A Comparison of Tail Dependence Estimators*

Hendrik Supper† *TU Dortmund University*

Felix Irresberger‡ *University of Leipzig*

Gregor N.F. Weiß§ *University of Leipzig*

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ABSTRACT

We review several commonly used methods for estimating the tail dependence in a given data sample. In simulations, we show that especially static estimators produce severely biased estimates of tail dependence when applied to samples with time-varying extreme dependence. In some instances, using static estimators for time-varying data leads to estimates more than twice as high as the true tail dependence. Our findings attenuate the need to account for the time-variation in extreme dependence by using dynamic models. Taking all simulations into account, the dynamic tail dependence estimators perform best with the Dynamic Symmetric Copula (DSC) taking the lead. We test our findings in an empirical study and show that the choice of estimator significantly affects the importance of tail dependence for asset prices.

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†e-mail: *hendrik.supper@tu-dortmund.de*.

‡Corresponding author: Leeds University Business School, University of Leeds, Maurice Keyworth Building, Moorland Rd, LS61AN, Leeds, UK; e-mail: *F.Irresberger@leeds.ac.uk*.

§Grimmaische Str. 12, D-04109 Leipzig, Germany, telephone: +49 341 97 33821, e-mail: *weiss@wifa.uni-leipzig.de*.

1 Introduction

Tail dependence modeling using copulas has gained significant interest from researchers and practitioners across various fields, including finance, insurance, hydrology, engineering, and energy. Of particular interest are (upper and lower) tail dependence coefficients, which are defined as the asymptotic probability that two extreme events occur simultaneously. Estimates of such probabilities are useful for decision makers that need to efficiently allocate resources, e.g., for portfolio tail diversification, catastrophe management, or large investment project appraisals. All of these applications require an accurate estimate of tail dependence coefficients and therefore confidence in the performance of the employed copula models. Many of those models estimate tail dependence statically, which can be misleading when the dependence structure of underlying data changes dynamically over time. In this paper, we determine how fatal such model risk can be with respect to the estimation of lower tail dependence coefficients.[1]

We review various commonly used techniques for estimating the tail dependence of a joint distribution and show that several of these techniques produce severely biased estimates of tail dependence in simulations. We then apply these estimators in an empirical setting in which tail dependence coefficients have been previously used to model extreme dependence. As our key finding, we show that the systematic overestimation of tail dependence found in the simulation study translates to financial data, i.e., joint crash probabilities of equities are likely to be severely overestimated by static estimators employed in previous studies. Consequently, our results imply that findings from the related (finance) literature need to be interpreted with care and critically depend on the choice of estimator.

Tail dependence can be used to estimate the likelihood of extreme events occurring at the same time, which in turn may be employed for practical purposes. For example, in insurance, extreme losses are modeled using copulas to account for tail dependence ("ruin probability"), which ultimately influences insurer solvency requirements (see Eckert and Gatzert, 2018). Portfolio managers may be most interested in whether prices of two (or more) assets crash at the same time. As pointed out by Poon et al. (2004), investors can achieve better portfolio tail diversification by choosing asymptotically independent assets and therefore reduce the need to hedge positions with the use of options. When considering credit risk and contagion effects (see, e.g., Ye et al., 2012; Weiß et al., 2014), one would like to accurately estimate the joint probability of default.[2] Outside of finance, there are various fields that may want to employ non-linear measures of dependence instead of simple correlations to capture the uncertainty around extreme outcomes (see, e.g., Wang and Dyer, 2012; Werner et al., 2017, for brief overviews). For example, Wu (2014) models dependence in warranties of car manufacturers, while Bassetti et al. (2018) consider energy markets. Investors in or managers of oil and gas exploration projects, which are huge in size and long in duration,

[1] We concentrate on lower tail dependence in our paper as the tail risk of two assets jointly experiencing extreme losses is arguably the greatest concern to portfolio and risk managers. Moreover, the models we employ in our study can be easily adapted to measure upper tail dependence (which could be used to detect price bubbles) as well.

[2] Similarly, De Jonghe (2010) or Oh and Patton (2017, 2018) employ measures of tail dependence to proxy for systemic fragility in the financial sector.

benefit from modeling the dependence structure and account for respective tail risk (see Accioly and Chiyoshi, 2004; Al-Harthy et al., 2007). Silbermayr et al. (2017) argue that tail dependence measurement is a useful tool for inventory planning, e.g., to capture simultaneous extreme demands for production in two locations. In hydrology, the probability of the simultaneous occurence of extreme river flow volumes, rainfalls, or drought periods, in several locations, is modeled using tail dependence estimators (see, e.g., Poulin et al., 2007; Serinaldi et al., 2015) and helps risk managers in natural disaster management. Similarly, Elberg and Hagspiel (2015) compare the performance of several copulas when trying to capture (spatial) tail correlation between wind power stations and its impact on electricity spot prices and grid planning. Choosing the wrong model to estimate (time-varying) tail dependence, such as the Gaussian copula with asymptotic tail independence, will over- or understate respective probabilities and therefore have consequences for pricing and planning in all of the applications mentioned above.

Despite the consensus in the literature on the importance of accounting for extreme dependence for numerous applications, authors have employed many different models to estimate tail dependence without considering the model risk in choosing one approach over another. This is especially true in the finance literature, which we focus on in our study, but extends to other areas as well. The set of models in our comparative study is motivated by the literature on classical problems in asset pricing (see, e.g., Meine et al., 2016; Chabi-Yo et al., 2018; Irresberger et al., 2018), credit risk (see Oh and Patton, 2017; Christoffersen et al., 2018), financial intermediation (Oh and Patton, 2018), and portfolio management (see Christoffersen et al., 2012), where linear correlations are substituted by measures of extreme dependence. The consensus underlying these studies is that joint extreme co-movements in equity prices, default intensities, and liquidity are not adequately captured by correlation, but should rather be modeled using estimates of tail dependence. Most of these studies comprise a parametric copula model from which the estimates of tail dependence are derived. For example, in the early studies of Rodriguez (2007), Okimoto (2008), and Garcia and Tsafack (2011), estimates of the lower tail dependence in equity returns are extracted from simple static and regime-switching copula models. More recent work, such as in Patton (2006), Christoffersen et al. (2012), and Oh and Patton (2017, 2018), proposes to use dynamic copula models to account for possibly time-varying extreme dependence in financial data, the need of which is empirically confirmed by, e.g., Grundke and Polle (2012).[3] Furthermore, the statistical literature includes additional nonparametric estimators like the one proposed by Schmidt and Stadtmueller (2006), which eliminates the model risk of selecting a non-optimal parametric model at the expense of being purely data-driven and static. Finally, some asset pricing studies such as Chabi-Yo et al. (2018) and Ruenzi et al. (2018) use convex combinations of different static parametric copulas to estimate the tail dependence between equity returns and liquidity, respectively. Interestingly, the literature still lacks a comparison of these different estimators of a distribution's tail dependence. But even more importantly, the empirical relevance of selecting the right estimator for a data sam-

[3]Such dynamic models are found to be useful for applications in, e.g., portfolio optimization (see Al Janabi et al., 2017).

ple's tail dependence for applications in financial economics remains completely unacknowledged. [4]

The findings from both our simulations as well as our application to equities have highly relevant consequences for our understanding of extreme dependence. As our main contribution, we show in this paper that several tail dependence estimators which have been proposed in the literature are severely biased. Especially when applied to data samples with time-varying extreme dependence, static estimators tend to significantly overestimate the actual level of tail dependence in the data. This finding casts reasonable doubt on the frequent finding that extreme dependence in financial markets has increased and is high (especially during a time of crisis). What we find most striking is that this tendency to overestimate extreme dependence is common to almost all estimators that we identified from previous empirical studies in financial economics and econometrics. As this paper's second main contribution, we show in our empirical application that the choice of the correct tail dependence estimator has significant effects on the outcomes of asset pricing studies which rely on tail dependence estimates. The implications of these findings are straightforward: The role of extreme dependence in financial assets, which often exhibit dynamic dependence structures, requires to be reassessed in several areas of interest (stock returns, liquidity, systemic risk of banks, etc.) whenever empirical findings have been based on tail dependence estimates stemming from inaccurate static estimators.

The rest of this paper is organized as follows. Section 2 quickly reviews the most popular estimators of the coefficient of lower tail dependence that have been proposed in the literature. In Section 3, we present the results of our comprehensive simulation study on the finite sample properties of the various estimators of tail dependence. In Section 4, we discuss the economic importance of our findings by applying several tail dependence estimators to equity data. Section 5 concludes.

2 Copulas and Tail Dependence

The lower tail dependence (LTD) estimators included in our simulation study are based on copulas.[5] Thus, in this section we provide a brief overview of copulas and show how they can be used to measure tail dependence. Further details and a complete introduction to copulas can be found in Nelsen (2006) and Joe (1997).

Loosely speaking, a copula is a function that specifies the link between a multivariate distribution function and its one-dimensional marginal distribution functions. Formally, a copula can be defined as a multivariate distribution function with standard uniform margins. With $\boldsymbol{X} = (X_1, X_2)$

[4] The empirical finance literature is far from agreeing on the question how extreme dependence should be measured. To better understand how researchers deal with the estimation of extreme dependence we provide a survey table of recent studies on extreme dependence published in the *Review of Financial Studies*, the *Journal of Financial and Quantitative Analysis*, the *Journal of Banking and Finance*, and others in the period starting from 2006. As one can easily see from the table in the Internet Appendix, existing studies employ a great variety of different extreme dependence estimators, reaching from nonparametric to fully parametric and from static to dynamic estimators.

[5] Another popular way of measuring tail risk in finance and portfolio management that is based on *univariate* distributions is given by the estimation of a sample's tail risk index.

denoting a two-dimensional random vector with joint density $\boldsymbol{f} = (f_1, f_2)$ and distribution function $\boldsymbol{F} = (F_1, F_2)$, the copula $\boldsymbol{C}$ of the distribution $\boldsymbol{F}$ is given by

$$\boldsymbol{C}(u_1, u_2) = \boldsymbol{F}(F_1^{-1}(u_1), F_2^{-1}(u_2)) \tag{1}$$

where F_i^{-1} is the generalized inverse of F_i and $u_i \in [0, 1]$, $i = 1, 2$.

The theoretical framework of copulas goes back to the work of Sklar (1959) who shows that, under certain conditions, every copula is a joint distribution function and vice versa. More precisely, Sklar's (1959) Theorem states that, if F_1 and F_2 are continuous, $\boldsymbol{C}$ exists and is unique. Conversely, if $\boldsymbol{C}$ is a copula, the theorem states that $\boldsymbol{F}$ is a joint distribution function with margins F_i, $i = 1, 2$.[6]

Using (1), the joint density, $\boldsymbol{f}$, can be expressed as

$$\boldsymbol{f}(x_1, x_2) = \boldsymbol{c}(F_1(x_1), F_2(x_2)) \cdot f_1(x_1) f_2(x_2) \tag{2}$$

where $\boldsymbol{c}$ denotes the density of $\boldsymbol{C}$. Hence, the dependence structure can be separated from the marginal structure implying the following important applications of Sklar's (1959) Theorem. On the one hand, we can characterize the complete dependence structure in a multivariate data set and, on the other hand, are able to generate highly flexible multivariate models.

In our simulation study, however, we shall use copulas to simulate and estimate coefficients of (lower) tail dependence. Thus, in the following, we discuss the concept of tail dependence and the computation of tail dependence coefficients.

Intuitively, the concept of tail dependence refers to the amount of dependence in the lower-left or upper-right quadrant of the joint distribution, $\boldsymbol{F}$, and thus provides measures for the dependence between extreme realizations of X_1 and X_2. More precisely, the coefficient of lower (upper) tail dependence is defined as the conditional probability that X_1 takes on a realization in the left (right) tail of F_1 given that X_2 has already realized a value in the left (right) tail of F_2. In our simulation study, we are merely interested in the coefficient of lower tail dependence so that we will exclude the coefficient of upper tail dependence from the further discussion.[7]

Formally, the LTD coefficient, τ^L, is given by

$$\tau^L = \lim_{u \downarrow 0} \Pr\left[X_1 \leq F_1^{-1}(u) \middle| X_2 \leq F_2^{-1}(u) \right]. \tag{3}$$

According to McNeil et al. (2005), we can express τ^L in terms of the copula $\boldsymbol{C}$ of the joint distribution $\boldsymbol{F}$ if the marginal distributions F_1 and F_2 are continuous, and obtain the following simple formula

$$\tau^L = \lim_{u \downarrow 0} \frac{\boldsymbol{C}(u, u)}{u}. \tag{4}$$

[6]Note that Sklar's (1959) Theorem is not restricted to dimension two but holds for arbitrarily high dimensions. A general presentation and a formal proof can be found in Schweizer and Sklar (1983).

[7]Note that the properties and formulas for the LTD coefficient given in this section can be easily transferred to the coefficient of upper tail dependence. See, e.g., McNeil et al. (2005).

Hence, tail dependence can be viewed as a copula property where the copula $\boldsymbol{C}$ is said to have lower tail dependence if $\tau^L \in (0, 1]$. In case of τ^L being equal to zero, $\boldsymbol{C}$ has no lower tail dependence implying that X_1 and X_2 are asymptotically independent in the lower tail.[8]

3 Simulation Study

We now turn to a comparison of various copula-based LTD estimators that are frequently used in the financial economics literature. We conduct a comprehensive Monte-Carlo simulation study to investigate the performance of the estimators with respect to different performance metrics as well as varying simulation environments. We start with a brief overview of the models under study. A formal description of the models and details on estimation procedures can be found in the Internet Appendix.

3.1 Models under study

The LTD estimators included in our simulation study comprise three dynamic models allowing for time-varying LTD coefficients and eight static models which assume that LTD coefficients are constant over time.[9]

The dynamic models are based on the t copula which has received much recent attention in financial modeling and has been shown to be superior to other copulas such as, e.g., the Gaussian copula (see Demarta and McNeil, 2004). The method of dynamizing the t copula, however, differs across the three models. The t copula is parameterized by the degree of freedom parameter, ν, and the correlation parameter, ρ, with the implied LTD coefficient being given in closed form. The first dynamic model we consider is Patton's (2006) model that parameterizes time variation in the t copula by assuming an ARMA(1,10)-type process for the correlation parameter, ρ, to capture both persistence in correlation and any variation in dependence. We refer to this model as the *Patton model* hereafter. The second model dynamizes the t copula by applying Engle's (2002) Dynamic Conditional Correlation (DCC) model to copula correlations, which are correlations between the copula shocks implied by the t copula. This model is denoted as the *DCC model* in our study. In the same manner, we also apply the Dynamic Symmetric Copula (DSC) model as proposed by Christoffersen et al. (2012) to the copula correlations of the t copula and call this model the *DSC*

[8] It is worth noting that tail dependence as a concept, i.e., an inherent feature of the copula model that does not depend on the marginal distributions, is different from tail risk measures such as the Value-at-Risk (VaR) or Expected Shortfall (ES), which try to capture actual losses and not probabilities in the tail. In particular, tail dependence, an asymptotic probability, is not the same as moving the quantile $\tau \searrow 0$ when calculating VaR/ES measures, i.e., calculating joint losses when going deeper into the left tail. The introduction in Section 1 references a number of applications in finance and other fields such as hydrology, manufacturing, or energy markets, in which decision makers are interested in the likelihood of extreme events occurring simultaneously, not necessarily pure joint losses. There are some approaches in the finance literature that employ both VaR/ES estimates and tail dependence to construct a more complete measure of financial tail risk (see Agarwal et al., 2017; Chabi-Yo et al., 2019).

[9] The Internet Appendix provides an overview of the basic copulas underlying the dynamic and static LTD models.

model in the following. Hence, the dynamic LTD estimators can be expressed as

$$\tau_t^L = 2t_{\nu+1}\left(-\frac{\sqrt{\nu+1}\sqrt{1-\rho_t}}{\sqrt{1+\rho_t}}\right) \tag{5}$$

with the correlation dynamics being given by

$$\rho_t = \Lambda\left(\omega + \beta\rho_{t-1} + \alpha\frac{1}{10}\sum_{i=1}^{10} t_\nu^{-1}(u_{1,t-i})t_\nu^{-1}(u_{2,t-i})\right) \quad \textit{(Patton)} \tag{6}$$

$$\rho_t = \frac{Q_{12,t}}{\sqrt{Q_{11,t}Q_{22,t}}},\ Q_t = (1-\phi-\psi)\Omega + \psi Q_{t-1} + \phi\bar{z}_{t-1}^c\bar{z}_{t-1}^{c\top} \quad \textit{(DCC)} \tag{7}$$

$$\rho_t = \frac{\tilde{Q}_{12,t}}{\sqrt{\tilde{Q}_{11,t}\tilde{Q}_{22,t}}},\ \tilde{Q}_t = (1-\tilde{\phi}-\tilde{\psi})\left[(1-\kappa)\Omega + \kappa D_t\right] + \tilde{\psi}\tilde{Q}_{t-1} + \tilde{\phi}\bar{z}_{t-1}^c\bar{z}_{t-1}^{c\top} \quad \textit{(DSC)} \tag{8}$$

where ω, β, α, ϕ, ψ, $\tilde{\phi}$, $\tilde{\psi}$, and κ are scalar parameters, $\Lambda(x) \equiv (1-e^{-x})(1+e^{-x})^{-1}$ is a normalizing function, $u_{1,t}$ and $u_{2,t}$ denote the ranks of the residuals from univariate GARCH processes, Ω and D_t are two-by-two correlation matrices containing constant correlations and time trends, respectively, and $\bar{z}_t^c$ denotes a vector of (modified) copula shocks.[10]

Turning to the static LTD estimators, we first include two mixture copulas in our simulation study which are based on two different convex combinations of the basic copulas.[11] In the spirit of Chabi-Yo et al. (2018), Rodriguez (2007), and Hong et al. (2007), we select the basic copulas such that the resulting mixture copula allows for the maximum possible flexibility and is capable of modeling upper and lower tail dependence as well as independence and asymmetry in the tails. Accordingly, the first mixture is based on the Joe, Rotated-Joe, and the F-G-M copula and is given by

$$\boldsymbol{C}_{\text{mix},1} = w_1\boldsymbol{C}_{\text{Joe}} + w_2\boldsymbol{C}_{\text{rJoe}} + w_3\boldsymbol{C}_{\text{FGM}} \tag{9}$$

where $w_i \in [0,1]$ for $i = 1,2,3$ with $\sum_{i=1}^3 w_i = 1$. Following the same line of reasoning, the second mixture is composed of the t copula as well as the Clayton and Frank copula, and can be expressed as

$$\boldsymbol{C}_{\text{mix},2} = w_1\boldsymbol{C}_t + w_2\boldsymbol{C}_{\text{Cl}} + w_3\boldsymbol{C}_{\text{Fr}}. \tag{10}$$

The corresponding constant LTD coefficients can then be computed as

$$\tau_{\text{mix},1}^L = w_2\left(2-2^{\frac{1}{\theta}}\right) \qquad \text{and} \qquad \tau_{\text{mix},2}^L = 2w_1t_{\nu+1}\left(-\frac{\sqrt{\nu+1}\sqrt{1-\rho}}{\sqrt{1+\rho}}\right) + 2^{-\frac{1}{\theta}}w_2. \tag{11}$$

Following existing empirical studies in the finance literature, both mixture models are estimated in

[10] Technical details can be found in the Internet Appendix. Note that the DSC model incorporates a time trend into copula correlations and that setting $\kappa = 0$ in the DSC model yields the DCC model.

[11] Tawn (1988) shows that any convex combination of a given (finite) set of copulas is again a copula.

two different ways, respectively. On the one hand, we estimate the mixtures via maximum likelihood (ML) where the likelihood is maximized with respect to both copula parameters and the weights at the same time (see, e.g., Ruenzi et al., 2018; Chabi-Yo et al., 2018). The respective models are denoted as $\mathit{Mix1}_{ML}$ and $\mathit{Mix2}_{ML}$. On the other hand, we estimate the mixtures via maximizing the log likelihood function via the Expectation-Maximization (EM) algorithm as proposed by Dempster et al. (1977) and call the respective models $\mathit{Mix1}_{EM}$ and $\mathit{Mix2}_{EM}$ (Okimoto, 2008; Chollete et al., 2009). Note, however, that estimating mixture copulas by maximizing the log likelihood with respect to both the copula parameters and the weights implicitly assumes that the latter are observable, which is not the case here. The estimation of mixtures constitutes an incomplete-data problem which needs to be estimated via the EM algorithm. Being aware of this fact, in our simulation study we shall investigate how this potential bias translates into the calculation of LTD coefficients.

Further, we include a static LTD estimator that is based on a regime-switching copula model and referred to as the *RS model*. More precisely, we follow Okimoto (2008) and Garcia and Tsafack (2011) and identify two regimes where we assume the first regime to be Gaussian and the second regime to be specified by the Clayton copula. Formally, the LTD estimator is based on a mixture of the regime copulas and thus given by

$$\boldsymbol{C}_{\mathrm{RS}} = s_t \boldsymbol{C}_{\mathrm{GA}} + (1 - s_t)\boldsymbol{C}_{\mathrm{Cl}} \tag{12}$$

where $\boldsymbol{C}_{\mathrm{GA}}$ and $\boldsymbol{C}_{\mathrm{Cl}}$ denote the Gaussian and the Clayton copula, respectively. The variable s_t is a latent state variable taking the values 0 (Gaussian regime) and 1 (Clayton regime) and follows a Markov chain with a constant transitional probability matrix

$$P = \begin{pmatrix} p_{00} & 1 - p_{00} \\ 1 - p_{11} & p_{11} \end{pmatrix}, \quad p_{ii} = \Pr[s_t = i | s_{t-1} = i] \text{ for } i = 0, 1. \tag{13}$$

Since the Gaussian copula is asymptotically independent in the tails, the LTD coefficients generated by this model are based on the LTD coefficient of the Clayton copula which is given in closed form.

Moreover, we include two simple static LTD estimators that are based on the Clayton copula. The difference between the two estimators lies in the method used for modeling the margins. While the first estimator is based on a nonparametric approach and uses the empirical distribution function, the second estimator exploits results from Extreme Value Theory (EVT) and models the margins semi-parametrically by assuming the Generalized Pareto Distribution (GPD) for the distribution of excesses and the empirical distribution for the remaining portion. The two estimators are called CL and CL_{EVT}, respectively.

Finally, we follow Schmidt and Stadtmueller (2006) and include a nonparametric LTD estimator in our simulation study, denoted as *Nonparam*. Schmidt and Stadtmueller (2006) build on the concept of empirical tail copulas and introduce tail dependence estimators that are based on the empirical copula. Formally, with X_1 and X_2 denoting two n-dimensional random vectors and with

$\boldsymbol{R}_{m,1} = (R^j_{m,1})_{j=1,\dots,n}$ and $\boldsymbol{R}_{m,2} = (R^j_{m,2})_{j=1,\dots,n}$ denoting the rank of X_1 and X_2, respectively, they propose the following empirical LTD estimator

$$\tau^L_m = \frac{1}{k}\sum_{j=1}^{n} \mathbf{1}_{\left\{R^j_{m,1}\le k \text{ and } R^j_{m,2}\le k\right\}} \tag{14}$$

where the parameter k needs to be specified adequately. In our simulations and empirical study, we follow Schmidt and Stadtmueller (2006) and choose k via a plateau-finding algorithm applied to LTD estimates for successive k (similar to the Hill estimator).

The LTD estimators included in our simulation study are summarized in Table I along with the expressions for the corresponding LTD coefficients and the correlation dynamics for the time-varying estimators.

Table I: Lower tail dependence estimators under study.

The table presents the lower tail dependence (LTD) estimators included in our simulation study along with the expressions for the corresponding LTD coefficients and the correlation dynamics for the time-varying estimators. We consider eight static LTD estimators (Mix1$_{\text{ML}}$, Mix1$_{\text{EM}}$, Mix2$_{\text{ML}}$, Mix2$_{\text{EM}}$, RS, CL, CL$_{\text{EVT}}$, Nonparam) and three dynamic estimators based on different dynamizations of the Student's t copula (Patton, DCC, DSC). The notation is as follows: t_ν and t_ν^{-1} denote the univariate distribution and quantile function of the Student's t distribution with degrees of freedom parameter ν, respectively; w_1 and w_2 denote the weights of the mixture copulas. Regarding the correlation dynamics, ω, β, α, ϕ, ψ, $\tilde{\phi}$, $\tilde{\psi}$, and κ are scalar parameters, $\Lambda(x) \equiv (1-e^{-x})(1+e^{-x})^{-1}$ is a normalizing function, $u_{1,t}$ and $u_{2,t}$ denote the ranks of the residuals from univariate GARCH processes, Ω and D_t are two-by-two correlation matrices containing constant correlations and time trends, respectively, and $\bar{z}^c_t$ denotes a vector of (modified) copula shocks. The DSC model incorporates a time trend into copula correlations and nests the DCC model in case of $\kappa = 0$. Technical details can be found in the Internet Appendix.

Model	LTD estimator	Correlation dynamics
Patton	$\tau^L_t = 2t_{\nu+1}\left(-\frac{\sqrt{\nu+1}\sqrt{1-\rho_t}}{\sqrt{1+\rho_t}}\right)$	$\rho_t = \Lambda\left(\omega + \beta\rho_{t-1} + \alpha\frac{1}{10}\sum_{i=1}^{10} t_\nu^{-1}(u_{1,t-i})t_\nu^{-1}(u_{2,t-i})\right)$
DCC		$\rho_t = \frac{Q_{12,t}}{\sqrt{Q_{11,t}Q_{22,t}}},\ Q_t = (1-\phi-\psi)\Omega + \psi Q_{t-1} + \phi\bar{z}^c_{t-1}\bar{z}^{c\top}_{t-1}$
DSC		$\rho_t = \frac{Q_{12,t}}{\sqrt{\tilde{Q}_{11,t}\tilde{Q}_{22,t}}},\ \tilde{Q}_t = (1-\tilde{\phi}-\tilde{\psi})[(1-\kappa)\Omega + \kappa D_t] + \tilde{\psi}\tilde{Q}_{t-1} + \tilde{\phi}\bar{z}^c_{t-1}\bar{z}^{c\top}_{t-1}$
Mix1$_{\text{ML}}$	$\tau^L = w_2\left(2-2^{\frac{1}{\theta}}\right)$	–
Mix1$_{\text{EM}}$		–
Mix2$_{\text{ML}}$	$\tau^L = 2w_1 t_{\nu+1}\left(-\frac{\sqrt{\nu+1}\sqrt{1-\rho}}{\sqrt{1+\rho}}\right) + 2^{-\frac{1}{\theta}}w_2$	–
Mix2$_{\text{EM}}$		–
RS	$\tau^L = 2^{-\frac{1}{\theta}}$	–
CL		–
CL$_{\text{EVT}}$		–
Nonparam	$\tau^L = \frac{1}{k}\sum_{j=1}^{n}\mathbf{1}_{\left\{R^j_{m,1}\le k \text{ and } R^j_{m,2}\le k\right\}}$	–

3.2 Simulation Design

We now present the setup of our simulation study. To investigate the performance of the LTD estimators introduced in the previous section, we organize each simulation trial into two steps, a simulation step and an estimation step. In the first step, we simulate copula data and LTD coefficients from a specified data-generating process (DGP) and generate artificial price return data on the basis of the simulated copula data. In the second step, we then apply the LTD estimators to the artificial return data and evaluate the performance by comparing the estimated

LTD coefficients to the true LTD coefficients from the simulation step in terms of an appropriate performance metric. We repeat these steps a large number of times and evaluate the performance in each simulation trial, resulting in a vector of values for the corresponding performance metric.[12] In the following, we discuss the two steps in more detail.

The simulation step comprises two tasks, simulating LTD coefficients and generating artificial price return data to embed the simulation into an environment that is comparable to real-data applications. To simulate LTD coefficients which will be assumed to describe the true LTD inherent to the data, we identify the three dynamic LTD estimators as the DGPs throughout the simulation study.[13] To simulate from the dynamic models, we first need to specify the parameters driving the correlation dynamics in equations (6) to (8).[14] For increased comparability with real-data applications, parameter choices are based on the empirical studies in Engle (2002), Patton (2006), and Christoffersen et al. (2012). Having determined the parameters, we are now able to conduct the simulation of true LTD coefficients. Using the notation introduced in the previous section, the simulation involves the following steps.[15] First, as a starting point we randomly draw $\boldsymbol{u}^{(0)} = (u_{1,0}, u_{2,0})^\top$ from a bivariate standard uniform distribution, $\mathcal{U}_{[0,1]}$. Then, we calculate ρ_1 and τ_1^L using $\boldsymbol{u}^{(0)}$ and, finally, simulate $\boldsymbol{u}^{(1)}$ from the t copula, $\boldsymbol{C}_{t^2_{\nu,\rho_1}}$, implied by a bivariate t distribution with correlation parameter ρ_1. We repeat the latter steps for $t = 2, ..., T$ and generate true LTD coefficients, $(\tau_t^L)_{t=1}^T$, as well as copula data, $(\boldsymbol{u}^{(t)})_{t=1}^T$. Estimation of LTD coefficients in the second step is based on the the series $(\boldsymbol{u}^{(t)})_{t=1}^T$. Since copula data are not directly observable, we transform the series $(\boldsymbol{u}^{(t)})_{t=1}^T$ into artificial price return data before moving on to the estimation step. As is standard in the econometrics literature, we assume that the returns come from a GARCH(1,1) process with zero mean and t-distributed innovations. With $\boldsymbol{r}^{(t)} = (r_{1,t}, r_{2,t})^\top$ denoting the (artificial) return corresponding to $\boldsymbol{u}^{(t)}$, we thus define

$$r_{i,t} = \sqrt{h_{i,t}} z_{i,t}, \quad z_{i,t} | \mathcal{F}_{i,t-1} \sim t_{\nu_i} \tag{15}$$

$$h_{i,t} = c_i + a_i r_{i,t-1}^2 + b_i h_{i,t-1} \tag{16}$$

where $\mathcal{F}_{i,t}$ denotes the information available on the ith series up to and including the tth observation, $i = 1, 2$ and $t = 1, ..., T$. With $\theta_i = (c_i, a_i, b_i, \nu_i)^\top$ being the parameter vector of the GARCH processes, we follow the empirical applications in Engle (2002), Kang et al. (2010), Christoffersen et al. (2012) and set $\theta_1 = (0.0005, 0.1, 0.85, 5)^\top$ and $\theta_2 = (0.0001, 0.05, 0.9, 10)^\top$ to generate artificial returns in line with the stylized facts on real price return series. To simulate return data from the copula data, we set $r_{1,0} = r_{2,0} = 0$ and $\sigma_{1,0} = \sigma_{2,0} = 0$ as starting points and compute return

[12] Note that in our baseline simulation approach we simulate 500 data points from the DGP, use the mean squared error to evaluate performance, and repeat the simulation and estimation step for a total of 1000 trials. Further details are provided in Section 3.3.

[13] Note that, due to the time-varying nature of LTD, simulating from the dynamic LTD estimators will provide simulated LTD coefficients that are comparable to the LTD coefficients implied by real data.

[14] The parameter choices as well as the resulting expressions for the correlation dynamics are given in the Internet Appendix.

[15] Technical details on the simulation from the Patton, DCC, and DSC model can be found in the Internet Appendix.

innovations via $z_{i,t} = t_{\nu_i}^{-1}(u_{i,t})$.

Having simulated return data $(\boldsymbol{r}^{(t)})_{t=1}^{T}$ with (true) LTD coefficients $(\tau_t^L)_{t=1}^T$, the second step of our simulation study deals with computing estimated LTD coefficients, $(\hat{\tau}_t^L)_{t=1}^T$, according to the models discussed in the previous section. Since our LTD estimators are based on copulas and copula theory requires white-noise residuals for the computation of unbiased LTD coefficient estimates, we first apply the GARCH(1,1) filter to transform the marginal return series, $(r_{i,t})_{t=1}^T$, into white-noise series, $(\hat{u}_{i,t})_{t=1}^T$, where $\hat{\boldsymbol{u}}^{(t)} = (\hat{u}_{1,t}, \hat{u}_{2,t})^\top$. Then, we apply our LTD estimators summarized in Table I to $(\hat{\boldsymbol{u}}^{(t)})_{t=1}^T$ to obtain the series $(\hat{\tau}_t^L)_{t=1}^T$ of estimated LTD coefficients. To evaluate the performance of the LTD estimators, we apply an appropriate performance metric, Π, to the true and the estimated LTD coefficients. Thus, with $\boldsymbol{\tau} = (\tau_t^L)_{t=1}^T$ and $\hat{\boldsymbol{\tau}} = (\hat{\tau}_t^L)_{t=1}^T$, the performance of the corresponding LTD estimator is given by $\Pi = \Pi(\boldsymbol{\tau}, \hat{\boldsymbol{\tau}})$.

Altogether, our simulation study is organized into the following steps:

1. Simulation step

   1.1. Draw $\boldsymbol{u}^{(0)} = (u_{1,0}, u_{2,0})^\top \sim \mathcal{U}_{[0,1]}$.

   1.2. Calculate ρ_1 and τ_1^L using $\boldsymbol{u}^{(0)}$.

   1.3. Simulate $\boldsymbol{u}^{(1)}$ from $\boldsymbol{C}_{t^2_{\nu,\rho_1}}$.

   1.4. Repeat steps 1.2. and 1.3. for $t = 2, ..., T$ and obtain $(\tau_t^L)_{t=1}^T$ and $(\boldsymbol{u}^{(t)})_{t=1}^T$.

   1.5. Calculate $z_{i,t} = t_{\nu_i}^{-1}(u_{i,t})$, $i = 1, 2$.

   1.6. Compute $r_{i,t} = \sqrt{h_{i,t}} z_{i,t}$, where $h_{i,t} = c_i + a_i r_{i,t-1}^2 + b_i h_{i,t-1}$ and $r_{i,0} = h_{i,0} = 0$.

2. Estimation step

   2.1. Apply the GARCH(1,1) filter to $(r_{i,t})_{t=1}^T$ and obtain $(\hat{u}_{i,t})_{t=1}^T$, $i = 1, 2$.

   2.2. Apply LTD estimators to $(\hat{\boldsymbol{u}}^{(t)})_{t=1}^T$ and obtain $(\hat{\tau}_t^L)_{t=1}^T$.

   2.3. Apply the performance metric to $(\tau_t^L)_{t=1}^T$ and $(\hat{\tau}_t^L)_{t=1}^T$ and obtain $\Pi = \Pi(\boldsymbol{\tau}, \hat{\boldsymbol{\tau}})$.

These two steps are repeated for a total of N simulation trials resulting in the performance vector $\boldsymbol{\Pi} = (\Pi_n)_{n=1}^N$, where $\Pi_n = \Pi(\boldsymbol{\tau}_n, \hat{\boldsymbol{\tau}}_n)$ with $\boldsymbol{\tau}_n$ and $\hat{\boldsymbol{\tau}}_n$ denoting the true and estimated LTD coefficients drawn from the nth simulation trial.

3.3 Simulation Results

We now turn to the results of our simulation study. We first introduce our baseline approach and discuss the corresponding results. In the following, we then extend our baseline approach with respect to the sample size and performance metric and check the robustness of the conclusions drawn from the baseline approach. Finally, we conduct a ranking approach to identify the best performing LTD estimator across all simulation settings (i.e., across all sample sizes and performance metrics).

3.3.1 Does the choice of estimator matter? The baseline approach.

The baseline approach is based on the following simulation setting. Given the notation introduced in the previous section, we set

$$T = 500, \quad N = 1000, \quad \text{and} \quad \Pi = \Pi(\boldsymbol{\tau}, \hat{\boldsymbol{\tau}}) = T^{-1} \sum_{t=1}^{T} (\tau_t - \hat{\tau}_t)^2. \tag{17}$$

That is, performance is measured in terms of the mean squared error (MSE). Thus, we simulate 500 LTD coefficients from the Patton, the DCC, and the DSC model, respectively, and then apply the LTD models presented in Section 3.1 to the resulting series of artificial returns to generate estimated LTD coefficients. For each of the three DGPs, these steps are repeated for a total of 1000 trials.

The panels of Table II report descriptive statistics of true and estimated LTD coefficients separately for each DGP.[16]

[16]Note that the estimation of most LTD models included in our study requires removing pseudo observations equal to 1. To preserve comparability of true and estimated LTD coefficients, we remove the corresponding value from the series of true LTD coefficients as well, resulting in $499 \times 1000 = 499,000$ true and estimated LTD coefficients for the majority of LTD models.

Table II: Descriptive statistics of true and estimated lower tail dependence.

The table presents descriptive statistics of true and estimated lower tail dependence (LTD) coefficients. True LTD coefficients are simulated from the Patton, DCC, and DSC model and the corresponding results are shown separately for each of these data-generating processes (DGP) throughout the panels of the table. To compute estimated LTD coefficients, we first generate artificial return data on the basis of the true LTD coefficients and then apply the different LTD estimators to the artificial returns. The descriptive statistics listed in the table arise from the baseline simulation approach, which is based on a sample size of $T = 500$ and a number of simulation trials equal to $N = 1000$, i.e., we estimate LTD coefficients on the basis of 500 simulated returns and repeat the simulation and estimation step for a total of 1000 trials. Except for the number of observations, skewness, and (excess) kurtosis, all entries are denominated in %. In case of the DGP and the LTD estimator being identical, corresponding statistics are printed in bold type. The names of the LTD estimators are abbreviated according to the notation introduced in Section 3.1.

Panel A: DGP Patton				Percentiles								Moments			
		Number	Min	1st	5th	25th	Median	75th	95th	99th	Max	Mean	St. Dev.	Skewness	Exc. Kurt.
Patton	**True LTD**	**499,000**	**0.00**	**2.55**	**5.27**	**9.46**	**13.73**	**20.02**	**33.98**	**47.59**	**93.65**	**15.90**	**9.30**	**1.57**	**6.87**
	Est. LTD	**499,000**	**0.00**	**0.01**	**0.46**	**7.41**	**15.36**	**24.21**	**38.80**	**51.65**	**85.18**	**16.84**	**12.03**	**0.80**	**3.68**
DCC	True LTD	499,000	0.00	2.55	5.31	9.47	13.66	19.86	33.83	47.51	93.19	15.83	9.23	1.58	6.93
	Est. LTD	499,000	0.00	0.01	0.43	8.24	16.13	24.14	36.50	46.70	76.52	16.88	11.15	0.54	3.05
DSC	True LTD	499,000	0.00	2.55	5.32	9.48	13.77	20.06	33.95	48.03	97.47	15.94	9.33	1.60	7.12
	Est. LTD	499,000	0.00	0.32	4.73	15.71	23.27	30.41	41.32	51.16	78.82	23.22	10.99	0.20	3.11
Mix1_{ML}	True LTD	499,000	0.00	2.55	5.28	9.44	13.66	19.80	33.45	47.38	96.67	15.76	9.15	1.60	7.17
	Est. LTD	1,000	15.00	21.66	27.74	31.49	33.11	34.29	38.25	42.09	52.07	32.92	3.49	-0.32	8.33
Mix1_{EM}	True LTD	499,000	0.00	2.55	5.31	9.49	13.79	20.06	33.92	47.86	94.26	15.93	9.29	1.59	7.02
	Est. LTD	1,000	13.46	17.82	20.96	24.34	27.06	29.62	33.48	37.77	41.05	27.07	3.95	0.12	3.37
Mix2_{ML}	True LTD	499,000	0.00	2.55	5.30	9.49	13.71	20.01	33.77	47.19	96.60	15.88	9.22	1.56	6.90
	Est. LTD	1,000	0.01	0.08	5.90	14.12	20.61	29.39	40.14	42.31	54.20	21.89	10.45	0.24	2.32
Mix2_{EM}	True LTD	499,000	0.00	2.55	5.27	9.46	13.74	19.99	33.90	48.09	88.42	15.89	9.30	1.58	6.89
	Est. LTD	1,000	0.04	0.68	3.56	12.39	19.86	26.37	34.63	39.34	43.67	19.47	9.46	0.00	2.29
RS	True LTD	499,000	0.00	2.55	5.27	9.45	13.63	19.85	33.84	47.74	96.86	15.81	9.27	1.61	7.14
	Est. LTD	1,000	0.00	0.00	0.00	3.45	20.10	44.46	71.63	87.26	95.78	25.30	23.64	0.76	2.67
CL	True LTD	500,000	0.00	2.55	5.27	9.43	13.68	19.86	33.59	47.23	96.16	15.79	9.18	1.58	7.07
	Est. LTD	1,000	25.23	34.15	39.77	47.41	51.37	55.05	59.98	62.06	65.82	50.98	5.94	-0.49	3.57
CL_{EVT}	True LTD	500,000	0.00	2.55	5.30	9.51	13.77	20.01	33.83	47.49	98.60	15.90	9.25	1.59	7.12
	Est. LTD	1,000	14.82	23.70	29.53	36.06	40.23	44.15	49.51	52.15	55.42	39.96	6.21	-0.39	3.23
Nonparam	True LTD	500,000	0.00	2.55	5.27	9.47	13.74	20.06	33.94	47.44	96.64	15.89	9.25	1.55	6.79
	Est. LTD	1,000	0.00	12.50	20.00	46.02	57.00	64.36	72.80	76.92	84.45	53.07	15.73	-0.95	3.35

As can be seen from Panel A of the table, specifying the Patton model as the DGP (according to the parameterization displayed in the Internet Appendix) leads to true LTD coefficients ranging from 0% to 98.60%, where the means are close to 16%.[17] Comparing the means of the true and estimated LTD coefficients provides first evidence on the performance of the different estimators. Regarding the dynamic estimators, the Patton and DCC model show an exceptionally good performance, with the means of the estimated LTDs deviating by approximately 1% from the means of the true LTDs in absolute terms.[18] Not surprisingly, the Patton model, when determined to be the DGP, is the best performing LTD estimator. The DSC model, however, shows a somewhat worse performance, with the mean true and estimated LTD differing by more than 7% (in absolute terms). Turning to the static estimators, the performance deteriorates considerably for most estimators, with the differences in the means increasing dramatically to levels ranging between 3.58% to more than 37% in absolute terms. Interestingly, the Mix2_{ML} and Mix2_{EM} model outperform the DSC model as well as all other static estimators (including the Mix1_{ML} and Mix1_{EM} model) in terms of the differences between the means. Further, with respect to the CL and the CL_{EVT} model, the table shows that modeling the excess distributions of the marginals by the GPD substantially improves the estimates and decreases the differences in the means from 35% to 24%. The worst performing estimator is the Nonparam model, with the difference being more than 37% in absolute terms. These results are supported by the percentiles and the higher moments captured in Panel A of Table II, which show the superior ability of the dynamic estimators to reproduce the distributional properties of the true LTD coefficients. Panels B and C show corresponding results for the cases in which the DCC and the DSC model are specified as the DGP. As can be seen from the panels, in these cases the true LTDs are 44% and 42% on average, respectively, indicating that the parameterization of these models leads to remarkably higher LTDs than that of the Patton model. The main conclusions, however, remain the same.

To further study the performance of the different estimators, we compute and compare the MSEs across all estimators for each of the three DGPs and report the corresponding results in Table II. This table presents descriptive statistics of the MSEs and splits up the MSEs into mean squared positive deviations (denoted as MSE^+) and mean squared negative deviations (denoted as MSE^-) to assess whether MSEs result from underestimation or overestimation of true LTD coefficients. As can be seen from the table, the MSE results confirm the first evidence and support the above conclusions. Irrespective of the choice of DGP, the dynamic LTD estimators consistently outperform the static estimators in terms of MSE. Interestingly, when determined to be the DGP, the Patton model has the lowest average MSE (0.0099) and is the best performing LTD estimator, whereas the DSC model has a considerably worse (average) MSE of 0.0152 and is the most inaccurate dynamic LTD estimator. In case of the DCC and the DSC model being the DGP, the DSC model clearly outperforms the Patton and the DCC model, with the average MSE being around 0.0080 in both cases. Turning to the static LTD estimators, the mixture copula models dominate

[17] True LTD coefficients are simulated independently and separately for each LTD estimator in each simulation trial.

[18] In the following, we will use the terms LTD coefficient and LTD interchangeably.

Table I: Descriptive statistics of true and estimated lower tail dependence (continued).

Panel B: DGP DCC				Percentiles								Moments			
		Number	Min	1st	5th	25th	Median	75th	95th	99th	Max	Mean	St. Dev.	Skewness	Exc. Kurt.
Patton	True LTD	499,000	0.00	11.64	22.78	37.48	45.27	51.82	60.21	66.60	84.60	43.97	11.37	-0.59	3.57
	Est. LTD	499,000	0.00	11.82	23.87	36.98	43.92	50.30	59.55	67.95	89.80	43.18	10.96	-0.44	3.94
DCC	**True LTD**	**499,000**	**0.01**	**11.85**	**22.94**	**37.48**	**45.17**	**51.75**	**60.19**	**66.24**	**87.92**	**43.95**	**11.29**	**-0.60**	**3.57**
	Est. LTD	**499,000**	**0.00**	**6.26**	**17.95**	**32.95**	**41.87**	**49.79**	**60.17**	**67.51**	**88.54**	**40.88**	**12.80**	**-0.39**	**3.15**
DSC	True LTD	499,000	0.10	11.51	23.18	37.66	45.34	51.87	60.43	66.89	85.51	44.12	11.34	-0.59	3.62
	Est. LTD	499,000	0.02	6.31	17.96	34.40	43.82	51.68	61.41	68.33	92.49	42.34	13.16	-0.50	3.16
Mix1_{ML}	True LTD	499,000	0.00	11.81	23.34	37.61	45.23	51.66	60.27	66.94	89.90	44.03	11.25	-0.57	3.65
	Est. LTD	1,000	6.31	22.09	30.14	33.08	34.37	36.39	46.04	54.77	69.72	35.54	5.61	1.34	10.93
Mix1_{EM}	True LTD	499,000	0.00	12.70	23.39	37.48	45.21	51.69	60.22	66.46	87.11	44.01	11.18	-0.55	3.49
	Est. LTD	1,000	21.81	23.96	27.85	32.34	35.60	38.92	44.18	47.58	51.70	35.76	4.95	0.10	2.89
Mix2_{ML}	True LTD	499,000	0.01	12.22	23.27	37.61	45.25	51.73	60.23	66.57	94.62	44.04	11.25	-0.57	3.63
	Est. LTD	1,000	0.06	11.75	20.80	32.24	39.47	47.18	58.07	62.43	66.10	39.36	11.08	-0.21	3.07
Mix2_{EM}	True LTD	499,000	0.01	12.25	23.45	37.59	45.27	51.68	60.12	66.32	93.68	44.03	11.19	-0.58	3.62
	Est. LTD	1,000	1.54	5.84	15.36	25.84	33.16	41.62	48.26	50.96	53.93	33.02	10.23	-0.33	2.58
RS	True LTD	499,000	0.00	12.13	23.10	37.37	45.08	51.54	59.96	66.19	87.00	43.85	11.21	-0.58	3.56
	Est. LTD	1,000	0.00	0.00	0.00	15.22	32.81	46.35	61.00	78.69	94.01	31.28	19.80	0.10	2.40
CL	True LTD	500,000	0.06	12.18	23.02	37.43	45.25	51.77	60.23	66.47	87.23	43.98	11.29	-0.57	3.52
	Est. LTD	1,000	36.29	49.19	58.28	64.67	68.68	72.05	76.05	78.00	80.35	68.00	5.84	-1.05	5.64
CL_{EVT}	True LTD	500,000	0.03	11.89	23.06	37.45	45.24	51.80	60.35	66.56	88.81	43.99	11.33	-0.57	3.51
	Est. LTD	1,000	25.54	41.32	47.59	54.29	59.00	62.77	67.71	69.92	73.26	58.32	6.31	-0.62	3.86
Nonparam	True LTD	500,000	0.00	12.44	23.57	37.60	45.15	51.64	60.17	66.44	88.97	44.01	11.15	-0.56	3.58
	Est. LTD	1,000	12.50	28.57	45.45	60.52	66.00	71.16	77.13	82.71	91.07	64.60	10.08	-1.36	6.25

Panel C: DGP DSC				Percentiles								Moments			
		Number	Min	1st	5th	25th	Median	75th	95th	99th	Max	Mean	St. Dev.	Skewness	Exc. Kurt.
Patton	True LTD	499,000	0.00	5.51	13.54	30.67	43.90	53.48	62.88	68.66	85.45	41.55	15.28	-0.43	2.45
	Est. LTD	499,000	0.00	6.35	18.18	33.79	42.04	49.52	60.27	69.16	90.57	41.15	12.62	-0.37	3.45
DCC	True LTD	499,000	0.00	5.48	13.42	30.68	43.63	53.18	62.92	68.86	85.55	41.40	15.25	-0.42	2.47
	Est. LTD	499,000	0.00	4.93	13.09	29.35	40.25	50.17	62.93	70.57	89.87	39.46	14.92	-0.18	2.61
DSC	**True LTD**	**499,000**	**0.10**	**5.91**	**13.87**	**30.89**	**43.93**	**53.45**	**63.04**	**68.85**	**88.62**	**41.65**	**15.21**	**-0.42**	**2.45**
	Est. LTD	**499,000**	**0.00**	**3.03**	**9.32**	**28.06**	**41.50**	**51.89**	**63.62**	**70.86**	**89.34**	**39.52**	**16.42**	**-0.32**	**2.42**
Mix1_{ML}	True LTD	499,000	0.00	5.64	13.74	30.69	43.81	53.35	63.09	69.20	93.03	41.55	15.27	-0.41	2.47
	Est. LTD	1,000	7.62	21.16	28.07	32.46	34.14	37.02	44.53	50.86	62.67	34.84	5.16	0.29	7.22
Mix1_{EM}	True LTD	499,000	0.00	5.36	13.59	30.83	44.03	53.54	63.22	68.93	90.19	41.68	15.33	-0.43	2.47
	Est. LTD	1,000	21.01	23.95	26.51	31.03	33.87	37.33	42.44	45.98	49.35	34.18	4.81	0.22	2.89
Mix2_{ML}	True LTD	499,000	0.01	5.57	13.39	30.75	43.79	53.34	62.92	68.78	91.19	41.48	15.30	-0.42	2.46
	Est. LTD	1,000	0.30	8.31	19.58	30.78	38.46	47.05	56.54	61.58	72.24	38.50	11.47	-0.23	3.00
Mix2_{EM}	True LTD	499,000	0.01	5.34	13.35	30.81	43.89	53.31	62.91	68.99	95.31	41.52	15.31	-0.43	2.49
	Est. LTD	1,000	1.38	5.29	14.11	24.33	31.25	39.48	47.08	50.96	53.70	31.27	10.25	-0.24	2.63
RS	True LTD	499,000	0.02	5.15	13.27	30.64	43.99	53.60	63.22	69.13	92.44	41.59	15.45	-0.43	2.46
	Est. LTD	1,000	0.00	0.00	0.00	11.91	29.15	42.34	57.83	70.51	90.49	28.06	18.76	0.14	2.29
CL	True LTD	500,000	0.02	5.31	13.44	30.79	43.81	53.31	62.92	68.89	87.04	41.49	15.28	-0.43	2.48
	Est. LTD	1,000	32.86	46.17	54.50	61.17	65.62	69.24	73.52	76.25	81.17	64.89	6.16	-0.79	4.37
CL_{EVT}	True LTD	500,000	0.03	5.98	14.01	31.00	44.12	53.71	63.30	69.08	88.26	41.83	15.27	-0.42	2.44
	Est. LTD	1,000	17.59	36.47	42.15	50.72	55.77	60.08	66.15	69.42	74.87	55.19	7.21	-0.55	3.86
Nonparam	True LTD	500,000	0.05	5.67	13.52	30.60	43.84	53.44	62.98	68.86	87.94	41.52	15.31	-0.42	2.44
	Est. LTD	1,000	0.00	23.05	36.35	57.14	64.00	69.00	75.34	81.82	89.75	61.46	11.86	-1.47	6.14

Table II: Performance of lower tail dependence estimators.

The table shows descriptive statistics on mean squared errors (MSE) for the lower tail dependence (LTD) estimators included in our simulation study. MSE is computed according to $\text{MSE} = \Pi(\boldsymbol{\tau}, \hat{\boldsymbol{\tau}}) = T^{-1}\sum_{t=1}^{T}(\tau_t - \hat{\tau}_t)^2$, where $\boldsymbol{\tau} = (\tau_t)_{t=1}^{T}$ and $\hat{\boldsymbol{\tau}} = (\hat{\tau}_t)_{t=1}^{T}$ denote the series of true and estimated LTD coefficients, respectively. The statistics in the table result from the baseline approach, which is determined by setting the sample size, T, to 500 and the number of simulation trials, N, to 1000. MSE is computed in each simulation trial, resulting in a total of 1000 MSEs for each combination of data-generating process (DGP) and LTD estimator. In addition to the mean, median, minimum, and maximum MSE, the table reports statistics on mean squared positive and negative deviations (denoted as MSE^{+} and MSE^{-}, respectively), where $\text{MSE}^{+} = T^{-1}\sum_{t=1}^{T}[\max\{0; \tau_t - \hat{\tau}_t\}]^2$ and $\text{MSE}^{-} = T^{-1}\sum_{t=1}^{T}[\min\{0; \tau_t - \hat{\tau}_t\}]^2$. The first column of the statistics on MSE^{+} and MSE^{-} reports the corresponding average across the simulation trials, the second column shows the average of the ratios $\text{MSE}^{+}/\text{MSE}$ and $\text{MSE}^{-}/\text{MSE}$, and the third column reports the average of the numbers $T^{-1}\sum_{t=1}^{T}\mathbf{1}_{\{\tau_t > \hat{\tau}_t\}}$ and $T^{-1}\sum_{t=1}^{T}\mathbf{1}_{\{\tau_t < \hat{\tau}_t\}}$ (with $\mathbf{1}$ denoting the indicator function), respectively. In case of the DGP and the LTD estimator being identical, corresponding statistics are printed in bold type. The names of the LTD estimators are abbreviated according to the notation introduced in Section 3.1.

Panel A: DGP Patton	MSE				MSE^{+}			MSE^{-}		
	Mean	Median	Min	Max	Mean	% of MSE	# Underest. (in %)	Mean	% of MSE	# Overest. (in %)
Patton	**0.0099**	**0.0069**	**0.0004**	**0.0560**	**0.0048**	**51.69**	**55.69**	**0.0051**	**48.15**	**44.31**
DCC	0.0115	0.0077	0.0012	0.0548	0.0054	49.57	55.51	0.0061	50.28	44.49
DSC	0.0152	0.0121	0.0016	0.0634	0.0020	22.77	78.86	0.0132	77.07	21.14
Mix1$_{\text{ML}}$	0.0388	0.0384	0.0059	0.1379	0.0008	2.66	94.14	0.0379	97.18	5.86
Mix1$_{\text{EM}}$	0.0220	0.0208	0.0050	0.0633	0.0016	9.04	88.16	0.0203	90.80	11.84
Mix2$_{\text{ML}}$	0.0220	0.0139	0.0034	0.1336	0.0049	41.20	68.37	0.0171	58.63	31.63
Mix2$_{\text{EM}}$	0.0179	0.0138	0.0034	0.0772	0.0061	45.86	62.60	0.0118	53.98	37.40
RS	0.0739	0.0299	0.0040	0.6382	0.0114	48.11	56.02	0.0624	51.73	43.98
CL	0.1343	0.1344	0.0219	0.2328	0.0001	0.06	99.36	0.1342	99.94	0.64
CL$_{\text{EVT}}$	0.0687	0.0677	0.0077	0.1416	0.0004	0.77	97.34	0.0684	99.23	2.66
Nonparam	0.1705	0.1746	0.0046	0.4606	0.0005	4.18	96.15	0.1700	95.82	3.85
Panel B: DGP DCC	MSE				MSE^{+}			MSE^{-}		
	Mean	Median	Min	Max	Mean	% of MSE	# Underest. (in %)	Mean	% of MSE	# Overest. (in %)
Patton	0.0117	0.0095	0.0033	0.1564	0.0066	49.48	46.49	0.0052	50.40	53.51
DCC	**0.0095**	**0.0049**	**0.0007**	**0.2038**	**0.0075**	**59.22**	**38.50**	**0.0020**	**40.65**	**61.50**
DSC	0.0081	0.0056	0.0007	0.1433	0.0060	60.02	45.85	0.0021	39.86	54.15
Mix1$_{\text{ML}}$	0.0223	0.0214	0.0052	0.1515	0.0191	82.89	21.97	0.0032	16.99	78.03
Mix1$_{\text{EM}}$	0.0199	0.0184	0.0032	0.0647	0.0174	84.57	21.77	0.0025	15.30	78.23
Mix2$_{\text{ML}}$	0.0236	0.0186	0.0052	0.2036	0.0169	62.59	35.97	0.0067	37.29	64.03
Mix2$_{\text{EM}}$	0.0319	0.0240	0.0043	0.1578	0.0291	80.99	21.02	0.0028	18.88	78.98
RS	0.0609	0.0309	0.0044	0.2471	0.0533	70.47	29.03	0.0075	29.41	70.97
CL	0.0698	0.0705	0.0141	0.1097	0.0000	0.07	98.96	0.0698	99.93	1.04
CL$_{\text{EVT}}$	0.0334	0.0330	0.0090	0.0673	0.0004	1.66	90.88	0.0330	98.34	9.12
Nonparam	0.0629	0.0605	0.0064	0.2654	0.0011	4.48	92.86	0.0618	95.52	7.14
Panel C: DGP DSC	MSE				MSE^{+}			MSE^{-}		
	Mean	Median	Min	Max	Mean	% of MSE	# Underest. (in %)	Mean	% of MSE	# Overest. (in %)
Patton	0.0157	0.0140	0.0069	0.0633	0.0078	48.61	46.69	0.0079	51.25	53.31
DCC	0.0106	0.0073	0.0018	0.1587	0.0067	48.22	41.62	0.0040	51.65	58.38
DSC	**0.0080**	**0.0049**	**0.0008**	**0.1586**	**0.0059**	**60.42**	**42.05**	**0.0021**	**39.45**	**57.95**
Mix1$_{\text{ML}}$	0.0300	0.0287	0.0079	0.1235	0.0226	73.25	32.18	0.0074	26.62	67.82
Mix1$_{\text{EM}}$	0.0296	0.0289	0.0098	0.0789	0.0233	76.95	30.38	0.0062	22.92	69.62
Mix2$_{\text{ML}}$	0.0336	0.0296	0.0075	0.1707	0.0200	54.79	41.46	0.0135	45.08	58.54
Mix2$_{\text{EM}}$	0.0411	0.0354	0.0065	0.1765	0.0349	76.68	27.76	0.0062	23.19	72.24
RS	0.0719	0.0431	0.0074	0.3680	0.0618	72.56	28.20	0.0099	27.31	71.80
CL	0.0780	0.0791	0.0178	0.1119	0.0002	0.31	95.99	0.0778	99.69	4.01
CL$_{\text{EVT}}$	0.0417	0.0413	0.0109	0.0734	0.0015	4.51	79.53	0.0402	95.49	20.47
Nonparam	0.0744	0.0718	0.0105	0.2619	0.0026	7.18	87.32	0.0718	92.82	12.68

the remaining LTD models irrespective of the DGP. When we specify the Patton model as the DGP, the results are as expected; due to their greater flexibility, the Mix2_{ML} and Mix2_{EM} model outperform the Mix1_{ML} and Mix1_{EM} model as indicated by the consistently lower average MSEs. Moreover, estimating the mixtures via the EM algorithm yields considerably better results than ML estimation for both mixture models. Somewhat surprisingly, these results do not hold anymore for the Mix2_{ML} and Mix2_{EM} model when specifying either the DCC or the DSC model as the DGP. As can be seen from the table, in these cases the corresponding average MSEs of the Mix2_{ML} and Mix2_{EM} model are greater than those of the Mix1_{ML} and Mix1_{EM} model and increase from 0.0236 to 0.0319 and from 0.0336 to 0.0411 when changing from ML to the EM algorithm, respectively. Furthermore, the most inaccurate LTD estimates are generated by the CL and the Nonparam model, with the average MSEs ranging from 0.0698 to 0.1343 and from 0.0629 to 0.1705 across the DGPs, respectively. Remarkably, with the Patton model being the DGP, the average MSEs for the two estimators are substantially greater than those resulting from choosing the DCC or the DSC model as the DGP (e.g., the average MSEs for the CL and Nonparam model decrease from 0.1343 to 0.0698 and from 0.1705 to 0.0629 when switching from the Patton model to the DCC model, respectively). Further, confirming the evidence from Table II, the CL_{EVT} model has a much lower average MSE than the CL model across all DGPs, indicating that the EVT approach of applying the GPD to the marginal distributions prior to estimating the copula model results in a material improvement in the accuracy of LTD estimates. More precisely, as shown in Table II, MSEs of the CL_{EVT} model are roughly half the MSEs of the CL model on average, irrespective of the DGP. With respect to under- and overestimation, Table II shows that there is no specific pattern in the statistics of MSE^+ and MSE^- for most of the LTD estimators. In case of the Patton model, however, approximately 50% of MSE results, on average, from under- or overestimation of true LTD coefficients across all DGPs. Interestingly, in case of the CL, CL_{EVT}, and the Nonparam model, the percentages of MSE that on average result from underestimation are consistently low across all DGPs, ranging from 0.06% (CL, DGP Patton) to 7.18% (Nonparam, DGP DSC) and indicating that these models systematically overestimate LTD.[19]

3.3.2 How important is sample size? Extending the baseline approach.

When estimating copula models, sample size is a critical issue. In this section, we extend our baseline approach and examine the performance of the LTD estimators with respect to varying sample sizes. More precisely, we include two additional simulation specifications that arise from the baseline approach by altering the number of simulated (true) LTD coefficients, T, from 500 to

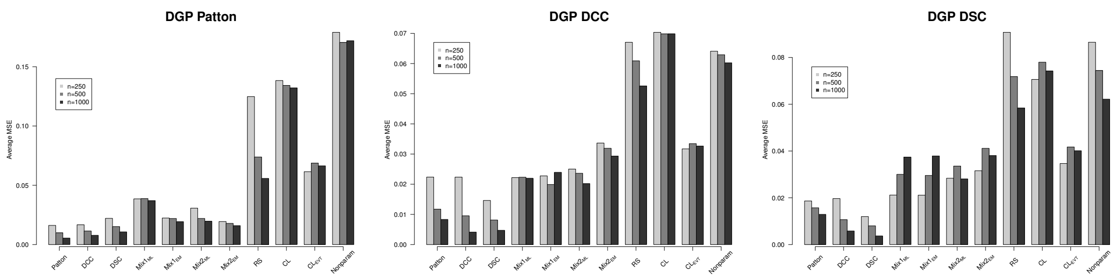
[19]The results discussed above are illustrated and supported by additional figures reported in the Internet Appendix that plot MSEs separately for each of the three DGPs as well as for each of the LTD estimators studied in our simulation approach. MSEs remain relatively flat for the dynamic models with sporadic peaks across the simulation replications for some of the DGP specifications. The MSEs for the static LTD estimators, on the other hand, are for the most part characterized by considerable fluctuations and a generally higher level than that of the dynamic estimators' MSEs. Supporting the evidence from Table II, the mixture copula models show the best performance among the static estimators, whereas the MSEs of the remaining static models exhibit an increased variability and magnitude.

250 and to 1,000, respectively.

The results of the extended simulation approach are illustrated in Figure 1. As can be seen

Figure 1: Average mean squared errors for different sample sizes.

The figure shows average mean squared errors (MSE) for the lower tail dependence (LTD) estimators with respect to different sample sizes and separately for each of the three data-generating processes (Patton, DCC, and DSC model). MSE is computed according to MSE $= \Pi(\boldsymbol{\tau}, \hat{\boldsymbol{\tau}}) = T^{-1} \sum_{t=1}^{T} (\tau_t - \hat{\tau}_t)^2$, where $\boldsymbol{\tau} = (\tau_t)_{t=1}^{T}$ and $\hat{\boldsymbol{\tau}} = (\hat{\tau}_t)_{t=1}^{T}$ denote the series of true and estimated LTD coefficients, respectively. For each LTD estimator, the figure plots three bars showing the average MSE for each of the three sample sizes considered ($T = 250; 500; 1000$), where the average is calculated across a total of $N = 1000$ simulation replications. The names of the LTD estimators are abbreviated according to the notation introduced in Section 3.1.



from the figure, the general conclusions drawn in the previous section remain valid when varying the sample size, i.e., the dynamic models are the best performing LTD estimators and the mixture copula models clearly dominate the remaining static models across all three sample sizes. Further, the figure shows that the performance of the dynamic LTD estimators substantially improves with increasing sample size, irrespective of the specified DGP. This effect is particularly pronounced when the DCC model is specified as the DGP. In this case, when increasing the sample size, T, from 250 to 1,000, the average MSE for the Patton, the DCC, and the DSC model decreases considerably from 0.0224 to 0.0083, from 0.0224 to 0.0041, and from 0.0146 to 0.0047, respectively. Put differently, reducing sample size from 1,000 to 250 (that is, by a factor of 4.00) increases the average MSE by a factor of 2.94 for the Patton model, a factor of 5.46 for the DCC model, and a factor of 3.11 for the DSC model, leading to a remarkable deterioration in performance. Hence, we find clear evidence of consistency for the dynamic LTD estimators studied in our simulation approach so that the dynamic models provide statistically consistent estimates of LTD coefficients.

However, the pattern is not as pronounced for the static estimators. In fact, for most of the static models, increasing the sample size does not necessarily result in a better performance, i.e., decreasing MSEs. Except for the RS and the Nonparam models, which exhibit decreasing (average) MSEs for increasing sample sizes across all DGP specifications, the relation between performance and sample size is not as clear for the remaining static estimators.[20] Consequently, we do not find evidence of consistency for most of the static LTD estimators in our simulation approach so that

[20] In case of the Mix1_{ML} model, for example, we can see from results tabulated in the Internet Appendix that, when the Patton model is determined to be the DGP, the average MSE slightly decreases from 0.0386 to 0.0370 when increasing sample size from 250 to 1,000. When specifying the DSC model as the DGP, the average MSE increases substantially from 0.0212 to 0.0374, implying a worse performance for a greater sample size.

most static models in our study seem to deliver inconsistent estimates of LTD coefficients.

Overall, the extended baseline approach shows the robustness of our results with respect to sample size on the one hand, and demonstrates the importance of considering sample size when estimating LTD models on the other hand. Based on our results, the issue of sample size is particularly relevant for the dynamic estimators. Increasing the sample size results in a material improvement in the performance of the estimators, or put the other way round, decreasing sample size deteriorates LTD estimates substantially.

3.3.3 Is performance measurement crucial? Reevaluating simulation results.

One concern about our simulation study might be the choice of performance metric we used to evaluate the accuracy of the LTD estimates. Up to this point, performance evaluation exclusively relied on the mean squared error criterion and neglected any other performance measures. Hence, in this section we introduce additional performance metrics and check the robustness of the results presented in the preceding sections with respect to performance measurement. More precisely, we include three additional performance metrics in the evaluation of our simulation results, namely a slight variation of MSE (denoted as MSE_2) and two metrics based on the absolute deviation between true and estimated LTD coefficients (denoted as MAD_1 and MAD_2). The additional performance metrics are computed according to the following formulas

$$\text{MSE}_2 = T^{-1} \sum_{t=1}^{T} \left(\tau_t^2 - \hat{\tau}_t^2\right)^2 \tag{18}$$

$$\text{MAD}_1 = T^{-1} \sum_{t=1}^{T} \left|\tau_t - \hat{\tau}_t\right| \tag{19}$$

$$\text{MAD}_2 = T^{-1} \sum_{t=1}^{T} \left|\tau_t^2 - \hat{\tau}_t^2\right| . \tag{20}$$

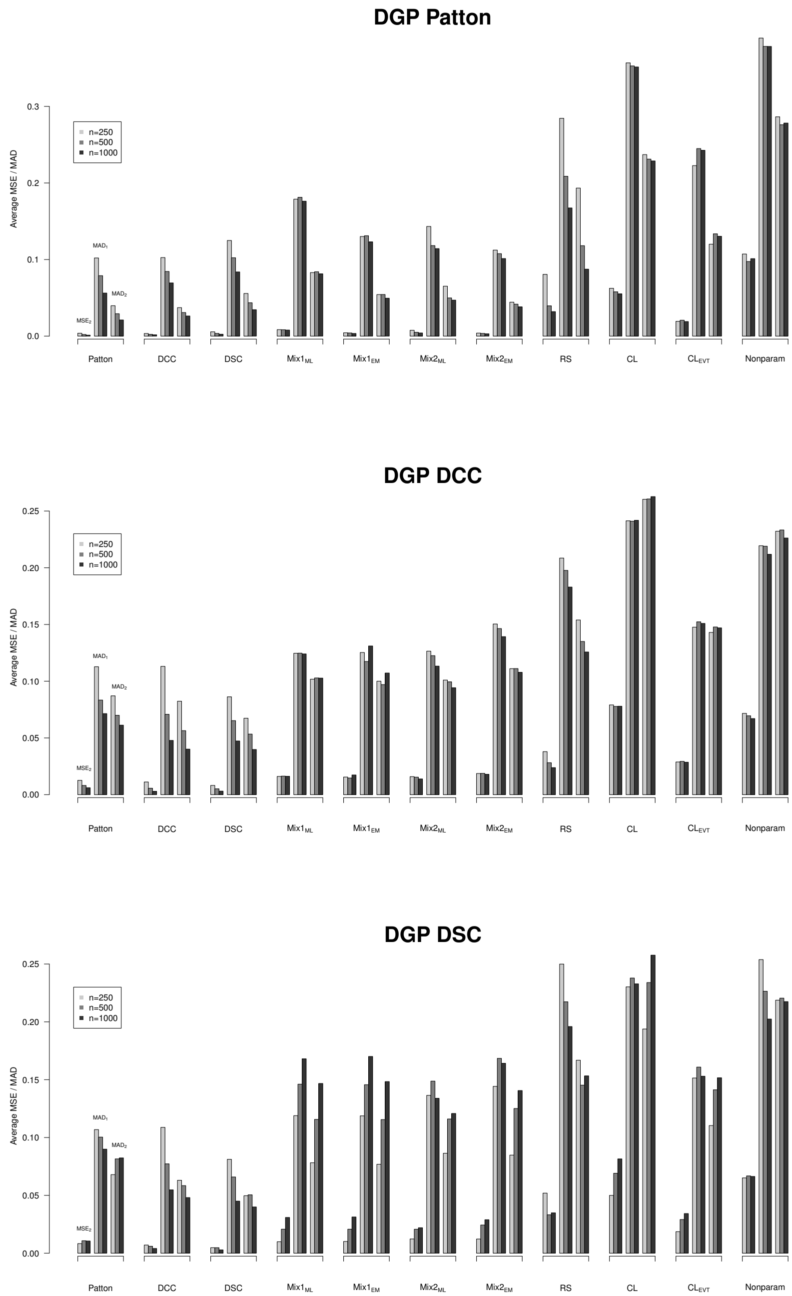
Results on average values of the performance metrics are illustrated in Figure 2 separately for each DGP, performance metric, and each sample size ($T = 250; 500; 1,000$).[21] Figure 2 demonstrates that the main results and conclusions drawn in the previous sections remain valid when altering the performance metric, indicating that our findings from above are robust towards performance measurement and do not depend on the specific properties of MSE. More precisely, we can see from the figure that the dynamic LTD estimators clearly outperform the static models across all DGPs and across all performance metrics, with the superiority becoming increasingly evident as sample size grows. Further, as in the case of MSE being the performance metric, both MSE_2 and the MAD measures decrease with increasing sample size, indicating better performance for larger sample sizes.[22]

[21] We report a comprehensive result table in the Internet Appendix, from which we retrieve several numerical examples discussed below.

[22] An exception to this pattern is constituted by the Patton model when the DSC model is specified as the DGP. As shown in the results table in the Internet Appendix, in this setting, average MSE_2 and MAD_2 increase from 0.0083

Figure 2: Average alternative performance metrics for different sample sizes.

The figure shows average values of the alternative performance metrics for the lower tail dependence (LTD) estimators with respect to different sample sizes and separately for each of the three data-generating processes (Patton, DCC, and DSC model). The alternative performance measures include a modified version of the mean squared error (denoted as MSE_2) and two mean absolute deviation measures (MAD_1 and MAD_2), which are computed according to $\mathrm{MSE}_2 = T^{-1}\sum_{t=1}^{T}\left(\tau_t^2 - \hat{\tau}_t^2\right)^2$, $\mathrm{MAD}_1 = T^{-1}\sum_{t=1}^{T}\left|\tau_t - \hat{\tau}_t\right|$, and $\mathrm{MAD}_2 = T^{-1}\sum_{t=1}^{T}\left|\tau_t^2 - \hat{\tau}_t^2\right|$, where $\boldsymbol{\tau} = (\tau_t)_{t=1}^{T}$ and $\hat{\boldsymbol{\tau}} = (\hat{\tau}_t)_{t=1}^{T}$ denote the series of true and estimated LTD coefficients, respectively. For each LTD estimator, the figure plots three bars for each performance metric showing the average MSE_2, MAD_1, and MAD_2 for each of the three sample sizes considered ($T = 250; 500; 1000$), where the average is calculated across a total of $N = 1000$ simulation replications. The names of the LTD estimators are abbreviated according to the notation introduced in Section 3.1.



Moreover, the mixture copula models are the dominating static LTD estimators across all DGPs, performance metrics, and sample sizes.

Regarding the mixture copula models, the results do not provide evidence of one of the two mixture models being superior to the other or of the EM algorithm leading to more accurate LTD estimates. Further, similar to the results in the previous sections, the CL and the Nonparam model are the worst performing LTD estimators across all simulation settings, with the performance metrics being substantially higher than those of the other estimators.[23] Assuming the Patton model as the DGP and a sample size equal to 1,000, for example, average MSE_2 for the CL and the Nonparam model is approximately 42 (0.0552/0.0013) and 77 (0.1012/0.0013) times the average MAD_2 of the Patton model. As expected, the effect of the EVT approach remains significant across all DGPs and sample sizes as can be seen from the considerable reduction in the average values of the MSE and MAD measures for the CL_{EVT} model when compared to the CL model in Figure 2.

3.3.4 Which estimator performs best? Summary and conclusions.

This section summarizes the results from our simulation study and shortly reviews the most important conclusions. Table III provides a ranking of the LTD estimators included in our study for each of the simulation specifications investigated in the previous sections (i.e., for each performance metric, DGP, and sample size), where each estimator is assigned a number between 1 (best performer) and 11 (worst performer). For each performance metric and LTD estimator, the rankings are summed up and the values of the corresponding performance metric are averaged across all DGPs and sample sizes (see the last two columns in Table III), with low sums and average values implying global superior performance (that is, across all simulation settings). The rankings, sums, and averages reported in the table summarize our general findings, which can be stated as follows.

First, the dynamic LTD estimators clearly dominate the static estimators, with the superiority of the former becoming increasingly evident with growing sample sizes.[24] Among the dynamic estimators, the Patton model is the best performing model only when at the same time assumed to be the DGP. Otherwise, the DSC model outperforms the Patton and the DCC model.[25] Second, the mixture copula models are the best performing static LTD estimators, irrespective of the

to 0.0105 and from 0.0679 to 0.0824, respectively.

[23] In some settings, the RS model performs even worse than the CL or the Nonparam model. These settings are, however, restricted to the small sample size specifications. When sample size is increased, the performance metrics consistently decrease to values below those of the CL and Nonparam model.

[24] As mentioned before, tail dependence coefficients, i.e., asymptotic probabilities that are an inherent feature of a copula, are different from other tail risk measures such as VaR or ES. Therefore, the best performing copula model with respect to tail dependence estimation may not necessarily be the best estimator for VaR and ES figures. In unreported simulation results for different sample sizes, we see that in fact the mixture models may sometimes provide better accuracy with regard to VaR and ES measures than dynamic models. However, these differences in performance, as indicated by MSEs, are not as striking as the differences in MSEs of LTD estimates as shown before. In short, the performance of dynamic models with respect to LTD coefficient estimates is relatively better than the performance of static models for VaR and ES.

[25] Note the corresponding pattern in Table III. When specified as the DGP, the Patton model ranks on first place for the most part, while the rankings of the DSC model range between the third and fifth place. Changing the DGP from the Patton to the DCC or DSC model, however, results in the Patton model ranking between the second and fourth place and the DSC model ranking on first place for most specifications.

Table III: Ranking and overall performance of lower tail dependence estimators.

The table provides a ranking of the lower tail dependence (LTD) estimators for each of the simulation specifications investigated in our simulation study, i.e., for each performance metric, data-generating process (Patton, DCC, DSC), and sample size ($T = 250; 500; 1000$). Each LTD estimator is assigned a number between 1 (best performer) and 11 (worst performer) and for each performance metric and estimator the rankings are summed up and the values of the corresponding performance metric are averaged across all data-generating processes (DGP) and sample sizes (see the last two columns in the table). The performance metrics included are two versions of the mean squared error (denoted as MSE and MSE_2) and mean absolute deviation (MAD_1 and MAD_2), which are computed according to $\text{MSE} = T^{-1}\sum_{t=1}^{T}(\tau_t - \hat{\tau}_t)^2$, $\text{MSE}_2 = T^{-1}\sum_{t=1}^{T}\left(\tau_t^2 - \hat{\tau}_t^2\right)^2$, $\text{MAD}_1 = T^{-1}\sum_{t=1}^{T}|\tau_t - \hat{\tau}_t|$, and $\text{MAD}_2 = T^{-1}\sum_{t=1}^{T}\left|\tau_t^2 - \hat{\tau}_t^2\right|$, where $\boldsymbol{\tau} = (\tau_t)_{t=1}^{T}$ and $\hat{\boldsymbol{\tau}} = (\hat{\tau}_t)_{t=1}^{T}$ denote the series of true and estimated LTD coefficients, respectively. In case of the DGP and the LTD estimator being identical, corresponding statistics are printed in bold type. The names of the LTD estimators are abbreviated according to the notation introduced in Section 3.1.

Panel A: MSE	DGP Patton			DGP DCC			DGP DSC			Sum	Average MSE
	$n=250$	$n=500$	$n=1000$	$n=250$	$n=500$	$n=1000$	$n=250$	$n=500$	$n=1000$		
Patton	**1**	**1**	**1**	4	3	3	2	3	3	21	**0.0162**
DCC	2	2	2	3	2	**1**	3	2	2	19	0.0167
DSC	4	3	3	**1**	**1**	2	**1**	**1**	**1**	**17**	0.0221
Mix1_{ML}	7	7	7	2	5	5	5	5	5	48	0.0386
Mix1_{EM}	5	5	5	5	4	6	4	4	6	44	0.0224
Mix2_{ML}	6	6	6	6	6	4	6	6	4	50	0.0307
Mix2_{EM}	3	4	4	8	7	7	7	7	7	54	0.0194
RS	9	9	8	10	9	9	**11**	9	9	83	0.1248
CL	10	10	10	**11**	**11**	**11**	9	**11**	**11**	**94**	0.1384
CL_{EVT}	8	8	9	7	8	8	8	8	8	72	0.0615
Nonparam	**11**	**11**	**11**	9	10	10	10	10	10	92	**0.1790**
Panel B: MSE_2	DGP Patton			DGP DCC			DGP DSC			Sum	Average MSE_2
	$n=250$	$n=500$	$n=1000$	$n=250$	$n=500$	$n=1000$	$n=250$	$n=500$	$n=1000$		
Patton	2	**1**	**1**	3	3	3	3	3	3	22	0.0038
DCC	**1**	2	2	2	2	**1**	2	2	2	**16**	**0.0034**
DSC	5	4	3	**1**	**1**	2	**1**	**1**	**1**	19	0.0056
Mix1_{ML}	7	7	7	6	6	5	4	6	6	54	0.0084
Mix1_{EM}	4	5	5	4	4	6	5	5	7	45	0.0044
Mix2_{ML}	6	6	6	5	5	4	7	4	4	47	0.0077
Mix2_{EM}	3	3	4	7	7	7	6	7	5	49	0.0039
RS	10	9	9	9	8	8	10	9	9	81	0.0806
CL	9	10	10	**11**	**11**	**11**	9	**11**	**11**	93	0.0625
CL_{EVT}	8	8	8	8	9	9	8	8	8	74	0.0193
Nonparam	**11**	**11**	**11**	10	10	10	**11**	10	10	**94**	**0.1072**
Panel C: MAD_1	DGP Patton			DGP DCC			DGP DSC			Sum	Average MAD_1
	$n=250$	$n=500$	$n=1000$	$n=250$	$n=500$	$n=1000$	$n=250$	$n=500$	$n=1000$		
Patton	**1**	**1**	**1**	2	3	3	2	3	3	19	**0.1021**
DCC	2	2	2	3	2	2	3	2	2	20	0.1026
DSC	4	3	3	**1**	**1**	**1**	**1**	**1**	**1**	**16**	0.1249
Mix1_{ML}	7	7	8	4	6	5	5	5	7	54	0.1788
Mix1_{EM}	5	6	6	5	4	6	4	4	8	48	0.1301
Mix2_{ML}	6	5	5	6	5	4	6	6	4	47	0.1432
Mix2_{EM}	3	4	4	8	7	7	7	8	6	54	0.1123
RS	9	8	7	9	9	9	10	9	9	79	0.2845
CL	10	10	10	**11**	**11**	**11**	9	**11**	**11**	**94**	0.3568
CL_{EVT}	8	9	9	7	8	8	8	7	5	69	0.2227
Nonparam	**11**	**11**	**11**	10	10	10	**11**	10	10	**94**	**0.3892**
Panel D: MAD_2	DGP Patton			DGP DCC			DGP DSC			Sum	Average MAD_2
	$n=250$	$n=500$	$n=1000$	$n=250$	$n=500$	$n=1000$	$n=250$	$n=500$	$n=1000$		
Patton	2	**1**	**1**	3	3	3	3	3	3	22	0.0397
DCC	**1**	2	2	2	2	2	2	2	2	**17**	**0.0372**
DSC	5	4	3	**1**	**1**	**1**	**1**	**1**	**1**	18	0.0557
Mix1_{ML}	7	7	7	6	6	5	5	5	6	54	0.0830
Mix1_{EM}	4	6	6	4	4	6	4	4	7	45	0.0543
Mix2_{ML}	6	5	5	5	5	4	7	6	4	47	0.0652
Mix2_{EM}	3	3	4	7	7	7	6	7	5	49	0.0443
RS	9	8	8	9	8	8	9	9	9	77	0.1933
CL	10	10	10	**11**	**11**	**11**	10	**11**	**11**	**95**	0.2370
CL_{EVT}	8	9	9	8	9	9	8	8	8	76	0.1201
Nonparam	**11**	**11**	**11**	10	10	10	**11**	10	10	94	**0.2864**

specification of the mixture and the estimation method. Occasionally, when sample size is low ($T = 250$), the mixture copula models outperform some of the dynamic LTD estimators, but as sample size increases, the mixtures considerably underperform the dynamic models.[26] Third, neither the specification of the mixture nor the estimation method has a distinct impact on the accuracy of LTD estimates. The two mixture models provide similarly accurate LTD estimates and, somewhat surprisingly, the bias arising from using the two different maximization strategies for the log likelihood (one step and the EM algorithm) for estimation of the mixtures does not translate into a consistent deterioration in performance. Fourth, the worst performing LTD estimators are the CL and the Nonparam model, where the performance of the former improves significantly when modified by the EVT approach of applying the GPD to the marginal distributions prior to estimating the copula. The resulting CL_{EVT} model as well as the RS model fall somewhere in between the mixture models and the CL and Nonparam model regarding their performance, with the lowest values of their corresponding performance metrics reaching those of the mixtures and the highest reaching those of the CL and Nonparam model.

4 Tail dependence in finance

We now apply selected tail dependence estimators to the universe of U.S. equities from 1980 to 2011 to investigate the economic implications of the choice of tail dependence estimator used in many empirical asset pricing studies.[27] We estimate the Mix1_{EM} model[28] (as used in, e.g., Chabi-Yo et al., 2018), the Patton model as one that outperformed the mixture model, and the underperforming Clayton (EVT) model, and then compute corresponding tail dependence coefficients for each stock and year.[29]

Figures 3 and 4 depict and compare the time series of aggregate LTD and the range between the 25th and 75th percentile of LTD across the sample for the three tail dependence estimators. We define aggregate LTD as the yearly cross-sectional and equal-weighted average LTD over all stocks in our sample (cf. Chabi-Yo et al., 2018). As can be seen from the panels in Figure 3, the general patterns in the *temporal* variation of aggregate LTD are similar across all estimators. Peaking in 1987 (the year of Black Monday), aggregate LTD stayed relatively flat during the 1990s and has been on a strong and stable upward trend since the turn of the millennium.[30] However,

[26] Note that this difference in performance is also economically relevant. For example, the mean absolute deviation (MAD_1) in Panel C of Table III shows that the average error across all settings for the Patton model is 10.2%, while the corresponding average deviations for mixture models are between 11.2% (Mix2_{EM}) and 17.9% (Mix1_{ML}), i.e., a spread of 1% to 7.7% (on average).

[27] As an example, we further replicate the study by Chabi-Yo et al. (2018) to show that the choice of tail dependence estimators and the potential, economically large bias can lead to different results in empirical asset pricing studies involving extreme dependence measures. We refer to the Internet Appendix for a detailed discussion of the replication procedure and additional portfolio sorts and regression results confirming our prior.

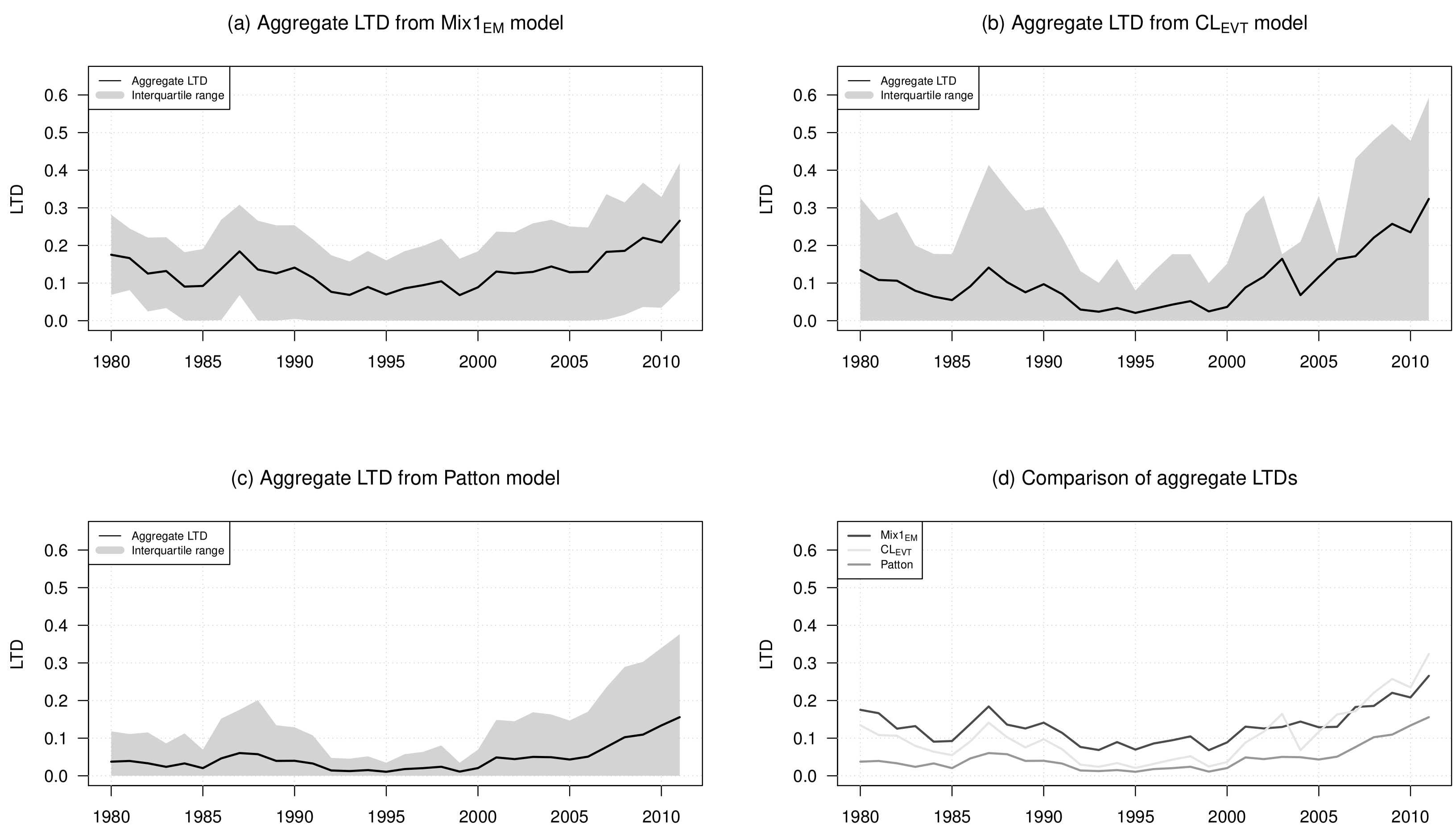
[28] As found in Chabi-Yo et al. (2018), the Mix1 model, consisting of the Joe, F-G-M, and the Rotated-Joe copula, is the most frequently selected convex combination.

[29] Here, we concentrate on those dynamic and static LTD estimators/models that have been used in previous studies on asset pricing to allow for a direct comparison of the results.

[30] Note that Chabi-Yo et al. (2018) find no specific pattern in aggregate LTD and the estimates of aggregate LTD

Figure 3: Aggregate lower tail dependence over time.

The figure depicts the time evolution of aggregate lower tail dependence (LTD) and the range between the 25th and 75th percentile of LTD across the sample estimated from the three tail dependence models employed in our empirical study, including the Mix1_{EM}, CL_{EVT}, and the Patton model. Aggregate LTD is defined as the cross-sectional, equal-weighted average of the individual LTD coefficients computed between stock returns and market returns over all stocks and years in the sample. Our sample encompasses all U.S. common stocks trading on the NYSE, AMEX, and NASDAQ from January 1, 1980 to December 31, 2011.



the panels show considerable differences in the *amount* and *variation* of aggregate LTD across the estimators, with the less sophisticated estimators implying a greater and more volatile amount of tail dependence.

Figure 4 investigates the differences in aggregate LTD across the three tail dependence estimators in more detail. As can be seen from the panels, there are considerable differences between the estimates from the Patton model and the two static models, whereas the differences between the estimates from the Mix1_{EM} and CL_{EVT} model are somewhat less pronounced but still significant. For example, when comparing the CL_{EVT} and Patton model in the lower panel we observe that the estimated tail dependence is more than twice as high for the former estimator than for the latter in most of the years.

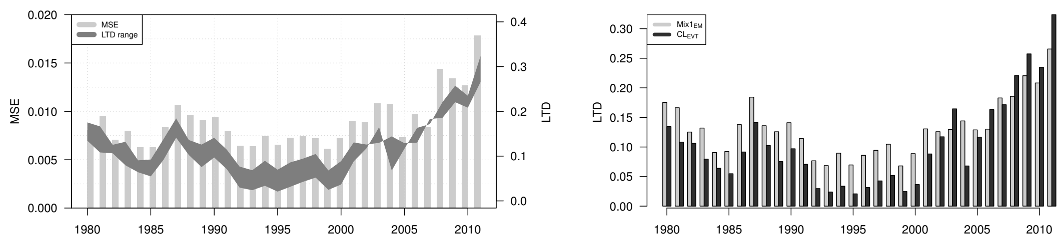
In line with the results from our simulations, neglecting intra-year time dynamics appears to have severe consequences for the tail dependence estimates.

are somewhat more erratic and characterized by occasional spikes.

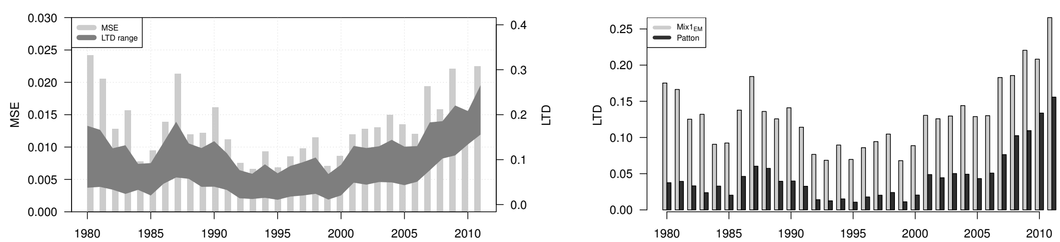
Figure 4: Comparing aggregate lower tail dependence across estimators.

The panels of the figure compare the time evolution of aggregate lower tail dependence (LTD) across the LTD estimators included in our empirical study. Aggregate LTD is defined as the cross-sectional, equal-weighted average of the individual LTD coefficients computed between stock returns and market returns over all stocks and years in the sample. The estimators included in our study comprise the Mix1_{EM}, CL_{EVT}, and the Patton model. The left-hand panels show the range between the aggregate LTD coefficients computed from the different estimators (shaded area) as well as corresponding mean squared errors (MSE, light-gray bars) calculated according to the formula in (17) for each stock and year in the sample. The right-hand panels directly compare the amounts of tail dependence over time by means of bar plots. Our sample encompasses all U.S. common stocks trading on the NYSE, AMEX, and NASDAQ from January 1, 1980 to December 31, 2011.

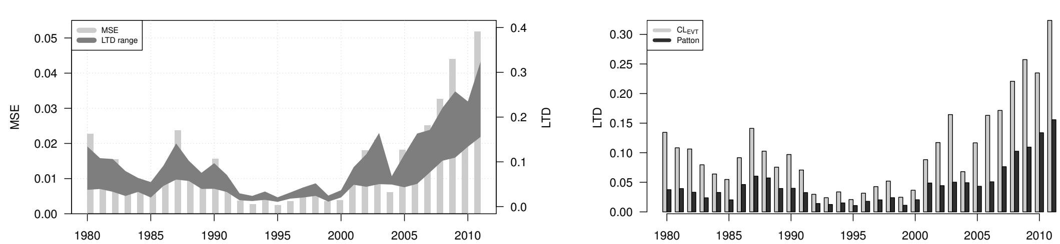
(a) Aggregate LTD: Mix1_{EM} vs. CL_{EVT}



(b) Aggregate LTD: Mix1_{EM} vs. Patton



(c) Aggregate LTD: CL_{EVT} vs. Patton



5 Conclusion

In this paper, we have demonstrated that several estimators of tail dependence used in the literature produce severely biased estimates, especially when static models are used to describe time-varying extreme dependence in data samples. Estimators that do not account for time-varying tail dependence or that are incorrectly used (e.g., using Maximum Likelihood in finite mixture models), and nonparametric estimators regularly overestimate the actual level of tail dependence in simulated samples.

Within financial economics, our empirical findings suggest that several key results from the literature (e.g., Okimoto, 2008; Kang et al., 2010; Garcia and Tsafack, 2011; Chabi-Yo et al., 2018; Ruenzi et al., 2018) need to be treated with care as the actual extreme dependence in asset prices, which often have time-varying dependence structures (see, e.g., Christoffersen et al., 2012, 2018), could be lower than stated. We confirm this conjecture from our Monte Carlo experiments in an empirical analysis of the factors that drive the cross-sectional variation of U.S. stocks between 1980 and 2011. Several estimators of tail dependence that have been extensively used in the previous literature significantly overestimate the level of lower tail dependence inherent in stock returns.

The implications of our article for future investigations into the role of extreme dependence are simple, yet important. Choosing a static, nonparametric, or statistically incorrectly estimated model for measuring extreme dependence in random variables invalidates any conclusions drawn from potential applications. Economic intuition and previous findings in the literature (even from those studies that later on employ static models) state that extreme dependence in most financial data (stock, bond, option, CDS prices) is time-varying. Consequently, future studies in this field need to account for the time-variation in extreme dependence by using sophisticated dynamic models, of which some have been proposed almost a decade ago.

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