Participation and Welfare in Auctions with Default *

Joosung Lee[†] Daniel Z. Li[‡]

Abstract

In an auction with costly participation, we show that bidder default may cause social welfare loss through (i) the possibility of no trade and (ii) the under participation of bidders in equilibrium. We also provide closed-form solutions to the model.

Keywords: Auction; Default; Participation cost; Welfare loss
JEL Classification: D44; D82

1 Introduction

The past decades has witnessed an increase in consumer rights, along with the rapid growth of internet sales. For instance, the latest EU law (Directive 2011/83/EU) on distance sales contracts has extended consumers' right to withdraw from a sale from 7 to 14 days. The rationale behind is that internet consumers are usually informationally disadvantaged when compared with those shopping in bricks-and-mortar stores. In this paper, however, we would argue that the (over) protection of consumer rights may induce some socially undesirable outcomes in a competitive bidding environment.

The literature typically models consumer protection as a kind of limited liability constraints; that is, there exists an upper bound for a consumer's ex-post loss. It takes the usual forms of withdrawal rights, return policies, and defaults as special cases. Here we will consider an auction with a participation cost and default possibility. Specifically, a bidder needs to pay an entry cost to attend the auction, and he makes the entry decision before learning his private signal on the product. After the auction, uncertainty about the product value is revealed, and a winning bidder then decides whether to proceed with the auction payment or default.

If there is no default, a well-known result in the literature is that equilibrium outcomes are socially efficient (McAfee and McMillan, 1987). However, when a default is possible, we show that it may cause social welfare loss through two channels. The first is obvious, as the product is not allocated when the default occurs. The second is more subtle. We

*We are grateful to the Editor, Joseph Harrington, and an anonymous referee for the valuable comments.

†Business School, University of Edinburgh, 29 Buccleuch Place, Edinburgh EH8 9JS, UK. Email: Joosung.Lee@ed.ac.uk.

‡Corresponding author, Durham University Business School, Mill Hill Lane, Durham DH1 3LB, UK. Email: daniel.li@durham.ac.uk, Tel: +44(0)1913346335

show that the possibility of default reduces the expected winning rent and hence deter entries. The resulted under-participation of bidders also induces welfare loss.

The other contribution of this paper is that, by applying some properties of order statistics, we provide closed-form solutions to the default probability, the expected winning rent and social welfare, in a model of auctions with default. These closed-form results can be conveniently applied to other related studies.

Our paper is mainly related to two strands of literature. The first is on auctions with costly participation, where bidders make entry decisions before knowing their valuations (McAfee and McMillan, 1987). Li (2017) further compares the degrees of equilibrium competition in three common forms of auctions with costly participation. The second is on auctions with a default possibility. This literature produces similar results that the possibility of default induces more aggressive bids and reduces bidders' payoffs. Specifically, Waehrer (1995) considers an auction with a non-refundable deposit, which acts as a measure to protect the seller from default. Zheng (2001) analyzes a common-value auction where budget-constrained bidders can default. Board (2007) compares the outcomes of the first and the second-price auction, where a winning bidder may default by declaring bankruptcy. To our best knowledge, our paper is the first one to study costly participation in auctions with default. Finally, our paper is also related the literature on auctions with contingent payments where the auction payment is contingent on the realization of future states (Skrzypacz, 2013, for a review).

2 The Model

A seller allocates an indivisible product among a finite set N of potential bidders. Both the seller and the bidders are risk-neutral. Bidder i's valuation of the product is

$$V_i = S + X_i, \tag{1}$$

where X_i is bidder i's idiosyncratic signal, and the random variable

$$S = \begin{cases} 0 & \Pr = q \\ \bar{s} & \Pr = 1 - q, \end{cases}$$

with $\bar{s} > 0$. X_i's are independent draws from the same distribution F on $[0, \infty)$ with density $f > 0$. F has increasing failure rate and a finite mean. S and X_i's are independent from each other. We consider a second price auction with participation cost and default possibility, and the timing of the game is as follows:

- $t = 1$: Bidders make entry decisions before learning the realizations of S and X_i, and the entry cost is c. They can observe how many bidders have entered the auction.
- $t = 2$: A participating bidder i learns the realization of X_i, denoted by x_i, upon which he submits a bid $\beta_i = \beta(x_i)$. All the bids are then revealed, and the winning bidder and the auction payment are determined according to the auction rule.

- $t = 3$: The realization of S is observed, and the winning bidder decides whether to proceed with the payment, or default where he needs to pay $\eta > 0$ to the seller.

If a default occurs, we assume there is no further action taken by the seller. For example, she cannot reallocate the product to other bidders. We consider a symmetric equilibrium where all the bidders adopt the same bidding strategy. We first derive a bidder's equilibrium bidding strategy, and then solve for his optimal entry decision.

2.1 Equilibrium Bidding Strategy

It is clear that a bidder with private signal x never bids more than $x + \bar{s}$, and therefore, when $S = \bar{s}$, a winning bidder never defaults. When $S = 0$, the winning bidder will default if $x - p < -\eta$, where p is his auction payment. The following assumption insures the possibility of default in equilibrium.

Assumption 1 *$\eta < (1-q)\,\bar{s}$.*

Lemma 1 *Under assumption 1, a bidder's equilibrium bidding strategy is*

$$\beta(x) = x + \bar{s} - \frac{q}{1-q}\eta. \tag{2}$$

Proof. Let $U(p, x)$ denote a bidder's expected payoff conditional on winning with a bid p and that both his and the highest private signal of other bidders are x. In a second-price auction, it is a weakly dominant strategy to bid $\beta(x)$ such that $U(\beta(x), x) = 0$. Assuming a winning bidder will default if $S = 0$, it then follows that $U(\beta(x), x) = \Pr(S = \bar{s})(\bar{s} + x - \beta(x)) - \Pr(S = 0)\,\eta = 0$. The equilibrium bidding strategy is thus $\beta(x) = x + \bar{s} - q\eta/(1-q)$. We still need to verify the condition that a winning bidder with $U(\beta(x), x) = 0$ does default when $S = 0$, that is, $x + 0 - \beta(x) < -\eta$ which is equivalent to $\eta < (1-q)\,\bar{s}$. ■

The equilibrium bidding strategy (2) is already known in the literature, e.g., Waehrer (1995, Example 1) and Board (2007, Section A.1), and we have the following observations. When $\eta \geq (1-q)\,\bar{s}$, default never occurs and a bidder just bids his expected valuation of the product, that is, $\beta^e(x) = x + (1-q)\,\bar{s}$. We henceforth use the upper script "e" to denote the case of no default, where the equilibrium outcomes are "efficient." When $\eta < (1-q)\,\bar{s}$, it is clear that $\beta(x) > \beta^e(x)$, which indicates that the possibility of default encourages bidders to bid more aggressively.

2.2 Default Probability

Denote n the number of participating bidders, and $X_n^{(k)}$ the k-th largest order statistics of n independent draws from F. The condition for a winning bidder to default is that: i) $S = 0$, and ii) the ex-post payoff of making payment is smaller than $-\eta$, which is equivalent to $Z_n := X_n^{(1)} - X_n^{(2)} < \bar{s} - \frac{\eta}{1-q}$. Denote $G_n(z)$ the distribution of Z_n, and we have

Lemma 2 *The distribution function of* $Z_n \in \mathbb{R}_+$ *is*

$$G_n(z) = 1 - \int_0^{+\infty} \frac{f(y+z)}{f(y)} \mathrm{d}F^n(y). \tag{3}$$

Moreover, the probability of default is $qG_n(\bar{s} - \eta/(1-q))$, *which is (i) decreasing in* η, *and (ii) increasing (decreasing) in* n *if* $f(y+z)/f(y)$ *is decreasing (increasing) in* y.

Proof. According to Arnold et al. (2008, Result (2.3.2)), by replacing $j = n$, $i = n-1$ and that $x_i = x_j - z$, we have the joint density function of $\left(X_n^{(1)}, Z_n\right)$ as

$$h(x,z) = n(n-1) F(x-z)^{n-2} f(x-z) f(x), \tag{4}$$

where x denotes the realization of $X_n^{(1)}$, and $z \in [0,x]$ denotes that of Z_n. Integrating out x, we have the marginal density function of Z_n,

$$g_n(z) = n(n-1) \int_z^{+\infty} F(x-z)^{n-2} f(x-z) f(x) \mathrm{d}x = n \int_0^{+\infty} f(y+z) \mathrm{d}F^{n-1}(y).$$

The cumulative distribution of Z_n is thus

$$\begin{aligned} G_n(z) &= n \int_0^z \left[\int_0^{+\infty} f(y+t) \mathrm{d}F(x)^{n-1}\right] \mathrm{d}t = n \int_0^{+\infty} [F(y+z) - F(y)] \mathrm{d}F(y)^{n-1} \\ &= -n \int_0^{+\infty} F(y)^{n-1} \mathrm{d}[F(y+z) - F(y)] = -n \int_0^{+\infty} F(y)^{n-1} \mathrm{d}F(y+z) + 1 \\ &= 1 - \int_0^{+\infty} \frac{f(y+z)}{f(y)} \mathrm{d}F^n(y). \end{aligned}$$

Finally, result (i) is self-evident, and (ii) is based on the fact that, with increasing n, $F^n(y)$'s are ordered according to first-order stochastic dominance. ■

3 Equilibrium Entry and Welfare

This section derives the main results of this paper. First, we show that the possibility of default induces under-participation in equilibrium. Second, we identify two channels through which the possibility of default may cause social welfare loss.

3.1 Under-Participation

Denote $\pi(n)$ the expected winning rent with n participating bidders, and it is equal to

$$(1-q)\mathbb{E}\left[Z_n + \frac{q\eta}{1-q}\right] + q\left[\int_{\bar{s}-\frac{\eta}{1-q}}^{\infty} \left(z + \frac{q\eta}{1-q} - \bar{s}\right) \mathrm{d}G_n(z) - \int_0^{\bar{s}-\frac{\eta}{1-q}} \eta \mathrm{d}G_n(z)\right]. \tag{5}$$

It follows from simple calculations (please refer to Lemma 3 proof for details) that

$$\pi(n) = \mathbb{E}[Z_n] - q\mathbb{E}\left[\min\left\{Z_n, \bar{s} - \frac{\eta}{1-q}\right\}\right]. \tag{6}$$

Note that $\pi^e(n) := \mathbb{E}[Z_n]$ is the expected winning rent in the benchmark case of no default. When $\eta < (1-q)\bar{s}$, the second term on the RHS of (6) is strictly negative, which implies $\pi(n) < \pi^e(n)$. Therefore, the possibility of default reduces the expected winning rent of the bidders. In a symmetric equilibrium, a bidder's expected winning rent is simply $U(n) = \pi(n)/n$, and we have the following result.

Lemma 3 *Under Assumption 1, a bidder's expected winning rent*

$$U(n) = \int_0^\infty \left[1 - qF\left(x + \bar{s} - \frac{\eta}{1-q}\right) - (1-q)F(x)\right] F^{n-1}(x)\,\mathrm{d}x, \tag{7}$$

which is i) increasing in η, and ii) strictly decreasing in n with $\lim_{n\to\infty} U(n) = 0$.

Proof. From (5), we have

$$\begin{aligned}\pi(n) &= (1-q)\mathbb{E}[Z_n] + q\int_{\bar{s}-\frac{\eta}{1-q}}^\infty \left(z + \frac{\eta}{1-q} - \bar{s}\right)\mathrm{d}G_n(z) \\ &= \mathbb{E}Z_n - q\left\{\int_0^{\bar{s}-\frac{\eta}{1-q}} z\mathrm{d}G_n(z) + \int_{\bar{s}-\frac{\eta}{1-q}}^\infty \left(\bar{s} - \frac{\eta}{1-q}\right)\mathrm{d}G_n(z)\right\} \quad \text{implies (6)} \\ &= \mathbb{E}Z_n - q\int_0^{\bar{s}-\frac{\eta}{1-q}} [1 - G_n(z)]\,\mathrm{d}z.\end{aligned}$$

Substituting $G_n(z)$ in (3) and inter-changing the integration order, we have

$$\begin{aligned}\int_0^{\bar{s}-\frac{\eta}{1-q}} [1 - G_n(z)]\,\mathrm{d}z &= \int_0^{\bar{s}-\frac{\eta}{1-q}} \left[\int_0^{+\infty} \frac{f(x+z)}{f(x)}\mathrm{d}F^n(x)\right]\mathrm{d}z \\ &= \int_0^{+\infty} \left[\int_0^{\bar{s}-\frac{\eta}{1-q}} \frac{f(x+z)}{f(x)}\mathrm{d}z\right]\mathrm{d}F^n(x) \\ &= n\int_0^{+\infty} \left[F\left(x + \bar{s} - \frac{\eta}{1-q}\right) - F(x)\right] F^{n-1}(x)\,\mathrm{d}x.\end{aligned}$$

Moreover, it is a standard result in order statistics that

$$\frac{\mathbb{E}Z_n}{n} = \mathbb{E}\left[X_n^{(1)} - X_{n-1}^{(1)}\right] = \int_0^\infty [1 - F(x)]\,F^{n-1}(x)\,\mathrm{d}x.$$

Substituting both results into $U(n) = \pi(n)/n$, we then have (7). The proofs of result i) and ii) are self-evident. ■

Lemma 3 shows that a bidder's expected winning rent $U(n)$ is increasing in η. For comparison purpose, when $\eta \geq (1-q)\bar{s}$, we return to the case of no default and

$$U^e(n) := \frac{\mathbb{E}Z_n}{n} = \int_0^\infty [1 - F(x)]\,F^{n-1}(x)\,\mathrm{d}x.$$

It is clear from (7) that $U(n) < U^e(n)$ when $\bar{s} < \eta/(1-q)$. We next denote n^* and n^{**} respectively as the equilibrium number of participating bidders, in the two cases of with and without default. Proposition 1 below then states that the possibility of default induces under-participation of bidders in equilibrium.

Proposition 1 *Under Assumption 1, $n^* \leq n^{**}$.*

Proof. From Lemma 3, the optimization conditions for n^* and n^{**} are $U(n^*) \geq c > U(n^*+1)$ and $U^e(n^{**}) \geq c > U^e(n^{**}+1)$ respectively. Given that $U^e(n) > U(n)$ under Assumption 1, and that both $U(n)$ and $U^e(n)$ are strictly decreasing in n and converges to 0 when $n \to \infty$, the result $n^* \leq n^{**}$ is then straightforward. ■

3.2 Welfare Loss

Social welfare is the sum of the seller revenue and the bidder payoff. The seller revenue is the auction payment if no default occurs, and η otherwise. The bidder payoff is the winning rent net of the total entry costs. The expected social welfare $W(n)$ is thus

$$\begin{aligned} W(n) &= (1-q)\,\mathbb{E}\left[X_n^{(1)}+\bar{s}\right]+q\Pr\left(Z_n\geq \bar{s}-\frac{\eta}{1-q}\right)\mathbb{E}\left[X_n^{(1)}\middle|Z_n\geq \bar{s}-\frac{\eta}{1-q}\right] \\ &\quad -q\Pr\left(Z_n<\bar{s}-\frac{\eta}{1-q}\right)(\eta-\eta)-nc. \end{aligned}$$

Simple transformation gives

$$W(n)=\mathbb{E}\left[X_n^{(1)}+S\right]-q\Pr\left(Z_n<\bar{s}-\frac{\eta}{1-q}\right)\mathbb{E}\left[X_n^{(1)}\middle|Z_n<\bar{s}-\frac{\eta}{1-q}\right]-nc. \tag{8}$$

We note that, in the benchmark case of no default, the expected social welfare is

$$W^e(n)=\mathbb{E}\left[X_n^{(1)}+S\right]-nc.$$

Proposition 2 reveals that the possibility of default causes social welfare loss.

Proposition 2 *Under Assumption 1, the expected social welfare is*

$$W(n)=W^e(n)-nq\int_0^{\bar{s}-\frac{\eta}{1-q}}\left[\int_z^{\infty}xf(x)\,\mathrm{d}F^{n-1}(x-z)\right]\mathrm{d}z, \tag{9}$$

which is increasing in η.

Proof. We have derived the joint density function of $\left(X_n^{(1)},Z_n\right)$ in (4). Denoting the term $\Pr(Z_n<\bar{s}-\eta/(1-q))\,\mathbb{E}\left[X_n^{(1)}\,|Z_n<\bar{s}-\eta/(1-q)\right]$ in (8) by D, we then have

$$\begin{aligned} D &= \int_0^{\bar{s}-\frac{\eta}{1-q}}\left[\int_z^{\infty}xn(n-1)F(x-z)^{n-2}f(x-z)f(x)\,\mathrm{d}x\right]\mathrm{d}z \\ &= \int_0^{\bar{s}-\frac{\eta}{1-q}}\left[n\int_z^{\infty}xf(x)\,\mathrm{d}F^{n-1}(x-z)\right]\mathrm{d}z. \end{aligned}$$

It is clear that $W(n)$ is increasing in η. ■

For given n, the reason for welfare loss is that the product is not allocated when a default occurs, as shown by the second term on the RHS of (9). Furthermore, Proposition 1 indicates that the possibility of default induces under-participation in equilibrium, e.g., $n^*\leq n^{**}$, which also causes welfare loss. Let $W(n^*)$ denote the equilibrium social welfare with a default possibility, and $W^e(n^{**})$ that in the case of no default. Note that the entry number n^{**} is efficient and $W^e(n^{**})$ is the maximum expected social welfare. We then can provide a decomposition result on welfare loss, as follows

$$W^e(n^{**})-W(n^*)=\underbrace{\left[W^e(n^{**})-W^e(n^*)\right]}_{A\text{: under participation}}+\underbrace{\left[W^e(n^*)-W(n^*)\right]}_{B\text{: possibility of no trade}}. \tag{10}$$

Equation (10) indicates the two channels through which the possibility of default may cause social welfare loss. First, it induces under-participation in equilibrium, as shown in Proposition 1 and term A of (10). Second, it implies a strictly positive probability that the product is not allocated, as shown in Proposition 2 and term B of (10).

References

Arnold, B. C., N. Balakrishnan, and H. N. Nagaraja (2008): *A first course in order statistics*, vol. 54, Siam.

Board, S. (2007): "Bidding into the red: A model of post-auction bankruptcy," *The Journal of Finance*, 62, 2695–2723.

Li, D. Z. (2017): "Ranking equilibrium competition in auctions with participation costs," *Economics Letters*, 153, 47–50.

McAfee, R. P. and J. McMillan (1987): "Auctions with entry," *Economics Letters*, 23, 343–347.

Skrzypacz, A. (2013): "Auctions with contingent payments—an overview," *International Journal of Industrial Organization*, 31, 666–675.

Waehrer, K. (1995): "A model of auction contracts with liquidated damages," *Journal of Economic Theory*, 67, 531–555.

Zheng, C. Z. (2001): "High bids and broke winners," *Journal of Economic theory*, 100, 129–171.