Anisotropic bidispersive convection

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Abstract

This paper investigates thermal convection in an anisotropic bidisperse porous medium. A bidisperse porous medium is one which possesses the usual pores, but in addition, there are cracks or fissures in the solid skeleton and these give rise to a second porosity known as micro porosity. The novelty of this paper is that the macro permeability and the micro permeability are each diagonal tensors but the three components in the vertical and in the horizontal directions may be distinct in both the macro and micro phases. Thus, there are six independent permeability coefficients. A linear instability analysis is presented and a fully nonlinear stability analysis is inferred. Several Rayleigh number and wave number calculations are presented and it is found that novel cell structures are predicted which are not present in the single porosity case.

1 Introduction

Porous materials which have a double porosity structure are occupying significant research attention. Such materials, which are also known as bidisperse or bidispersive porosity media. contain normal pores as present in a single porosity material, but the solid skeleton contains cracks or fissures which lead to a second porosity which is known as a microporosity.

The fundamental theory for thermal convection in a bidisperse porous medium was presented by Nield & Kuznetsov [1], and these writers further developed the area in Nield & Kuznetsov [2, 3, 4, 5] and in Nield [6]. The theories proposed by Nield and Kuznetsov are based upon independent velocity, temperature and pressure fields in both the macro and micro phases.

Falsaperla *et al.* [7] and Gentile & Straughan [8] employed the Nield - Kuznetsov theory, but they restrict attention to a single temperature field, T. This theory still has independent velocity and pressure fields in both the macro and micro phases and is thus still capable of analysing different behaviour in both structures. For many classes of problem a single temperature field should suffice. If it is known that widely different fluid and solid temperatures are expected such as when a hot fluid is injected into a cold skeleton, cf. Rees *et*

al. [9], then perhaps one should consider a theory with separate temperatures in the solid and in the fluid macro and micro phases, cf. Nield & Kuznetsov [1], Franchi *et al.* [10].

The single temperature theory has been further investigated by Franchi *et al.* [11], Gentile & Straughan [12], and by Straughan [13, 14]. In particular, Straughan [13, 14] analyses thermal convection in two types of horizontally isotropic (transversely isotropic) bidisperse porous media. Bidisperse porous media are increasingly important in the Chemical Engineering field, see e.g. Enterria *et al.* [15], Huang *et al.* [16], Ly *et al.* [17], Said *et al.* [18] and they can be man made designer materials and so it is advantageous to allow fully anisotrpic permeabilities at both macro and micro levels. This is the goal of this work. Both the macro and micro permeabilities are taken to be diagonal tensors, but the entries in each are independent and thus there are six permeability coefficients in total.

Thermal convection in anisotropic single porosity media has been intensely studied, see e.g. Capone *et al.* [19], Harfash [20], Harfash & Hill [21], Karmakar & Raja Sekhar [22], Kuznetsov & Nield [23], Nield & Kuznetsov [24], Rees & Postelnicu [25], Rees & Barletta [26], Rees *et al.* [27, 28], Rees & Tyvand [29]. Since many real rocks are anisotropic, see e.g. Fazelalani [30], Widarsono *et al.* [31], this attention is understandable. However, thermal convection in anisotropic bidisperse media has the potential to be much richer than the single porosity case. It is worth pointing out that thermal effects in bidisperse anisotropic porous media have applications in many important areas, see e.g. in chemical engineering, Enterria *et al.* [15], Huang *et al.* [16], Ly *et al.* [17], Said *et al.* [18]; in landslides, Borja *et al.* [32], Montrasio *et al.* [33], Scotto di Santolo & Evangelista [34]; in gas storage in shale, Alnoaimi & Kovscek [35]; in coal stockpiling, Hooman & Maas [36]; in hydraulic fracturing for natural gas, Kim & Moridis [37]; in bone replacement technology, Svanadze & Scalia [38], Zhou *et al.* [39]; in clinical applications, Dejaco *et al.* [40], Dufresne *et al.* [42]; in heat pipe technology, Lin *et al.* [43], Mottet & Prat [44]; and in many other areas.

The goal in this paper is to present an analysis of linear instability and global nonlinear stability for thermal convection in a fully anisotropic bidispersive porous medium, in the sense that the macro and micro permeabilities may be different in the vertical direction, and in each of the horizontal directions. Details of the linear instability analysis are given while the nonlinear stability results are inferred. The global nonlinear stability boundary is found to be the same as the linear instability one and so our results are optimal and demonstrate that the linear theory captures correctly the physics of the onset of convection.

2 Governing equations

Throughout we follow Nield & Kuznetsov [1] and employ a sub or superscript f to denote the macro phase whereas a sub or superscript p indicates the micro phase. We use standard indicial notation throughout.

Let the permeability tensors in the macro and microphases be K^f_{ij} and K^p_{ij}, let μ be the dynamic viscosity of the saturating fluid, and define the tensors M^f_{ij} and M^p_{ij} by

$$M^f_{ij} = \mu(K^f_{ij})^{-1}, \qquad M^p_{ij} = \mu(K^p_{ij})^{-1}. \tag{1}$$

In this work K^f_{ij} and K^p_{ij} are diagonal tensors of form

$$\mathbf{K}^f = \begin{pmatrix} K^f_{11} & 0 & 0 \\ 0 & K^f_{22} & 0 \\ 0 & 0 & K^f_{33} \end{pmatrix} \qquad \mathbf{K}^p = \begin{pmatrix} K^p_{11} & 0 & 0 \\ 0 & K^p_{22} & 0 \\ 0 & 0 & K^p_{33} \end{pmatrix} \tag{2}$$

The tensors M^f_{ij} and M^p_{ij} take the form

$$\mathbf{M}^f = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \qquad \mathbf{M}^p = \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix} \tag{3}$$

The governing equations for thermal convection in an anisotropic bidispersive porous medium then have the same form as (2.6), (2.7) of Straughan [14], although the forms for $\mathbf{M}^f$ and $\mathbf{M}^p$ are different. Thus, the basic equations are

$$\begin{aligned} &- M^f_{ij}U^f_j - \zeta(U^f_i - U^p_i) - p^f_{,i} + \rho_F \alpha g T k_i = 0, \qquad U^f_{i,i} = 0\,, \\ &- M^p_{ij}U^p_j - \zeta(U^p_i - U^f_i) - p^p_{,i} + \rho_F \alpha g T k_i = 0, \qquad U^p_{i,i} = 0\,, \\ &(\rho c)_m T_{,t} + (\rho c)_f (U^f_i + U^p_i) T_{,i} = \kappa \Delta T. \end{aligned} \tag{4}$$

As in Straughan [14], U^f_i, U^p_i are the velocities in the macro and micro pores, p^f, p^p are the pressures in the macro and micro pores, ζ is an interaction coefficient due to the macro and micro pores, ρ_F, α, g are a relative density, coefficient of thermal expansion, and gravity. The terms $(\rho c)_m$ and the thermal conductivity κ have form

$$\begin{aligned} &(\rho c)_m = (1-\phi)(1-\epsilon)(\rho c)_s + \phi(\rho c)_f + \epsilon(1-\phi)(\rho c)_p\,, \\ &\kappa = (1-\phi)(1-\epsilon)\kappa^s + \phi\kappa^f + \epsilon(1-\phi)\kappa^p \end{aligned} \tag{5}$$

where c denotes specific heat at constant pressure, s denotes the solid skeleton, ϕ is the micro porosity and ϵ is the macro porosity.

We suppose the bidisperse porous material is contained in the horizontal layer $\{(x,y) \in \mathbb{R}^2\} \times \{0 < z < d\}$ and the boundary conditions are

$$U^f_3 = 0,\ U^p_3 = 0, \quad \text{on} \quad z = 0, d, \qquad T = T_L,\ z = 0, \quad T = T_U,\ z = d, \tag{6}$$

for constants $T_L > T_U$.

The basic steady conduction solution to (4) and (6) has form

$$\bar{U}^f_i \equiv 0, \quad \bar{U}^p_i \equiv 0, \quad \bar{T} = -\beta z + T_L\,, \tag{7}$$

where $\beta = (T_L - T_U)/d$ is the temperature gradient. Our aim is to investigate the instability and nonlinear stability of this solution.

Introduce the perturbation variables $\{u_i^f, u_i^p, \theta, \pi^f, \pi^p\}$ to the base solution $\{\bar{U}_i^f, \bar{U}_i^p, \bar{T}, \bar{p}^f, \bar{p}^p\}$. We define the variables ω, λ, m_α, s_α, $\alpha = 1, 2$, by

$$b_{33} = \omega a_{33}, \qquad \lambda = \frac{\zeta}{a_{33}}, \qquad m_1 = \frac{a_{11}}{a_{33}}, \qquad m_2 = \frac{a_{22}}{a_{33}}, \qquad s_1 = \frac{b_{11}}{b_{33}}, \qquad s_2 = \frac{b_{22}}{b_{33}}. \tag{8}$$

Introduce the tensors D_{ij}^f and D_{ij}^p as

$$D_{ij}^f = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad D_{ij}^p = \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9}$$

Define the time, velocity, pressure and temperature scales as

$$\mathcal{T} = \frac{(\rho c)_m d^2}{\kappa}, \qquad U = \frac{\kappa}{(\rho c)_f d}, \qquad P = d a_{33} U, \qquad T^\sharp = U\sqrt{\frac{a_{33}\beta d^2}{k\rho_F \alpha g}},$$

where $k = \kappa/(\rho c)_f$. In addition, the Rayleigh number $Ra = R^2$, is

$$R^2 = \frac{\beta d^2 \rho_F \alpha g}{k a_{33}}. \tag{10}$$

(Note that the non-dimensionalization is necessarily different from that of Straughan [14]).

The fully nonlinear non-dimensional perturbation equations governing the behaviour of the variables $u_i^f, u_i^p, \theta, \pi^f$ and π^p are then found to be

$$\begin{aligned} &D_{ij}^f u_j^f + \lambda(u_i^f - u_i^p) = -\pi_{,i}^f + R\theta k_i, \qquad u_{i,i}^f = 0, \\ &\omega D_{ij}^p u_j^p + \lambda(u_i^p - u_i^f) = -\pi_{,i}^p + R\theta k_i, \qquad u_{i,i}^p = 0, \\ &\theta_{,t} + (u_i^f + u_i^p)\theta_{,i} = R(w^f + w^p) + \Delta\theta, \end{aligned} \tag{11}$$

where $w^f = u_3^f, w^p = u_3^p$. Equations (11) hold in the domain $\{(x, y) \in \mathbb{R}^2\} \times \{0 < z < 1\} \times \{t > 0\}$. We suppose the perturbation solution satisfies a plane tiling periodicity in the (x, y) directions, and we denote the periodicity cell by V. The boundary conditions are

$$w^f = w^p = \theta = 0, \qquad \text{on} \qquad z = 0, 1. \tag{12}$$

3 Thermal convection

In Straughan [13], pp. 4–6, it is shown that for a system of equations more general than (11), (12), exchange of stabilities holds and the global nonlinear stability threshold for R is the *same* as the one obtained by linear instability

theory. Thus, we employ this result for (11), (12), and hence the nonlinear stability boundary may be obtained by ignoring the $\theta_{,t}$ and nonlinear terms in $(11)_5$. Full details of the proof of equivalence of the linear instability and nonlinear stability boundaries is given in Straughan [13], pp. 4–6.

It is now advantageous to write out explicitly equations (11). Thus, let $\mathbf{u}^f = (u^f, v^f, w^f)$ and let $\mathbf{u}^p = (u^p, v^p, w^p)$. In this section we dispense with the comma notation for a derivative and denote partial derivatives with respect to x, y or z by a subscript x, y or z. For example, $\pi^f_x = \partial\pi^f/\partial x$, or $\pi^f_{xz} = \partial^2\pi^f/\partial x\partial z$, etc. Then, equations $(11)_{1-4}$ and the reduced form of $(11)_5$ after dropping $\theta_{,t}$ and the nonlinear terms become

$$m_1 u^f + \lambda u^f - \lambda u^p = -\pi^f_x\,, \qquad m_2 v^f + \lambda v^f - \lambda v^p = -\pi^f_y\,, \tag{13}$$

$$\omega s_1 u^p + \lambda u^p - \lambda u^f = -\pi^p_x\,, \qquad \omega s_2 v^p + \lambda v^p - \lambda v^f = -\pi^p_y\,, \tag{14}$$

and

$$w^f + \lambda w^f - \lambda w^p = -\pi^f_z + R\theta\,, \qquad \omega w^p + \lambda w^p - \lambda w^f = -\pi^p_z + R\theta\,, \tag{15}$$

and

$$u^f_x + v^f_y + w^f_z = 0\,, \qquad u^p_x + v^p_y + w^p_z = 0\,, \tag{16}$$

together with

$$Rw^f + Rw^p + \Delta\theta = 0. \tag{17}$$

Define M and S by

$$M = m_1 s_1 \omega + \lambda(\omega s_1 + m_1), \qquad S = \omega m_2 s_2 + \lambda(\omega s_2 + m_2).$$

We then solve equations (13) for u^f and u^p and we solve equations (14) for v^f and v^p to see that

$$\begin{aligned} u^f &= \frac{1}{M}\big[-(\omega s_1 + \lambda)\pi^f_x - \lambda\pi^p_x\big]\,, \qquad & u^p &= \frac{1}{M}\big[-(m_1 + \lambda)\pi^p_x - \lambda\pi^f_x\big]\,, \\ v^f &= \frac{1}{S}\big[-(\omega s_2 + \lambda)\pi^f_y - \lambda\pi^p_y\big]\,, \qquad & v^p &= \frac{1}{S}\big[-(m_2 + \lambda)\pi^p_y - \lambda\pi^f_y\big]\,. \end{aligned} \tag{18}$$

One now differentiates equations (18) with respect to z and likewise one differentiates equations (15) with respect to x and y. Eliminate the terms $\pi^f_{xz}, \pi^p_{xz}, \pi^f_{yz}$ and π^p_{yz} between the results to find

$$u^f_z = \Big[\frac{\omega s_1(1+\lambda) + \lambda}{M}\Big] w^f_x + \Big[\frac{\lambda\omega(1 - s_1)}{M}\Big] w^p_x - R\theta_x\Big(\frac{\omega s_1 + 2\lambda}{M}\Big) \tag{19}$$

and

$$u^p_z = \Big[\frac{\omega m_1 + \lambda(m_1 + \omega)}{M}\Big] w^p_x + \Big[\frac{\lambda(1 - m_1)}{M}\Big] w^f_x - R\theta_x\Big(\frac{m_1 + 2\lambda}{M}\Big) \tag{20}$$

and

$$v^f_z = \Big[\frac{\omega s_2 + \lambda(1 + \omega s_2)}{S}\Big] w^f_y + \Big[\frac{\lambda\omega(1 - s_2)}{S}\Big] w^p_y - R\theta_y\Big(\frac{\omega s_2 + 2\lambda}{S}\Big) \tag{21}$$

and finally

$$v_z^p = \Big[\frac{\omega m_2 + \lambda(m_2+\omega)}{S}\Big] w_y^p + \Big[\frac{\lambda(1-m_2)}{S}\Big] w_y^f - R\theta_y\Big(\frac{m_2+2\lambda}{S}\Big). \tag{22}$$

Now differentiate equations (19) - (22) to form the quantities $u_{zx}^f, v_{zy}^f, u_{zx}^p$ and v_{zy}^p and use the incompressibility conditions (16) to see that

$$u_{zx}^f + v_{zy}^f = -w_{zz}^f\,, \qquad u_{zx}^p + v_{zy}^p = -w_{zz}^p\,. \tag{23}$$

Upon substitution of the relevant forms for $u_{zx}^f, v_{zy}^f, u_{zx}^p$ and v_{zy}^p into (23) we arrive at the equations

$$\begin{aligned} -w_{zz}^f =& w_{xx}^f\Big[\frac{\omega s_1(1+\lambda)+\lambda}{M}\Big] + w_{xx}^p\Big[\frac{\lambda\omega(1-s_1)}{M}\Big] - R\theta_{xx}\Big(\frac{\omega s_1+2\lambda}{M}\Big) \\ &+ w_{yy}^f\Big[\frac{\omega s_2+\lambda(1+\omega s_2)}{S}\Big] + w_{yy}^p\Big[\frac{\lambda\omega(1-s_2)}{S}\Big] - R\theta_{yy}\Big(\frac{\omega s_2+2\lambda}{S}\Big) \end{aligned} \tag{24}$$

and

$$\begin{aligned} -w_{zz}^p =& w_{xx}^f\Big[\frac{\lambda(1-m_1)}{M}\Big] + w_{xx}^p\Big[\frac{m_1\omega+\lambda(m_1+\omega)}{M}\Big] - R\theta_{xx}\Big(\frac{m_1+2\lambda}{M}\Big) \\ &+ w_{yy}^f\Big[\frac{\lambda(1-m_2)}{S}\Big] + w_{yy}^p\Big[\frac{m_2\omega+\lambda(\omega+m_2)}{S}\Big] - R\theta_{yy}\Big(\frac{m_2+2\lambda}{S}\Big) \end{aligned} \tag{25}$$

These equations coupled with (17) yield a system of three equations for w^f, w^p and θ. One now employs normal modes in w^f, w^p and θ with forms such as $w^f = \hat{w}^f \sin n\pi z \exp(\sigma t + ilx + imy)$ where $\hat{w}^f$ is a constant and we may take $\sigma = 0$.

This leads to the system of equations

$$\begin{aligned} &c_{11}\hat{w}^f + c_{12}\hat{w}^p - c_{13}R\hat{\theta} = 0, \\ &c_{21}\hat{w}^f + c_{22}\hat{w}^p - c_{23}R\hat{\theta} = 0, \\ &R\hat{w}^f + R\hat{w}^p - H\hat{\theta} = 0, \end{aligned} \tag{26}$$

where

$$H = n^2\pi^2 + l^2 + m^2,$$

and where

$$
\begin{aligned}
c_{11} &= n^2\pi^2 + l^2\Big[\frac{\omega s_1(1+\lambda)+\lambda}{M}\Big] + m^2\Big[\frac{\omega s_2+\lambda(1+\omega s_2)}{S}\Big],\\
c_{12} &= l^2\Big[\frac{\lambda\omega(1-s_1)}{M}\Big] + m^2\Big[\frac{\omega\lambda(1-s_2)}{S}\Big],\\
c_{13} &= l^2\Big[\frac{\omega s_1+2\lambda}{M}\Big] + m^2\Big[\frac{\omega s_2+2\lambda}{S}\Big],\\
c_{21} &= l^2\Big[\frac{\lambda(1-m_1)}{M}\Big] + m^2\Big[\frac{\lambda(1-m_2)}{S}\Big],\\
c_{22} &= n^2\pi^2 + l^2\Big[\frac{\omega m_1+\lambda(m_1+\omega)}{M}\Big] + m^2\Big[\frac{\omega m_2+\lambda(\omega+m_2)}{S}\Big],\\
c_{23} &= l^2\Big[\frac{m_1+2\lambda}{M}\Big] + m^2\Big[\frac{m_2+2\lambda}{S}\Big].
\end{aligned}
\tag{27}
$$

We require a zero determinant for (26) and this leads to the expression for R^2,

$$
R^2 = \frac{H(c_{11}c_{22}-c_{12}c_{21})}{[c_{11}c_{23}+c_{22}c_{13}-c_{21}c_{13}-c_{23}c_{12}]}\,. \tag{28}
$$

The critical Rayleigh number is found by minimizing R^2 in n^2, l^2 and m^2. One may show that the numerator is greater than or equal to zero and we have also shown that the denominator is strictly positive. For all the computations we have performed, the minimum has been achieved for $n = 1$.

The next section discusses numerical results for the minimization of R^2 in l and m with R^2 given by (28) with $n = 1$.

4 Numerical results and conclusions

4.1 One porosity

In this section we report numerical solutions for the minimization of R^2 given by (28). However, it is instructive to recollect results for the thermal convection problem in a single porous material when the anisotropic permeability is due to a diagonal permeability tensor of form $K_{ij} = \mathrm{diag}\,\{K_{11}, K_{22}, K_{33}\}$. This problem was analysed in detail by Kvernvold & Tyvand [45]. If we write $M_{ij} = \mu(K_{ij})^{-1}$ and set $M_{ij} = \mathrm{diag}\,(a_{11}, a_{22}, a_{33})$ then for a single porosity material the governing equations are

$$
M_{ij}v_j = -p_{,i} + \alpha g\rho_F T k_i, \qquad v_{i,i} = 0, \qquad T_{,t} + v_i T_{,i} = \kappa\Delta T. \tag{29}
$$

With the analogous boundary conditions to (12) one may show exchange of stabilities holds and so the stability problem is governed by the system of perturbation equations

$$
D_{ij}u_j = -\pi_{,i} + R\theta k_i, \qquad u_{i,i} = 0, \qquad 0 = Rw + \Delta\theta, \tag{30}
$$

where $D_{ij} = \text{diag}\,(m_1, m_2, 1)$, $m_1 = a_{11}/a_{33}$, $m_2 = a_{22}/a_{33}$, and $R^2 = \beta d^2 \alpha g \rho_F / \kappa a_{33}$ is the Rayleigh number. One shows from (30) that R^2 is given by

$$R^2 = (\pi^2 + l^2 + m^2)\left[1 + \frac{\pi^2}{\left(\frac{l^2}{m_1} + \frac{m^2}{m_2}\right)}\right]. \tag{31}$$

As noted by Kvernvold & Tyvand [45], when $m_1 = m_2$, R^2 depends only on $a^2 = l^2 + m^2$ and then the critical wave number and Rayleigh number values are given by

$$a_c = \pi m_1^{1/4}, \qquad Ra_c = \pi^2(1 + \sqrt{m_1})^2. \tag{32}$$

Suppose $0 < m_1 < m_2$. By differentiating (31) with respect to l and assuming $m = 0$ one sees that on the $m = 0$ boundary R^2 has a minimum when $l^2 = \pi^2\sqrt{m_1}$ and when $R^2_{min} = \pi^2(1 + \sqrt{m_1})^2$. Then, for $m \neq 0$, $m^2/m_2 < m^2/m_1$ and so $1/(l^2/m_1 + m^2/m_2) > 1/(a^2/m_1)$. Therefore, from (31),

$$Ra = R^2 > (\pi^2 + a^2)\Big(1 + \frac{\pi^2 m_1}{a^2}\Big) = Ra_1.$$

From (32) we know the minimum value of Ra_1 is $\pi^2(1 + \sqrt{m_1})^2$ and so

$$Ra > \pi^2(1 + \sqrt{m_1})^2.$$

Thus, when $m_1 < m_2$ the minimum value of Ra from (31) is given when $m = 0$ and when

$$l = \pi m_1^{1/4}, \qquad Ra_c = \pi^2(1 + \sqrt{m_1})^2. \tag{33}$$

A similar argument may be applied with $0 < m_2 < m_1$, noting then for $l \neq 0$, $1/(m^2/m_2 + l^2/m_1) > 1/(a^2/m_2)$ and one may deduce

$$Ra \geq (\pi^2 + a^2)\Big(1 + \frac{\pi^2 m_2}{a^2}\Big) = Ra_2.$$

The minimum of Ra_2 is when $l = 0$ and $m = \pi m_2^{1/4}$ and $Ra_c = \pi^2(1 + \sqrt{m_2})^2$.

Thus, when $m_1 = m_2$, $a = m_1^{1/4}\pi$ and $Ra_c = \pi^2(1+\sqrt{m_1})^2$; when $m_1 < m_2$, $m = 0, l = m_1^{1/4}\pi$, $Ra_c = \pi^2(1 + \sqrt{m_1})^2$; and when $m_2 < m_1$, $l = 0$ and $Ra_c = \pi^2(1 + \sqrt{m_2})^2$. Thus, when $m_1 < m_2$ instability is initiated with rolls with axis in the $y-$direction. Let $\mathbf{K} = \text{diag}\,\{K_{11}, K_{22}, K_{33}\}$ be the permeability tensor in the one porosity case, then $m_1 < m_2$ is equivalent to $K_{22} < K_{11}$ and so the horizontal permeability is larger in the $x-$direction and as a consequence it is easier for the fluid to move in the $x-$direction rather than in the $y-$direction. Thus, rolls aligned along the $y-$axis are to be expected physically. When $m_2 < m_1$ instability is initiated with rolls aligned along the $x-$direction. The condition $m_2 < m_1$ is equivalent to $K_{11} < K_{22}$ and with the horizontal permeability being greater in the $y-$direction one expects movement in the y and z directions and, therefore, rolls along the $x-$axis are expected physically. These are results of Kvernvold & Tyvand [45], but they are important to understand the bidisperse case studied here.

4.2 Dual porosity

The case where $m_1 = m_2 = s_1 = s_2$ but $\omega = b_{33}/a_{33} = K^f_{33}/K^p_{33} \neq 1$ is analysed in Straughan [13]. The more general case, $m_1 = m_2$, $s_1 = s_2$ but $m_1 \neq s_1$ and $\omega \neq 1$ is covered in Straughan [14]. However, care must be taken when comparing the results of those articles with the present one since the Rayleigh number, Ra, and interaction parameter, λ, in those works differs from the ones used here in (8) and (10).

The results for the minimization of R^2 given by (28) over l and m are reported in tables 1 - 12. Tables 1 and 2 take $m_1 = 10, m_2 = 1, s_1 = 0.1$ and $s_2 = 1$ and vary ω with $\lambda = 0.1$ or 0.5. Tables 4 - 6 choose $m_1 = 0.1, m_2 = 1, s_1 = 10$, $s_2 = 1$ and vary ω with $\lambda = 0.01, 0.1$ and 0.5. In table 8 we set $m_1 = 5, m_2 = 0.9, s_1 = 0.3, s_2 = 1.1, \lambda = 0.1$ and ω is varied. Tables 9 - 12 take $\omega = 5$, $\lambda = 0.1$ and we choose the same m_α and s_α values as in table 8 except we vary m_1, m_2, s_1 and s_2 in turn to assess the effect each variable has upon the Rayleigh number, Ra, and the critical wavenumbers l and m.

In the tables the numbers $\hat{x}$ and $\hat{y}$ are the wavelength in the x and y directions given by $\hat{x} = 2\pi/l$ and $\hat{y} = 2\pi/m$. When $\hat{y}/\hat{x} = 0$ then $l = 0$, the solution is a function of y and z and so the convection cells are rolls in the $x-$direction. When $\hat{y}/\hat{x} \to \infty$ then $m = 0$ and the solution is a function of x and z. In this case the cells are rolls in the $y-$direction.

It is useful to recollect the relations

$$\omega = \frac{b_{33}}{a_{33}} = \frac{K^f_{33}}{K^p_{33}}, \quad m_1 = \frac{K^f_{33}}{K^f_{11}}, \quad m_2 = \frac{K^f_{33}}{K^f_{22}}, \quad s_1 = \frac{K^p_{33}}{K^p_{11}}, \quad s_2 = \frac{K^p_{33}}{K^p_{22}}. \quad (34)$$

4.3 Relatively large values of m_α

In both tables 1 and 2 we observe that for ω small enough the critical value of l is zero and so the convection patterns are rolls with the axis in the $x-$direction. As ω increases there is a transition to where the convection patterns become cells with non-zero l and m values. Once ω increases further eventually the convection patterns become rolls along the $x-$axis again. The transition values depend on ω, but also on what value the interaction parameter λ has. For $\lambda = 0.1$, the smaller value, we see that the first transition from a roll to a cell is for $\omega = 0.04$, whereas the transition back to a roll occurs when $\omega = 10.00$; the transition values of ω are generally reported to 2 decimal places.

The equivalent transition values for rolls to cells and then from cells to rolls when $\lambda = 0.5$ occur for $\omega = 1.00$ and $\omega = 10.00$, respectively. The maximum value of $\hat{y}/\hat{x}$ when $\lambda = 0.1$ occurs for ω near 0.3, while for $\lambda = 0.5$ the maximum is for ω near 2.5, although the values at this maximum differ a lot. When $\omega = 0.3$ and $\lambda = 0.1$ we see that $\hat{y}/\hat{x} = 1.279$ whereas when $\omega = 2.5$ and $\lambda = 0.5$ we note $\hat{y}/\hat{x} = 0.310$. Since $\omega = K^f_{33}/K^p_{33}$ it means that when this ratio is small or it is large rolls in the $x-$direction are found. For the parameters in tables 1 and 2 one has $K^f_{33} = K^f_{22} = 10K^f_{11}$ and $K^p_{33} = K^p_{22} = 0.1K^p_{11}$, thus $K^f_{22} >> K^f_{11}$ whereas $K^p_{22} << K^p_{11}$. Thus, when ω is small or large it appears

that the macro permeability is playing a leading role since the larger value of K_{22}^f allows the fluid to move easier in the y and z directions creating rolls along the $x-$axis. When ω has values closer to 1 then K_{33}^f and K_{33}^p are closer and it would appear that the macro permeability favours convective movement in the y and z directions whereas the micro permeability is tending to create convective movement in the x and z directions. Thus, rolls are not predicted and a three-dimensional cellular pattern is expected. We might expect a hexagonal pattern (with generally elongated hexagonal cells), but the analysis given here cannot confirm this.

To interpret the Rayleigh number effect physically we employ a different measure because from (10) one observes that the Rayleigh number depends on a_{33} and, consequently, invoking (1), (2) and (3) it depends on K_{33}^f. Upon using (10), (1), (2), (3) and (34), one observes that for fixed K_{33}^p, we may write

$$\frac{Ra}{\omega} = C\,\Delta T, \tag{35}$$

where the constant C is given by $C = \rho_F \alpha g K_{33}^p / k\mu$, and $\Delta T = T_L - T_U$. For a fixed depth of layer the measure Ra/ω indicates the temperature difference required to initiate convective overturning. Table 3 shows how Ra/ω varies with different values of λ. One sees that as λ increases the equivalent Ra/ω values increase indicating that the layer is more stable. This is to be expected since the λ term is stabilizing and increasing Ra/ω means it is less easy for the layer to initiate convective overturning.

From tables 1 and 2 we also note that as ω increases $Ra/\omega = C\,\Delta T$ decreases substantially. Since $\omega = K_{33}^f / K_{33}^p$ we might expect increasing K_{33}^f relative to K_{33}^p will increase the likelihood of convective overturning and so the layer convects more easily as ω increases. This is precisely what is seen in tables 1 and 2.

4.4 Relatively small values of m_α

Tables 4 - 6 concentrate on another set of values for permeability ratios m_1, m_2, s_1 and s_2, namely when $m_1 = 0.1$, $m_2 = 1$, $s_1 = 10$ and $s_2 = 1$. This means that $0.1K_{11}^f = K_{33}^f = K_{22}^f$ and $K_{22}^p = K_{33}^p = 10K_{11}^p$, and so $K_{11}^f >> K_{22}^f$ and $K_{22}^p >> K_{11}^p$. Here, we allow λ to take the values 0.01, 0.1 and 0.5 and we analyse the effect of varying $\omega = K_{33}^f / K_{33}^p$.

For all three cases of λ reported the qualitative pattern of convection structure is similar. When ω is small the critical value of l is 0 and so the solution depends on y and z which corresponds to rolls aligned along the $x-$axis. As ω increases the rolls disappear and both l and m are non-zero corresponding to three - dimensional convection cells. Then for ω large enough the critical value of m becomes zero and the solution depends on x and z which means the cells transform to rolls aligned along the $y-$axis. The transition value for ω when the $x-$axis rolls change to cells is $\omega = 0.10$, $\omega = 0.10$ and $\omega = 1.00$, for $\lambda = 0.01, 0.1$ and 0.5, respectively. The transition from the cell to rolls along

the $y-$axis occurs when $\omega = 1.83$, $\omega = 2.69$ and $\omega = 8.95$, for $\lambda = 0.01, 0.1$ and 0.5, respectively.

It is noteworthy that the transition pattern is different with the values in tables 4 - 6 to that observed with the values in tables 1 and 2. For the case of tables 4 - 6 we have $K_{33}^f = K_{22}^f << K_{11}^f$ and $K_{33}^p = K_{22}^p >> K_{11}^p$. For the transitions at the larger values of ω we have (all three cases of λ) $K_{33}^f > K_{33}^p$. Thus,

$$K_{11}^f >> K_{22}^f = K_{33}^f > K_{33}^p = K_{22}^p >> K_{11}^p.$$

In this case we might expect the macro permeabilities to dominate. This would mean the fluid finds it easier to move in the $x-$direction and the solution will depend on x and z with rolls along the $y-$axis. This is precisely what we find numerically as noted above for $\omega = 1.83$, $\omega = 2.69$ and $\omega = 8.95$.

It appears trickier to interpret the transition for the smaller values of ω. In this case we have $K_{33}^f = K_{22}^f << K_{11}^f$ and $K_{33}^p = K_{22}^p >> K_{11}^p$ but for the values of $\lambda = 0.01$ and 0.1, $\omega = 0.10$ and $\omega = 0.10$. Thus, in this situation, at the transition

$$10K_{22}^f = 10K_{33}^f = 10K_{11}^p = K_{11}^f = K_{22}^p = K_{33}^p.$$

Hence,

$$K_{22}^f = K_{33}^f = K_{11}^p << K_{11}^f = K_{22}^p = K_{33}^p.$$

The buoyancy force contribution in this case is determined from K_{33}^p which is much greater than the macro vertical permeability K_{33}^f. Thus, the terms which appear to dominate are K_{22}^p and K_{33}^p. This encourages motion in the y and z directions and gives rise to solutions which are rolls aligned along the $x-$axis. This is seen in tables 4 and 5 where $l = 0$ for $\omega \leq 0.10$. Thus, the micro permeability appears to dominate the convection cell pattern.

For the value of $\lambda = 0.5$ as in table 6 the lower transition value for ω is $\omega = 1.00$. In this case at the transition we have the situation

$$K_{11}^f = 10K_{22}^f = 10K_{33}^f = 10K_{33}^p = 10K_{22}^p = 100K_{11}^p.$$

Thus,

$$K_{11}^f >> K_{22}^f = K_{33}^f = K_{33}^p = K_{22}^p >> K_{11}^p. \tag{36}$$

Even though K_{11}^f is the dominant permeability coefficient we see from table 6 that for $\omega \leq 1.00$ one has $l = 0$. To interpret this physically we note from (36) that $K_{22}^f = K_{22}^p$ and $K_{33}^f = K_{33}^p$ and these are equal. Thus, the buoyancy force is aided by the K_{33}^f and K_{33}^p permeabilities and the combined K_{22}^f and K_{22}^p coefficients lead to convective motion in the y and z directions and thus one finds $l = 0$ and rolls aligned with the $x-$axis. At the transition value of $\omega = 1.00$ we find $Ra = 2\pi^2$, $m = \pi$, as may be confirmed by minimization of R^2 in (28), taking $l = 0$ and minimizing in m.

In all cases the critical Rayleigh number increases as ω increases. Table 7 shows how Ra/ω varies with ω for different values of λ. One observes that as

λ increases, the ΔT threshold is increased. Since λ is a stabilizing term this is expected physically and corresponds to the layer being less easy to convect.

Furthermore, from tables 4 - 6, as ω increases, Ra/ω decreases strongly indicating that a small value of ΔT is required to initiate convective overturning. Thus, increasing K^f_{33} promotes convective overturning and effectively makes the layer less stable. This effect is also observed in table 8.

4.5 Different m_α, s_α values

In table 8 we select $m_1 = 5$, $m_2 = 0.9$, $s_1 = 0.3$ and $s_2 = 1.1$, with $\lambda = 0.1$. We show the variation of Ra, l and m as ω is varied. However, we include tables 9 - 12 to examine the effect of varying m_1, m_2, s_1 and s_2 in turn, keeping $\omega = 5$ but the other parameters as in table 8. The results display a complex relationship between the macro and micro permeability values and the critical Rayleigh and wave numbers.

Table 8 shows that the critical Rayleigh number increases with increasing ω. For the m_α and s_α values chosen in table 8 we have

$$K^f_{33} = 5K^f_{11},\ K^f_{33} = 0.9K^f_{22},\ K^p_{33} = 0.3K^p_{11} \quad \text{and} \quad K^p_{33} = 1.1K^p_{22}\,.$$

Thus,

$$K^f_{22} > K^f_{33} >> K^f_{11} \qquad \text{and} \qquad K^p_{11} >> K^p_{33} > K^p_{22}. \tag{37}$$

For the transition at the larger values of ω from table 8 we observe that when $\omega \geq 3.15$ one finds $l = 0$. Since $\omega = K^f_{33}/K^p_{33}$ this means $K^f_{33}/K^p_{33} \geq 3.15$ and so $K^f_{33} >> K^p_{33}$. Thus, in this case, using (37) it appears that the macro permeabilities are dominating the situation and K^f_{22} and K^f_{33} lead to the solution being a function of y and z and so to rolls aligned along the $x-$axis with $l = 0$. This is what is seen in table 8.

The transition at $\omega = 0.042$ is somewhat akin to that discussed above for table 6. At the transition value $\omega = 0.042$ one has, coefficients to 2 decimal places,

$$K^p_{11} = 3.33K^p_{33} = 3.67K^p_{22} = 71.36K^f_{22} = 79.29K^f_{33} = 396.44K^f_{11}. \tag{38}$$

The K^p_{11} permeability value is dominant. However, the numerical results indicate $l = 0$ and rolls along the $x-$axis. However, from (38) we see that K^p_{33} and K^p_{22} are close together, as are K^f_{22} and K^f_{33}, with K^f_{11} being much smaller. Thus, it appears the y and z values of $\mathbf{K}^p$ and $\mathbf{K}^f$ are reinforcing each other and ensuring the convective motion is in the form of a solution in y and z and so rolls aligned along the $x-$axis. For values of ω between 0.042 and 3.15 there is an interaction effect of both the macro and micro permeabilities which leads to cells involving non-zero values of both l and m.

4.6 Variation of m_1 and m_2

In table 9 ω is fixed at value 5, hence $K_{33}^f = 5K_{33}^p$. Also, $m_2 = 0.9, s_1 = 0.3$ and $s_2 = 1.1$. Thus

$$0.9K_{22}^f = K_{33}^f = 5K_{33}^p = 1.5K_{11}^p = 5.5K_{22}^p.$$

This means that

$$K_{22}^f > K_{33}^f > K_{11}^p >> K_{33}^p > K_{22}^p. \tag{39}$$

When $m_1 \geq 2.20$ it is seen numerically that $l = 0$ and rolls along the $x-$axis form. This inequality on m_1 means $K_{33}^f \geq 2.20K_{11}^f$ so

$$K_{22}^f > K_{33}^f >> K_{11}^f.$$

Since these values are larger than the micro permeability values, see (39), it follows that the fluid moves in the macro pores in the y and z directions and this is consistent with rolls aligned along the $x-$axis.

At the other transition in table 9, $m_1 \leq 1.041$. Inequalities (39) still hold but now $K_{33}^f \leq 1.041K_{11}^f$, and $K_{22}^f \leq 1.157K_{11}^f$. From (39) $K_{11}^p >> K_{33}^p > K_{22}^p$ and with the K_{11}^f values now being close to or much smaller than the K_{22}^f values the solution switches to one depending on x and z, and so rolls along the $y-$axis are witnessed. For m_1 in the region (1.041,2.2) the macro and micro effects counterbalance each other and three-dimensional cells are found.

For table 10, $K_{33}^f = 5K_{11}^f,\ K_{33}^f = 5K_{33}^p,\ K_{33}^p = 0.3K_{11}^p$ and $K_{33}^p = 1.1K_{22}^p$. Thus,

$$K_{33}^f = 5K_{11}^f = 5K_{33}^p = 1.5K_{11}^p = 5.5K_{22}^p.$$

Hence,

$$K_{33}^f > K_{11}^p >> K_{11}^f = K_{33}^p > K_{22}^p. \tag{40}$$

For the transition to $x-$rolls, $m_2 \leq 2.095$, or $5K_{11}^f = K_{33}^f \leq 2.095K_{22}^f$. Therefore,

$$K_{22}^f \geq \frac{5}{2.095}K_{11}^f \approx 2.39K_{11}^f.$$

Since from (40) K_{33}^f dominates the buoyancy effect it appears K_{33}^f and K_{22}^f will play the major role in determining solution structure and so this in turn will depend on y and z and lead to $x-$rolls as seen in table 10.

For the next transition where $y-$rolls are seen, $m_2 \geq 4.255$. Hence, $K_{33}^f \geq 4.255K_{22}^f$ and $K_{33}^f = 5K_{11}^f$. This suggests the micro permeabilities also come into play. Relations (40) still hold and so since $K_{11}^p >> K_{22}^p$ we expect the solution to depend on x and z and so $y-$rolls are found, as seen in table 10. For $m_2 \in (2.095, 4.255)$ three-dimensional cells are observed according to table 10.

4.7 Variation of s_1 and s_2

In table 11 we have $K_{33}^f = 5K_{33}^p$, $K_{33}^f = 5K_{11}^f$, $K_{33}^f = 0.9K_{22}^f$ and $K_{33}^p = 1.1K_{22}^p$. This means

$$K_{22}^f > K_{33}^f >> K_{11}^f.$$

This favours the formation of $x-$rolls. The transition is when $s_1 \geq 0.22$. Thus, we have

$$1.1K_{22}^p = K_{33}^p \geq 0.22K_{11}^p$$

and so

$$K_{22}^p \geq 0.2K_{11}^p.$$

This also favours the production of $x-$rolls. It should be observed, however, that when $s_1 \leq 0.21$ the micro permeabilities are having a pronounced effect and lead to the predicition of three-dimensional cells.

In table 12 the values are such that

$$K_{33}^f = 5K_{33}^p = 5K_{11}^f = 0.9K_{22}^f = 1.5K_{11}^p.$$

Thus,

$$K_{22}^f > K_{33}^f > K_{11}^p >> K_{33}^p = K_{11}^f.$$

This would suggest that the macro permeabilties are dominating convection, $K_{33}^f >> K_{33}^p$, and since $K_{22}^f > K_{33}^f$ one might expect $x-$rolls. This is seen in table 12 provided $K_{33}^p \leq 1.5K_{22}^p$. However, once $K_{33}^p > 1.5K_{22}^p$ the micro permeability effects become important and the solution transfers to three-dimensional convection cells.

There are two special cases worth mentioning. The first is where a) $s_1 = s_2$ and m_α vary, or where b) $m_1 = m_2$ and s_α vary. In this case the solution behaves as in the one porosity situation in that for case a) with $m_1 = m_2$ and then b) with $s_1 = s_2$ the switch is from $x-$rolls to $y-$rolls without any intermediate region where three-dimensional cells form, excepting for one special value where the minimization depends only on $a^2 = l^2 + m^2$. For example, if we take $s_1 = s_2 = 1$, $m_2 = 0.9$, $\lambda = 0.1$, $\omega = 3$ when we select $m_1 = 0.9$ and perform the minimization we find $Ra = 29.440$ and $a^2 = 9.518$. As another example, take $m_1 = m_2 = 1$, $s_2 = 0.9$, $\lambda = 0.1$, $\omega = 3$. Then when $s_1 = 0.9$ we find $Ra = 30.048$ and $a^2 = 9.716$.

In tables 9-11 one observes that for $m_1 \geq 2.2$, for $m_2 \geq 4,255$ and for $s_1 \geq 0.22$, respectively, then Ra and the appropriate non-zero value of m or l do not change. This may be explained by inspection of (28) and taking $l = 0$ for the situation of tables 9 and 11 and $m = 0$ for table 10. It is seen in that case that when $m = 0$ certain c_{ij} terms contain no m_2 or s_2, or when $l = 0$, c_{ij} terms contain no m_1 or s_1.

We conclude that the fully anisotropic bidispersive problem contains many interesting effects which await experimental observation. What is now very much required are numerical values for real materials for the interaction coefficient ζ and the macro and micro permeability coefficients. An analysis akin to that of Rees [46, 47] where he cleverly produced estimates for coefficients in a local thermal non-equilibrium porous material would be very welcome in the bidispersive case.

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Ra	ω	Ra/ω	l	m	$\hat{x}$	$\hat{y}$	$\hat{y}/\hat{x}$
3.108	0.01	310.8	0	3.142	∞	2.000	0
3.948	0.04	98.7	0	3.142	∞	2.000	0
3.975	0.041	96.951	0.431	3.112	14.569	2.019	0.139
4.082	0.045	90.711	0.921	3.001	6.819	2.093	0.307
4.210	0.05	84.2	1.238	2.879	5.075	2.182	0.430
5.319	0.1	53.19	2.093	2.179	3.002	2.884	0.961
8.802	0.3	29.34	2.162	1.690	2.907	3.718	1.279
11.759	0.5	23.518	2.010	1.812	3.127	3.470	1.111
17.666	1	17.666	1.736	2.186	3.619	2.874	0.794
34.203	5	6.841	0.868	2.987	7.236	2.104	0.291
37.953	9	4.217	0.321	3.125	19.578	2.011	0.103
38.209	9.5	4.022	0.224	3.134	28.099	2.005	0.0714
38.395	9.9	3.878	0.100	3.140	62.832	2.001	0.0318
38.435	9.99	3.847	0.316	3.141	198.692	2.000	0.0101
38.440	10	3.844	0	3.142	∞	2.000	0
39.960	15	2.664	0	3.142	∞	2.000	0

Table 1: Critical values of Ra, l and m for quoted values of ω. The numbers $\hat{x}$ and $\hat{y}$ represent the critical wavelengths. Here $\lambda = 0.1$, $m_1 = 10, m_2 = 1$, $s_1 = 0.1, s_2 = 1$.

Ra	ω	Ra/ω	l	m	$\hat{x}$	$\hat{y}$	$\hat{y}/\hat{x}$
8.278	0.1	82.777	0	3.142	∞	2.000	0
14.099	0.5	28.198	0	3.142	∞	2.000	0
19.739	1.0	19.739	0	3.142	∞	2.000	0
19.838	1.01	19.641	0.155	3.138	40.558	2.002	0.0494
20.226	1.05	19.262	0.339	3.123	18.528	2.012	0.109
21.607	1.20	18.006	0.616	3.079	10.206	2.041	0.200
24.073	1.50	16.049	0.822	3.025	7.642	2.077	0.272
27.519	2.0	13.759	0.919	2.990	6.835	2.100	0.307
30.353	2.5	12.141	0.925	2.986	6.791	2.104	0.310
32.733	3.0	10.911	0.900	2.993	6.981	2.099	0.301
39.369	5.0	7.874	0.723	3.048	8.688	2.061	0.237
46.054	9.0	5.117	0.285	3.128	22.077	2.008	0.091
47.061	9.99	4.711	0.0316	3.142	198.692	2.000	0.0101
47.070	10.0	4.707	0	3.142	∞	2.000	0
50.445	15.0	3.363	0	3.142	∞	2.000	0

Table 2: Critical values of Ra, l and m for quoted values of ω. The numbers $\hat{x}$ and $\hat{y}$ represent the critical wavelengths. Here $\lambda = 0.5$, $m_1 = 10, m_2 = 1$, $s_1 = 0.1, s_2 = 1$.

ω	Ra/ω ($\lambda = 0.1$)	Ra/ω ($\lambda = 0.5$)
0.1	53.19	82.777
0.5	23.518	28.198
1	17.666	19.739
5	6.841	7.874
10	3.844	4.707

Table 3: Critical values of Ra/ω vs. ω for quoted values of λ. Here $m_1 = 10, m_2 = 1, s_1 = 0.1, s_2 = 1$.

Ra	ω	Ra/ω	l	m	$\hat{x}$	$\hat{y}$	$\hat{y}/\hat{x}$
2.191	0.05	43.82	0	3.142	∞	2.000	0
3.844	0.1	38.44	0	3.142	∞	2.000	0
4.157	0.11	37.791	0.305	3.126	20.603	2.010	0.0975
5.340	0.15	35.6	0.653	3.062	9.627	2.052	0.213
8.914	0.3	29.713	1.145	2.811	5.490	2.235	0.407
12.090	0.5	24.18	1.421	2.488	4.421	2.525	0.572
15.829	1.0	15.829	1.703	1.749	3.690	3.592	0.974
16.993	1.5	11.329	1.811	1.000	3.469	6.283	1.820
17.191	1.8	9.551	1.845	0.285	3.406	22.077	6.483
17.196	1.82	9.448	1.847	0.158	3.403	39.738	11.679
17.199	1.83	9.398	1.847	0	3.401	∞	∞
17.215	1.9	9.061	1.846	0	3.404	∞	∞

Table 4: Critical values of Ra, l and m for quoted values of ω. The numbers $\hat{x}$ and $\hat{y}$ represent the critical wavelengths. Here $\lambda = 0.01$, $m_1 = 0.1, m_2 = 1$, $s_1 = 10, s_2 = 1$.

Ra	ω	Ra/ω	l	m	$\hat{x}$	$\hat{y}$	$\hat{y}/\hat{x}$
4.220	0.05	84.4	0	3.142	∞	2.000	0
5.527	0.1	55.27	0	3.142	∞	2.000	0
5.778	0.11	52.527	0.214	3.134	29.296	2.005	0.0684
6.745	0.15	44.967	0.503	3.099	12.492	2.028	0.162
7.874	0.2	39.37	0.723	3.048	8.688	2.061	0.237
9.878	0.3	32.927	1.010	2.939	6.221	2.138	0.344
13.034	0.5	26.068	1.345	2.715	4.670	2.314	0.496
17.666	1.0	17.666	1.736	2.187	3.619	2.874	0.794
20.839	2.0	10.420	2.016	1.211	3.116	5.188	1.665
21.264	2.5	8.506	2.079	0.598	3.022	10.516	3.479
21.332	2.68	7.960	2.096	0.122	2.998	51.302	17.111
21.335	2.69	7.931	2.096	0	2.997	∞	∞
21.411	3.0	7.137	2.092	0	3.004	∞	∞

Table 5: Critical values of Ra, l and m for quoted values of ω. The numbers $\hat{x}$ and $\hat{y}$ represent the critical wavelengths. Here $\lambda = 0.1$, $m_1 = 0.1, m_2 = 1$, $s_1 = 10, s_2 = 1$.

Ra	ω	Ra/ω	l	m	$\hat{x}$	$\hat{y}$	$\hat{y}/\hat{x}$
14.099	0.5	28.198	0	3.142	∞	2.000	0
19.739	1.0	19.739	0	3.142	∞	2.000	0
19.838	1.01	19.642	0.158	3.138	39.738	2.002	0.0504
20.226	1.05	19.263	0.349	3.122	17.989	2.013	0.119
24.022	1.5	16.015	1.057	2.944	5.945	2.134	0.359
27.241	2.0	13.561	1.420	2.752	4.424	2.284	0.516
34.173	4.0	8.543	2.068	2.063	3.039	3.046	1.002
36.792	6.0	6.132	2.338	1.446	2.687	4.344	1.616
37.801	8.0	4.725	2.488	0.757	2.525	8.300	3.287
37.999	8.9	4.270	2.536	0.161	2.478	38.967	15.725
38.005	8.94	4.251	2.538	0.0548	2.476	114.715	46.329
38.007	8.95	4.247	2.538	0	2.476	∞	∞
38.015	9.0	4.224	2.537	0	2.476	∞	∞

Table 6: Critical values of Ra, l and m for quoted values of ω. The numbers $\hat{x}$ and $\hat{y}$ represent the critical wavelengths. Here $\lambda = 0.5$, $m_1 = 0.1, m_2 = 1$, $s_1 = 10, s_2 = 1$.

ω	Ra/ω ($\lambda = 0.01$)	Ra/ω ($\lambda = 0.1$)	Ra/ω ($\lambda = 0.5$)
0.05	43.82	84.4	
0.1	38.44	55.27	
0.5	24.18	26.068	28.198
1	15.829	17.666	19.739
2	8.618	10.420	13.561

Table 7: Critical values of Ra/ω vs. ω for quoted values of λ. Here $m_1 = 0.1, m_2 = 1, s_1 = 10, s_2 = 1$.

Ra	ω	Ra/ω	l	m	$\hat{x}$	$\hat{y}$	$\hat{y}/\hat{x}$
3.077	0.01	307.7	0	3.125	∞	2.011	0
4.002	0.042	95.286	0	3.140	∞	2.001	0
4.030	0.043	93.721	0.375	3.118	16.733	2.015	0.120
5.427	0.1	54.27	2.088	2.275	3.009	2.762	0.918
12.711	0.5	25.422	2.081	2.123	3.019	2.959	0.980
18.981	1	18.981	1.711	2.520	3.671	2.493	0.679
29.844	3	9.948	0.345	3.086	18.214	2.036	0.112
30.213	3.14	9.622	0.0447	3.103	140.496	2.025	0.0144
30.238	3.15	9.599	0	3.103	∞	2.025	0
31.057	3.5	8.873	0	3.101	∞	2.026	0
37.033	10	3.703	0	3.082	∞	2.039	0

Table 8: Critical values of Ra, l and m for quoted values of ω. The numbers $\hat{x}$ and $\hat{y}$ represent the critical wavelengths. Here $\lambda = 0.1$, $m_1 = 5.0, m_2 = 0.9$, $s_1 = 0.3, s_2 = 1.1$.

Ra	m_1	l	m	$\hat{x}$	$\hat{y}$	$\hat{y}/\hat{x}$
17.489	0.1	2.040	0	3.080	∞	∞
24.570	0.5	2.587	0	2.428	∞	∞
31.020	1.0	2.903	0	2.164	∞	∞
31.490	1.041	2.922	0	2.150	∞	∞
31.501	1.042	2.916	0.195	2.154	32.232	14.960
31.534	1.045	2.892	0.446	2.173	14.085	6.482
32.039	1.1	2.519	1.578	2.495	3.981	1.596
33.299	1.5	1.305	2.790	4.816	2.252	0.468
33.524	2	0.533	3.048	11.790	2.062	0.175
33.533	2.15	0.251	3.083	25.033	2.038	0.0814
33.534	2.19	0.110	3.091	57.357	2.033	0.0354
33.534	2.2	0	3.093	∞	2.031	0
33.534	2.5	0	3.093	∞	2.031	0
33.534	10	0	3.093	∞	2.032	0

Table 9: Critical values of Ra, l and m for varying values of m_1. The numbers $\hat{x}$ and $\hat{y}$ represent the critical wavelengths. Here $\lambda = 0.1$, $\omega = 5.0$, $m_2 = 0.9$, $s_1 = 0.3, s_2 = 1.1$.

Ra	m_2	l	m	$\hat{x}$	$\hat{y}$	$\hat{y}/\hat{x}$
19.573	0.1	0	2.132	∞	2.947	0
34.877	1	0	3.156	∞	1.991	0
46.420	2	0	3.585	∞	1.753	0
47.389	2.095	0	3.613	∞	1.739	0
47.440	2.1	0.0632	3.614	99.346	1.739	0.0175
55.494	3	1.495	3.417	4.202	1.839	0.438
61.055	4	2.851	2.057	2.204	3.055	1.387
61.421	4.25	3.335	0.405	1.884	15.515	8.235
61.421	4.255	3.353	0	1.874	∞	∞
61.421	5	3.353	0	1.874	∞	∞
61.421	10	3.353	0	1.874	∞	∞

Table 10: Critical values of Ra, l and m for varying values of m_2. The numbers $\hat{x}$ and $\hat{y}$ represent the critical wavelengths. Here $\lambda = 0.1$, $\omega = 5.0$, $m_1 = 5.0$, $s_1 = 0.3, s_2 = 1.1$.

Ra	s_1	l	m	$\hat{x}$	$\hat{y}$	$\hat{y}/\hat{x}$
30.169	10^{-4}	0.947	2.628	6.638	2.391	0.360
30.225	10^{-3}	0.954	2.631	6.587	2.388	0.363
30.708	0.01	1.009	2.662	6.224	2.361	0.379
32.904	0.1	1.019	2.873	6.167	2.187	0.355
33.522	0.2	0.451	3.061	13.945	2.053	0.147
33.531	0.21	0.300	3.079	20.944	2.041	0.097
33.534	0.22	0	3.093	∞	2.031	0
33.534	0.3	0	3.093	∞	2.031	0
33.534	1	0	3.093	∞	2.031	0
33.534	10	0	3.093	∞	2.031	0

Table 11: Critical values of Ra, l and m for varying values of s_1. The numbers $\hat{x}$ and $\hat{y}$ represent the critical wavelengths. Here $\lambda = 0.1$, $\omega = 5.0$, $m_1 = 5.0$, $m_2 = 0.9, s_2 = 1.1$.

Ra	s_2	l	m	$\hat{x}$	$\hat{y}$	$\hat{y}/\hat{x}$
26.887	0.01	0	2.475	∞	2.538	0
28.058	0.1	0	2.657	∞	2.365	0
33.217	1	0	3.079	∞	2.040	0
34.582	1.5	0	3.128	∞	2.009	0
34.605	1.51	0.110	3.127	57.357	2.009	0.0350
34.692	1.55	0.293	3.120	21.425	2.014	0.0940
35.418	2	0.835	3.057	7.521	2.055	0.273
36.224	3	1.164	2.990	5.400	2.102	0.389
36.849	5	1.359	2.938	4.623	2.138	0.463
37.306	10	1.483	2.901	4.236	2.166	0.511
37.709	100	1.583	2.868	3.969	2.191	0.552

Table 12: Critical values of Ra, l and m for varying values of s_2. The numbers $\hat{x}$ and $\hat{y}$ represent the critical wavelengths. Here $\lambda = 0.1$, $\omega = 5.0$, $m_1 = 5.0$, $m_2 = 0.9, s_1 = 0.3$.