}>0$, so this is indeed possible. The result of our choices is that when $s\ge B\ge 1$ and $\tau\le\tau_1$

$$\left|\det(1-\mathcal{L}_{\tau,s,\rho}^2)\right| = \exp\left(\mathrm{Re}\left(-\sum_{k=1}^{\infty}\frac{1}{k}\mathrm{Tr}\mathcal{L}_{\tau,s,\rho}^{2k}\right)\right) \ge \exp\left(-(\dim V)\sum_{k=1}^{\infty}\left(\frac{\tau}{2}\right)^{sk}\right),$$

so

$$-\log|\zeta_{\tau,\rho}(s)| \le (\dim V)\sum_{k=1}^{\infty}\left(\frac{\tau}{2}\right)^{sk} \le (\dim V)\sum_{k=1}^{\infty}\left(\frac{\tau}{2}\right)^{k} \le (\dim V)\tau.$$

□

5. The expectation of the Hilbert-Schmidt norm of the transfer operator

5.1. **Statement of the main probabilistic estimate.** The main estimate we wish to prove in this Section 5 is the following.

Proposition 5.1. *Given $H_1 > 0$, $\sigma_1 > \frac{3\delta}{4}$, and $Q > \sigma_1$ there are constants $\epsilon = \epsilon(\Gamma, H_1, Q, \sigma_1) > 0$, and $n_0 = n_0(\Gamma, H_1, Q, \sigma_1) > 0$ such that if $\tau = n^{-\frac{2}{\delta}}$, $n \geq n_0$, $s = \sigma + it$ with $\sigma \in [\sigma_1, Q]$ and $|t| \leq H_1$ we have*

$$\mathbb{E}_n \|\mathcal{L}_{\tau,s,\rho_n^0}\|_{\text{H.S.}}^2 \leq n^{-\epsilon}.$$

5.2. **The expected value of the trace of a word.** The key probabilistic estimate for ρ_n^0 that we use in this paper is essentially due to Broder-Shamir [BS87], and in the stronger form that we use it can be deduced from the work of Puder [Pud15]. We will explain how to deduce the result below.

Theorem 5.2 (Broder-Shamir, Puder)**.** *Let $\gamma \in \Gamma$ have reduced word length t. Then for any $n > t^2$*

$$\left|\mathbb{E}_n(\mathrm{Tr}[\rho_n^0(\gamma)])\right| \leq \begin{cases} n-1 & \text{if } \gamma = \mathrm{id}, \\ d(q) - 1 + \frac{t^4}{n-t^2} & \text{if } \gamma = \gamma_0^q,\ q \geq 2 \text{ and } q \text{ maximal}, \\ \frac{t^4}{n-t^2} & \text{otherwise.} \end{cases}$$

Here $d(q)$ is the number of divisors of q.

Remark 5.3. Broder and Shamir [BS87] only prove upper bounds for $\mathbb{E}_n(\mathrm{Tr}[\rho_n^0(\gamma)])$, whereas it is crucial for us to have upper and lower bounds, since we deal with *complex* weighted sums of the random variables $\mathrm{Tr}[\rho_n^0(\gamma)]$.

Deduction of Theorem 5.2. Let γ be an element of the non-abelian free group Γ with reduced word length t. Note that Theorem 5.2 is trivial if $\gamma = \mathrm{id}$, so we assume this is not the case. Puder proves in [Pud15, pg. 885] that for $n > t$ one has an absolutely convergent Laurent series

$$\mathbb{E}_n(\mathrm{Tr}[\rho_n(\gamma)]) = \sum_{S=0}^{\infty} \frac{a_S(\gamma)}{n^S} \tag{5.1}$$

where each $a_S(\gamma) \in \mathbf{Z}$. Puder associates to γ a quantity $\pi(\gamma) \in \mathbf{N}_0 \cup \{\infty\}$ called the *primitivity*[7] *rank* of γ. For our purposes, the only thing we need to know is that $\pi(\gamma) = 0$ if and only if $\gamma = \mathrm{id}$, and $\pi(\gamma) = 1$ if and only if γ is a proper power. Puder also considers a certain finite set $\mathrm{Crit}(\gamma)$ of subgroups of the free group. Again, the only thing we need to know is that if if $\gamma = \gamma_0^q$, $q \geq 2$ and q maximal, then $|\mathrm{Crit}(\gamma)| = d(q) - 1$ [PP15, pg. 67].

The following facts are proven by Puder in [Pud15, pp. 885-887]:

- We have $a_0(\gamma) = 1$, unless $\pi(\gamma) = 1$, in which case

$$a_0(\gamma) = |\mathrm{Crit}(\gamma)| + 1.$$

[7]For good reasons, 'primitivity' in the setting of [Pud15] does not coincide with the notion of primitive closed geodesics, although they are related. However, this is not relevant to the current proof.

- If $1 \leq S < \pi(\gamma) - 1$ then
$$a_S(\gamma) = 0.$$
- If $\pi(\gamma) \neq 1$ then
$$a_{\pi(\gamma)-1} = |\mathrm{Crit}(\gamma)|.$$
- For any $S \geq 0$
$$|a_S(\gamma)| \leq t^{2S+2}.$$

Since $\mathrm{Tr}[\rho_n(\gamma)] = 1 + \mathrm{Tr}[\rho_n^0(\gamma)]$, if $\gamma = \gamma_0^q$, $q \geq 2$ and q maximal, we have from (5.1)

$$|\mathbb{E}_n(\mathrm{Tr}[\rho_n^0(\gamma)])| \leq d(q) - 1 + \sum_{S=1}^{\infty} \frac{t^{2S+2}}{n^S} = d(q) - 1 + \frac{t^4}{n - t^2}.$$

If γ is neither a proper power nor the identity then the estimate is similar, but there is no $d(q) - 1$ term since $\pi(\gamma) \geq 2$. □

5.3. **Majorization of the expectation of the Hilbert-Schmidt norm.**

Lemma 5.4. *Given $Q, H_1 > 0$ there is a constant $C = C(\Gamma, H_1, Q)$ such that if $\tau \leq \tau_0$ and $s = \sigma + it$ with $\sigma \in (0, Q]$ and $|t| \leq H_1$,*

$$\mathbb{E}_n \|\mathcal{L}_{\tau,s,\rho_n^0}\|_{\mathrm{H.S.}}^2 \leq C\tau^{2\sigma} \sum_{\mathbf{a},\mathbf{b} \in \overline{Z}(\tau)} |\mathbb{E}_n[\mathrm{Tr}(\rho_n^0(\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1}))]|. \tag{5.2}$$

Proof. Suppose we are given H_1 as in the statement of the lemma. Taking the expectation of the expression given in Lemma 4.7 gives

(5.3)
$$\mathbb{E}_n \|\mathcal{L}_{\tau,s,\rho_n^0}\|_{\mathrm{H.S.}}^2 = \sum_{a,b \in \mathcal{A}} \sum_{\substack{\mathbf{a}_1,\mathbf{a}_2 \in \overline{Z}(\tau) \\ a \rightsquigarrow \mathbf{a}_1, \mathbf{a}_2 \rightsquigarrow b}} \mathbb{E}_n[\mathrm{Tr}(\rho_n^0(\gamma_{\mathbf{a}_1'}\gamma_{\mathbf{a}_2'}^{-1}))] \int_{D_b} \gamma_{\mathbf{a}_1'}'(x)^s \overline{\gamma_{\mathbf{a}_2'}'(x)^s} B_{D_a}(\gamma_{\mathbf{a}_1'}(x), \gamma_{\mathbf{a}_2'}(x)) dm(x).$$

We wish to estimate the modulus of all quantities appearing in the integral on the right hand side. Firstly the assumption that $\tau \leq \tau_0$ ensures $\overline{Z}(\tau) \subset \mathcal{W}_{\geq 2}$, and so each $\gamma_{\mathbf{a}_1'}, \gamma_{\mathbf{a}_2'}$ maps D_b into a compact subset of D_a. It then follows from the explicit expression for the Bergman kernel in (2.1) that there is $K = K(\Gamma) > 0$ such that

$$B_{D_a}(\gamma_{\mathbf{a}_1'}(x), \gamma_{\mathbf{a}_2'}(x)) \leq K \tag{5.4}$$

for all $a, x, \mathbf{a}_1, \mathbf{a}_2$ as in (5.3).

By definition, if $s = \sigma + it$,

$$(\gamma_{\mathbf{a}_1'}'(x))^s = \exp\Big((\sigma + it)(\log|\gamma_{\mathbf{a}_1'}'(x)| + i\arg(\gamma_{\mathbf{a}_1'}'(x))\Big)$$

where arg is the principal value of the argument, $\arg : \mathbf{C} - \mathbf{R}_{\leq 0} \to (-\pi, \pi)$. Hence

$$|(\gamma_{\mathbf{a}_1'}'(x))^s| = \exp(\sigma \log|\gamma_{\mathbf{a}_1'}'(x)| - t\arg(\gamma_{\mathbf{a}_1'}'(x)) \leq e^{\pi|t|}|\gamma_{\mathbf{a}_1'}'(x)|^\sigma.$$

Therefore by Lemma 3.6 for some $c = c(H_1, Q) > 0$ we have for $|t| \leq H_1$

$$|(\gamma_{\mathbf{a}_1'}'(x))^s| \leq c\tau^\sigma. \tag{5.5}$$

for all $\mathbf{a}_1'$, x in (5.3), and the same for $\mathbf{a}_2$ in place of $\mathbf{a}_1$. Hence applying the triangle inequality to (5.3) and using (5.4) and (5.5), together with the fact that the D_b have finite Lebesgue measure gives

$$\begin{aligned}\mathbb{E}_n\|\mathcal{L}_{\tau,s,\rho_n^0}\|_{\text{H.S.}}^2 &\leq C_0\tau^{2\sigma}\sum_{a,b\in\mathcal{A}}\sum_{\substack{\mathbf{a}_1,\mathbf{a}_2\in\overline{Z}(\tau)\\ a\rightsquigarrow\mathbf{a}_1,\mathbf{a}_2\rightsquigarrow b}}|\mathbb{E}_n[\mathrm{Tr}(\rho_n^0(\gamma_{\mathbf{a}_1'}\gamma_{\mathbf{a}_2'}^{-1}))]|\\ &\leq C\tau^{2\sigma}\sum_{\mathbf{a},\mathbf{b}\in\overline{Z}(\tau)}|\mathbb{E}_n[\mathrm{Tr}(\rho_n^0(\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1}))]|\end{aligned}$$

for some $C = C(\Gamma, H_1, Q)$ whenever $|t| \leq H_1$ and $\tau \leq \tau_0$. □

The next step is to input the estimates of Theorem 5.2 into the estimate of Lemma 5.4. To organize the result we introduce, for each $q \in \mathbf{Z}_{\geq 2}$, the set

$$\text{PowerPairs}(\tau; q) \stackrel{\text{def}}{=} \{(\mathbf{a}, \mathbf{b}) \in \overline{Z}(\tau) \times \overline{Z}(\tau),\ \gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1} \text{ is a } q^{th} \text{ power in } \Gamma \text{ with } q \text{ maximal}\},$$

and

$$\text{PowerPairs}(\tau) \stackrel{\text{def}}{=} \bigcup_{q\geq 2}\text{PowerPairs}(\tau; q).$$

Notice that in the above, $\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1} \neq \text{id}$. We will show

Lemma 5.5. *Given $Q, H_1, \alpha, \epsilon > 0$, there are constants $C = C(\Gamma, H_1, Q) > 0$ and $n_0 = n_0(\Gamma, \epsilon, \alpha)$ such that if $\tau = n^{-\alpha}$ and $s = \sigma + it$ with $\sigma \in (0, Q]$, $|t| \leq H_1$, and $n \geq n_0$, we have*

$$\mathbb{E}_n\|\mathcal{L}_{\tau,s,\rho_n^0}\|_{\text{H.S.}}^2 \leq C\tau^{2\sigma}\left(n\tau^{-\delta} + n^{\epsilon}|\text{PowerPairs}(\tau)| + \frac{1}{n^{1-\epsilon}}\tau^{-2\delta}\right).$$

Proof. We will input Theorem 5.2 into Lemma 5.4. For this to be valid we need to control the word lengths of elements of $\overline{Z}(\tau)$. By Lemma 3.7, all $\mathbf{a} \in \overline{Z}(\tau)$ have $|\mathbf{a}| \leq c\log\tau^{-1} + \kappa$, so if $\tau = n^{-\alpha}$ with $\alpha > 0$,

$$|\mathbf{a}| \leq c\alpha\log n + \kappa < \frac{1}{2}n^{\frac{1}{2}}$$

for n sufficiently large, say $n \geq n_0$. In this case, if $\mathbf{a}, \mathbf{b} \in \overline{Z}(\tau)$ the reduced word length of $\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1}$ is

$$< n^{\frac{1}{2}}$$

so we may apply Theorem 5.2 to $\mathbb{E}_n[\mathrm{Tr}(\rho_n^0(\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1}))]$. Moreover, if t is the reduced word length of $\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1}$, we have $t \leq 2c\alpha\log n + \kappa$ so for any $\epsilon > 0$, we have

$$\frac{t^2}{n - t^2} \leq \frac{1}{n^{1-\epsilon}}$$

when $n \geq n_0$, after increasing n_0 if necessary. Finally, in the case $\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1}$ is a q^{th} power in the free group Γ, with $q \geq 2$ we must have $q \leq t$ and so $d(q) \leq t \leq 2c\alpha\log n + \kappa \leq n^{\epsilon}$ for any $\epsilon > 0$ and $n \geq n_0(\epsilon)$ (here we increase n_0 again if necessary).

With these estimates in hand, we partition the range of the sum of the right hand of (5.2) according to the following three cases:

- $\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1}$ is the identity; if this is the case then $|\mathbb{E}_n[\mathrm{Tr}(\rho_n^0(\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1}))]| = n-1 \leq n$. We observe that $\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1} = \mathrm{id}$ implies $\gamma_{\mathbf{a}'} = \gamma_{\mathbf{b}'}$, but since the map $\mathbf{a} \to \gamma_{\mathbf{a}}$ is one-to-one, this forces $\mathbf{a}' = \mathbf{b}'$. Therefore the number of pairs $(\mathbf{a}, \mathbf{b})$ of this type is $\leq |\mathcal{A}||\bar{Z}(\tau)| \leq |\mathcal{A}|C_2\tau^{-\delta}$ by Lemma 3.9. So in total, these pairs contribute at most

$$C|\mathcal{A}|C_2\tau^{2\sigma}n\tau^{-\delta} \tag{5.6}$$

to the bound for $\mathbb{E}_n\|\mathcal{L}_{\tau,s,\rho_n^0}\|_{\mathrm{H.S.}}^2$ given in (5.2).

- $\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1}$ is a qth power with q maximal, $q \geq 2$. In this case, Theorem 5.2 gives

$$|\mathbb{E}_n[\mathrm{Tr}(\rho_n^0(\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1}))]| \leq d(q) - 1 + \frac{1}{n^{1-\epsilon}} \leq 2n^{\epsilon}$$

for $n \geq n_0$. The total number of these pairs (for all possible q) is $|\mathrm{PowerPairs}(\tau)|$ so in total, these pairs contribute at most

$$2C\tau^{2\sigma}n^{\epsilon}|\mathrm{PowerPairs}(\tau)| \tag{5.7}$$

to (5.2).

- If $\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1}$ is not the identity and not a proper power, then Theorem 5.2 gives

$$|\mathbb{E}_n[\mathrm{Tr}(\rho_n^0(\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1}))]| \leq \frac{1}{n^{1-\epsilon}}.$$

We overestimate how many pairs of this kind there are by counting all pairs, of which there are $|\bar{Z}(\tau)|^2 \leq C_2^2\tau^{-2\delta}$ by Lemma 3.9. So in total, these pairs contribute at most

$$CC_2^2\tau^{2\sigma}\frac{\tau^{-2\delta}}{n^{1-\epsilon}} \tag{5.8}$$

to (5.2).

Summing up the bounds (5.6), (5.7), and (5.8) gives the result. □

In the next section, we will estimate $|\mathrm{PowerPairs}(\tau)|$.

5.4. **Estimating the size of** $\mathrm{PowerPairs}(\tau)$**.** Our goal is now to prove the following proposition controlling the size of $\mathrm{PowerPairs}(\tau)$.

Proposition 5.6. *For any $\epsilon > 0$, there is $\tau_2 = \tau_2(\Gamma, \epsilon)$ such that for $\tau \leq \tau_2$*

$$|\mathrm{PowerPairs}(\tau)| \leq \tau^{-\delta-\epsilon}.$$

In the remainder of this §5.4 we prove Proposition 5.6.

We decompose $\mathrm{PowerPairs}(\tau)$ as follows. We introduce integer parameters $L, R \geq 0$, $M_1, M_2 \geq 0$, and $q \geq 2$. For such parameters, let $\mathrm{PowerPairs}(\tau, L, M_1, M_2, R; q)$ be the subset of $\mathrm{PowerPairs}(\tau; q)$ consisting of those $(\mathbf{a}, \mathbf{b}) \in \mathrm{PowerPairs}(\tau; q)$ with

$$|\mathbf{a}'| = N_1 \stackrel{\mathrm{def}}{=} L + M_1 + R,$$

$$|\mathbf{b}'| = N_2 \stackrel{\mathrm{def}}{=} L + M_2 + R,$$

$$\mathbf{a}' = (a_1, \ldots, a_{N_1}), \quad \mathbf{b}' = (b_1, \ldots, b_{N_2}),$$

$$a_1 = b_1, a_2 = b_2, \dots, a_L = b_L, a_{L+1} \neq b_{L+1},$$
$$a_{N_1} = b_{N_2}, a_{N_1-1} = b_{N_2-1}, \dots, a_{N_1-R+1} = b_{N_2-R+1}, a_{N_1-R} \neq b_{N_2-R}.$$

Since every element of PowerPairs(τ) belongs to some PowerPairs(τ,$L, M_1, M_2, R; q$), we have

$$|\text{PowerPairs}(\tau)| \leq \sum_{L,M_1,M_2,R\geq 0, q\geq 2} |\text{PowerPairs}(\tau,L, M_1, M_2, R; q)|. \tag{5.9}$$

We will estimate |PowerPairs(τ,$L, M_1, M_2, R; q$)| in the following lemma.

Lemma 5.7. *There are constants $D > 0$ and $\kappa \in \mathbf{R}$ depending only on Γ such that* PowerPairs(τ,$L, M_1, M_2, R; q$) *is empty unless*

$$D^{-1} \log \tau^{-1} - \kappa \leq N_1,\ N_2 \leq D \log \tau^{-1} + \kappa, \tag{5.10}$$

and

$$2 \leq q \leq D \log \tau^{-1} + \kappa. \tag{5.11}$$

Under the same assumptions as Proposition 5.6, and assuming (5.10) holds, there is a constant $K = K(\Gamma) > 0$ such that

$$|\text{PowerPairs}(\tau,L, M_1, M_2, R; q)| \leq K\tau^{-\delta}. \tag{5.12}$$

Proof. For the first statement of the lemma, if $(\mathbf{a}, \mathbf{b}) \in$ PowerPairs(τ,$L, M_1, M_2, R; q$), since $|\mathbf{a}'| = N_1$, $|\mathbf{b}'| = N_2$, and $\mathbf{a}, \mathbf{b} \in \overline{Z}(\tau)$, Lemma 3.7 implies (5.10) must hold; therefore PowerPairs(τ,$L, M_1, M_2, R; q$) is empty if (5.10) does not hold. Moreover since $2 \leq q \leq N_1 + N_2$, after doubling D and κ, (5.11) must hold also.

Now we prove (5.12) assuming (5.10) holds. For $(\mathbf{a}, \mathbf{b}) \in$ PowerPairs(τ,$L, M_1, M_2, R; q$) we have

$$\mathbf{a}' = \mathbf{cAd}, \quad \mathbf{b}' = \mathbf{cBd}$$

where $|\mathbf{c}| = L$, $|\mathbf{d}| = R$, $|\mathbf{A}| = M_1$ and $|\mathbf{B}| = M_2$. Therefore

$$\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1} = \gamma_{\mathbf{c}}\gamma_{\mathbf{A}}\gamma_{\mathbf{d}}\gamma_{\mathbf{d}}^{-1}\gamma_{\mathbf{B}}^{-1}\gamma_{\mathbf{c}}^{-1} = \gamma_{\mathbf{c}}\gamma_{\mathbf{A}}\gamma_{\mathbf{B}}^{-1}\gamma_{\mathbf{c}}^{-1}.$$

Since $\gamma_{\mathbf{a}'}\gamma_{\mathbf{b}'}^{-1}$ is a qth power, with q maximal, $\gamma_{\mathbf{A}}\gamma_{\mathbf{B}}^{-1}$ is also a qth power, with q maximal, as they are conjugate in Γ. Since the first letters of $\mathbf{A}$ and $\mathbf{B}$ are not the same, and the last letters of $\mathbf{A}$ and $\mathbf{B}$ are not the same, we have $\mathbf{A} \to \overline{\mathbf{B}} \to \mathbf{A}$, in other words, the word $\mathbf{A}\overline{\mathbf{B}}$ is cyclically reduced. It now follows that there is some $\mathbf{u} \in \mathcal{W}^\circ$ with $|\mathbf{u}| = \frac{M_1+M_2}{q}$ and $\mathbf{u} \to \mathbf{u}$ (i.e. $\mathbf{u}$ is also cyclically reduced) such that

$$\mathbf{A}\overline{\mathbf{B}} = \underbrace{\mathbf{uu}\dots\mathbf{u}}_{q};$$

$\mathbf{A}\overline{\mathbf{B}}$ is q repeated copies of $\mathbf{u}$. Therefore

$$\mathbf{A} = \underbrace{\mathbf{uu}\dots\mathbf{u}}_{q_1}\mathbf{v}_1, \quad \mathbf{B} = \underbrace{\bar{\mathbf{u}}\bar{\mathbf{u}}\dots\bar{\mathbf{u}}}_{q_2}\mathbf{v}_2 \tag{5.13}$$

with $\mathbf{v}_1 \to \overline{\mathbf{v}_2}$ and

$$\mathbf{v}_1\overline{\mathbf{v}_2} = \mathbf{u}, \tag{5.14}$$

$$q_1 + q_2 = q - 1.$$

Our estimates will crucially rely on the observation that for fixed L, M_1, M_2, R, q, choosing $\mathbf{c}, \mathbf{d}, \mathbf{u}$ specifies $\mathbf{a}'$ and $\mathbf{b}'$ and hence specifies $\mathbf{a}$ and $\mathbf{b}$ except for their last letters.

We will use the shorthand $\mathbf{u}^m \stackrel{\text{def}}{=} \underbrace{\mathbf{u}\mathbf{u}\dots\mathbf{u}}_{m}$ for $m \in \mathbf{N}$. From (5.13), using Lemma 3.4 three times gives

$$\Upsilon_{\mathbf{a}'} \le K_2^3 \Upsilon_{\mathbf{c}} \Upsilon_{\mathbf{u}^{q_1}} \Upsilon_{\mathbf{v}_1} \Upsilon_{\mathbf{d}}$$

and using the same estimate in addition to the mirror estimate of Lemma 3.5 gives

$$\begin{aligned}\Upsilon_{\mathbf{b}'} &\le K_2^3 \Upsilon_{\mathbf{c}} \Upsilon_{\overline{\mathbf{u}}^{q_2}} \Upsilon_{\overline{\mathbf{v}_2}} \Upsilon_{\mathbf{d}} \\ &\le K_2^3 K_3 \Upsilon_{\mathbf{c}} \Upsilon_{\mathbf{u}^{q_2}} \Upsilon_{\overline{\mathbf{v}_2}} \Upsilon_{\mathbf{d}}\end{aligned}$$

Therefore, now using Lemma 3.4 in the opposite direction together with (5.14) we obtain

$$\begin{aligned}\Upsilon_{\mathbf{a}'}\Upsilon_{\mathbf{b}'} &\le K_2^6 K_3 \Upsilon_{\mathbf{c}}^2 \Upsilon_{\mathbf{u}^{q_1}} \Upsilon_{\mathbf{u}^{q_2}} \Upsilon_{\mathbf{v}_1} \Upsilon_{\overline{\mathbf{v}_2}} \Upsilon_{\mathbf{d}}^2 \\ &\le K_2^7 K_3 \Upsilon_{\mathbf{c}}^2 \Upsilon_{\mathbf{u}^{q_1}} \Upsilon_{\mathbf{u}^{q_2}} \Upsilon_{\mathbf{u}} \Upsilon_{\mathbf{d}}^2 \\ &\le K_2^9 K_3 \Upsilon_{\mathbf{c}}^2 \Upsilon_{\mathbf{u}^q} \Upsilon_{\mathbf{d}}^2. \qquad (5.15)\end{aligned}$$

To exploit these estimates, we note that by Lemma 3.3, we have

$$\begin{aligned}|\mathrm{PowerPairs}(\tau, L, M_1, M_2, R; q)| &\le K_1^{\frac{\delta}{2}} \tau^{-\delta} \sum_{(\mathbf{a},\mathbf{b}) \in \mathrm{PowerPairs}(\tau, L, M_1, M_2, R; q)} \Upsilon_{\mathbf{a}}^{\frac{\delta}{2}} \Upsilon_{\mathbf{b}}^{\frac{\delta}{2}} \\ &\le K_1^{\frac{\delta}{2}} \tau^{-\delta} \sum_{(\mathbf{a},\mathbf{b}) \in \mathrm{PowerPairs}(\tau, L, M_1, M_2, R; q)} \Upsilon_{\mathbf{a}'}^{\frac{\delta}{2}} \Upsilon_{\mathbf{b}'}^{\frac{\delta}{2}}\end{aligned}$$

where the last inequality is by (3.4). Let

$$\Sigma = \Sigma(\tau, L, M_1, M_2, R, q) \stackrel{\text{def}}{=} \sum_{(\mathbf{a},\mathbf{b}) \in \mathrm{PowerPairs}(\tau, L, M_1, M_2, R; q)} \Upsilon_{\mathbf{a}'}^{\frac{\delta}{2}} \Upsilon_{\mathbf{b}'}^{\frac{\delta}{2}},$$

so we have

$$|\mathrm{PowerPairs}(\tau, L, M_1, M_2, R; q)| \le K_1^{\frac{\delta}{2}} \tau^{-\delta} \Sigma. \tag{5.16}$$

By (5.15)

$$\begin{aligned}\Sigma &= \sum_{(\mathbf{a},\mathbf{b})\in \mathrm{PowerPairs}(\tau,L,M_1,M_2,R;q)} \Upsilon_{\mathbf{a}'}^{\frac{\delta}{2}}\Upsilon_{\mathbf{b}'}^{\frac{\delta}{2}} \\ &\leq |\mathcal{A}|^2 \sum_{\mathbf{c},\mathbf{u},\mathbf{d}} (K_2^9K_3)^{\frac{\delta}{2}}\Upsilon_{\mathbf{c}}^{\delta}\Upsilon_{\mathbf{u}^q}^{\frac{\delta}{2}}\Upsilon_{\mathbf{d}}^{\delta} \\ &\leq |\mathcal{A}|^2 (K_2^9K_3)^{\frac{\delta}{2}} \left(\sum_{\mathbf{c}\in\mathcal{W}_L}\Upsilon_{\mathbf{c}}^{\delta}\right)\left(\sum_{\mathbf{u}\in\mathcal{W}_{(M_1+M_2)/q}}\Upsilon_{\mathbf{u}^q}^{\frac{\delta}{2}}\right)\left(\sum_{\mathbf{d}\in\mathcal{W}_R}\Upsilon_{\mathbf{d}}^{\delta}\right). \qquad (5.17)\end{aligned}$$

By Lemma 3.10 there is some $C = C(\Gamma) > 0$ such that

$$\sum_{\mathbf{c}\in\mathcal{W}_L}\Upsilon_{\mathbf{c}}^{\delta} \leq C\exp(LP(\delta)) = C, \quad \sum_{\mathbf{d}\in\mathcal{W}_R}\Upsilon_{\mathbf{d}}^{\delta} \leq C\exp(RP(\delta)) = C. \qquad (5.18)$$

To deal with $\sum_{\mathbf{u}\in\mathcal{W}_{(M_1+M_2)/q}}\Upsilon_{\mathbf{u}^q}^{\frac{\delta}{2}}$, we write

$$q = 2\tilde{q} + r$$

where $r = 1$ if q is odd and $r = 0$ if q is even. Now using Lemma 3.4 twice and the uniform bound for $\Upsilon_{\mathbf{u}}$ from (3.5) we obtain

$$\Upsilon_{\mathbf{u}} \leq K_2^2\Upsilon_{\mathbf{u}^{\tilde{q}}}\Upsilon_{\mathbf{u}^{\tilde{q}}}\Upsilon_{\mathbf{u}^r} \leq cK_2^2\Upsilon_{\mathbf{u}^{\tilde{q}}}^2.$$

Therefore

$$\begin{aligned}\sum_{\mathbf{u}\in\mathcal{W}_{(M_1+M_2)/q}}\Upsilon_{\mathbf{u}^q}^{\frac{\delta}{2}} &\leq (cK_2^2)^{\frac{\delta}{2}} \sum_{\mathbf{u}\in\mathcal{W}_{(M_1+M_2)/q}}\Upsilon_{\mathbf{u}^{\tilde{q}}}^{\delta} \\ &\leq (cK_2^2)^{\frac{\delta}{2}} \sum_{\mathbf{U}\in\mathcal{W}_{\tilde{q}(M_1+M_2)/q}}\Upsilon_{\mathbf{U}}^{\delta} \\ &\leq C(cK_2^2)^{\frac{\delta}{2}}\exp\left(\frac{\tilde{q}(M_1+M_2)}{q}P(\delta)\right) \\ &= C(cK_2^2)^{\frac{\delta}{2}}. \qquad (5.19)\end{aligned}$$

where the final inequality is by Lemma 3.10 and $C = C(\delta)$ is the constant provided there. Therefore in total, inputting our bounds (5.18) and (5.19) into (5.17) we get

$$\Sigma \leq \tilde{K}$$

for $\tilde{K} = \tilde{K}(\Gamma) > 0$. Hence by (5.16)

$$|\mathrm{PowerPairs}(\tau,L,M_1,M_2,R;q)| \leq K\tau^{-\delta}$$

for some $K = K(\Gamma) > 0$. □

Now we can prove Proposition 5.6.

Proof of Proposition 5.6. Combining (5.9) with Lemma 5.7 we obtain for constants $D, \kappa > 0$

$$\begin{aligned} |\text{PowerPairs}(\tau)| &\leq \sum_{0 \leq L, M_1, M_2, R, q \leq D \log \tau^{-1} + \kappa} K \tau^{-\delta} \\ &\leq K \left(D \log \tau^{-1} + \kappa \right)^5 \tau^{-\delta} \\ &\leq \tau^{-\delta - \epsilon} \end{aligned}$$

for any $\epsilon > 0$ and τ sufficiently small. □

5.5. **Proof of Proposition 5.1.** Suppose we are given parameters Q, H_1, σ_1 as in Proposition 5.1. We assume $\sigma_1 > \frac{3\delta}{4}$, $\sigma \in [\sigma_1, Q]$, and $|t| \leq H_1$. Let

$$\epsilon \stackrel{\text{def}}{=} \min \left(\frac{\delta}{4}, \frac{1}{4} \left(\frac{4\sigma_1}{\delta} - 3 \right) \right) > 0.$$

Let $\alpha = \frac{2}{\delta}$ and let $\tau = n^{-\alpha} = n^{-\frac{2}{\delta}}$. Let n_0 be such that for $n \geq n_0$, $\tau \leq \tau_2$ where τ_2 is the one provided by Proposition 5.6 for the current ϵ, and n_0 is at least the one provided by Lemma 5.5 for the current ϵ and α.

Combining Proposition 5.6 and Lemma 5.5 gives for $n \geq n_0$

$$\begin{aligned} \mathbb{E}_n \| \mathcal{L}_{\tau, s, \rho_n^0} \|_{\text{H.S.}}^2 &\leq C \tau^{2\sigma} \left(n \tau^{-\delta} + n^{\epsilon} \tau^{-\delta - \epsilon} + \frac{1}{n^{1-\epsilon}} \tau^{-2\delta} \right) \\ &\leq 2C \left(n^{1 - (2\sigma - \delta)\alpha} + n^{-1 + \epsilon - 2(\sigma - \delta)\alpha} \right) \\ &\leq 4C n^{\left(3 - \frac{4\sigma}{\delta}\right) + \epsilon} \\ &\leq 4C n^{\left(3 - \frac{4\sigma_1}{\delta}\right) + \epsilon} \leq 4C n^{-3\epsilon} \leq n^{-\epsilon} \end{aligned}$$

after possibly increasing n_0. This completes the proof of Proposition 5.1. □

6. Proof of Theorems 1.1 and 1.8

As explained in the Introduction, Theorem 1.8 implies Theorem 1.1, so we will prove Theorem 1.8. A direct proof of Theorem 1.1 would use most of the same ideas and not be significantly shorter.

Let $n \in \mathbf{N}$, ϕ_n be a random homomorphism $\phi_n : \Gamma \to S_n$, and (ρ_n^0, V_n^0) be the random representation described in §2.3. Let X_n be the random convex co-compact hyperbolic surface described in the Introduction.

Let $\sigma_0 \in (\frac{3}{4}\delta, \delta)$ and H be the number given in the assumptions of Theorem 1.8. Let τ_1 and $B > 2\delta$ be the constants provided by Proposition 4.8 and choose $b \geq B$ such that the open disc $D_{b - \frac{3}{4}\delta}(b)$ contains $\text{Rect}(\sigma_0, H)$. We let $\tau = n^{-\frac{2}{\delta}}$.

Since as n varies in $\mathbf{N}$ and ϕ_n runs over all homomorphisms from $\Gamma \to S_n$, the countable collection of holomorphic functions ζ_{τ, ρ_n^0} have amongst them all, a countable number of zeros in the closed disc $\overline{D_{b - \frac{3}{4}\delta}(b)}$, it is possible to find a $\sigma_1 \in (\frac{3}{4}\delta, \sigma_0)$ such that

- no ζ_{τ, ρ_n^0} has a zero s with $|s - b| = b - \sigma_1$, and
- the open disc $D_{b - \sigma_1}(b)$ contains the closed rectangle $\text{Rect}(\sigma_0, H)$

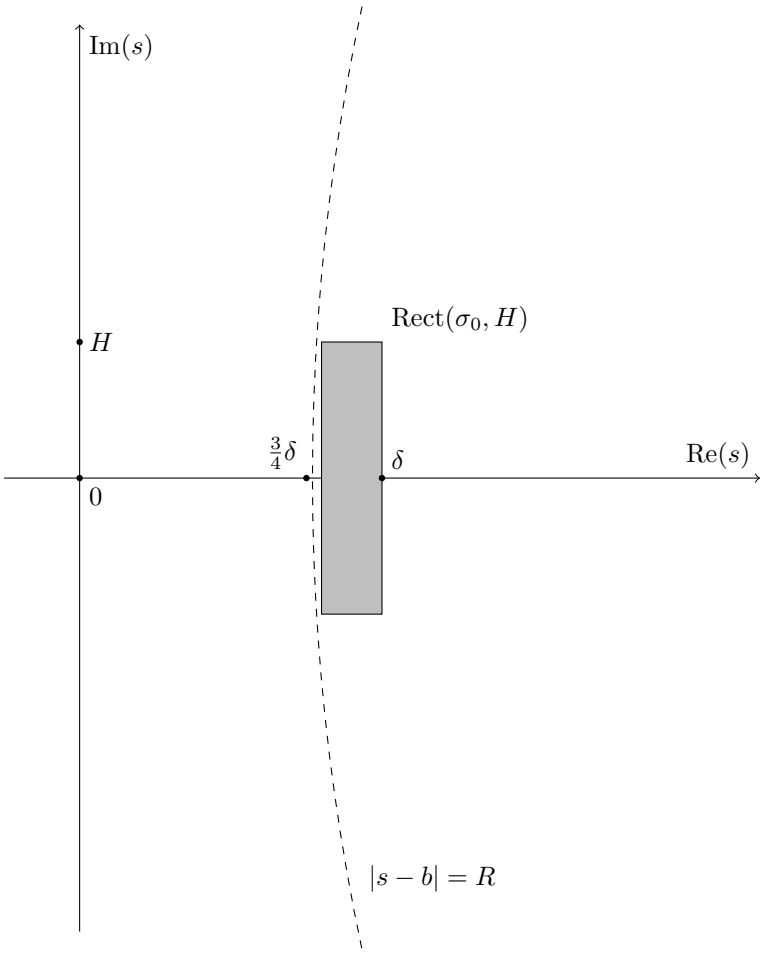

FIGURE 6.1. Illustration of the contour used in Jensen's formula.

We pick such a σ_1. Now we let

$$R \stackrel{\text{def}}{=} b - \sigma_1, \quad R' \stackrel{\text{def}}{=} \sup_{s \in \text{Rect}(\sigma_0, h)} |b - s| < R.$$

We will shortly apply Proposition 5.1 with $Q = b + R$, σ_1 as is it is in the current context, and $H_1 = R$. Let ϵ and n_0 be the positive constants provided by these inputs to Proposition 5.1. We pick $n_1 \geq n_0$ such that for $n \geq n_1$, $\tau \leq \tau_1$. This sets up all the constants for the proof.

If for $\sigma > 0$, σ is an resonance for X_n, and is either not a resonance of X or a resonance of X with a lower multiplicity, then by Theorem 2.2 combined with Corollary 4.6, $\zeta_{\tau,\rho_n^0}(\sigma) = 0$. *Therefore it suffices to show that a.a.s. there are no zeros of* ζ_{τ,ρ_n^0} *in* $\text{Rect}(\sigma_0, H)$.

Let $\mathcal{N}(\phi_n)$ be the number of zeros of ζ_{τ,ρ_n^0} in $\mathrm{Rect}(\sigma_0, H)$. Note that $\mathrm{Rect}(\sigma_0, H) \subset \overline{D_{R'}(b)}$. By Jensen's formula [Bor16, Thm. A.2] applied to the translate of ζ_{τ,ρ_n^0} by b we have

$$\sum_{\substack{z\in D_R(b)\\ \zeta_{\tau,\rho_n^0}(z)=0}}^{*} \log\left(\frac{R}{|z-b|}\right) = \mathcal{M}(\phi_n) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} \log|\zeta_{\tau,\rho_n^0}(b+Re^{i\theta})|d\theta - \log|\zeta_{\tau,\rho_n^0}(b)|. \tag{6.1}$$

The star on the sum means zeros are repeated according to their multiplicity. Note that $b \geq B$ so Proposition 4.8 ensures $\zeta_{\tau,\rho_n^0}(b) \neq 0$, and the choice of σ_1 ensures $\zeta_{\tau,\rho_n^0}(b+Re^{i\theta})$ is never zero. These conditions were needed for Jensen's formula. Now (6.1) implies

$$\mathcal{N}(\phi_n) \leq \log\left(\frac{R}{R'}\right)^{-1} \mathcal{M}(\phi_n). \tag{6.2}$$

Next we majorize $\mathcal{M}(\phi_n)$. By Weyl's inequality (cf. [Bor16, (A36)]) we have for any $s \in \mathbf{C}$

$$\log|\zeta_{\tau,\rho_n^0}(s)| = \log|\det(1-\mathcal{L}^2_{\tau,s,\rho_n^0})| \leq \|\mathcal{L}^2_{\tau,s,\rho_n^0}\|_1 \leq \|\mathcal{L}_{\tau,s,\rho_n^0}\|^2_{\text{H.S.}}, \tag{6.3}$$

where $\|\bullet\|_1$ and $\|\bullet\|_{\text{H.S.}}$ stand for the trace and Hilbert-Schmidt norms, respectively. This was the reason for the square in the definition of $\zeta_{\tau,\rho_n^0}(s)$. Also, by Proposition 4.8 we have

$$-\log|\zeta_{\tau,\rho_n^0}(b)| \leq (n-1)\tau \leq n^{1-\frac{2}{\delta}} \leq n^{-1} \tag{6.4}$$

since $\delta \in (0,1)$. Using (6.3) and (6.4) gives

$$\mathcal{M}(\phi_n) \leq \mathcal{M}^*(\phi_n) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} \|\mathcal{L}_{\tau,b+Re^{i\theta},\rho_n^0}\|^2_{\text{H.S.}} d\theta + n^{-1}. \tag{6.5}$$

Combining (6.2) and (6.5) and taking expectations gives

$$\begin{aligned}\mathbb{E}_n[\mathcal{N}(\phi_n)] &\leq \log\left(\frac{R}{R'}\right)^{-1} \mathbb{E}_n[\mathcal{M}^*(\phi_n)] \\ &= \log\left(\frac{R}{R'}\right)^{-1} \frac{1}{2\pi}\int_0^{2\pi} \mathbb{E}_n[\|\mathcal{L}_{\tau,b+Re^{i\theta},\rho_n^0}\|^2_{\text{H.S.}}]d\theta + n^{-1}.\end{aligned}$$

By Proposition 5.1 we have $\mathbb{E}_n[\|\mathcal{L}_{\tau,b+Re^{i\theta},\rho_n^0}\|^2_{\text{H.S.}}] \leq n^{-\epsilon}$ for all $\theta \in [0, 2\pi]$. Hence

$$\mathbb{E}_n[\mathcal{N}(\phi_n)] \leq \log\left(\frac{R}{R'}\right)^{-1} \left(n^{-\epsilon} + n^{-1}\right) \tag{6.6}$$

for $n \geq n_1$. By Markov's inequality, the probability that ζ_{τ,ρ_n^0} has at least one zero in $\mathrm{Rect}(\sigma_0, H)$ is bounded by the right hand side of (6.6); since this $\to 0$ as $n \to \infty$, a.a.s. ζ_{τ,ρ_n^0} has no zeros in $\mathrm{Rect}(\sigma_0, H)$. Hence by the previous arguments, a.a.s.

$$\mathcal{R}_{X_n} \bigcap \mathrm{Rect}(\sigma_0, H) = \mathcal{R}_X \bigcap \mathrm{Rect}(\sigma_0, H)$$

and the multiplicities on both sides are the same. □

7. Proof of Theorem 1.10 about high frequency resonances

This part is largely independent from the previous sections. Although we use the technique of induced representations to keep track of resonances in covers, we prove a spectral estimate on transfer operators twisted by any unitary representation which implies Theorem 1.10 via induced representations. We will prove the following completely general fact. Let $\rho : \Gamma \to \mathcal{U}(V)$ be a unitary representation of Γ on a complex Hilbert[8] space V. Here $\mathcal{U}(V)$ is the set of unitary operators on V. Recall that $I = \cup_{j=1}^{2r} I_j$. Let $C^1(I, V)$ denote the Banach space of V-valued functions, C^1 on $\overline{I}$, endowed with the norm ($t \neq 0$)

$$\|f\|_{(t),V} := \|f\|_{\infty,V} + \frac{1}{|t|}\|f'\|_{\infty,V},$$

where as usual

$$\|f\|_{\infty,V} = \sup_{x\in I} \|f(x)\|_V,$$

where $\|.\|_V$ is the Hilbert space norm on V. We recall that the action of the "basic" transfer operator $\mathcal{L}_{s,\rho}$, now on the function space $C^1(I, V)$, is given by

$$\mathcal{L}_{s,\rho}(F)(x) \stackrel{\text{def}}{=} \sum_{j\to i} (\gamma_j')^s(x)\rho(\gamma_j^{-1})F(\gamma_j x), \text{ if } x \in I_i.$$

We will use the notation $\mathcal{W}_N^j \stackrel{\text{def}}{=} \{\mathbf{a} \in \mathcal{W}_N \ : \ \mathbf{a} \to j\}$. Given the previously defined notations and $F \in C^1(I, V)$, we have for all $x \in I_j$ and $N \in \mathbf{N}$,

$$\mathcal{L}_{s,\rho}^N(F)(x) = \sum_{\mathbf{a}\in\mathcal{W}_N^j} (\gamma_{\mathbf{a}}'(x))^s \rho(\gamma_{\mathbf{a}}^{-1})F(\gamma_{\mathbf{a}}(x)).$$

We mention here that we could also alternatively use the "refined" transfer operator $\mathcal{L}_{\tau,s,\rho}$ here in place of $\mathcal{L}_{s,\rho}^N$, but it wouldn't change the final result, nor it would make the size of the gap explicit. We will need in this section some standard distortion estimates. Some of them (bounded distortion) have already been used in previous sections, but we recall them for the convenience of the reader.

- *(Uniform hyperbolicity).* There exists $C > 0$ and $0 < \overline{\theta} < \theta < 1$ such that for all N and all j such that $\mathbf{a} \in \mathcal{W}_N^j$, then for all $x \in I_j$ we have
$$C^{-1}\overline{\theta}^N \leq |\gamma_{\mathbf{a}}'(x)| \leq C\theta^N.$$
- *(Bounded distortion).* There exists $M_1 > 0$ such that for all N, j and all $\mathbf{a} \in \mathcal{W}_N^j$,
$$\sup_{I_j} \left|\frac{\gamma_{\mathbf{a}}''}{\gamma_{\mathbf{a}}'}\right| \leq M_1.$$
- *(Bounded distortion for the third derivatives).* There exists $Q > 0$ such that for all n, j and all $\mathbf{a} \in \mathcal{W}_N^j$,
$$\sup_{I_j} \left|\frac{\gamma_{\mathbf{a}}'''}{\gamma_{\mathbf{a}}'}\right| \leq Q.$$

[8] We do not assume that it is finite dimensional here.

Notice that the "bounded distortion for the third derivatives" follows directly from differentiating two times $\log(\gamma'_{\mathbf{a}})$, and using bounded distortion and uniform hyperbolicity several times, see for example [BV05, §3] for a previous occurrence of this condition in the literature. We now state the Ruelle-Perron-Frobenius Theorem, which will be used below. The statement of this theorem in the symbolic setting can be found in [PP90, Thm. 2.2]. The version we use can be obtained via the work of Liverani [Liv95] as in [Nau05a, Thm. 5.1].

Theorem 7.1. *Set $\mathcal{L}_\sigma = \mathcal{L}_{\sigma,\mathrm{Id}}$ where σ is real and $\rho = \mathrm{Id}$ means the trivial one-dimensional representation.*

1. *The spectral radius of $\mathcal{L}_\sigma$ on $C^1(I,\mathbf{C})$ is $e^{P(\sigma)}$ which is a simple eigenvalue associated to a strictly positive eigenfunction $h_\sigma > 0$ in $C^1(I,\mathbf{C})$.*
2. *The operator $\mathcal{L}_\sigma$ on $C^1(I,\mathbf{C})$ is quasi-compact with essential spectral radius smaller than $\kappa(\sigma)e^{P(\sigma)}$ for some $\kappa(\sigma) < 1$.*
3. *There are no other eigenvalues on $|z| = e^{P(\sigma)}$. Moreover, the spectral projector $\mathbb{P}_\sigma$ on $\{e^{P(\sigma)}\}$ is given by*
$$\mathbb{P}_\sigma(f) = h_\sigma \int_{\Lambda(\Gamma)} f d\mu_\sigma,$$
where μ_σ is the unique probability measure on Λ that satisfies $\mathcal{L}^*_\sigma(\mu_\sigma) = e^{P(\sigma)}\mu_\sigma$, and the eigenfunction h_σ is normalized so that
$$\int h_\sigma d\mu_\sigma = 1.$$

We continue with a basic a priori estimate.

Lemma 7.2. *Fix some $\sigma_0 < \delta$, then there exists $C_0 > 0$, $\rho < 1$ such that for all N, all unitary representations (ρ, V) and all $s = \sigma + it$ with $\sigma \geq \sigma_0$, we have*
$$\| \left(\mathcal{L}^N_{s,\rho}(f)\right)' \|_{\infty,V} \leq C_0 e^{NP(\sigma_0)} \left\{(1+|t|)\|f\|_{\infty,V} + \theta^N \|f'\|_{\infty,V}\right\}.$$

Proof. Differentiate the formula for $\mathcal{L}^n_{s,\rho}(f)$: since the representation factor is locally constant, we don't need to differentiate it. Use the bounded distortion property plus the uniform contraction, combined with the pressure estimate in Lemma 3.10. Uniformity with respect to (ρ, V) follows from triangle inequality plus the fact that for all $\gamma \in \Gamma$, we have $\|\rho(\gamma)\|_V = 1$. □

The main fact of this section is the following. It is essentially a vector-valued version of a result stated in [JNS19]. This type of estimate is called a *Dolgopyat estimate* by reference to Dolgopyat's work on Anosov flows [Dol98] where these type of bounds appeared for the first time.

Proposition 7.3. *There exist $\varepsilon > 0$, $T_0 > 0$ and $C_1, \beta > 0$ such that for all $N = N(t) = [C_1 \log|t|]$ with $s = \sigma + it$ satisfying $|\sigma - \delta| \leq \varepsilon$ and $|t| \geq T_0$, we have*
$$\int_{\Lambda(\Gamma)} \|\mathcal{L}^N_{s,\rho}(f)\|^2_V d\mu_\delta \leq \frac{\|f\|^2_{(t),V}}{|t|^\beta}.$$
All the constants here are uniform with respect to ρ, V.

A particular case of this estimate was proved in [OW16, MOW17] for the case of congruence subgroups, where

$$\rho : \Gamma \to \mathcal{U}\left(L^2(\mathrm{SL}_2(\mathbb{F}_p))\right),$$

is obtained after reduction mod p via the regular representation of $\mathrm{SL}_2(\mathbb{F}_p)$. The proof was an adaptation of the arguments of [Nau05a]. We will present below a shorter, more direct version of this estimate which allows to prove this generalization without much effort.

Let us first briefly explain why this actually implies Theorem 1.10. We set $\rho = \mathrm{Ind}_{\widetilde{\Gamma}}^{\Gamma}$, where $\widetilde{\Gamma}$ is an arbitrary, finite index subgroup of Γ, and $\mathrm{Ind}_{\widetilde{\Gamma}}^{\Gamma}$ is the induced representation to Γ of the trivial representation of $\widetilde{\Gamma}$. We work by contradiction. Assume that $Z_{\widetilde{\Gamma}}(s) = 0$, then according to the induction formula of Venkov-Zograf [VZ82, FP17], we have for $s = \sigma + it$,

$$\mathcal{L}_{s,\rho}(F_s) = F_s,$$

for some $F_s \not\equiv 0 \in C^1(I, V)$. We can definitely normalize F_s so that $\|F_s\|_{(t),V} = 1$. Write $N = N_1 + N(t)$, where $N(t)$ is given by Proposition 7.3. Take $\sigma_0 \leq \sigma \leq \delta$. Using the triangle inequality for $\|.\|_V$ and unitarity of ρ, we have (by Cauchy-Schwarz) and the pressure estimate (Lemma 3.10),

$$\|F_s\|_{\infty,V} \leq C_0 e^{\frac{N_1}{2}P(2\sigma_0-\delta)} \left(\mathcal{L}_{\delta}^{N_1}\left(\|\mathcal{L}_{s,\rho}^{N(t)}(F_s)\|_V^2\right)\right)^{1/2}.$$

We need to estimate the C^1-norm of $x \mapsto \|\mathcal{L}_{s,\rho}^{N(t)}(F_s)\|_V^2(x)$ on I. Since we work with a Hilbert norm, the square of the norm is differentiable and we can compute

$$\frac{d}{dx}\|\mathcal{L}_{s,\rho}^{N(t)}(F_s)\|_V^2 = 2\mathrm{Re}\left(\langle(\mathcal{L}_{s,\rho}^{N(t)}(F_s))', \mathcal{L}_{s,\rho}^{N(t)}(F_s)\rangle_V\right),$$

and use the V-valued Lasota-Yorke estimate from Lemma 7.2 and Cauchy-Schwarz to obtain

$$\|\|\mathcal{L}_{s,\rho}^{N(t)}(F_s)\|_V^2\|_{C^1(I)} \leq Ce^{2N(t)P(\sigma_0)}(1+|t|).$$

Using the Ruelle-Perron-Frobenius Theorem (Theorem 7.1), and the fact that $P(\delta) = 0$, we get

$$\|F_s\|_{\infty,V}^2 \leq Ce^{N_1P(2\sigma_0-\delta)}\left(\int_{\Lambda(\Gamma)}\|\mathcal{L}_{s,\rho}^{N(t)}(F_s)\|_V^2 d\mu_\delta + \kappa_\delta^{N_1}e^{2N(t)P(\sigma_0)}(1+|t|)\right),$$

with $\kappa_\delta < 1$. Assuming that $\sigma_0 \geq \delta - \epsilon$ and $|t| \geq T_0$, we can apply Proposition 7.3 and set $N_1 = N_1(t) = [C_2 \log|t|]$ to get

$$\|F_s\|_{\infty,V}^2 \leq C\left(|t|^{C_2P(2\sigma_0-\delta)-\beta} + |t|^{-C_2|\log\kappa|+2C_1P(\sigma_0)+1}\right).$$

We then take C_2 large enough and fix σ_0 close enough to δ so that $C_2P(2\sigma_0-\delta)-\beta < 0$ and we get

$$\|F_s\|_{\infty,V}^2 \leq C|t|^{-\widetilde{\beta}},$$

for some $\widetilde{\beta} > 0$. The same calculation can be performed to obtain similarly

$$\|F_s'\|_{\infty,V}^2 \leq C|t|^{-\widetilde{\beta}+2},$$

and we reach a contradiction for all $|t|$ large since $1 = \|F_s\|_{(t),V} \leq C'|t|^{-\widetilde{\beta}/2}$. Once again, all the constants are uniform with respect to (ρ, V).

The proof of the key Proposition 7.3 will rest on the following result of Bourgain-Dyatlov [BD17].

Theorem 7.4. *There exist constants $\beta_1, \beta_2 > 0$ such that the following holds. Given $g \in C^1(I)$ and $\Phi \in C^2(I)$, consider the integral*

$$\mathcal{I}(\xi) \overset{\text{def}}{=} \int_{\Lambda(\Gamma)} e^{-i\xi\Phi(x)} g(x) d\mu_\delta(x).$$

If we have

$$\inf_{\Lambda(\Gamma)} |\Phi'| \geq |\xi|^{-\beta_1},$$

and $\|\Phi\|_{C^2} \leq M$, then for all $|\xi| \geq 1$, we have

$$|\mathcal{I}(\xi)| \leq C_M |\xi|^{-\beta_2} \|g\|_{C^1},$$

where $C_M > 0$ does not depend on ξ, g.

For comments on this version of the Bourgain-Dyatlov decay estimate, see [JNS19]. Let us just mention that μ_δ, up to a smooth density, is the Patterson-Sullivan measure, see [JNS19]. To be able to use this estimate, we will use the following fact from [JNS19], which is referred there as the "uniform-non-integrability property" (UNI), see Proposition 4.10.

Proposition 7.5. *(UNI) For all $\mathbf{a}, \mathbf{b} \in \mathcal{W}_N^j$ set*

$$\mathscr{D}(\mathbf{a}, \mathbf{b}) := \inf_{x \in I_j} \left| \frac{\gamma_{\mathbf{a}}''(x)}{\gamma_{\mathbf{a}}'(x)} - \frac{\gamma_{\mathbf{b}}''(x)}{\gamma_{\mathbf{b}}'(x)} \right|.$$

There exist constants $M > 0$ and $\eta_0 > 0$ such that for all n and all $\epsilon = e^{-\eta N}$ with $0 < \eta < \eta_0$, we have for all $\mathbf{a} \in \mathcal{W}_N^j$,

$$\sum_{\mathbf{b} \in \mathcal{W}_N^j,\ \mathscr{D}(\mathbf{a},\mathbf{b}) < \epsilon} \|\gamma_{\mathbf{b}}'\|_{I_j,\infty}^{\delta} \leq M \epsilon^{\delta}.$$

For a proof of that fact, see [JNS19, §4]. We are now ready to conclude this section by the proof of Proposition 7.3. Pick $f \in C^1(I, V)$. We set $s = \sigma + it$ and we assume that σ is close to δ. Let us write

$$\mathcal{S}_{\sigma,N}(t) := \int_{\Lambda(\Gamma)} \|\mathcal{L}_{s,\rho}^N(f)\|_V^2 d\mu_\delta = \sum_{j=1}^{2r} \sum_{\mathbf{a},\mathbf{b} \in \mathcal{W}_N^j} \int_{\Lambda(\Gamma)} e^{it\Phi_{\mathbf{a},\mathbf{b}}(x)} g_{\mathbf{a},\mathbf{b}}^{(j)}(x) d\mu_\delta(x),$$

with

$$\Phi_{\mathbf{a},\mathbf{b}}(x) = \log \gamma_{\mathbf{a}}'(x) - \log \gamma_{\mathbf{b}}'(x),$$

and

$$g_{\mathbf{a},\mathbf{b}}^{(j)}(x) = \begin{cases} (\gamma_{\mathbf{a}}'(x))^\sigma \, (\gamma_{\mathbf{b}}'(x))^\sigma \, \langle \rho(\gamma_{\mathbf{a}}^{-1}) f \circ \gamma_{\mathbf{a}}(x), \rho(\gamma_{\mathbf{b}}^{-1}) f \circ \gamma_{\mathbf{b}}(x) \rangle_V \text{ if } x \in I_j, \\ 0 \text{ otherwise.} \end{cases}$$

Notice that $g_{\mathbf{a},\mathbf{b}}^{(j)}$ is indeed a C^1 function on a neighborhood of $\Lambda(\Gamma)$. By using the bounded distortion property and Cauchy-Schwarz we have easily:

$$\sup_{I_j} |g_{\mathbf{a},\mathbf{b}}^{(j)}| \leq C_1 \sup_{I_j} |\gamma_{\mathbf{a}}'|^\sigma \sup_{I_j} |\gamma_{\mathbf{b}}'|^\sigma \|f\|_{(t),V}^2. \tag{7.1}$$

Differentiating inside the inner product $\langle ., . \rangle_V$ and using the bounded distortion plus the uniform contraction (with Cauchy-Schwarz again) gives also

$$\sup_{I_j} \left| \frac{d}{dx} g^{(j)}_{\mathbf{a},\mathbf{b}} \right| \leq C_2 \sup_{I_j} |\gamma'_{\mathbf{a}}|^{\sigma} \sup_{I_j} |\gamma'_{\mathbf{b}}|^{\sigma} (1 + |t|\theta^N) \|f\|^2_{(t),V}. \tag{7.2}$$

Both estimates (7.1) and (7.2) can be combined to yield

$$\|g^{(j)}_{\mathbf{a},\mathbf{b}}\|_{C^1} \leq \widetilde{C_2} \sup_{I_j} |\gamma'_{\mathbf{a}}|^{\sigma} \sup_{I_j} |\gamma'_{\mathbf{b}}|^{\sigma} (2 + |t|\theta^N) \|f\|^2_{(t),V}. \tag{7.3}$$

We also observe that $\inf_{x \in I_j} |\Phi'_{\mathbf{a},\mathbf{b}}(x)| = \mathscr{D}(\mathbf{a}, \mathbf{b})$, and that by using the bounded distortion for the second and third derivatives we have for some uniform $C_3 > 0$,

$$\|\Phi_{\mathbf{a},\mathbf{b}}\|_{C^2} \leq C_3.$$

The plan is now to split $\mathcal{S}_{\sigma,N}(t)$ as

$$\mathcal{S}_{\sigma,N}(t) = \mathcal{S}^{(1)}_{\sigma,N}(t) + \mathcal{S}^{(2)}_{\sigma,N}(t),$$

with the "near-diagonal" sum

$$\mathcal{S}^{(1)}_{\sigma,N}(t) := \sum_{j=1}^{2r} \sum_{\mathscr{D}(\mathbf{a},\mathbf{b}) \leq \epsilon} \int_{\Lambda(\Gamma)} e^{it\Phi_{\mathbf{a},\mathbf{b}}(x)} g^{(j)}_{\mathbf{a},\mathbf{b}}(x) d\mu_{\delta}(x),$$

and the "off-diagonal" sum

$$\mathcal{S}^{(2)}_{\sigma,N}(t) := \sum_{j=1}^{2r} \sum_{\mathscr{D}(\mathbf{a},\mathbf{b}) > \epsilon} \int_{\Lambda(\Gamma)} e^{it\Phi_{\mathbf{a},\mathbf{b}}(x)} g^{(j)}_{\mathbf{a},\mathbf{b}}(x) d\mu_{\delta}(x),$$

with $\epsilon > 0$. We now assume that $\sigma_0 \leq \sigma \leq \delta$ and $N = [\kappa \log |t|]$, with $\epsilon = e^{-\eta N}$ with $0 < \eta < \eta_0$. We *fix* κ large enough so that $|t|\theta^N$ stays uniformly bounded as $|t| \to \infty$, and pick $\eta > 0$ small enough such that $\epsilon = e^{-\eta[\kappa \log |t|]} > |t|^{-\beta_1}$, so that we can apply Theorem 7.4. Combining estimate (7.3) with the pressure bound from Lemma 3.10, we get

$$|\mathcal{S}^{(2)}_{\sigma,N}(t)| \leq C \frac{\|f\|^2_{(t),V}}{|t|^{\beta_2}} e^{2NP(\sigma_0)}.$$

On the other hand we have

$$|\mathcal{S}^{(1)}_{\sigma,N}(t)| \leq C \|f\|^2_{(t),V} \sum_j \sum_{\mathbf{a} \in \mathcal{W}^j_N} \sup_{I_j} |\gamma'_{\mathbf{a}}|^{\sigma} \sum_{\mathbf{b}: \mathscr{D}(\mathbf{a},\mathbf{b}) < \epsilon} \sup_{I_j} |\gamma'_{\mathbf{b}}|^{\sigma},$$

which by using Proposition 7.5 and the pressure estimate combined with the uniform hyperbolicity (the lower bound) gives

$$|\mathcal{S}^{(1)}_{\sigma,N}(t)| \leq C \|f\|^2_{(t),V} e^{NP(\sigma_0)} \overline{\theta}^{N(\sigma_0 - \delta)} \epsilon^{\delta}.$$

because $P(\sigma_0) \to 0$ as $\sigma_0 \to \delta$, we can definitely pick $\sigma_0 < \delta$ so that for all $|t| \geq 1$, we have

$$|\mathcal{S}_{\sigma,N}(t)| \leq |\mathcal{S}^{(1)}_{\sigma,N}(t)| + |\mathcal{S}^{(2)}_{\sigma,N}(t)| \leq \widetilde{C} \|f\|^2_{(t),V} |t|^{-\widetilde{\beta}},$$

for some uniform $\widetilde{C} > 0$ and $\widetilde{\beta} > 0$. This ends the proof.

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