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On spectral sequences from Khovanov homology

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There are a number of homological knot invariants, each satisfying an unoriented skein exact sequence, which can be realised as the limit page of a spectral sequence starting at a version of the Khovanov chain complex. Compositions of elementary 1–handle movie moves induce a morphism of spectral sequences. These morphisms remain unexploited in the literature, perhaps because there is still an open question concerning the naturality of maps induced by general movies.

Here we focus on the spectral sequence due to Kronheimer and Mrowka from Khovanov homology to instanton knot Floer homology, and on that due to Ozsváth and Szabó to the Heegaard Floer homology of the branched double cover. For example, we use the 1–handle morphisms to give new information about the filtrations on the instanton knot Floer homology of the $(4,5)$–torus knot, determining these up to an ambiguity in a pair of degrees; to determine the Ozsváth–Szabó spectral sequence for an infinite class of prime knots; and to show that higher differentials of both the Kronheimer–Mrowka and the Ozsváth–Szabó spectral sequences necessarily lower the delta grading for all pretzel knots.

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1 Introduction

Recent work in the area of the 3–manifold invariants called *knot homologies* has illuminated the relationship between Floer-theoretic knot homologies and "quantum" knot homologies. The relationships observed take the form of spectral sequences starting with a quantum invariant and abutting to a Floer invariant. A primary example is due to Ozsváth and Szabó [18], in which a spectral sequence is constructed from Khovanov homology of a knot (with $\mathbb{Z}/2$–coefficients) to the Heegaard Floer homology of the 3–manifold obtained as branched double cover over the knot. A later example is due to Kronheimer and Mrowka, which gives a spectral sequence [12; 13] from Khovanov homology to an instanton knot Floer homology.

There are automatically naturality questions about such spectral sequences. Both the quantum homology and the Floer homology involved exhibit some functoriality with

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respect to link cobordism, and one can ask if the spectral sequences behave well with respect to this functoriality. The project of demonstrating such naturality is important (and is addressed, at least with $\mathbb{Z}/2$–coefficients, in Baldwin, Hedden and Lobb [2]), but in this paper we use the limited naturality already available (essentially naturality for cobordisms presented as a movie of elementary 1–handle additions) to make some computations. The basic idea is that if we are interested in the Floer homology of a knot K, we find a cobordism to a knot K' with a simple spectral sequence and then use the Khovanov homology of K' to draw conclusions on the Floer homology of K.

We are restricting ourselves to the Ozsváth–Szabó and Kronheimer–Mrowka spectral sequences, but the technique should have wider applicability. In the next section we review these Floer homologies; Section 3 then deals with the spectral sequences; and Section 4 contains the computations.

A word of warning: as a matter of notational convenience, our Floer-theoretic invariant of a knot or link or 3–manifold is really what in the literature would be the Floer invariant of the mirror image of a knot or link or 3–manifold. This avoids permanent use of the word "mirror" in the spectral sequences that we study.

1.1 Summary of results

We give three of the results that we deduce in the final section of this paper. We start with a result in instanton knot Floer homology, specifically concerning the spectral sequences due to Kronheimer and Mrowka from reduced Khovanov homology to a flavour of instanton homology. There is no such nontrivial spectral sequence whose structure is entirely known: the filtration on the instanton knot Floer homology is only known for those knots whose spectral sequence collapses at the Khovanov page. The most understood case is that of the torus knot $T(4,5)$, for which the number of possible spectral sequences is known to be at most eight; see [13, Section 11]. We manage to restrict this from eight to two. Specifically we have the following proposition:

Proposition 1.1 *The differential in the Kronheimer–Mrowka spectral sequence for $T(4,5)$ either goes from the generator at bigrading $(2,13)$ to the generator at $(9,16)$ or goes from $(4,13)$ to $(9,16)$.*

(Our bigrading conventions are given in a later section.) This means, for example, that the spectral sequence of $T(4,5)$ corresponding to the quantum filtration has a nontrivial differential either on page 8 or page 10.

Next we turn to a general result that holds for all the spectral sequences under our consideration. It has long been conjectured that the spectral sequence of Ozsváth and Szabó should have differentials strictly lowering the delta grading (this is defined precisely later on); see Greene [6, Conjecture 8.1]. We give a universal proof that this holds for pretzel knots — the proof works for any of the spectral sequences under our consideration. We state it here for the Kronheimer–Mrowka spectral sequences (our filtration conventions are given in Section 2.1) and indicate the extension to the Ozsváth–Szabó spectral sequence in the proof of Theorem 4.5.

Theorem 1.2 *Let $2 \leq p < \min\{q, r\}$. Then for any filtration determined by numbers a and b, the Kronheimer–Mrowka spectral sequence, starting from the reduced Khovanov homology $\mathrm{Khr}(P(-p,q,r))$ and abutting to the instanton knot Floer homology $I^{\natural}(P(-p,q,r))$, can only have nontrivial differentials that strictly lower the δ–grading.*

To be slightly more precise, let E_K be the Khovanov page of the spectral sequence. Then, for any page E_s with $s \geq K$ of this spectral sequence, we have a decomposition $E_s = E_s^u \oplus E_s^l$, where at E_K this is the decomposition into the subspace with the upper and the lower δ–grading, and the differential decomposes as

$$d_s = \begin{pmatrix} 0 & 0 \\ d_s^{ul} & 0 \end{pmatrix} \tag{1}$$

according to this decomposition, and hence induces a δ grading at any page inductively.

To state the theorem in other language, at each page E_s for $s \geq K$ we have that the s–boundaries are contained in E_s^l, and the s–cycles contain E_s^l.

Finally, we mention here a constructive application to the Ozsváth–Szabó spectral sequence. We show that for an infinite class of knots we can determine the (nontrivial) spectral explicitly.

Proposition 1.3 *The Ozsváth–Szabó spectral sequence for the knot $P(-2,3,2n+1)$ is obtained by shifting the spectral sequence for $P(-2,3,5)$ by q^{2n-4} and taking the direct sum with a trivial spectral sequence given by $T_n = E_2 = E_\infty$.*

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2 Review of Heegaard Floer and instanton Floer homology

While Khovanov homology is very simply defined and Heegaard Floer homology for many is a relatively comfortable object, instanton Floer homology is far less known. Therefore we are going to assume familiarity with Khovanov homology and devote the first subsection merely to quoting a result from Heegaard Floer, while the remaining subsections give a review of the relevant instanton Floer homology. We will work with the reduced homology theories.

Remark All our results in this paper will be statements about the respective homology theories with $\mathbb{Q}$–coefficients in instanton Floer homology, and with $\mathbb{Z}/2$–coefficients in Heegaard Floer homology, although some statements should extend over to more general coefficients.

2.1 Khovanov homology, and grading conventions

We are assuming familiarity with reduced Khovanov homology. Given a marked link K with a diagram D we shall denote the reduced Khovanov chain complex by $(C(D), d_{\mathrm{Khr}}(D))$ whose homology is the reduced Khovanov homology $\mathrm{Khr}(K)$ of K. The vector space $C(D)$ is a bigraded complex $(C(D)^{i,j})$, where i denotes the *homological* and j denotes the *quantum* grading. The differential d_{Khr} is bigraded of degree $(1,0)$. The two gradings also define a descending filtration $\mathscr{F}^{i,j}C(D)$ indexed by $\mathbb{Z}\oplus\mathbb{Z}$. With respect to these filtrations a morphism $\phi\colon C(D)\to C(D')$ is said to be of order $\geq (s,t)$ if $\phi(\mathscr{F}^{i,j}C(D))\subseteq \mathscr{F}^{i+s,j+t}C(D')$, where D' is possibly a different diagram, but not necessarily so.

We follow the standard convention that gives the reduced Khovanov chain complex as a subcomplex of the Khovanov chain complex. This has the unfortunate effect that the reduced Khovanov homology of the unknot is one copy of the ground ring (for us either $\mathbb{Q}$ or $\mathbb{Z}/2$) supported in gradings $i=0$ and $j=-1$ (where one might think $j=0$ more natural). Nevertheless this brings us in line with most current usage.

Finally, we recall that the *δ–grading* in Khovanov homology is defined to be $\delta = j-2i$ in terms of the homological grading i and the quantum grading j.

2.2 Heegaard Floer homology

We are concerned with $\widehat{\mathrm{HF}}$, the "hat" version of Heegaard Floer homology [17]. This is an invariant of a closed 3–manifold equipped with a Spin^c–structure and takes the form of a finitely generated vector space over $\mathbb{Z}/2$. We are interested in 3–manifolds $\Sigma(L)$ that are obtained as branched double covers over the mirror images of links $L \subset S^3$, and, taking the sum over all Spin^c–structures, we regard $\widehat{\mathrm{HF}}$ simply as a vector space.

Theorem 2.1 (Ozsváth and Szabó [18]) *Given a link $L \subset S^3$, there is a spectral sequence* (which a priori depends on a choice of link diagram) *abutting to* $\widehat{\mathrm{HF}}(\Sigma(L))$ *with E_1 page equal to the reduced Khovanov chain complex and E_2 page equal to the reduced Khovanov homology* $\mathrm{Khr}(L)$ (where everything has been taken with $\mathbb{Z}/2$–coefficients).

In general this theorem implies that the rank of $\widehat{\mathrm{HF}}(\Sigma(K))$ is bounded above by the rank of $\mathrm{Khr}(K)$.

For a knot K the number of Spin^c structures on $\Sigma(K)$ is equal to $|\det(K)|$, from which by an Euler characteristic argument it follows that the rank of $\widehat{\mathrm{HF}}(\Sigma(K))$ is bounded below by $|\det(K)|$, and, when this bound is tight, $\Sigma(K)$ is called an L–space. It is a quick check that if K is a knot with thin Khovanov homology then the rank of $\mathrm{Khr}(K)$ is exactly $|\det(K)|$ and hence the spectral sequence collapses at the E_2 page.

Computations of nontrivial spectral sequences for specific prime knots were given by Baldwin [1], and he observed that the spectral sequences he found had differentials that strictly decreased the δ–grading on Khovanov homology. Later in this paper we extend Baldwin's examples to an infinite class of prime knots and furthermore show that the Ozsváth–Szabó spectral sequence has differentials which strictly decrease the δ–grading for all pretzel knots.

2.3 Instanton knot Floer homology

Instanton knot Floer homology as constructed by Kronheimer and Mrowka [11; 12; 13] is an invariant of pairs consisting of links in 3–manifolds. In the manifestation that interests us, we shall be restricting our attention to the case of knots and links inside the 3–sphere $K \subset S^3$.

Reduced instanton knot Floer homology $I^\natural(K)$ of a link K with a marked component in the 3–sphere S^3 is, roughly speaking, defined via the Morse homology of a Chern–Simons functional on a space of connections that have a prescribed asymptotic holonomy around the link K [13]. It is an abelian group with an absolute $\mathbb{Z}/4$–grading [12] (usually instanton Floer homology comes with relative $\mathbb{Z}/4$–gradings, but in [12, Sections 4.5 and 7.4] absolute gradings are given). We denote by $(C(K)^\natural, d^\natural)$ the $\mathbb{Z}/4$–graded complex whose homology is $I^\natural(K)$. The differential $d^\natural$ lowers the $\mathbb{Z}/4$–grading by 1. Involved in the construction of this complex are various choices of perturbations one has made, but we have suppressed these in the notation as our computations will not use the definition.

Remark For a matter of notation we denote by $I^\natural(K)$ what Kronheimer and Mrowka denote as $I^\natural(\overline{K})$, the reduced instanton Floer homology of the mirror image of K.

Kronheimer and Mrowka have shown in [12; 13] that this admits a definition in which the underlying chain complex has a description via a cube of resolutions, just as in the case of Khovanov homology. Furthermore, the differentials have close ties with those of the Khovanov complex, as we shall now recall.

Theorem 2.2 (Kronheimer and Mrowka) *Let D be a diagram of a knot or link K in S^3. When working with Khovanov homology and instanton Floer homology with $\mathbb{Z}$–coefficients we have the following*:

(i) *There is a differential $d_\natural(D)$ on the module $C(D)$ whose homology is isomorphic to the reduced instanton knot Floer homology $I^\natural(K)$. More precisely*, *the bigrading (i,j) gives a $\mathbb{Z}/4$–grading on $C(D)$ by $j-i-1 \bmod 4$. The differential $d_\natural(D)$ lowers this $\mathbb{Z}/4$–grading by 1. With these gradings*, *there is a quasi-isomorphism of $\mathbb{Z}/4$–graded chain complexes $(C(K)^\natural, d^\natural) \to (C(D), d_\natural(D))$.*

(ii) *The difference $d_\natural(D) - d_{\mathrm{Khr}}(D)$ is filtered of order $\geq (1,2)$.*

This statement appears, with the exception of the explicit identification of the $\mathbb{Z}/4$–grading, as [13, Theorem 1.1] for the *unreduced* theory which computes the unreduced instanton knot Floer homology $I^\#(K)$ from unreduced Khovanov homology. The remark after Proposition 1.5 in the same reference states that exactly the same statement also holds for the *reduced* theory, realising $I^\natural(K)$ as the final page of a spectral sequence starting at $\mathrm{Khr}(K)$.

With the exception of the additional statement about the filtration coming from the bigrading, Theorem 1.1 of [13] is a consequence of [12, Theorem 6.8, Corollary 6.9 and Proposition 7.8]. For the sake both of completeness and exposition, we briefly review how this result is derived by Kronheimer and Mrowka.

Theorem 6.8 in [12] is essentially an iterated skein exact triangle.

If we inscribe a regular tetrahedron inside a 3–ball, each pair of opposite edges yields the three "skein configurations" of unknotted edges. One then starts with a knot or link K in a 3–manifold Y with n chosen balls where each ball intersects K in an unknotted pair of strands corresponding to one of the skein configurations. One may think of these as "crossing balls".

One then considers the knots obtained from K by replacing the skein configuration inside the n 3–balls with one of the other two skein configurations.

Given n crossing balls, there are 2^n configurations one can derive from K through crossing changes, where any crossing is resolved in one of the two possible ways (denoted by K_0 or K_1 in Figure 1 in the case of a single crossing, with $K = K_2$). These can be thought of as sitting on the vertices $\{0,1\}^n$ of an n–dimensional cube $[0,1]^n$. One denotes by K_u the resolution of K corresponding to $u \in \{0,1\}^n$. There is a natural cobordism S_{vu} (which is a product cobordism outside the crossing balls) between any two of them, where $u, v \in \{0,1\}^n$ denote the crossing change data. (But motivated by Khovanov homology, one just considers cobordisms if $v \geq u$ later on in the construction.)

To any resolution K_u there is an associated singular instanton knot homology group $I^\omega(Y, K_u)$. Here ω is an arc from some component of K to some other, and the bundle in the construction is chosen to have a second Stiefel–Whitney class Poincaré dual to ω. (This is to avoid reducible connections.)

Kronheimer and Mrowka define the module

$$\boldsymbol{C} := \bigoplus_{u \in \{0,1\}^n} \boldsymbol{C}_u$$

"on the cube", where $\boldsymbol{C}_u := I^\omega(Y, K_u)$, together with a differential $\boldsymbol{F}$ constructed as follows. A cobordism S_{vu}, in which k crossings are changed, comes with a $(k-1)$–dimensional family of metrics. A map

$$m_{vu}\colon \boldsymbol{C}_v \to \boldsymbol{C}_u$$

is defined by making a count of a 0–dimensional moduli space of antiselfdual connections singular along the cobordism S_{vu}, and parametrised by the $(k-1)$–dimensional family of metrics. This map m_{vu} is then corrected by a sign to produce a map $f_{vu}\colon C_v \to C_u$. The map $\boldsymbol{F}\colon \boldsymbol{C} \to \boldsymbol{C}$ is then defined as the direct sum of the maps f_{vu}. In this setup, $\boldsymbol{F}^2 = 0$, so $(\boldsymbol{C}, \boldsymbol{F})$ is a complex.

Kronheimer and Mrowka then prove that the homology of this complex is isomorphic to the singular instanton knot Floer homology $I^\omega(Y, K)$ of the original knot or link

$K \subseteq Y$ one has started with,

$$I^{\omega}(Y, K) \cong H((\boldsymbol{C}, \boldsymbol{F})). \tag{2}$$

The proof is by induction on the number of crossings and quite involved. For a single crossing, this reduces to a long exact sequence induced by the mapping cone of a skein cobordism. On the algebraic side it makes use of the *exact triangle detection lemma* first appearing in Ozsváth and Szabó's work [18] as Lemma 4.2 for $\mathbb{Z}/2$–coefficients, and as [12, Lemma 7.1] for $\mathbb{Z}$–coefficients.

This general approach is applied to the situation in Theorem 2.2 as follows: One starts with an n–crossing diagram D of a knot or link $K \subseteq S^3$. Neighborhoods of these n crossings of D appear as the crossing balls in the description above.

In order to make the singular instanton knot Floer homology well defined, one needs a suitable SO(3)–bundle to work with. To this end, one adds the boundary of a meridional disk of some component of K, resulting in a link denoted by $K^{\natural}$; see for instance [12, Figure 2]. Then the arc ω is chosen to be a radial arc in the chosen meridional disk from K to the boundary of this meridional disk, which we call an *earring*.

In this setup, all 2^n resolutions of $K^{\natural}$ result in unlinks, one of whose components carries an earring (and this 2–component link is a Hopf link). If $u \in \{0,1\}^n$ labels a possible resolution of $K^{\natural}$, then $I^{\omega}(S^3, K^{\natural}_u)$ is identified with the same $\mathbb{Z}$–module that appears in the complex constructing reduced Khovanov homology. This is shown in [12, Section 8], and in particular in Section 8.7 of this reference for the reduced theory. Hence, in the above notation, $(\boldsymbol{C}, \boldsymbol{F})$ is a complex with the same underlying $\mathbb{Z}$–module structure as Khovanov homology, which we have denoted by $C(D)$ in the statement of Theorem 2.2. The differential $\boldsymbol{F}$ is denoted by $d_{\natural}(D)$ since it a priori depends on the chosen diagram through the crossing regions.

The differential $d_{\natural}(D)$ is filtered with respect to the homological degree in Khovanov homology (equal to $|u|_1$, the l^1–norm, at the vertex $u \in \{0,1\}^n$). To the leading homological order, $d_{\natural}(D)$ is equal to the (reduced) Khovanov differential. This is shown in [12, Section 8]. Hence, in the spectral sequence associated to this homological filtration, the E^2 page is identified with reduced Khovanov homology. The more refined statement with respect to the bidegree is the content of [13]. Finally, the statement about the $\mathbb{Z}/4$–grading appears as [12, Proposition 7.8].

As a consequence of the second point in Theorem 2.2, both the homological and the quantum filtrations on the Khovanov complex induce a filtration on the instanton knot

Floer homology. This is a novum of [13] compared to [12]. A priori these filtrations might depend on the chosen diagram, but it is not the case. In fact, Kronheimer and Mrowka have shown that the induced filtrations are invariants of the link K [13, Theorem 1.2 and Corollary 1.3], therefore yielding the following result:

Theorem 2.3 [13, Theorem 1.2 and Corollary 1.3] *Let K be a link in S^3 and let D be a diagram of K. Let $a, b \geq 1$. The descending filtrations induced by $ai + bj$ on $C(D)$ is preserved by $d_\natural(D)$, and the induced filtration on $I^\natural(K)$ depends on the link K only. The pages of the associated Leray spectral sequence (E_r, d_r), converging to $I^\natural(K)$, are invariants of K for $r \geq a+1$. There are no differentials before the E_a page, and the page E_{a+1} is the reduced Khovanov homology of K.*

For instance, the homological filtration induces a spectral sequence abutting to $I^\natural(K)$ whose E_2 page is Khovanov homology, and the quantum filtration induces a spectral sequence whose E_1 page is Khovanov homology.

The statement in the last sentence is not explicit in [13] but is easily checked from Kronheimer and Mrowka's Theorem 2.2.

2.4 The Alexander polynomial

In [11], Kronheimer and Mrowka developed an instanton Floer homology of sutured manifolds, yielding a $\mathbb{Z}/4$–graded link homology group $\mathrm{KHI}(K)$ of a link K. Kronheimer and Mrowka [10], and independently Lim [14], show that this is related to the Alexander polynomial. In fact, $\mathrm{KHI}(K)$ carries two commuting operators whose common eigenspace decompositions give $\mathrm{KHI}(K)$ a $(\mathbb{Z}\oplus\mathbb{Z}/2)$–grading. For a knot K and working with rational coefficients, the "graded Euler characteristic" of $\mathrm{KHI}(K;\mathbb{Q})$ is equal to minus the Conway-normalised Alexander polynomial Δ_K [10, Theorem 1.1]:

$$-\Delta_K(t) = \sum_{h\in\mathbb{Z},\, i\in\mathbb{Z}/2} (-1)^i t^h \dim(\mathrm{KHI}^{i,h}(K;\mathbb{Q})). \tag{3}$$

On the other hand, by [12, Proposition 1.4] there is an isomorphism between the rational instanton homology $I^\natural(K;\mathbb{Q})$ and the rational sutured instanton homology of the knot complement, $\mathrm{KHI}(K;\mathbb{Q})$. The following proposition is then immediate and was used by Kronheimer and Mrowka in [13, Section 11] when considering $I^\natural(T(4,5);\mathbb{Q})$ for $T(4,5)$ the $(4,5)$ torus knot.

Proposition 2.4 *The dimension of $I^\natural(K;\mathbb{Q})$ is bounded below by the sum of the absolute values of the coefficients of the Alexander polynomial Δ_K.*

2.5 Thin Khovanov homology

The reduced Khovanov homology of an oriented link L is a bigraded vector space over the rational numbers $\mathrm{Khr}(L)$ which categorifies the Jones polynomial $V_L(q)$, normalised such that for the unknot U one has $V_U(q) = q^{-1}$. More precisely, one has the formula

$$\sum_{i,j \in \mathbb{Z}} (-1)^i q^j \operatorname{rk}(\mathrm{Khr}^{i,j}(L)) = V_L(q); \tag{4}$$

see for instance [9].

A link L is said to have *thin* Khovanov homology if all nontrivial vector spaces $\mathrm{Khr}^{i,j}(L)$ occur on one line where $j - 2i$ is constant. Kronheimer and Mrowka have shown that their spectral sequence from reduced Khovanov homology $\mathrm{Khr}(K)$ to reduced instanton knot Floer homology $I^\natural(K)$ has no nontrivial differential after the Khovanov page if K is a quasialternating knot; see [12, Corollary 1.6]. Their result can easily be strengthened a little bit.

Proposition 2.5 *Suppose that K is a knot that has thin reduced Khovanov homology. Then the Kronheimer–Mrowka spectral sequence $\mathrm{Khr}(K) \Rightarrow I^\natural(K)$ has no nonzero differential (over $\mathbb{Q}$). The total rank of $\mathrm{Khr}(K)$ and $I^\natural(K)$ then agree with the determinant of K given by $|\Delta_K(-1)| = |V_K(-1)|$.*

Proof Suppose the reduced Khovanov homology $\mathrm{Khr}^{i,j}(K)$ of K is supported on the line $j = 2i + s$ for some even integer s. Then from formula (4) above it follows that

$$\begin{aligned} V_K(-1) = V_K(\sqrt{-1}^{\,2}) &= \sum_{i \in \mathbb{Z}} (-1)^i (-1)^{i+s/2} \operatorname{rk}(\mathrm{Khr}^{i,2i+s}(K)) \\ &= (-1)^{s/2} \operatorname{rk}(\mathrm{Khr}(K)). \end{aligned}$$

Therefore, the determinant is equal to the total rank of the reduced Khovanov homology of K. On the other hand, Proposition 2.4 above gives the same lower bound. As therefore the rank of reduced Khovanov homology $\mathrm{Khr}(K)$ and $I^\natural(K)$ have to coincide, there is no nonzero differential in the spectral sequence. □

An analogous result holds in Heegaard Floer homology: knots with thin Khovanov homology have branched double covers which are Heegaard Floer L–spaces.

As a consequence of the spectral sequence, the total rank of Khovanov homology provides an upper bound for the rank of instanton homology. In the case where these

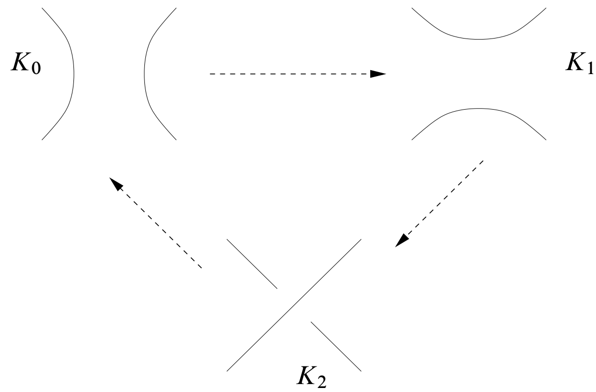


Figure 1: The links K_0, K_1 and K_2 form an unoriented skein triple.

ranks agree, all the information about the filtration on instanton homology is contained in Khovanov homology.

2.6 Unoriented skein exact triangles

Both Khovanov homology and instanton knot Floer homology have unoriented skein exact triangles, of which we shall make extensive use in our computational section. The statement for instanton Floer homology is a corollary of [12, Theorem 6.8]; see the paragraph after the statement of Theorem 6.8 in this reference. The cobordisms inducing the maps in knot Floer homology are the subject of [12, Section 6.1], while the maps themselves are constructed in [12, Section 6.2].

The statement of the skein exact sequence in Khovanov homology is already present in [8].

Proposition 2.6 *Suppose K_0, K_1 and K_2 are three links with diagrams D_0, D_1 and D_2, respectively, that look the same except near a crossing of D_2, where they differ as in Figure 1. Then there is a long exact triangle relating the groups $I^\natural(K_0)$, $I^\natural(K_1)$ and $I^\natural(K_2)$, and likewise for the reduced Khovanov homology groups $\mathrm{Khr}(K_0)$, $\mathrm{Khr}(K_1)$ and $\mathrm{Khr}(K_2)$:*

$$\begin{tikzcd}[column sep=small] I^\natural(K_0) \arrow[rr] & & I^\natural(K_1) \arrow[dl] \\ & I^\natural(K_2) \arrow[ul] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] \mathrm{Khr}(K_0) \arrow[rr] & & \mathrm{Khr}(K_1) \arrow[dl] \\ & \mathrm{Khr}(K_2) \arrow[ul] & \end{tikzcd}$$

All maps are induced by standard cobordisms corresponding to 1–handle attachment in both theories.

2.7 Knot cobordisms and functoriality

The instanton knot Floer homology $I^\natural(K)$ groups are functorial for decorated knot and link cobordisms [12]. Given two decorated oriented embedded links K_0 and K_1, and an isotopy class (rel boundary) of decorated cobordism $S \subseteq [0,1]\times S^3$ from $K_0 \subseteq \{0\}\times S^3$ to $K_1 \subseteq \{1\}\times S^3$, there is an induced morphism $I^\natural(S)\colon I^\natural(K_0) \to I^\natural(K_1)$ which is well defined up to an overall sign. Furthermore, the morphism induced by a composite decorated cobordism is the composition of the morphisms of the decorated cobordisms.

The decorations referred to consist of basepoints of the boundary links together with normal directions at the basepoints and a path on the cobordism between the basepoints also equipped with a normal direction at each point on the path. These decorations are important to make sense of functoriality — for example a *module* should be associated to a decorated link, rather than just an *isomorphism class* of module. For the computational results of this paper, however, it will be enough to identity the rank and nullity of maps induced by cobordism (when working over a field), instead of determining the maps completely.

We make use of the following proposition:

Proposition 2.7 [13, Proposition 1.5] *Let S be a cobordism from a link K_0 to a link K_1. Let D_0 and D_1 be diagrams for K_0 and K_1. Then the map $I^\natural(S)\colon I^\natural(K_0) \to I^\natural(K_1)$ is induced by a chain map $c\colon C(D_0) \to C(D_1)$ which has order*

$$\geq \left(\tfrac{1}{2}(S\cdot S), \chi(S) + \tfrac{3}{2}(S\cdot S)\right),$$

where $\chi(S)$ denotes the Euler characteristic of S and $S\cdot S$ denotes the self-intersection number of S with the boundary condition that a pushoff at the ends is required to have linking number 0 with K_0, respectively K_1.

In general, a movie M between diagrams D_0 and D_1 consisting of 0–, 1– and 2– handle attachments, and of Reidemeister moves, induces a cobordism S_M between the corresponding knots K_0 and K_1. Such a movie induces a morphism $c(M)\colon C(D_0) \to C(D_1)$ between the corresponding Khovanov complexes by composing the Khovanov morphisms from handle attachments and the chain homotopy equivalences coming from the Reidemeister moves in the respective order. In particular, there is a resulting map $\mathrm{Khr}(M)$ from the Khovanov homology of K_0 to K_1.

3 Constraints on Floer homology

In this section we show how conclusions on the Kronheimer–Mrowka or Ozsváth–Szabó spectral sequences for a specific knot or link might be made from link cobordisms.

3.1 Morphisms of spectral sequences

Given two spectral sequences (E_r, d_r) and (E'_r, d'_r), a collection of morphisms $(f_r\colon E_r \to E'_r)$ is said to be a morphism of spectral sequences if

- for any r the morphism f_r is a morphism of chain complexes from the complex (E_r, d_r) to (E'_r, d'_r), ie f_r intertwines the differentials d_r and d'_r, and
- the morphism f_{r+1} is the morphism induced by f_r on homology under the isomorphisms $H(E_r, d_r) \cong E_{r+1}$ and $H(E'_r, d'_r) \cong E'_{r+1}$ for any $r \in \mathbb{N}$.

For instance, having filtered complexes (C, d) and (C', d'), filtered by families $(\mathscr{F}^n C)_n$ and $(\mathscr{G}^n C')_n$, and a morphism of chain complexes $f\colon C \to C'$ that respects the filtrations — meaning that $f(\mathscr{F}^n C) \subseteq \mathscr{G}^n C'$ for all n — the map f induces a morphism between the two spectral sequences coming from the filtrations.

Definition 3.1 We say that an element $x \in E_s$ is an s–boundary if x is in the image of d_s, and we say that x is an s–cycle if $d_s(x) = 0$. We say that an element $x \in E_s$ is an ∞–cycle if x lies in the kernel of d_s and its homology class $[x]_t$ is a $(t+1)$–cycle for all $t \geq s$.

Lemma 3.2 *Let $(f_r)\colon (E_r, d_r) \to (E'_r, d'_r)$ be a morphism of spectral sequences.*

(i) *If $x \in E_s$ is an s–cycle then $f_s(x) \in E'_s$ is an s–cycle.*

(ii) *If $x \in E_s$ is an s–boundary then $f_s(x) \in E'_s$ is an s–boundary.*

(iii) *If $x \in E_s$ is an ∞–cycle then $f_s(x) \in E'_s$ is an ∞–cycle.*

Proof The result follows from the fact that morphisms of chain complexes preserve cycles and boundaries. □

A chain map $c\colon C(D_0) \to C(D_1)$ as in Proposition 2.7 respects the filtrations by $\mathbb{Z} \oplus \mathbb{Z}$ on the two complexes $C(D_0)$ and $C(D_1)$ up to a global shift. Therefore, such a chain map induces a graded morphism of spectral sequences $(c_r)\colon (E_r(D_0)) \to (E_r(D_1))$, where $(E_r(D_i))$ is the spectral sequence converging to the Floer homology group $I^\natural(K_i)$. The grading of this graded morphism agrees with the global shift.

In the spectral sequences $E_*(D_i)$, each page after the Khovanov pages is a topological invariant, ie depends on the links K_i only. In the proof of Proposition 2.7 above in [13], the chain map c is in fact obtained by representing the cobordism S by a movie M between diagrams D_0 and D_1 for K_0 and K_1, respectively, and by then checking the claim for the map induced on reduced instanton knot Floer homology by the particular handle and Reidemeister moves.

One is tempted to believe that at the Khovanov page, the corresponding morphism $c(M)$ between the instanton Floer chain complexes as in the last proposition is just equal to the map $\mathrm{Khr}(M)\colon \mathrm{Khr}(K_0) \to \mathrm{Khr}(K_1)$ in Khovanov homology. In fact, such a functoriality property remains open in [13]. What we *can* say, however, is that there is such a result in a particular situation.

Proposition 3.3 *Let D_0 and D_1 be diagrams of knots K_0 and K_1. Let S be a cobordism from K_0 to K_1 that is represented by a movie M between the diagrams D_0 and D_1. Let us assume this movie consists only of isotopies of the diagrams (outside of balls containing the crossings) and handle attachment of index 1 (excluding Reidemeister moves). Then the map $I^\natural(S)\colon I^\natural(K_0) \to I^\natural(K_1)$ is induced by a morphism of chain complexes*

$$c(M)\colon (C(D_0), d_\natural(D_0)) \to (C(D_1), d_\natural(D_1))$$

respecting the bifiltration by $\mathbb{Z} \oplus \mathbb{Z}$, and this morphism induces the map

$$\mathrm{Khr}(M)\colon \mathrm{Khr}(K_0) \to \mathrm{Khr}(K_1)$$

at the Khovanov page of the Kronheimer–Mrowka spectral sequence.

Proof This is, in the language of [2], asking that the spectral sequence satisfy the second condition required of a Khovanov–Floer theory. An argument for this, valid for arbitrary coefficients, is given in Proposition 5.2 of [2]. □

The corresponding result for the Ozsváth–Szabó spectral sequence seems to be known and follows the same line of argument.

Proposition 3.4 *Let D_0, D_1, K_0, K_1, S and M be as above. Working now of course with $\mathbb{Z}/2$–coefficients, the map $\widehat{\mathrm{HF}}(\Sigma(S))\colon \widehat{\mathrm{HF}}(K_0) \to \widehat{\mathrm{HF}}(K_1)$ is induced by a morphism of filtered chain complexes inducing the map $\mathrm{Khr}(M)\colon \mathrm{Khr}(K_0) \to \mathrm{Khr}(K_1)$ at the E_2 page of the Ozsváth–Szabó spectral sequence.* □

Suppose now that S is a link cobordism between K_0 and K_1 with only index 1 critical points. In fact, it is not too hard to see that there exist diagrams D_0 and D_1 and a movie M (presenting K_0, K_1 and S) which satisfy the requirements of the propositions above. For those wanting details of how to construct such a movie M we refer them to the proof of Theorem 1.6 of [15].

For knots K_0, K_1 and K_2 related by the unoriented skein moves via

$$\begin{tikzcd} \mathrm{Khr}(K_0) \arrow[rr] & & \mathrm{Khr}(K_1) \arrow[dl] \\ & \mathrm{Khr}(K_2) \arrow[ul] & \end{tikzcd}$$

the maps in the long exact sequence on Khovanov homology are each induced by some 1–handle attachment up to Reidemeister-isomorphism. Applying Proposition 3.3 in this case gives us the following:

Proposition 3.5 *Suppose we are given knots or links K_0, K_1 and K_2 that only differ inside a ball by the unoriented skein moves; then there are obvious cobordisms S_{01} from K_0 to K_1, S_{12} from K_1 to K_2, and S_{20} from K_2 to K_0 such that each S_{ij} has a single critical point of index 1. Then, fixing $i \in \{0,1,2\}$, we can arrange that the map $I^\natural(S_{i,i+1})\colon I^\natural(K_i) \to I^\natural(K_{i+1})$ is induced by a filtered map on chain complexes*

$$c\colon (C(D_i), d_\natural(D_i)) \to (C(D_{i+1}), d_\natural(D_{i+1})),$$

with D_i and D_{i+1} being diagrams for K_i and K_{i+1}, and such that the induced morphism between the resulting Kronheimer–Mrowka spectral sequences fits into an exact triangle at the Khovanov page relating $\mathrm{Khr}(K_0)$, $\mathrm{Khr}(K_1)$ and $\mathrm{Khr}(K_2)$.

Proposition 3.6 *The analogue of the previous proposition holds for the Ozsváth–Szabó spectral sequence as well.* □

4 Computations

Essentially most of the arguments in this section progress by finding cobordisms between knots whose spectral sequences we wish to know and knots whose spectral sequences are necessarily trivial after the Khovanov page.

4.1 The $(4,5)$ torus knot and instanton homology

In this subsection we work exclusively with $\mathbb{Q}$–coefficients.

In [13, Section 11] the example of the $(4,5)$ torus knot $T(4,5)$ is analysed and it is determined that there is exactly one nontrivial differential in the spectral sequence after the Khovanov page. Furthermore, it is shown that this differential would cancel exactly one of eight explicit pairs of generators in the Khovanov homology.

The technique in this paper almost determines the spectral sequence completely: we are able reduce the number of possible pairs to two, enabling us to give the filtration on $I^\natural(K)$ in almost all degrees. More precisely, after proving Proposition 1.1 we can write $I^\natural(K) = V^6 \oplus W^1$ where V and W are filtered vector spaces of dimensions 6 and 1, respectively, and we know the filtration on V completely and there are two possibilities for the filtration on W.

We note that none of the techniques currently available to constrain the filtered instanton homology (our technique included) discriminates between the various filtrations corresponding to choices $(a,b) \in \mathbb{Z} \oplus \mathbb{Z}$ for $a,b \geq 1$. A priori it is possible that different choices of (a,b) give different spectral sequences, and so it may be the case that, of the two possible canceling pairs in the spectral sequence for $T(4,5)$, each pair does in fact occur for different choices of filtration.

In the plot below we show the reduced Khovanov homology over $\mathbb{Q}$ of $T(4,5)$ as the solid discs. The horizontal axis is the homological grading i, and we follow Kronheimer and Mrowka in making the vertical coordinate $j-i$, where j is the quantum grading. In [13] it is shown that there is exactly one nontrivial differential in their spectral sequence.

Using the $\mathbb{Z}/4$–grading on $I^\natural(K)$, Kronheimer and Mrowka showed that this differential will go from one of the three generators on line $j-i=13$ to one of the three generators on the line $j-i=16$. The exception is that a differential from $(6,13)$ to $(5,16)$ is impossible. In fact, as quoted in Theorem 2.2, the differential $d_\natural$ that computes the instanton Floer homology $I^\natural$ from a resolution cube in the spectral sequence preserves the descending quantum filtrations. Hence there are a priori eight possible differentials.

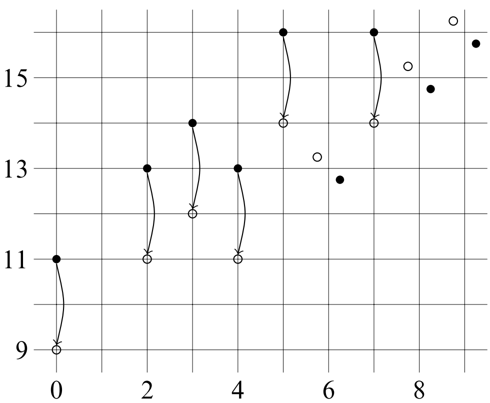
We now turn to the proof of Proposition 1.1, which we break into two lemmas.

Lemma 4.1 *The generators at bigradings $(5,16)$ and $(7,16)$ are never boundaries after the Khovanov page in the spectral sequence. Hence, since we know there is exactly one nontrivial differential, the generator at $(9,16)$ must be the boundary.*

Proof We consider the genus 1 knot cobordism Σ obtained as follows. First express $T(4,5)$ as a braid closure. Changing the sign of a crossing between the first two strands of the braid gives a knot, which we shall call K. Using time as the second coordinate, we have a cylinder embedded in $S^3 \times [0,1]$ with a single point of self-intersection which has boundary $T(4,5) \subset S^3 \times \{0\}$ and $K \subset S^3 \times \{1\}$. Replacing the point of self-intersection with a piece of genus gives a knot cobordism Σ between $T(4,5)$ and K.

We observe by computer calculation that the rank (over $\mathbb{Q}$) of the reduced Khovanov homology of K is 9 and the sum of the absolute values of the Alexander polynomial of K is also 9. Hence the Kronheimer–Mrowka spectral sequence associated to K collapses at the Khovanov page by Proposition 2.4.

In this plot we show the reduced Khovanov homology over $\mathbb{Q}$ of $T(4,5)$ and of K:



The discs correspond to generators of the homology of $T(4,5)$, the circles to generators of the homology of K. For the axes we take our conventions from [13, Section 11]. Kronheimer and Mrowka use the convention that the reduced Khovanov homology of the unknot is supported in bidegree $(i,j) = (0,-1)$, and in this plot the horizontal axis is i, while the vertical axis is $j-i$.

The cobordism Σ being oriented, it induces a map on the Khovanov homology which preserves the homological grading and lowers the quantum grading by 2. Hence the rank of this map is at most 6, and we have drawn the 6 possibly nonzero components of this map.

We split the cobordism Σ into the composition of two cobordisms Σ_1 and Σ_2 where Σ_1 is a cobordism obtained by adding a 1–handle to $T(4,5)$ to obtain a 2–component link L, and where Σ_2 is obtained by adding a 1–handle to L to obtain the knot K. We can think of L as being obtained by taking the vertical smoothing of a crossing between the first two strands of a standard braid presentation of $T(4,5)$. When we replace the crossing in question by the horizontal smoothing we obtain the trefoil knot.

So we can use Proposition 3.5 to see that there exists a movie presentation of Σ inducing a morphism of Kronheimer–Mrowka spectral sequences that at the Khovanov page is the composition of two maps

$$\mathrm{Khr}(T(4,5)) \xrightarrow{\mathrm{Khr}(\Sigma_1)} \mathrm{Khr}(L) \xrightarrow{\mathrm{Khr}(\Sigma_2)} \mathrm{Khr}(K),$$

each of which has cone equal to the reduced Khovanov homology of the trefoil. The rank of the reduced Khovanov homology of the trefoil knot is 3, hence if the rank of $\mathrm{Kh}(L)$ is $6+2b$ then the ranks of the maps $\mathrm{Khr}(\Sigma_1)$ and $\mathrm{Khr}(\Sigma_2)$ are both $6+b$ since

$$3 = \mathrm{rk}\big(\mathrm{Cone}(\mathrm{Khr}(\Sigma_i))\big) = 9+6+2b-2\,\mathrm{rk}(\mathrm{Khr}(\Sigma_i)).$$

Hence the rank of $\mathrm{Khr}(\Sigma)$ is at least $(6+b)+(6+b)-(6+2b)=6$, but we have already seen that the rank is at most 6.

So we see by Lemma 3.2 that, since the generators at $(5,16)$ and $(7,16)$ are mapped nontrivially under $\mathrm{Khr}(\Sigma)$, they are never boundaries after the Khovanov page, and

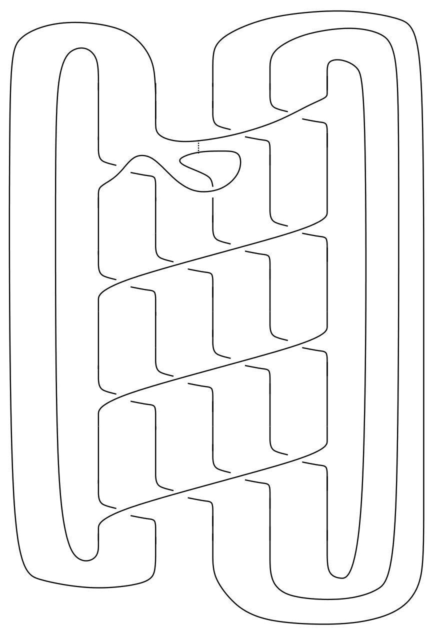


Figure 2: The dotted line represents a blackboard-framed 1–handle attachment which gives a knot cobordism from the knot 5_2 to $T(4,5)$.

hence they survive the spectral sequence. Thus the generator at $(9, 16)$ is the target of some nonzero differential. □

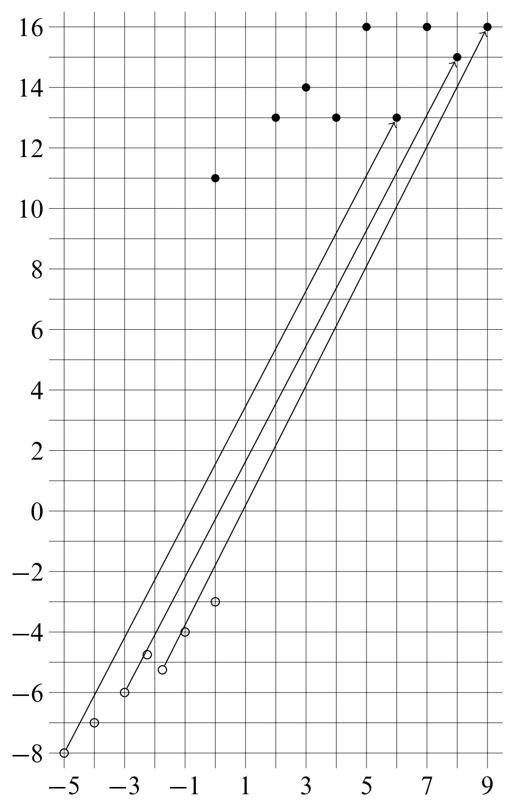
Next we try to narrow down the possible generators from which the differential of the spectral sequence emerges.

Lemma 4.2 *The generator at bigrading* $(6, 13)$ *on the Khovanov page is an* ∞*–cycle.*

Proof There is a cobordism topologically equivalent to a punctured Möbius band from the knot 5_2 in Rolfsen's knot table to $T(4, 5)$. This is presented as a single 1–handle attachment in Figure 2.

This 1–handle attachment induces a morphism of Kronheimer–Mrowka spectral sequences, which we are again able to compute explicitly on the Khovanov page.

On the Khovanov page the map raises the homological grading by 11 and raises the quantum grading by 32. The possible nonzero components of this map are shown below:



In fact, each component is nonzero, since the Khovanov homology of the cone (again, computed from the unoriented skein exact sequence) has homological width 2.

Since 5_2 is alternating, the spectral sequence has only trivial differentials past the Khovanov page. This implies, again by Lemma 3.2, that the generator of the Khovanov homology of $T(4,5)$ that occurs at grading $(i, j-i) = (6, 13)$ has to survive the spectral sequence (since it must be an ∞–cycle). $\square$

Thus we have shown that there are only two remaining possibilities for the nontrivial differential in the spectral sequence, hence verifying Proposition 1.1.

4.2 Three-stranded pretzel knots

We will now apply our method to draw conclusions about the Floer homology of 3–stranded pretzel knots $P(p,q,r)$. To avoid confusion we shall state this first about instanton homology and indicate at the end how the proof for the Heegaard Floer homology of the branched double cover differs. Firstly, then, we work over $\mathbb{Q}$.

To avoid trivialities, we assume all of p, q and r are nonzero. Notice also that $P(p,q,r)$ is invariant under permutation of the numbers p, q and r, and that reflection of $P(p,q,r)$ yields $P(-p,-q,-r)$. We will restrict ourselves to the cases where $P(p,q,r)$ is a knot, and this is so if and only if at most one of the numbers p, q and r is even.

Firstly, we identify some families of pretzel knots whose reduced Khovanov homology is supported in a single delta grading — those pretzel knots which are alternating or, more generally, quasialternating.

If the absolute value of one of the numbers p, q or r is 1 then $P(p,q,r)$ is easily seen to be a 2–bridge link, and hence alternating.

From now on we assume $p,q,r \geq 2$.

Note that $P(p,q,r)$ is alternating in its standard diagram. Moreover, by results of Greene [5] and Champanerkar and Kofman [3] the 3–stranded pretzel links $P(-p,q,r)$ are *nonquasialternating* if and only if $p \leq \min\{q,r\}$.

Starkston [21] has conjectured and Qazaqzeh [20] has shown that the Khovanov homology of $P(-p,q,r)$ is thin if $p = \min\{q,r\}$, and Manion [16] has proved that it is not thin if $p < \min\{q,r\}$.

As a consequence of these results, the only 3–stranded pretzel knots which do not have their reduced instanton knot Floer homology determined by the collapsing of the Kronheimer–Mrowka spectral sequence are the knots $P(-p,q,r)$ with $2 \le p < \min\{q,r\}$.

The following is a consequence of Manion's result [16, Theorem 1.1]:

Proposition 4.3 *Let $2 \le p < \min\{q,r\}$. The reduced Khovanov homology over $\mathbb{Q}$ of the pretzel knot $P(-p,q,r)$ is supported in two neighbouring δ–gradings and is given by*

$$\mathrm{Khr}(P(-p,q,r)) \cong \mathbb{Q}^{p^2-1} \oplus \mathbb{Q}^{(q-p)(r-p)-1}$$

if all of p, q and r are odd or only q or r is even, and is given by

$$\mathrm{Khr}(P(-p,q,r)) \cong \mathbb{Q}^{p^2} \oplus \mathbb{Q}^{(q-p)(r-p)}$$

if p is even. In both cases, the first summand denotes the reduced Khovanov homology of the upper δ–grading (say $\delta = u(P(-p,q,r))$) *and the second summand the one with the δ–grading which is two lower* (say $\delta = l(P(-p,q,r))$).

Manion also makes precise the respective δ–gradings u and l.

Our proof also requires understanding the Khovanov homology and spectral sequence of the 2–component torus link $T(2,2n)$ (for $n \neq 0$). For any $n \ge 1$ the $T(2,2n)$ torus link is an alternating nonsplit two component link, and has reduced rational Khovanov homology supported in delta grading $\delta = \frac{1}{2}(2n-1)$ (in other words it has thin homology).

Before stating our theorem we prove a simple lemma. Proposition 2.5 does not immediately apply to conclude that the spectral sequence to $I^\natural(T(2,2n))$ is trivial even though it has thin homology because the excision isomorphism $I^\natural(K) \cong \mathrm{KHI}(K)$ is just stated for knots in [12].

Lemma 4.4 *The $T(2,2n)$ torus link has trivial Kronheimer–Mrowka spectral sequence from $\mathrm{Khr}(T(2,2n))$ to $I^\natural(T(2,2n))$. Both groups have total rank $2n$.*

Proof We use the exact triangle from Proposition 2.6 twice. The torus knot $K = T(2,2n+1)$ in its standard diagram has a crossing such that the link K_0 resulting from the 0–resolution of that crossing is the torus link $T(2,2n)$, and the knot K_1 resulting from 1–resolution is the unknot U. For the ranks we have $\mathrm{rk}\big(\mathrm{Khr}(T(2,2n+1))\big) =$

$2n+1$ and $\mathrm{rk}(\mathrm{Khr}(U)) = 1$. Therefore, by the exact triangle the rank of $\mathrm{Khr}(T(2,2n))$ is either $2n+2$ or $2n$.

The torus link $L = T(2,2n)$ has a crossing in its standard diagram such that the two resolutions are the torus knot $T(2,2n-1)$ and the unknot U, respectively. The exact triangle implies this time that the rank of $\mathrm{Khr}(T(2,2n))$ is either $2n$ or $2n-2$. Thus the rank of $\mathrm{Khr}(T(2,2n))$ is $2n$.

The torus knots $T(2,2n+1)$ are alternating, so have trivial spectral sequence. The exact triangle argument just above, but this time applied to reduced instanton homology, implies the claim. $\square$

We are now able to prove Theorem 1.2.

Proof of Theorem 1.2 For the time being, we assume q and r are odd numbers, and without loss of generality we can assume $q \le r$. For any $p \ge 2$, the pretzel knots $P(-p,q,r)$, $P(-(p-1),q,r)$ and the torus link $T(2,q+r)$ are related by a skein triangle. By Proposition 3.5 there is a morphism of spectral sequences Ψ from the Kronheimer–Mrowka spectral sequence for $P(-(p-1),q,r)$ to that of $P(-p,q,r)$ such that at the Khovanov page the morphism ψ fits into an exact triangle

$$\begin{tikzcd} \mathrm{Khr}(P(-(p-1),q,r)) \arrow[rr, "\psi"] & & \mathrm{Khr}(P(-p,q,r)) \arrow[dl, "f"] \\ & \mathrm{Khr}(T(2,q+r)) \arrow[ul, "g"] & \end{tikzcd} \tag{5}$$

Let us here be explicit about the delta gradings. In this exact triangle, the map ψ takes the summand in grading $\delta = u(P(-(p-1),q,r))$ and $\delta = l(P(-(p-1),q,r))$ to the summand in grading $\delta = u(P(-p,q,r))$ and $\delta = l(P(-p,q,r))$, respectively. The map f takes the summand in grading $\delta = u(P(-p,q,r))$ to the summand in grading $\delta = \frac{1}{2}(2n-1)$ (recall that $\mathrm{Khr}(T(2,q+r))$ is supported in this grading). Finally, the map g takes the summand in grading $\delta = \frac{1}{2}(2n-1)$ to the summand in grading $\delta = l\big(P(-(p-1),q,r)\big)$.

Consider the exact triangle (5), and let the inductive hypothesis $H(p,s)$ for $1 \le p \le q$ and $s \ge K$ consist of the following statements:

(i) At page $E_s = E_s^u \oplus E_s^l$ we have that E_s^l contains the s–boundaries.

(ii) At page E_s we have that the s–cycles contain E_s^l.

(iii) If $p \geq 2$, then at page E_s the map Ψ_s splits as the direct sum of maps
$$\Psi_s^l\colon E_s^l(P(-(p-1),q,r)) \to E_s^l(P(-p,q,r))$$
and
$$\Psi_s^u\colon E_s^u(P(-(p-1),q,r)) \to E_s^u(P(-p,q,r)).$$

(iv) If $p \geq 2$, Ψ_s^l is surjective.

(v) If $p \geq 2$, Ψ_s^u is injective.

We start the induction at the Khovanov page.

First observe that at the Khovanov page E_K the long exact sequence (5) implies that ψ always satisfies (iii), (iv) and (v) of $H(p,K)$ due to the support of $\mathrm{Khr}(T(2,q+r))$ in a single delta grading and the gradings of the maps Ψ, f, and g.

Now observe that $P(-1,q,r)$ is 2–bridge and $P(-q,q,r)$ has thin Khovanov homology by Qazaqzeh's result [20] and hence both have trivial Kronheimer–Mrowka spectral sequence by Proposition 2.5. Hence we have that $H(1,K)$ and $H(q,K)$ are trivially satisfied.

Next see that if (i) of $H(p,K)$ is satisfied then (i) of $H(p-1,K)$ is implied by (v) of $H(p,K)$ and Lemma 3.2. Hence by induction on p decreasing with $p=q$ as the root case we have established (i) for all $H(p,K)$.

Finally see that if (ii) of $H(p,K)$ is satisfied then (ii) of $H(p+1,K)$ is implied by (iv) of $H(p+1,K)$ and Lemma 3.2. Hence by induction on p increasing with $p=1$ as the root case we have established (ii) for all $H(p,K)$.

Hence we have $H(p,K)$ for all p.

Next we notice that the shape of the differential (1) at the Khovanov page E_K implies that there is a well-defined delta grading on the homology of the Khovanov page, which is page E_{K+1} of the spectral sequence. We also realise that in the upper delta grading of this next page we obtain a subspace of the upper delta grading of Khovanov homology — the kernel of the differential — whereas the lower delta grading is a quotient of the lower delta grading of Khovanov homology — the cokernel of the differential. As a consequence the morphism Ψ_K, induced by the map Ψ above, maps the upper delta grading of this page for $P(-(p-1),q,r)$ injectively into the upper delta grading of this page for $P(-p,q,r)$, and likewise maps the lower delta grading surjectively onto the respective lower delta grading.

In other words, we see that (iii), (iv) and (v) hold for each $H(p, K+1)$, and again we have that $H(1, K+1)$ and $H(q, K+1)$ are trivially satisfied. Then the induction can proceed exactly as before, so that we see that $H(p, K+1)$ holds for all p. Then we take homology to move to the next page of the spectral sequence and so on. Hence we have $H(p, s)$ for all p and s.

So far we have proved the theorem for all cases where both q and r are odd numbers. Assume now without loss of generality that q is even and p and r are both odd.

The pretzel knots $P(-p,q,r)$ and $P(-p,q+1,r)$ and the torus link $T(2,r-p)$ also form an exact triangle to which we apply Proposition 3.5 and Lemma 3.2 another time. There is a morphism of spectral sequences Ψ from the Kronheimer–Mrowka spectral sequence of $P(-p,q+1,r)$ to that of $P(-p,q,r)$ such that the morphism ψ at the Khovanov page fits into the exact triangle

$$\begin{tikzcd}
\mathrm{Khr}(P(-p,q+1,r)) \arrow[rr, "\psi"] & & \mathrm{Khr}(P(-p,q,r)) \arrow[dl] \\
& \mathrm{Khr}(T(2,r-p)) \arrow[ul] &
\end{tikzcd}$$

Again, if we consider gradings, this morphism ψ has to map the summand with the lower delta grading of the group $\mathrm{Khr}(-p,q+1,r)$ onto the lower delta grading of $\mathrm{Khr}(P(-p,q,r))$, and we can use the theorem for the pretzel knot $P(-p,q+1,r)$ to draw the conclusion that there is no nontrivial differential when restricted to the summand with the lower delta grading, at any page of the Kronheimer–Mrowka spectral sequence for $P(-p,q,r)$.

Similarly, the pretzel knots $P(-p,q,r)$, $P(-p,q-1,r)$ and the torus link $T(2,r-p)$ also form an exact triangle to which we apply Proposition 3.5. There is a morphism of spectral sequences Ψ from the Kronheimer–Mrowka spectral sequence of $P(-p,q,r)$ to that of $P(-p,q-1,r)$ such that the morphism ψ at the Khovanov page fits into the exact triangle

$$\begin{tikzcd}
\mathrm{Khr}(P(-p,q,r)) \arrow[rr, "\psi"] & & \mathrm{Khr}(P(-p,q-1,r)) \arrow[dl] \\
& \mathrm{Khr}(T(2,r-p)) \arrow[ul] &
\end{tikzcd}$$

Again, this morphism has to map the summand with the upper delta grading of the group $\mathrm{Khr}(-p,q,r)$ injectively into the upper delta grading of $\mathrm{Khr}(P(-p,q-1,r))$. Using the same method as before, we can use the theorem for the pretzel knot $P(-p,q-1,r)$

to draw the conclusion that the differentials for $P(-p,q,r)$ will have no nontrivial projection onto the summand with the upper delta grading, at any page of the Kronheimer–Mrowka spectral sequence for $P(-p,q,r)$. □

Theorem 4.5 *Let $2 \le p < \min\{q,r\}$. Then the Ozsváth–Szabó spectral sequence, starting from the reduced Khovanov homology $\mathrm{Khr}(P(-p,q,r))$ and abutting to the Heegaard Floer homology of the branched double cover of the mirror $\widehat{\mathrm{HF}}(\Sigma(K))$, can only have nontrivial differentials that strictly lower the δ–grading.*

Proof The only substantial difference is that we are now working over the 2–element field $\mathbb{Z}/2$. With these coefficients one may be worried that the reduced Khovanov homology of a pretzel link may not be supported in two adjacent δ–gradings, but this turns out not to be the case, for example by appealing to Manion's result [16, Theorem 1.1], in which he proved that the reduced Khovanov homology of a pretzel knot over $\mathbb{Z}$ is torsion-free. □

Remark We have observed above that for the torus knot $T(4,5)$ the same conclusion holds: all possible nonzero differentials in the Kronheimer–Mrowka spectral sequence strictly lower the δ–grading.

Based on these results we state the following conjecture:

Conjecture 4.6 *For any knot, all nontrivial differentials in the Kronheimer–Mrowka spectral sequence strictly lower the δ–grading.*

In another direction, we make the following observation:

Proposition 4.7 *The suite of pretzel knots $P(-2,3,2n+1)$ all have trivial Kronheimer–Mrowka spectral sequence.*

Proof The sum of the absolute values of the coefficients of the Alexander polynomial of $P(-2,3,2n+1)$ is equal to $2n+3$. Therefore, $I^\natural(P(-2,3,2n+1))$ has rank bounded below by $2n+3$ by Proposition 2.4. On the other hand, Manion's result says that the rank of $\mathrm{Khr}(P(-2,3,2n+1))$ is also equal to $2n+3$. Hence, the Kronheimer–Mrowka spectral sequence is trivial. □

It is not the case that these knots $P(-2,3,2n+1)$ have trivial Ozsváth–Szabó spectral sequence. In fact, in Proposition 1.3 we determine explicitly the Ozsváth–Szabó spectral sequences for these knots.

4.3 The $(-2,3,2n+1)$ pretzel knots and the Ozsváth–Szabó spectral sequence

We have seen earlier that the Kronheimer–Mrowka spectral sequence collapses at the Khovanov page for all pretzel knots $P(-2,3,2n+1)$. This however is not the case for the Ozsváth–Szabó spectral sequence.

In this subsection we work over $\mathbb{Z}/2$. In [1], Baldwin considered the pretzel knot $P(-2,3,5)$ and determined the pages of the Ozsváth–Szabó spectral sequence from the reduced Khovanov homology of $P(-2,3,5)$ to the Heegaard Floer homology of the branched double cover (which in this case is the Poincaré homology 3–sphere).

The reduced Khovanov homology of $P(-2,3,5)$ is given by

$$\mathrm{Khr}(P(-2,3,5)) = t^0q^8 + t^2q^{12} + t^3q^{14} + t^4q^{14} + t^5q^{18} + t^6q^{18} + t^7q^{20},$$

where we have been cavalier about the distinction between the homology groups and the Poincaré polynomial (we shall continue to be cavalier). Baldwin showed that t^0q^8 survives the spectral sequence and the remaining six elements cancel in pairs

$$(t^2q^{12}, t^4q^{14}), \quad (t^5q^{18}, t^7q^{20}), \quad (t^3q^{14}, t^6q^{18}),$$

where the first two pairs cancel from the E_2 page to the E_3 page, and the third pair cancel from the E_3 to the E_4 page.

Manion's result [16] implies that the reduced Khovanov homology of $P(-2,3,2n+1)$ is supported in two adjacent delta gradings (where δ is defined as half the quantum grading minus the homological grading). It has rank $2n-1$ in delta grading $\delta = n+1$ and rank 4 in delta grading $\delta = n+2$. In fact, we can write

$$\begin{aligned}\mathrm{Khr}(&P(-2,3,2n+1)) \\ &= q^{2n-4}\,\mathrm{Khr}(P(-2,3,5)) + t^8q^{2n+18}(1 + tq^2 + (tq^2)^2 + \cdots + (tq^2)^{2n-5}),\end{aligned}$$

where $n \geq 3$. We write this as

$$\mathrm{Khr}(P(-2,3,2n+1)) = q^{2n-4}\,\mathrm{Khr}(P(-2,3,5)) \oplus T_n,$$

where T_n stands for *tail*. We note that for degree reasons each bihomogenous element of $\mathrm{Khr}(P(-2,3,2n+1))$ lies either in $q^{2n-4}\,\mathrm{Khr}(P(-2,3,5))$ or in T_n.

Proof of Proposition 1.3 Firstly we want to see that the element t^0q^{2n+4} in the homology $\mathrm{Khr}(P(-2,3,2n+1))$ has to survive the spectral sequence. To see this we just observe that Baldwin's argument for the case $n=2$ actually works for $n \geq 2$.

Essentially since $P(-2,3,2n+1)$ is a positive knot and the Khovanov homology is of rank 1 in homological degree 0, it follows that Plamenevskaya's element [19] is exactly the element t^0q^{2n+4}. Baldwin shows that Plamenevskaya's element represents a cycle in every page of the spectral sequence and, since the Ozsváth–Szabó differentials always increase the homological grading, this implies that t^0q^{2n+4} survives to the E_∞ page.

Next we note that there is a orientable knot cobordism induced by the addition of $2n-4$ 1–handles from $P(-2,3,2n+1)$ to $P(-2,3,5)$. Now this induces a map on Khovanov homologies

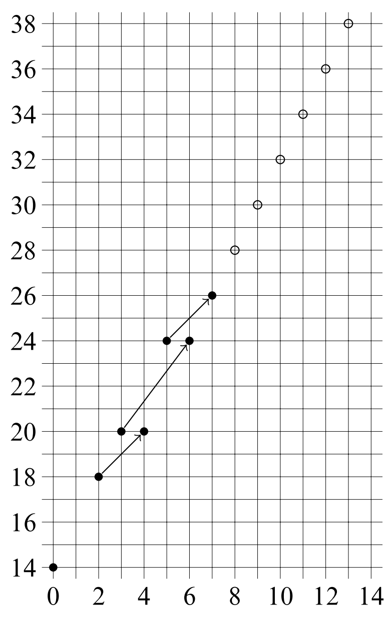
$$\phi\colon \mathrm{Khr}(P(-2,3,2n+1)) \to \mathrm{Khr}(P(-2,3,5))$$

such that

- ϕ is of bidegree $(0,4-2n)$,
- ϕ is the map on the E^2 pages of a morphism between the Ozsváth–Szabó spectral sequences of the two knots,
- ϕ is onto.

The last bullet point follows from the unoriented skein exact sequence in Khovanov homology and a comparison of ranks.

As an example, below we have drawn the reduced Khovanov homology (with $\mathbb{Z}/2$–coefficients) of $P(-2,3,11)$:



We have used the convention that the reduced Khovanov homology of the unknot should be supported in bidegree $(i,j)=(0,0)$, and in our plot the horizontal axis is i, while the vertical axis is j.

The discs correspond to generators whose image under ϕ is nonzero, the circles are generators in the kernel of ϕ. The arrows are the higher differentials of the spectral sequence which we are trying to prove exist.

Now each differential on the n^{th} page E_n in the Ozsváth–Szabó spectral sequence raises the homological grading by n. We know from the previous section that each differential in the spectral sequence for a pretzel knot has to lower the delta grading by 1. Hence each differential on the n^{th} page is of bidegree $(n, 2(n-1))$.

Let us now look at the map between the E_2 pages, we have the commutative diagram

$$\begin{tikzcd} t^2q^{2n+8} \arrow[r, "d_2"] \arrow[d, "\phi"] & t^4q^{2n+10} \arrow[d, "\phi"] \\ t^2q^{12} \arrow[r, "D_2"] & t^4q^{14} \end{tikzcd}$$

where the bottom row is part of the E_2 page for $P(-2,3,5)$ and the top row is part of the E_2 page for $P(-2,3,2n+1)$. The differential d_2 is forced to be nonzero since all other arrows are nonzero. Hence $(t^2q^{2n+8}, t^4q^{2n+10})$ is a canceling pair on the E_2 page for $P(-2,3,2n+1)$. A similar argument tells us that $(t^5q^{2n+14}, t^7q^{2n+16})$ is another canceling pair on the E_2 page.

Now we look at the E_3 page. Again the bottom row is $P(-2,3,5)$; the top row is $P(-2,3,2n+1)$:

$$\begin{tikzcd} {[t^3q^{2n+10}]_3} \arrow[r, "d_3"] \arrow[d, "{[\phi]_3}"] & {[t^6q^{2n+14}]_3} \arrow[d, "{[\phi]_3}"] \\ t^3q^{14} \arrow[r, "D_3"] & t^6q^{18} \end{tikzcd}$$

The bottom row is just the differential that we know exists on the E_3 page for $P(-2,3,5)$. The arrows labelled $[\phi]_3$ are components of the map induced by the map ϕ between the two E_2 pages. The terms labelled $[t^3q^{2n+10}]_3$ and $[t^6q^{2n+14}]_3$ are the images in the E_3 page of two generators of the E_2 page and d_3 is a potentially nonzero differential between them. In fact it is clear from the commutativity of the diagram that $d_3 \neq 0$ so long as both $[t^3q^{2n+10}]_3 \neq 0$ and $[t^6q^{2n+14}]_3 \neq 0$. And this is certainly true since there are no generators of the E_2 page of the spectral sequence for $P(-2,3,2n+1)$ with the correct bidegrees to cancel with these generators at that page. Hence $d_3 \neq 0$ and $(t^3q^{2n+10}, t^6q^{2n+14})$ is a canceling pair at the E_3 page.

We note that there is no homogenous generator in the tail T_n with the correct bidegree to cancel before the E_4 page.

It remains to see that this is where the spectral sequence for $P(-2,3,2n+1)$ ends: $E_4 = E_\infty$. We are left at the E_4 page with

$$E_4 = t^0q^{2n+4} \oplus T_n.$$

By Theorem 4.5, there can be no canceling pair entirely within T_n since T_n is supported in a single delta grading. Furthermore we already know that t^0q^{2n+4} survives the spectral sequence. □

4.4 The $(-3,5,7)$ pretzel knot

In this subsection we consider the problem of attempting to restrict the possible differentials of the Kronheimer–Mrowka spectral sequence of $P(-3,5,7)$ in order to deduce more information about the filtrations on $I^\natural(P(-3,5,7))$. This is to illustrate that the techniques of this paper can give more information on the Kronheimer–Mrowka spectral sequence of a pretzel knot than just that they decrease the δ–grading.

The pretzel knot $P(-3,5,7)$ has trivial Alexander polynomial $\Delta(P(-3,5,7)) = 1$. The rank of the reduced Khovanov homology $\mathrm{Khr}(P(-3,5,7))$ is 15, hence a priori the rank of $I^\natural(P(-3,5,7))$ is some odd integer between 1 and 15. Since $I^\natural$ detects the unknot we can immediately do a little better and exclude the possibility that $I^\natural(P(-3,5,7))$ has rank 1!

It is not too hard in fact to see that the rank of $I^\natural(P(-3,5,7))$ is at least 11, simply by using the long exact sequence

$$\begin{tikzcd} I^\natural(P(-3,6,7)) \arrow[rr] & & I^\natural(P(-3,5,7)) \arrow[dl] \\ & I^\natural(T(2,4)) \arrow[ul] & \end{tikzcd}$$

(where we write $T(2,4)$ for the $(2,4)$ torus link), and computing that the rank of $I^\natural(P(-3,6,7))$ has to be at least 15 since that is the sum of the absolute values of the coefficients of its Alexander polynomial.

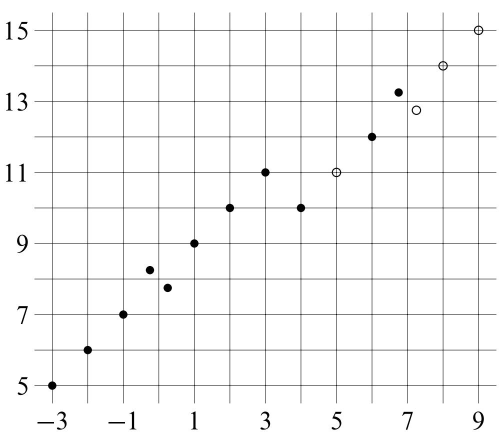
Let us now consider the unoriented skein long exact triangle in Khovanov homology induced by taking resolutions of a crossing in the second of the three twisted regions.

This induces a long exact triangle of the form

$$\begin{tikzcd}
\mathrm{Khr}(P(-3,5,7)) \arrow[rr] & & \mathrm{Khr}(P(-3,4,7)) \arrow[dl] \\
& \mathrm{Khr}(T(2,4)) \arrow[ul] &
\end{tikzcd}$$

We compute the ranks $|\mathrm{Khr}(P(-3,4,7))| = 11$ and $|\mathrm{Khr}(T(2,4))| = 4$ and the sums of absolute values of coefficients of the Alexander polynomial $|\Delta(P(-3,4,7))| = 11$. Hence we conclude that the spectral sequence for $P(-3,4,7)$ is trivial and moreover that the map $\mathrm{Khr}(P(-3,5,7)) \to \mathrm{Khr}(P(-3,4,7))$ is of rank 11. By considering the bidegree of this map we can write down the bigradings of 11 linearly independent bigraded generators of $\mathrm{Khr}(P(-3,5,7))$ which are mapped to nonzero elements of $\mathrm{Khr}(P(-3,4,7))$.

Below we have drawn the bigrading of $\mathrm{Khr}(P(-3,5,7))$. We have followed Kronheimer and Mrowka's conventions in taking i along the horizontal axis and $j-i$ in the vertical direction and normalising by taking the homology of the unknot to be supported in bidegree $(i,j) = (0,-1)$. We have indicated by solid discs these 11 generators:



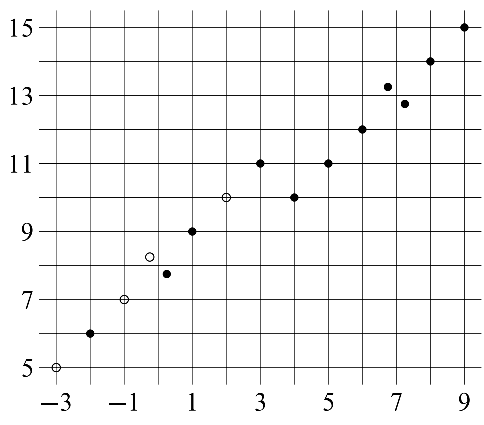
Since we know by Proposition 3.5 that we can realize this map $\mathrm{Khr}(P(-3,5,7)) \to \mathrm{Khr}(P(-3,4,7))$ as the induced map at the Khovanov page of a morphism between the two Kronheimer–Mrowka spectral sequences, we can apply Lemma 3.2. Since the spectral sequence for $P(-3,4,7)$ is trivial, none of these elements represented by solid discs can be the target of differentials in the spectral sequence for $P(-3,5,7)$. We have drawn circles to indicate the 4 remaining bigraded generators of $\mathrm{Khr}(P(-3,5,7))$ which may be targets of differentials in the spectral sequence.

Next we consider the long exact sequence in Khovanov homology obtained by resolving a crossing in the first of the three twisted regions of $P(-3,5,7)$. This gives a long exact sequence of the form

$$\begin{tikzcd} \mathrm{Khr}(P(-3,5,7)) \arrow[rr] & & \mathrm{Khr}(T(2,12)) \arrow[dl] \\ & \mathrm{Khr}(P(-2,5,7)) \arrow[ul] & \end{tikzcd}$$

We compute ranks $|\mathrm{Khr}(P(-2,5,7))| = 19$ and $|\mathrm{Khr}(T(2,12))| = 12$ and the sum of absolute values of the coefficients $|\Delta(P(-2,5,7))| = 19$. Hence we can conclude that the spectral sequence for $P(-2,5,7)$ is trivial and that the rank of the map $\mathrm{Khr}(P(-2,5,7)) \to \mathrm{Khr}(P(-3,5,7))$ is 11.

By considering the bigraded degree of this map we can give the bigradings of a bigraded basis for its image, none of whose elements can be sources of nontrivial differentials in the Kronheimer–Mrowka spectral sequence for $P(-3,5,7)$ (again by Proposition 3.5 and Lemma 3.2). In the diagram below we have indicated by circles the bigradings of the remaining 4 bigraded generators of $\mathrm{Khr}(P(-3,5,7))$ which may be the source of nontrivial differentials in the spectral sequence:



We now consider the $\mathbb{Z}/4$–grading which is just the reduction modulo 4 of the grading $j-i$. Any nontrivial differential in the spectral sequence changes this grading by 3 modulo 4. We can conclude that there are at most 4 possibilities for differentials in the spectral sequence, at most two of which can actually occur. We have drawn these four possibilities in Figure 3.

We observe that there are 8 generators which certainly survive the spectral sequence and whose bigradings we know explicitly.

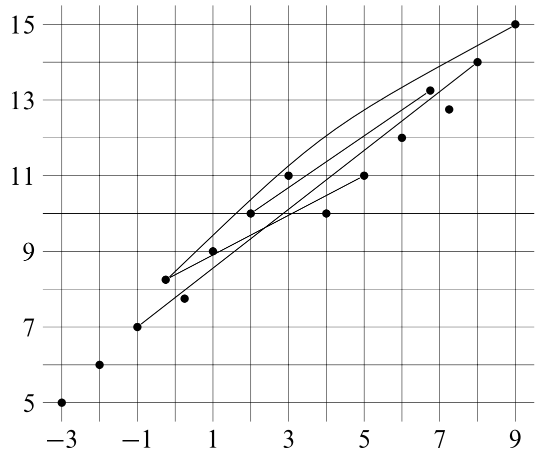


Figure 3

4.5 Relation to representation spaces

Given a knot K and a meridian m of K, one may define the space of representations

$$R(K;\boldsymbol{i}) = \{\rho \in \mathrm{Hom}(\pi_1(S^3 \setminus K), \mathrm{SU}(2)) \mid \mathrm{tr}(\rho(m)) = 0\}.$$

Here we also denote by m the class of a meridian in $\pi_1(S^3 \setminus K)$, well defined up to conjugacy, and a representation is required to send this element to the conjugacy class of traceless matrices in $\mathrm{SU}(2)$.

Reduced instanton knot Floer homology $I^\natural(K)$ of a knot K is by definition the homology of a complex $(C(K)^\natural, d^\natural)$. This is, in some sense, the Morse homology of a Chern–Simons functional, suitably perturbed so as to obtain transversality of the involved instanton moduli spaces. The critical space of the *unperturbed* functional is related to the space $R(K;\boldsymbol{i})$ as follows (see [12; 11; 7]): each conjugacy class of an irreducible representation in $R(K;\boldsymbol{i})$ accounts for a circle, and the conjugacy class of the reducible representation accounts for a point.

In the most generic situation, $R(K;\boldsymbol{i})$ consists of only finitely many conjugacy classes. In this situation, after perturbation of the Chern–Simons functional, each critical circle is expected to yield two critical points. This has been described explicitly by Hedden, Herald and Kirk in [7] in a quite general setting. In this situation, the complex $C(K)^\natural$ is a free $\mathbb{Q}$–vector space of dimension $1+2n$, where n is the number of conjugacy classes of irreducible representations in $R(K;\boldsymbol{i})$. The reduced instanton homology $I^\natural(K)$ is then bounded above by $1+2n$ as well.

It is an interesting fact that the upper bound from Khovanov homology seems to be better than the upper bound from the representation space for pretzel knots, whereas for torus knots the converse seems to be the case in general (except for the torus knots $T(3,n)$). We list a few cases explicitly. The claims on the representation spaces of pretzel knots can be found in [4; 22].

- For the pretzel knot $P(-3,5,7)$ we have $\mathrm{rk}\big(\mathrm{Khr}(P(-3,5,7))\big) = 15$, whereas $R(P(-3,5,7);\boldsymbol{i})$ contains the conjugacy class of the reducible and 16 conjugacy classes of irreducible nonbinary dihedral representations (see the table of the example in [22], where 3 errors occur that yield a total error of 1 which multiplied by two gave the wrong claim of 18 conjugacy classes).
- For the pretzel knots $P(-2,3,2n+1)$ we have $\mathrm{rk}\big(\mathrm{Khr}(P(-2,3,2n+1))\big) = 2n+3$. The representation space $R(P(-2,3,2n+1);\boldsymbol{i})$ contains the conjugacy class of the reducible representation, $2n-6$ irreducible binary dihedral representations, and $\lfloor \frac{8}{3}n \rfloor$ conjugacy classes of irreducible nonbinary dihedral representations, therefore yielding an upper bound to $I^{\natural}(P(-2,3,2n+1))$ by $\lfloor (4+\frac{2}{3})n-5 \rfloor$.
- Torus knots $T(p,q)$ with $p,q \geq 4$ seem to have a faster growth in reduced Khovanov homology than in the bound coming from representation spaces; see [7, Section 12.5].

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