

Internal habits formation and optimality

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Abstract

In a very influential model with internal habits, Carroll et al. [9, 10], establish that an increase in economic growth may cause a positive change in savings. The optimality of this result, and of many other contributions using a similar framework, has been questioned by some authors who have observed that the parametrization used in these models always implies a utility function not jointly concave in consumption and habits. In this paper, we revisit the optimality issue and, using advanced techniques in Dynamic Programming, we answer the following long-standing open questions: i) Is the solution found in [9, 10] optimal? ii) Is it also unique or do other optimal solutions exist?

JEL Classification: C61, D91, E21, O40

Keywords Endogenous Growth; Habit Formation, Sufficient Conditions of Optimality, Dynamic Programming, Viscosity Solution.

1 Introduction

In a couple of very influential works, Carroll et al. [9, 10] have studied the dynamics of an endogenous growth model with internal habits formation and they have shown that internal habit formation is a crucial channel to explain how an increase in economic growth may cause an increase in savings. This result was even more relevant since it was derived using a “multiplicative” form of habits instead of a “subtractive” form, whose drawbacks include an addictive behavior since consumption has to remain always higher than the habit stock and possibly an infinite negative utility (see Carroll [8]).

Since then, several authors have observed how “multiplicative” habits can be used to enhance the explanatory power of their models in several dimensions. For example, Fuhrer [22] has observed that habit formation improves significantly the effects of spending and inflation to monetary-policy actions; Diaz et al [16] have studied the role of habit formation in shaping precautionary savings and the wealth distribution in economies with heterogeneous agents; finally, multiplicative habits have been introduced also in portfolio choice models, see for example Gomes and Michaelides [23] among others.¹

However, other authors have recently questioned the optimality of the result found by Carroll et al. and, as a consequences, the optimality of all the previously mentioned contributions. In fact, it was observed that Carroll et al.’s result, as well as all the other results, was obtained in a parameters’ set, let us call it Θ , where the multiplicative utility function is never jointly concave in consumption and habits. In particular, they have noticed that joint concavity is never possible under the (realistic) assumptions of a greater than one coefficient of relative risk aversion, i.e. $\sigma > 1$, and of agents weighting habits less than consumption, i.e. $0 < \gamma < 1$ (e.g. Caballe et al. [6], page 1669, and more recently Yang and Zhang [33]).²

As a consequence, the sufficiency conditions for optimality are not satisfied;³ as clearly explained by Seierstad and Sydsaeter (see page 103 in [31]) the sufficiency conditions for

¹Further examples of contributions using “multiplicative” habits are Dennis [15], Gomez [24] and Weil [32] among others.

²Actually the problem of concavity still holds when $\sigma < 1$.

³More precisely, it will be shown later that neither the Hamiltonian nor the maximum value Hamiltonian are concave and therefore the Mangasarian’s and Arrow’s sufficiency conditions for optimality are not respected.

optimality are crucial because “(...) the Maximum Principle cannot by itself tell us whether a given candidate is optimal or not, nor does it tell us whether or not an optimal solution exists”. In other words, the solution candidate could be either a maximum or a minimum or neither of them.

In addition, it is even not clear, without the strict concavity of the objective function, if the optimal control is unique or not (e.g. Acemoglu [1], Theorem 6.4 and Corollary 6.1, page 189-190). Other candidate optimal controls cannot be excluded only looking at the analysis done in Carroll et al. [9, 10], because the authors have investigated the *local dynamics* around the steady state but not the global dynamics. Interestingly, there are several examples in the literature of models without a strictly concave Hamiltonian and multiple optimal controls, e.g. Dechert and Nishimura [14] and Kamihigashi and Roy [26] among others. Actually Alonso-Carrera et al [6] show that in a variation of the model with multiplicative habits originally proposed by Carroll et al, non-monotonic and discontinuous consumption paths are consistent with the necessary conditions for optimality.⁴ Such a finding was at odd with the smooth and monotonic consumption path found by Carroll et al. and reinforced the suspect that multiple optimal paths could actually exist.

In other words, Carroll et al. [9, 10], by looking at the first order conditions from the Maximum Principle, have found a *candidate* optimal control but they have fallen short of proving that this candidate is indeed the unique optimal control. As previously explained, the candidate could be either a maximum or a minimum or neither of them.⁵ Clearly, this is worrying especially because the model proposed by these authors have been extensively used and enriched by other authors and even worse some of these contributions give policy advices without actually knowing if the solution of their models is the unique optimum.

A simple way to restore optimality and uniqueness consists in imposing conditions on the parameters such that the utility function becomes strictly concave (e.g. Yang and Zhang [33]). Unfortunately, for a coefficient of relative risk aversion greater than one, i.e. $\sigma > 1$, concavity is restored only under the unattractive and rather implausible

⁴Although similar, the model in Alonso-Carrera et al [6] departs from the setting studied by Carroll et al by considering discrete time and a Jones-Manuelli production function instead of a linear technology.

⁵In the Appendix C we provide, for the reader’s convenience, a simple textbook example where the candidate optimal control is not actually a maximum.

assumption that agents care about the habits more than about consumption, i.e. $\gamma > 1$. Most importantly, under this assumption, Carroll et al.'s result on the relation between economic growth and saving does not hold anymore. It would be, therefore, important to understand whether optimality could be achieved even without this strong and implausible assumption.

In this short paper, we revisit this issue using a Dynamic Programming approach. The advantage of Dynamic Programming is that it investigates the *global dynamics* and, most importantly, it identifies *optimality conditions which are sufficient independently from any concavity assumption* and it provides the optimal strategies in feedback form (see among others Salsa and Squellati [30], Bardi and Capuzzo-Dolcetta [5] Chapter III, section 2.5, Fabbri et al [18] section 2.5). Therefore, using a Dynamic Programming approach seems a very natural choice in this context.⁶

However, the application of the Dynamic Programming approach to our specific problem is not straightforward because it requires a non trivial adaptation of the existing literature on optimal control and viscosity solutions.⁷ Although this literature was developed in the last four decades starting with the seminal work of Crandall and Lions [11, 12],⁸ its application to macroeconomic problems is more recent (see e.g. the recent contribution by Achdou et al. [3]).

In our framework, this theory can be applied after taking the homogeneity properties of the problem into account (see e.g. Freni et al. [21, 20]). Once this is done, we use the theory of viscosity solutions for the associated Hamilton-Jacobi-Bellman equation to prove the differentiability of the value function (see e.g. Cannarsa and Soner [7] or Bardi and Capuzzo-Dolcetta [5] for results of this type) and to solve the Closed Loop Equation (see e.g. Freni et al. [20]).

In particular, the existence and uniqueness of optimal feedback strategies can still be proved, even without a concave value function, because our problem has the follow-

⁶Augeraud-Veron et al [4] show how the dynamic programming approach can be used to solve an internal habit formation model where habits enters the utility function in a subtractive form.

⁷More precisely, the adaptation requires the following steps: first, a change of variables to reduce the dimension of the problem; second, a proof of the differentiability of the value function of the problem in the new variables; finally a proof of the existence of optimal feedback strategies. The second and last step require a non-trivial extension of existing methods.

⁸See also Crandall et al. [13] or Bardi and Capuzzo-Dolcetta [5] among others.

ing properties: linearity of the state equations, monotonicity of the utility function and differentiability of the value function.

Following this approach, we are therefore able to prove that a unique optimal control strategy exists. Such optimal strategy is given in feedback form and it is equal to the candidate optimal strategy found by Carroll et al. [10]. In this way, we provide a solid theoretical background to Carroll et al. [10] result on the relation between economic growth and savings, and most importantly we answer to the raising concerns on the validity/optimality of this prediction as well as of the predictions of all the contributions in the literature using multiplicative habits. Contrary to what found in previous contributions (e.g. Alonso-Carrera et al [6], page 1684), we also confirm that the uniqueness and optimality of the solution still holds for relative risk aversion coefficient's values lower than one and that such a solution converges without oscillations to the balanced growth path.

Moreover, our paper represents a neat example of how optimal control problems without a concave objective function can be dealt with using a Dynamic Programming approach. In particular, we show how much effective the dynamic programming approach can be in dealing with problems of optimality and uniqueness which would be otherwise unsolvable by simply looking at the Pontryagin's Maximum Principle.⁹

The paper is organized as it follows. In Section 2, the optimal control problem is introduced and it is shown that standard sufficiency conditions for optimality do not hold in the parameters set usually assumed in the literature. In Section 3, we move to the Dynamic Programming approach and we explain the key steps to solve the control problem and to prove the existence and uniqueness of an optimal control. Section 4 concludes the paper. Appendix A contains more details on the derivations to check the sufficiency conditions of optimality, while Appendix B contains more details on the Dynamic programming approach followed to solve our problem. Finally Appendix C contains, for the reader's convenience, a simple textbook example of an optimal control problem with a non-concave payoff function where a solution of the maximum principle's

⁹It is worth mentioning that this does not mean that the Dynamic Programming approach is more powerful than the Maximum Principle. This strongly depends on the problem under study and, in general, integrating the two approaches is the best strategy to deal with difficult problems like those characterized by non-concavity.

necessary conditions is not optimal.

2 The Optimal Control Problem

The optimal control problem studied by Carroll et al. [9, 10] is the following:

$$\max_c \int_0^\infty \frac{\left(\frac{c}{h^\gamma}\right)^{1-\sigma}}{1-\sigma} e^{-\theta t} dt \quad (1)$$

subject to

$$\dot{k} = Bk - c, \quad k(0) = k_0 \quad (2)$$

$$\dot{h} = \rho(c - h), \quad h(0) = h_0 \quad (3)$$

$$k_0 > 0 \text{ given}, \quad h_0 > 0 \text{ given} \quad (4)$$

$$k > 0, \quad h > 0, \quad c > 0, \quad (5)$$

where $B := A - \delta$, and the set of parameters conditions is:

$$\Theta := \{(A, \theta, \rho, \delta, \sigma, \gamma) : A, \theta > 0, \rho \in (0, 1), \delta \in [0, 1], \sigma \in (1, \infty), \gamma \in [0, 1]\}. \quad (6)$$

More formally, the optimal control can be described as it follows. Let us denote with $\mathcal{A}(k_0, h_0)$ the class of admissible controls containing all locally integrable functions $c(\cdot) : [0, +\infty) \rightarrow (0, +\infty)$ such that the unique solution of the initial value problem (2)-(3) is defined in $[0, \infty)$ and respects the inequality constraints (5), given the set of parameters' conditions Θ . Note that we underline the dependence of such set on the initial data $(k_0, h_0) \in \mathbb{R}_{++}^2$.

Then, the objective of this optimal control problem is to find the $c^* \in \mathcal{A}(k_0, h_0)$ which maximizes the functional

$$\mathcal{W}(k_0, h_0; c) := \int_0^\infty \frac{\left(\frac{c}{h^\gamma}\right)^{1-\sigma}}{1-\sigma} e^{-\theta t} dt,$$

i.e. to find $c^* \in \mathcal{A}(k_0, h_0)$ such that

$$\mathcal{W}(k_0, h_0; c^*) = \sup_{c \in \mathcal{A}(k_0, h_0)} \mathcal{W}(k_0, h_0; c)$$

From now on, we call **(P)** this problem. As usual, we define the value function of problem **(P)** as the map

$$V(k_0, h_0) := \sup_{c \in \mathcal{A}(k_0, h_0)} \mathcal{W}(k_0, h_0; c). \quad (7)$$

Carroll et al. [9, 10] use the Pontryagin's Maximum Principle (PMP) and find a balanced growth path, as well as paths converging to it, which solve PMP and are then solution candidates. However, the PMP provides the *necessary (but not sufficient) conditions* for optimality. In other words, the PMP cannot, by itself, tell us if a candidate is optimal or not, nor whether an optimal solution does exist.

On the other hand, if certain concavity/convexity conditions on the functions involved, are respected, then any solution candidate identified by the PMP is optimal.¹⁰ Such sufficiency conditions for optimality are mainly of two types.

The first, usually known as the Mangasarian sufficiency theorem for optimality, assumes the joint concavity in the control and state variables of the so-called “current value Hamiltonian” (see the original paper in finite horizon of Mangasarian [28] and, for the case treated here, Seierstad and Sydsaeter [31], Theorem 11, p.385). In our framework, the current value Hamiltonian is

$$H_{CV}(h, k, \psi, \lambda; c) \equiv \frac{\left(\frac{c}{h^\gamma}\right)^{1-\sigma}}{1-\sigma} + \psi[(A - \delta)k - c] + \lambda\rho(c - h), \quad (8)$$

which is concave as long as the utility function is (jointly) concave in (c, h) .¹¹ Unfortunately, the utility function is never jointly concave in c and h in the parameters' set Θ since concavity implies $\gamma \geq \frac{\sigma}{\sigma-1} > 1$ for $\sigma > 1$ (see Appendix A – “Mangasarian sufficiency conditions”, and also Caballe et al. [6], page 1669).¹²

The other sufficiency condition for optimality is based on the Arrow sufficiency theorem (e.g. Seierstad and Sydsaeter [31], Theorem 14, p.236-237 or also Theorem 11, p.385-386) which requires the joint concavity in the state variables of the “maximum value Hamiltonian”. In our framework, the maximum value Hamiltonian,

$$H_{MAX}(h, k, \psi, \lambda) \equiv \max_{c \in \mathcal{A}} H_{CV}(h, k, \psi, \lambda; c),$$

¹⁰To emphasize again the importance of the sufficiency conditions, we notice that, without them, it cannot be claimed that the solutions of first order conditions are actually optimal without further investigation. Appendix C contains a simple textbook example in this direction.

¹¹This is indeed the case because the state equations are linear in (c, h, k) and the Hamiltonian is separable in the state variables.

¹²It is worth mentioning that the problem of concavity still holds when $\sigma < 1$. Although this case is not directly dealt in this paper, a trivial adaptation of our analysis shows that even in this case the optimal control exists and it is unique. Similarly to our framework, it is also confirmed the no-oscillatory convergence to the BGP.

can be proved to be never jointly concave in (h, k) in the parameter set Θ and, therefore, the Arrow's sufficiency conditions for optimality do not hold (see Appendix A – “Arrow sufficiency conditions”). Exactly as before, the concavity can be restored when $\gamma \geq \frac{\sigma}{\sigma-1} > 1$ for $\sigma > 1$ which is not included in the parameter set Θ .

Therefore, further investigation is needed to verify whether the solution candidate identified by Carroll et al. [10] is optimal or not. In fact, the two types of sufficiency conditions for optimality, that we have checked above, are sufficient but not necessary, meaning that the parameter's condition, $\sigma > 1$ and $\gamma \geq \frac{\sigma}{\sigma-1} > 1$, suggested by these criteria, could be actually too restrictive.

Another issue related to the non-concavity of the utility function, is that the value function, $V(k_0, h_0)$, may be not strictly concave and multiple optimal controls could exist (e.g. Acemoglu [1], Theorem 6.4 and Corollary 6.1, page 189-190). Examples of models with a non-concave Hamiltonian and multiple optimal paths are Dechert and Nishimura [14] and Kamihigashi and Roy [26], among others.

All the above, as already observed in the introduction, explains why we decided to investigate the problem using the Dynamic Programming approach. This is done in the next section where we also provide the main results (Theorem 1 and Corollaries 1-2) which show that the strategy provided in Carroll et al. [10] is indeed the optimal one and we provide it in an appropriate feedback form.

3 Dynamic Programming

In this section we develop the Dynamic Programming approach to problem **(P)** and we show that, for every initial condition (k_0, h_0) , there exists a unique optimal consumption strategy $c_{(k_0, h_0)}^*(\cdot)$ which is given in closed-loop form.

The problem is not standard and cannot be solved using existing results in the literature, even the recent one. We then develop an ad hoc method to solve it.¹³ The steps of our method are the following.

- (I) We use the homogeneity of the utility function $u(c, h)$ to prove the homogeneity of the value function (see Appendix B.1) and to reduce the solution of the problem

¹³More details can be found in Appendix B.

(**P**) (which displays the two state variables k and h) to the solution of an auxiliary problem ($\tilde{\mathbf{P}}$) with one state variable z , corresponding to k/h ((see Appendix B.2)).

(II) We prove that the Hamilton-Jacobi-Bellman equation of problem ($\tilde{\mathbf{P}}$) admits a unique regular solution which can be used to find the unique optimal feedback strategy for it (see Appendix B.3).

(III) We go back to the problem (**P**) expressing its unique optimal strategy in terms of the optimal strategy of problem ($\tilde{\mathbf{P}}$) (see Appendix B.4).

We now provide a brief explanation of such steps and present our main results.

Roughly speaking, the auxiliary control problem ($\tilde{\mathbf{P}}$) is obtained by rewriting the original optimal control problem (**P**) in the new state-like variable and control-like variable which are respectively

$$z(t) := k(t)/h(t) \quad \text{and} \quad a(t) := c(t)/h(t).$$

The state equation becomes

$$z'(t) = (B + \rho)z(t) - (1 + \rho z(t))a(t), \quad z(0) = z_0 > 0, \quad (9)$$

as shown in Appendix B.2., while the functional in the new variables becomes

$$J(z_0; a) := \int_0^{+\infty} \frac{a(t)^{1-\sigma}}{1-\sigma} e^{-(\theta+\rho\eta)t+\rho\eta \int_0^t a(s)ds} dt = \mathcal{W}(k_0, 1; c), \quad (10)$$

where, for brevity, we set $\eta := (1 - \gamma)(1 - \sigma)$. The set of admissible controls, $\tilde{\mathcal{A}}(z_0)$, contains all locally integrable functions $a(\cdot) : [0, +\infty) \rightarrow (0, +\infty)$ such that the unique solution of the initial value problem (9) is strictly positive at any $t \geq 0$. The value function is

$$V_0(z_0) := \sup_{a(\cdot) \in \tilde{\mathcal{A}}(z_0)} J(z_0; a). \quad (11)$$

The HJB equation associated to this auxiliary problem ($\tilde{\mathbf{P}}$) is the following (with unknown solution v_0 and current variable z).

$$[\theta + \rho\eta] v_0(z) = (B + \rho)z v_0'(z) + \frac{\sigma}{1-\sigma} [v_0'(z)(1 + \rho z) - \rho\eta v_0(z)]^{1-\sigma^{-1}} \quad (12)$$

Using the Hamiltonian notation we can rewrite the above equation as

$$[\theta + \rho\eta] v_0(z) = \tilde{H}_{MAX}(z, v_0(z), v_0'(z)) \quad (13)$$

where

$$\tilde{H}_{CV}(z, q, p; a) := (B + \rho)zp - (1 + \rho z)ap + \rho\eta aq + \frac{a^{1-\sigma}}{1-\sigma}, \quad (14)$$

$$\tilde{H}_{MAX}(z, q, p) := \sup_{a>0} \tilde{H}_{CV}(z, q, p; a) = (B + \rho)zp + \frac{\sigma}{1-\sigma} [(1 + \rho z)p - \rho\eta q]^{1-\sigma^{-1}}. \quad (15)$$

The following proposition states the equivalence of the two problems and its proof can be found in Appendix B.2.

Proposition 1 *Let V and V_0 be the value functions defined in (7) and (11), respectively. Let the parameters $(A, \theta, \rho, \delta, \sigma, \gamma)$ belong to the set Θ defined in (6). Take any $k_0 > 0, h_0 > 0$. Then the following results hold.*

(i) *We have that*

$$V(k_0, h_0) = h_0^{(1-\gamma)(1-\sigma)} V_0(k_0/h_0). \quad (16)$$

(ii) *If a control c^* is optimal at (k_0, h_0) for problem (\mathbf{P}) , then the control $a^* := c^*/h^*$ (where h^* is the solution of (3)) is optimal for problem $(\tilde{\mathbf{P}})$ at $z_0 = k_0/h_0$.*

Viceversa if a control a^ is optimal at (z_0) for problem $(\tilde{\mathbf{P}})$, then the control $c^* := h^*a^*$, where h^* is the solution of*

$$(h^*)'(t) = \rho h^*(t)[a^*(t) - 1], \quad h(0) = h_0 \quad (17)$$

is optimal at $(h_0 z_0, h_0)$ for problem (\mathbf{P}) .

Problem $(\tilde{\mathbf{P}})$ is solved in Appendix B.3; using its solution we solve problem (\mathbf{P}) in Appendix B.4. The main result of this analysis is then summarized in the following theorem.

Theorem 1 *Let V and V_0 be the value functions defined in (7) and (11), respectively. Let the parameters $(A, \theta, \rho, \delta, \sigma, \gamma)$ belong to the set Θ defined in (6). Then V_0 and V are strictly negative and of class C^1 (in \mathbb{R}_{++} and \mathbb{R}_{++}^2 , respectively). Moreover, we have the following.*

(i) *Set, for all $z > 0$,*

$$G_0(z) := \operatorname{argmax}_{a>0} \tilde{H}_{CV}(z, V_0(z), V_0'(z); a) = [(1 + \rho z)V_0'(z) - \rho(1 - \gamma)(1 - \sigma)V_0(z)]^{-\sigma^{-1}}.$$

The closed loop equation

$$z'(t) = (B + \rho)z(t) - (1 + \rho z(t))G_0(z(t)), \quad z(0) = z_0 > 0 \quad (18)$$

admits a unique solution $z^*(\cdot)$. Such solution is strictly positive for all $t \geq 0$. The closed loop control strategy

$$a^*(t) := G_0(z^*(t)) \quad (19)$$

is the unique optimal control strategy for problem $(\tilde{\mathbf{P}})$ with initial state z_0 .

(ii) Set, for all $k, h > 0$,

$$G(k, h) := \operatorname{argmax}_{c>0} H_{CV}(k, h, V_k(k, h), V_h(k, h); c) = h^{\gamma-\gamma/\sigma} [V_k(k, h) - \rho V_h(k, h)]^{-1/\sigma}.$$

The closed loop system

$$\begin{cases} k'(t) = Bk(t) - G(k(t), h(t)), & k(0) = k_0 > 0 \\ h'(t) = -\rho h(t) + \rho G(k(t), h(t)), & h(0) = h_0 > 0 \end{cases} \quad (20)$$

admits a unique solution $(k^*(\cdot), h^*(\cdot))$. Its components are both strictly positive for all $t \geq 0$. The closed loop control strategy

$$c^*(t) := G(k^*(t), h^*(t)) \quad (21)$$

is the unique optimal control strategy for problem (\mathbf{P}) with initial state (k_0, h_0) .

As a consequence we get the following corollary.

Corollary 1 Let $k_0 > 0$, $h_0 > 0$, let c^* be the optimal control strategy at (k_0, h_0) and let k^* , h^* be the associated state trajectories. Set $\psi^*(t) := V_k(k^*(t), h^*(t))$ and $\lambda^*(t) := V_h(k^*(t), h^*(t))$. Then the 5-tuple $(k^*, h^*, c^*, \psi^*, \lambda^*)$ solves the system of conditions consisting of the state equations (2)-(3) and of (see equations (21)-(23) in Carroll et al. [9])

$$\begin{cases} \frac{\partial H_{CV}}{\partial c} = 0 \\ \psi'(t) = -\frac{\partial H_{CV}}{\partial k} + \theta \psi(t) \\ \lambda'(t) = -\frac{\partial H_{CV}}{\partial h} + \theta \lambda(t) \end{cases} \quad (22)$$

and satisfies the transversality conditions (see equations (24)-(27) in Carroll et al. [9])

$$\begin{cases} \psi(t) \geq 0 \\ \lim_{t \rightarrow +\infty} e^{-\theta t} \psi(t) k(t) = 0 \\ \lambda(t) \leq 0 \\ \lim_{t \rightarrow +\infty} e^{-\theta t} \lambda(t) h(t) = 0 \end{cases} \quad (23)$$

We observe that the conditions (21)-(27) in Section 3.2 of Carroll et al. [9], corresponding to conditions (22)-(23) in this paper, are presented as “necessary and sufficient conditions”: this is not at all obvious due to the lack of concavity of the problem. For the same reason it is not clear if the solution to such system is unique. We now show that the optimal strategies/trajectories/costate indeed coincide with the ones provided in Carroll et al. [10] and [9].

First of all we look at the case of Balanced Growth Path (BGP). In such case, using Corollary 1, we can explicitly compute the optimal control strategy and the associated optimal state path and immediately show that they are the same as in Carroll et al. [10] and [9].

Second, out of the BGP, the unique optimal control-state path given in Theorem 1 solves the above system (22) (plus the transversality conditions (23)). This can be used to prove that such optimal path indeed coincides with the one numerically computed in Carroll et al. [10] and [9].

Indeed, Carroll et al. [10] and [9], numerically compute what they claim to be the optimal strategy (without proving the optimality), by considering the system (7)–(9) in [10] with a generic initial condition (k_0, c_0) and selecting, among the initial controls, the one whose associated path converges to the steady state. The following corollary proves that such a path is indeed the unique optimal control.

Corollary 2 *Let $k_0 > 0$, $h_0 > 0$, and let c^* be the unique optimal control strategy as proved in Theorem 1. Then the unique optimal path (k^*, h^*, c^*) coincides with the one computed in Carroll et al. [10] and [9] because the following results hold:*

- *The associated variables $\frac{c^*}{h^*}$, $\frac{\dot{c}^*}{c^*}$, $\frac{k^*}{h^*}$ solve the system (7)–(9) in Carroll et al. [10] with initial conditions $k(0) = k_0$, $h(0) = h_0$, $c(0) = G(k_0, h_0)$, with G derived in Theorem 1.*
- *These associated variables converge to the steady state given in (10)–(13) in Carroll et al. [10].*
- *The initial value problem (7)–(9) in Carroll et al. [10] with initial condition $k(0) = k_0$, $h(0) = h_0$, $c(0) \neq G(k_0, h_0)$ has no solution converging to the steady state.*

Finally, we observe that our result can be used to find the optimal path of the original variables, i.e. consumption, habits and capital, and that their convergence to the balanced growth path is not oscillatory.

4 Concluding Remarks

In this paper, we have first shown that the Maximum Principle cannot be used to establish the optimality of the solution in the internal habit formation model studied by Carroll et al. [9, 10] since the usual sufficiency conditions for optimality do not hold. Then, we have re-investigated the optimality issue using a Dynamic Programming approach as well as a non-trivial modification of relatively new (from an economist's perspective) results on the regularity of viscosity solutions. In this way, we have successfully proved that the solution according to which an increase in economic growth may cause a positive change in savings is actually optimal.

This result fills an important gap in the habit formation literature since the optimality in the case of a multiplicative form of habit formation is now proved. This strengthens the rationale to use this form instead of the subtractive form with all its several well known drawbacks. At the same time, our optimality result represents a reassurance to all those contributions that have used multiplicative habits in different fields of the macro-finance literature.

Finally, the advanced techniques in Dynamic Programming presented in this paper can be adapted to solve similar optimal control problems where the existence and uniqueness of an optimal control cannot be derived from the Maximum Principle due to the lack of concavity of the objective function and the uninformative results emerging by checking the usual sufficiency conditions used in the literature.

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A Appendix A: Derivations

Mangasarian sufficiency condition

We need to check whether the current value Hamiltonian (8) is concave or not. Its concavity depends on the concavity of the utility function in (c, h, k) .¹⁴ The utility function $u(c, h) = \frac{(\frac{c}{h^\gamma})^{1-\sigma}}{1-\sigma}$, is concave in $\mathbb{R}_{++}^2 := \{(c, h) : c > 0, h > 0\}$ when its Hessian

$$D^2u(c, h) = \begin{bmatrix} -\sigma \frac{c^{-\sigma-1}}{h^{\gamma(1-\sigma)}} & \gamma(\sigma-1) \frac{c^{-\sigma}}{h^{\gamma(1-\sigma)+1}} \\ \gamma(\sigma-1) \frac{c^{-\sigma}}{h^{\gamma(1-\sigma)+1}} & -\gamma(\gamma(\sigma-1)-1) \frac{c^{1-\sigma}}{h^{\gamma(1-\sigma)+2}} \end{bmatrix}$$

is negative semi-definite. This is true if and only if all the principal minors of order 1 of $D^2u(c, h)$ are ≤ 0 and the determinant is ≥ 0 . This is clearly true in the case without habits, $\gamma = 0$, no matter what are the values of the other parameters. On the other hand, in the case with habits and specifically in the parameter's set Θ , where $\gamma \in [0, 1)$ and $\sigma \geq 1$, the condition for concavity are the following:

$$u_{cc}(c, h) = -\sigma \frac{c^{-\sigma-1}}{h^{\gamma(1-\sigma)}} < 0, \quad \text{always}, \quad (24)$$

$$u_{hh}(c, h) = -\gamma(\gamma(\sigma-1)-1) \frac{c^{1-\sigma}}{h^{\gamma(1-\sigma)+2}} \leq 0, \quad \Longleftrightarrow \quad \gamma \geq \frac{1}{\sigma-1}, \quad (25)$$

$$\det(D^2u(c, h)) = \frac{c^{-2\sigma}}{h^{2\gamma(1-\sigma)+2}} [\gamma(\gamma(\sigma-1)-\sigma)] \geq 0 \quad \Longleftrightarrow \quad \sigma > 1 \text{ and } \gamma \geq \frac{\sigma}{\sigma-1} \quad (26)$$

Therefore, the last inequality is never respected and the utility function is never jointly concave in c and h in the parameters' set Θ .¹⁵

Arrow sufficiency condition

To check if the Arrow sufficiency condition for optimality is respected, we start noticing that from the first order condition we have that

$$\frac{\partial H_{CV}(h, k, \psi, \lambda; c)}{\partial c} = 0 \quad \Leftrightarrow \quad \frac{c}{h^\gamma} = (\psi - \lambda\rho)^{-1/\sigma} h^{-\frac{\gamma}{\sigma}}$$

Substituting this into the Hamiltonian (8) leads to the maximum value Hamiltonian

$$H_{MAX}(h, k, \psi, \lambda) = a(\psi, \lambda) h^{\gamma-\gamma/\sigma} + \psi(A - \delta)k - \lambda\rho h, \quad (27)$$

¹⁴This is indeed the case because the state equations are linear in (c, h, k) and the Hamiltonian is separable in the state variables.

¹⁵It is also worth noticing that the utility function is also never convex because the Hessian is never positive semi-definite.

where $a(\psi, \lambda) = \frac{\sigma}{1-\sigma} (\psi - \lambda\rho)^{1-1/\sigma}$. Noting that a positive consumption-habit ratio requires $\psi - \lambda\rho > 0$ and that, in our parameter set Θ , we have $\sigma > 1$, we get $a(\psi, \lambda) < 0$.

Now we check whether $H_{MAX}(h, k, \lambda, \psi)$ is concave in (h, k) . Since k enters linearly, we have that $\frac{\partial^2 H_{MAX}}{\partial k^2} = 0$. Since H_{MAX} is separable in the variable h and k , we have also that $\frac{\partial^2 H_{MAX}}{\partial k \partial h} = 0$. Therefore, the determinant of the Hessian associated to H_{MAX} is zero and the only condition to check for concavity is the sign of $\frac{\partial^2 H_{MAX}}{\partial h^2}$. We have

$$\frac{\partial^2 H_{MAX}}{\partial h^2} = a(\psi, \lambda)(\gamma - \gamma/\sigma)(\gamma - \gamma/\sigma - 1)h^{\gamma-\gamma/\sigma-2}$$

Now, since $\sigma > 1$ and $\gamma < 1$ we get

$$\gamma - \gamma/\sigma > 0, \quad \gamma - \gamma/\sigma - 1 < 0$$

which implies, since $a(\psi, \lambda) > 0$, that

$$\frac{\partial^2 H_{MAX}}{\partial h^2} > 0, \quad \forall k > 0, h > 0.$$

Therefore, concavity does not hold.

B Appendix B: Dynamic Programming

B.1 First properties of the value function V

First of all we provide some useful properties of the value function V defined in (7).

Proposition 2 *Let V be the value function defined in (7) and let the parameters $(A, \theta, \rho, \delta, \sigma, \gamma)$ belong to the set Θ defined in (6). Then we have the following.*

(i) *We have*

$$0 \geq V(k_0, h_0) \geq \frac{\theta}{1-\sigma} (Bk_0)^{1-\sigma} [h_0 \vee (Bk_0)]^{-\gamma(1-\sigma)} > -\infty.$$

(ii) *V is increasing in the first variable (k_0) and decreasing in the second variable (h_0) .*

(iii) *V is $(1-\gamma)(1-\sigma)$ homogeneous in \mathbb{R}_{++}^2 .*

(iv) *Given any $\alpha > 0$ and $(k_0, h_0) \in \mathbb{R}_{++}^2$, if a control c^* is optimal at (k_0, h_0) then the control αc^* is optimal at $(\alpha k_0, \alpha h_0)$ and viceversa.*

Proof.

- (i) Obviously $V \leq 0$ since the utility function is negative. Moreover taking the control $c(t) \equiv Bk_0$ we get that $k(t) \equiv k_0$ and

$$h(t) = e^{-\rho t} h_0 + \rho \int_0^t e^{-\rho(t-s)} Bk_0 ds = e^{-\rho t} h_0 + Bk_0(1 - e^{-\rho t}) \leq h_0 \vee (Bk_0)$$

Hence

$$\begin{aligned} \int_0^{+\infty} e^{-\theta t} u(c(t), h(t)) dt &\geq \frac{1}{1-\sigma} \int_0^{+\infty} e^{-\theta t} (Bk_0)^{1-\sigma} [h_0 \vee (Bk_0)]^{-\gamma(1-\sigma)} dt = \\ &= \frac{\theta}{1-\sigma} (Bk_0)^{1-\sigma} [h_0 \vee (Bk_0)]^{-\gamma(1-\sigma)} > -\infty. \end{aligned}$$

This gives the required estimate of the value function.

- (ii) We can immediately see that the set $\mathcal{A}(k_0, h_0)$ gets bigger as k_0 increases while the utility does not depend on $k(t)$ (so it also does not depend on k_0). Hence V must increase in k_0 .

On the other hand the set $\mathcal{A}(k_0, h_0)$ does not change as h_0 changes while the utility decreases (since $\gamma > 0$ and $\sigma > 1$) in $h(t)$, hence in h_0 , as the equation for $h(t)$ is linear. So V must decrease in h_0 .

- (iii) and (iv) By linearity of the state equations (2)-(3) we get that, for every $\alpha > 0$ and $(k_0, h_0) \in \mathbb{R}_{++}^2$, $\mathcal{A}(\alpha k_0, \alpha h_0) = \alpha \mathcal{A}(k_0, h_0)$, $h(t; \alpha h_0, \alpha c) = \alpha h(t; h_0, c)$ and

$$\mathcal{W}(\alpha k_0, \alpha h_0; \alpha c) = \alpha^{(1-\gamma)(1-\sigma)} \mathcal{W}(k_0, h_0; c) \quad (28)$$

Hence, for every $\alpha > 0$, we have, taking the supremum in (28),

$$V(\alpha k_0, \alpha h_0) = \alpha^{(1-\gamma)(1-\sigma)} V(k_0, h_0).$$

The above also implies that if a control c^* is optimal at (k_0, h_0) then the control αc^* is optimal at $(\alpha k_0, \alpha h_0)$ and viceversa.

■

B.2 Equivalence of problem (P) with problem (\tilde{P})

Here we prove Proposition 1.

Call, for simplicity, $\eta := (1 - \sigma)(1 - \gamma)$ from now on. Let $(k_0, h_0) \in \mathbb{R}_{++}^2$ and let $c(\cdot) \in \mathcal{A}(k_0, h_0)$. Let $k(\cdot)$ and $h(\cdot)$ be the associated solutions of the state equations (2)-(3). We have

$$\frac{d}{dt} \frac{k(t)}{h(t)} = \frac{k'(t)h(t) - k(t)h'(t)}{h^2(t)} = \frac{Bk(t) - c(t)}{h(t)} - \frac{k(t)}{h(t)} \cdot \frac{\rho c(t) - \rho h(t)}{h(t)}.$$

Hence, setting $z(t) := k(t)/h(t)$ and $a(t) := c(t)/h(t)$, we get

$$z'(t) = Bz(t) - a(t) - z(t)(\rho a(t) - \rho) = (B + \rho)z(t) - (1 + \rho z(t))a(t).$$

Since $c(\cdot) \geq 0$ implies $a(\cdot) \geq 0$ and $k(\cdot) > 0$ implies $z(\cdot) > 0$ we immediately get that $a(\cdot) \in \tilde{\mathcal{A}}(z_0)$ for $z_0 = k_0/h_0$.

Viceversa, take any $z_0 > 0$ and $a(\cdot) \in \tilde{\mathcal{A}}(z_0)$, and, for $h_0 > 0$ let $h(\cdot)$ be the solution of

$$h'(t) = \rho h(t)[a(t) - 1], \quad h(0) = h_0.$$

Call $c(\cdot) := h(\cdot)a(\cdot)$ and $k(\cdot) := h(\cdot)z(\cdot)$. Then, reversing the above argument, we see that $c(\cdot) \in \mathcal{A}(k_0, h_0)$.

On the other hand the original objective functional in the variables $z(t)$ and $a(t)$ becomes

$$\mathcal{W}(k_0, h_0; c) = \int_0^\infty \frac{\left(\frac{c(t)}{h(t)} \cdot h(t)^{1-\gamma}\right)^{1-\sigma}}{1-\sigma} e^{-\theta t} dt = \int_0^\infty \frac{a(t)^{1-\sigma}}{1-\sigma} h(t)^\eta e^{-\theta t} dt.$$

Now, note that here $h'(t) = \rho c(t) - \rho h(t)$, which implies

$$\frac{h'(t)}{h(t)} = \rho a(t) - \rho \quad \Rightarrow \quad h(t) = h_0 e^{\rho \int_0^t (a(s)-1) ds}$$

and

$$\mathcal{W}(k_0, h_0; c) = h_0^\eta \int_0^\infty \frac{a(t)^{1-\sigma}}{1-\sigma} e^{\rho \eta \int_0^t a(s) ds} e^{-(\theta + \rho \eta)t} dt = h_0^\eta J(k_0; a). \quad (29)$$

By (29) and the above equivalence of admissible set of controls we get that

$$\sup_{c(\cdot) \in \mathcal{A}(k_0, h_0)} \mathcal{W}(k_0, h_0; c) = h_0^\eta \int_0^\infty \frac{a(t)^{1-\sigma}}{1-\sigma} e^{\rho \eta \int_0^t a(s) ds} e^{-(\theta + \rho \eta)t} dt = h_0^\eta \sup_{a(\cdot) \in \tilde{\mathcal{A}}(k_0/h_0)} J(k_0; a). \quad (30)$$

This proves point (i) of Proposition 1. Point (ii) immediately follows from the fact that the above argument on the equivalence of admissible controls works exactly in the same way to prove the equivalence of optimality of controls.

B.3 Solving the auxiliary problem using the properties of V_0

B.3.1 Properties of V_0

Proposition 3 *Let V_0 be the value function defined in (11) and let the parameters $(A, \theta, \rho, \delta, \sigma, \gamma)$ belong to the set Θ defined in (6). Then the following results hold.*

(i) *We have*

$$0 > V_0(z_0) \geq \frac{\theta}{1-\sigma} (Bz_0)^{1-\sigma} [1 \vee (Bz_0)]^{-\gamma(1-\sigma)} > -\infty.$$

(ii) $V_0(0^+) = -\infty$, $V_0(+\infty) = 0$ and V_0 is continuous and strictly increasing in $(0, +\infty)$.

(iv) V_0 is a viscosity solution, in $(0, +\infty)$, of the HJB equation (12). It is also the unique viscosity solution satisfying $V_0(0^+) = -\infty$ and $V_0(+\infty) = 0$.

(v) V_0 is continuously differentiable in $(0, +\infty)$ and hence also a classical solution of the HJB equation (12), in $(0, +\infty)$.

Proof. The proof of the above result is quite long and nontrivial. For brevity we only provide a sketch giving the main ideas. A more detailed proof can be provided under request to the interested reader.

(i) This point, when the first inequality is large, follows immediately from Proposition 1-(i) and Proposition 2-(i). The fact that $V_0 < 0$ is part of the proof of the subsequent point (ii).

(ii) The fact that $V_0(+\infty) = 0$ follows from point (i) simply taking the limits in the inequality. Moreover V_0 is increasing since the set of controls $\tilde{A}(z_0)$ increases when z_0 increases. Furthermore it can be proved, using an estimate from above like the one proved in [21, Lemma 4.2-(15)], that $V_0 < 0$ and $V_0(0^+) = +\infty$. The continuity follows from straightforward arguments which use the dynamic programming principle, as in [20, Proposition 4.5], or as in [27, Section 11]. The fact that V_0 is strictly increasing in $(0, +\infty)$ can be proved by using the same arguments of [2, Theorem 6.1].

(iv) The fact that V_0 is a viscosity solution is a standard consequence of the dynamic programming principle which here follows as in Section 4.2 of [20] (see in particular [20, Proposition 4.5]). See also [2, Section 7.2]

(v) This last fact follows using the same argument of [20, Theorem 4.12] (see also [7]).

B.3.2 Solution of $\tilde{\mathbf{P}}$

We provide here a sketch of the proof of part (i) of Theorem 1.

First of all the fact that V_0 is strictly negative and of class C^1 in \mathbb{R}_{++} follows from the above Proposition 3.

Second, consider the closed loop equation (18). Existence and uniqueness of a local solution z^* follows from the standard Cauchy-Lipschitz theory since V'_0 , and then G can be proved to be C^1 using the HJB equation (12) and the Implicit Function Theorem. To achieve global existence we use that $G(z) > 0$ for $z > 0$ and $G_0(0^+) = 0$, this last fact can be proved by using estimates like the ones of the proof of [2, Theorem 6.1] (see also [17, Corollary 4.20] for similar arguments). Strict positivity (and hence uniqueness) follows by a contradiction argument which uses the fact that optimal control strategies exist and must be strictly positive (which follows e.g. as in [2, Theorem 5.1] and [20, Proposition 4.16]).

Once the above is proved, the fact that (z^*, a^*) is an optimal couple follows from a Verification Theorem which can be proved here exactly as in [19, Section 3]. Uniqueness of the optimal control strategy follows since the Verification Theorem, in this case, also provides a necessary condition (as in [19, Section 4.3.2]).

B.4 Back to the problem \mathbf{P} : Proof of second part of Theorem 1 and of Corollaries 1-2

We provide first a sketch of the proof of part (ii) of Theorem 1. A more detailed proof can be provided under request to the interested reader.

First of all the fact that V is strictly negative and of class C^1 in \mathbb{R}_{++}^2 follows from the above Proposition 3 and from the fact that (see Proposition 1-(i))

$$V(k, h) = h^\eta V_0(k/h), \quad \forall k > 0, h > 0. \quad (31)$$

Moreover, by (31) we immediately get

$$V_k(k, h) = h^{\eta-1} V'_0(k/h), \quad V_h(k, h) = \eta h^{\eta-1} V_0(k/h) - k h^{\eta-2} V'_0(k/h), \quad \forall k > 0, h > 0.$$

Hence, by straightforward computations, we get that G is homogeneous of degree 1 and

$$G(k, h) = hG_0(k/h), \quad \forall k > 0, h > 0. \quad (32)$$

Take now $k_0 > 0$, $h_0 > 0$ and set $z_0 = k_0/h_0$. Let $z^*(\cdot)$ be the solution of (18) and let $a^*(t) = G_0(z^*(t))$ be the unique optimal control strategy for problem $\tilde{\mathbf{P}}$ starting at z_0 (as follows from part (i) of Theorem 1). Then, Proposition 1-(ii) implies that $c^*(t) := h^*(t)a^*(t)$, where $h^*(\cdot)$ is the unique solution to (17), is the unique optimal control for problem \mathbf{P} starting at (k_0, h_0) . Finally, using (32) we easily see that, setting $k^*(t) := h^*(t)z^*(t)$, the couple $(k^*(\cdot), h^*(\cdot))$ is the unique solution of the closed loop system (20). This gives the final claim.

Now we provide a sketch of proof of Corollary 1. The proof goes along the same line of the proof of [20, Theorem 5.5]. Differently from such theorem here we do not have concavity. However here we can perform the same computations since we know that V'_0 is C^1 using the HJB equation (12) and the Implicit Function Theorem.

Finally we provide a sketch of proof of Corollary 2.

Let $k_0 > 0$, $h_0 > 0$, let c^* be the unique optimal control strategy at (k_0, h_0) and let k^* , h^* be the associated state trajectories. To prove that the associated variables $\frac{c^*}{h^*}$, $\frac{\dot{c}^*}{c^*}$, $\frac{k^*}{h^*}$ solve the system (7)-(8)-(9) in [10] it is enough to perform straightforward computations using the state equations and system (22). The initial conditions $k(0) = k_0$, $h(0) = h_0$ are obviously satisfied, while $c(0) = G(k_0, h_0)$ follows from Theorem 1-(ii).

The fact that the paths $\frac{c^*}{h^*}$, $\frac{\dot{c}^*}{c^*}$, $\frac{k^*}{h^*}$ converge to the steady state given in (10)–(13) in [10], follows studying the asymptotic behavior¹⁶ of the optimal state-control paths (z^*, a^*) of problem $\tilde{\mathbf{P}}$ and then using that $\frac{c^*}{h^*} = a^*$, $\frac{k^*}{h^*} = z^*$ and $\frac{\dot{c}^*}{c^*} = \frac{\dot{a}^*}{a^*} + \rho(a - 1)$ ¹⁷.

The final statement follows from the fact that the system (7)-(8)-(9) in [10] is saddle path stable, hence when we start out of the stable manifold then the associated solution does not converge to the steady state.

¹⁶This is possible using the closed loop equation (18) and the feedback formula (19).

¹⁷This follows using that $c^* = a^*h^*$ and equation (17).

C Appendix C: An Example

Consider the following optimal control problem. The state equation is (x is the state variable and c is the control variable)

$$\dot{x}(t) = Bx(t) - c(t), \quad x(0) = x_0 \in \mathbb{R} \quad (33)$$

where $B > 0$ $\theta > 0$, and the set of admissible controls is $\mathcal{A}(x_0)$ which is the set of all locally integrable functions $c(\cdot) : [0, +\infty) \rightarrow \mathbb{R}$ such that the unique solution of the initial value problem (33) is defined in $[0, \infty)$.

The objective of this optimal control problem is to find the $c^* \in \mathcal{A}(x_0)$ which maximizes the functional

$$\mathcal{J}(x_0; c) := \int_0^\infty e^{-\theta t} [x(t)^3 - c(t)^2] dt \quad (34)$$

The system of optimality conditions is the following (here p is the costate variable)

$$\begin{cases} p(t) = -2c(t) \\ p'(t) = (\theta - B)p(t) - x(t)^2 \\ x'(t) = Bx(t) - c(t) \end{cases} \quad (35)$$

The transversality condition similar to the ones given in (23) is

$$\lim_{t \rightarrow +\infty} e^{-\theta t} p(t)x(t) = 0 \quad (36)$$

It is easy to see that, when $x_0 = 0$, the strategy $c^*(t) \equiv 0$ satisfies (35) and (36), with associated state $x(t) \equiv 0$ and costate $p(t) \equiv 0$. However this strategy is not optimal. Indeed, fix $c_0 < 0$ and take $\bar{c}(t) \equiv c_0$: it can be easily seen, by straightforward computations, that $\mathcal{J}(0; \bar{c}) > 0$ when the absolute value of c_0 is sufficiently high.