Statistical inference for the Arrhenius-Weibull accelerated life testing model with imprecision based on the likelihood ratio test

Abdullah A.H. Ahmadini

Department of Mathematics, Faculty of Science, Jazan University, Jazan, Saudi Arabia

Frank P.A. Coolen

Department of Mathematical Sciences, Durham University, Durham, United Kingdom

Abstract

In this paper, we present a new imprecise statistical inference method for accelerated life testing data, where nonparametric predictive inferences at normal stress levels are integrated with a parametric Arrhenius-Weibull model. The method includes imprecision based on the likelihood ratio test which provides robustness with regard to the model assumptions. We use the likelihood ratio test to obtain an interval for the parameter of the Arrhenius link function providing imprecision into the method. The imprecision leads to observations at increased stress levels being transformed into interval-valued observations at the normal stress level, where the width of an interval is larger for observations from higher stress levels. If the model fits well, our method has relatively little imprecision. However, if the model fits poorly, it leads to more imprecision. Simulation studies are presented to investigate the performance of the proposed method.

Keywords: Accelerated life testing, Arrhenius-Weibull model, failure data, imprecise probability, likelihood ratio test, lower and upper survival functions, nonparametric predictive inference

1. Introduction

Accelerated life testing (ALT) is frequently used to obtain information on the lifespan of devices. Testing items under normal conditions can require a great deal of time and expense. To determine the reliability of devices in a shorter period of time, and with lower costs, it may be possible to use ALT. In ALT, a unit is tested under levels of physical stress (e.g. temperature, voltage, or pressure) greater than normal operating conditions. Using this method, devices will fail more quickly, enabling estimation of the lifetime under normal conditions via extrapolation using an ALT model. There is a wide range of designs for ALT, including constant-, step- and progressive-stress testing [23]. Two components usually used to extrapolate a device's lifespan under normal use conditions from ALT results are the life time distribution (e.g. Weibull or log-normal) and the relationship between failure time and stress level (e.g. Arrhenius law or power law).

In recent decades, various methods for analyzing ALT data and a variety of applications for assessing the reliability with different ALT scenarios have been introduced. An excellent introduction to ALT was given by Nelson [23]. Since Nelson [23], there has been a large amount of literature on ALT and a variety of statistical methods have been devoloped, we mention several recent contributions. Fan et al. [14] developed the maximum likelihood estimation (MLE) and Bayesian inference on all parameters of ALT models under an exponential distribution, with a link function between the failure rate and the stress variables in a linear way under the Box-Cox transformation. Fard and Li [15] presented optimal step stress to obtain the optimal hold time at which the stress level is changed for step stress ALT design for reliability prediction. They assumed a Weibull distribution for the failure time at any constant stress level, and the scale parameter of the Wiebull distribution is assumed to be a log-linear function of the stress level. Elsayed and Zhang [13] proposed an optimal multiple-stress-type ALT plan using a proportional hazards model to obtain failure time data rapidly in a short period of time. Sha and Pan [27] introduced step-stress ALT with Bayesian analysis for the Weibull proportional hazard model. Nasir and Pan [22] posited a Bayesian optimal design criterion and presented acceleration model selection ALT studies, while Han [16] conducted research into temporally and financially constrained constant-stress and step-stress ALT.

Whilst ALT models typically consider failure times as the events of interest, there have also been important contributions with more detailed modelling, in particular exploiting methods to mathematically model degradation processes. For example, Liao and Tseng [19], proposed an optimal design for step-stress accelerated degradation tests with the degradation process modelled as a stochastic diffusion process. Pan et al. [25] presented a bivariate constant stress accelerated degradation model and related inference. They assumed a device which has two performance characteristics which are controlled by a Wiener process, to determine the relibility of high quality devices with a time scale transformation, and the Frank copula is used to model dependence of the two performance characteristics. Duan and Wang [12] proposed a bivariate constant stress accelerated degradation model with inference based on the Inverse Gaussian Process. It is important to note here that the modelling of degradation processes does require much information about the engineering process and physical properties of the equipment, which may come from detailed measurements of the process or expert judgements. While this is an important development for real world accelerated life testing, we do not address such approaches further in this paper and only assume information about the failure times to be present.

Yin et al. [29] introduced imprecise probability theory [4] for ALT data using the power-Weibull model. In this paper we follow the same approach used by Yin et al. [29] but we investigate the Arrhenius-Weibull model, so a different link function is used between different stress levels. The methods are basically the same, but Yin et al. [29] did not give an argument, other than simulation studies, for the imprecision in the parameter. We use a parametric statistical test between the pairwise stress levels, to obtain the intervals for the parameter of the link function for which we do not reject the null hypothesis. This use of a frequentist statistical test to determine the level of imprecision is the main novelty in this paper.

We develop a new likelihood ratio test based method for analysing ALT data with imprecise probabilities, where nonparametric predictive inferences (NPI) at the normal stress level is integrated with a parametric Arrhenius-Weibull model. This new method consist of two steps. First, we assume the Arrhenius-Weibull model for all levels and we derive an interval for the parameter of the Arrhenius link function by pairwise likelihood ratio tests, which allows observations of increased stress levels to be transformed to interval-valued observations at the normal stress level. Secondly, we use NPI at the normal stress level for predictive inference on the failure time of a future unit operating under normal stress, using the original data at the normal stress level and interval-valued data transformed from higher stress levels.

This paper is organized as follows: in Section 3, the main ideas of nonparametric predictive inference are briefly reviewed. In Section 4, our novel method of imprecise statistical inference is introduced. Section 5 presents an example with simulated data in order to illustrate our method and its main properties. In Section 6 the new method is applied to a data set from the literature. Section 7 presents results of simulation studies that investigate the performance of the proposed method. Finally, some concluding remarks are made in Section 8.

2. The model

In this paper, we consider the Arrhenius model and a Weibull lifetime distribution for a constant-stress ALT. The Arrhenius model is based on physical or chemical theory, and is often an appropriate model to use when the failure mechanism is driven by temperature [23]. The Weibull distribution is often suitable for examining component, system or product life. The Arrhenius-Weibull model is adopted to the current research on imprecise statistical approaches to establish the use of imprecision in modelling the nature of the relationship between stress levels and unit failure rates. The main issue here is how to extrapolate the failure data from units tested at higher-than-normal stress levels to units operating at the normal stress level.

The model at each stress level (the two-parameter Weibull distribution) is

$$f(t) = \frac{\beta}{\alpha_i}(\frac{t}{\alpha_i})^{\beta-1} \exp\Big[-(\frac{t}{\alpha_i})^{\beta}\Big]. \tag{1}$$

The unknown parameters of the Weibull distribution are the shape parameter β, and the scale parameters α_i at stress level i, where $\alpha_i > 0$ and $\beta > 0$, and its survival function is

$$P(T > t) = \exp\Big[-\Big(\frac{t}{\alpha_i}\Big)^{\beta}\Big].$$

The Arrhenius-Weibull model is specified as follows. K_0 represents the stress at the normal level. There are $m \geq 1$ increased stress levels, with stress K_i at level $i \in \{1, ..., m\}$, we assume that K_i increases as a function of i. In this paper, the Weibull distributions for different stress levels are assumed to have different scale parameters $\alpha_i > 0$ for level i, but the same shape parameter β. The Arrhenius link function for the scale parameters is

$$\alpha_i = \alpha_0 \exp\Big(\frac{\gamma}{K_i} - \frac{\gamma}{K_0}\Big). \tag{2}$$

Using this model with varying temperatures (in Kelvin) as stress levels, K_0 is the normal temperature at stress level 0, K_i is the higher temperature at stress level i, and $\gamma > 0$ is the parameter of the Arrhenius link function model.

Using this link function model, an observation t^i at the stress level i, subject to stress K_i, can be transformed to stress level 0. For fixed γ the transformed observation denoted by $t^{i\to 0}(\gamma)$ from level i to level 0 is given by the equation

$$t^{i\to 0}(\gamma) = t^i \exp\left(\frac{\gamma}{K_0} - \frac{\gamma}{K_i}\right). \tag{3}$$

Now, we define the model through the probability density function as:

$$f(t^i; \alpha, \beta, \gamma, \underline{K}) = \frac{\beta}{\alpha_i}\left(\frac{t^{i\to 0}(\gamma)}{\alpha_i}\right)^{\beta-1} \exp\left(-\left(\frac{t^{i\to 0}(\gamma)}{\alpha_i}\right)^{\beta}\right), \tag{4}$$

where the Arrhenius link function for scale parameters α_i should be identified to establish a connection between the different stress levels i.

Let all the data be denoted by $\underset{\sim}{t} = \{t_1^0, ...t_{n_0}^0, t_1^1, ...t_{n_1}^1, ..., t_1^m, ...t_{n_m}^m\}$ and let the stress levels be denoted by $\underline{K} = \{K_0, ..., K_m\}$. The likelihood function is defined as

$$L(\underset{\sim}{t}; \alpha, \beta, \gamma, \underline{K}) = \prod_{j=0}^{m}\prod_{i=1}^{n_j} f\left(t_i^j; \alpha, \beta, \gamma, \underline{K}\right).$$

So there are in total three parameters that need to be estimated to fit the complete model, α_0, β, and γ. In this paper, we will apply the pairwise likelihood ratio test to create an interval for the parameter γ of the link function, as presented in Section 4. We assume the same β for all stress levels, so differences between stress levels are only modelled through the α_i, and hence in the model through the α_0 and γ parameters of the link function between different stress levels. For ease of presentation, we assume that there are no right-censored observations and that there are failure observations at the normal stress level. We briefly comment on these assumptions in Section 8.

3. Nonparametric predictive inference

Nonparametric Predictive Inference (NPI) is a statistical method, which provides lower and upper survival functions for a future observation based on past data using imprecise probability [4, 7]. Hill [17] proposed an assumption which gives direct conditional probabilities for a future random quantity which depend on the values of related random qualities [3, 6, 7]. It proposes that the rank of a future observation among the values already observed will be equally likely to have each possible value $1, ..., n+1$. Suppose that $X_1, X_2, ..., X_n, X_{n+1}$ represent exchangeable and continuous real-valued possible random quanities, then the ranked observed values of $X_1, X_2, ..., X_n$ can be denoted by $x_{(1)} < x_{(2)} < ... < x_{(n)}$. Let $x_0 = 0$ and $x_{n+1} = \infty$. The assumption $A_{(n)}$ is

$$P(X_{n+1} \in (x_{(j-1)}, x_{(j)})) = 1/(n+1).$$

Here, no tied observations are included for convenience, any tied values can be dealt with by assuming that tied observations differ by a small amount which tends to zero [18].

Inferences which are based on $A_{(n)}$ are nonparametric and predictive. They can be considered suitable if there is hardly any knowledge about the random quantity of interest, except for the n observations, or if one does not want to use any such further information. The $A_{(n)}$ assumption is not sufficient to derive precise probabilities for many events of interest. However, this approach does yield optimal bounds for probabilities through the 'fundamental theorem of probability' [11], which are lower and upper probabilities in the imprecise probability theory [3, 4].

The lower and upper probabilities for event A are denoted by $\underline{P}(A)$ and $\overline{P}(A)$, respectively. These are open to interpretation in various ways [4]. For instance, $\underline{P}(A)$ can be assumed to be the supremum buying price for a gamble on event A, such that if A occurs then 1 is paid, if not then 0 is paid. This can also simply be interpreted as the maximum lower bound for the probability of A, which derives from the assumptions made. Similarly, $\overline{P}(A)$ can be interpreted as the minimum selling price for the gamble on A, or the minimum upper bound based on the assumptions made. We have $0 \leq \underline{P}(A) \leq \overline{P}(A) \leq 1$, and $\underline{P}(A) = 1 - \overline{P}(A^c)$ where, A^c is the complimentary event of A [3, 4].

The NPI lower and upper survival functions for a future observation X_{n+1} in case of no censoring are

$$\underline{S}_{X_{n+1}}(t) = \frac{n-j}{n+1}, \text{ for } x \in (x_j, x_{j+1}), \ j = 0, ..., n. \tag{5}$$

$$\overline{S}_{X_{n+1}}(t) = \frac{n+1-j}{n+1}, \text{ for } x \in (x_j, x_{j+1}), \ j = 0, ..., n. \tag{6}$$

The imprecision, the difference between the upper and lower survival functions, reflects the amount of information in the data. We will only use Equations (5) and (6) for the NPI method in this paper.

4. ALT inference using pairwise likelihood ratio tests

To investigate equality of two independent failure data groups, possibly including right-censored observations, the likelihood ratio test can be used [26]. This is a popular statistical test that can be applied to investigate equality of the probability distribution of two independent groups, which has been briefly introduced in Section 1 [2, 23].

In this section we present new predictive inference based on ALT data and the likelihood ratio test. We use NPI at the normal stress level, with the fully parametric model used in our new statistical method analysing data from ALT. The use of NPI here provides lower and upper survival functions for a future observation at the normal stress level, based on all failure data.

This new statistical method for data in ALT divides into two steps. First, the basic Arrhenius-Weibull model is adopted [23], and the pairwise likelihood ratio test is used between the stress levels K_i and stress level K_0, to obtain the intervals for the parameter γ for which we do not reject the null hypothesis that the data transformed from stress level i to normal stress level 0, and the original data obtained at the normal stress level 0, come from the same Weibull distribution. The hypothesis test we use in this paper is

$$\begin{cases} H_0 : \gamma = \gamma\prime \\ H_1 : \gamma \neq \gamma\prime, \end{cases}$$

where $\gamma \in \mathbb{R}$. The test statistic is defined as

$$LR = \frac{L(\underset{\sim}{t}; \tilde{\alpha}, \tilde{\beta}, \gamma\prime, \underline{K})}{L(\underset{\sim}{t}; \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \underline{K})}$$

where $\tilde{\alpha}, \tilde{\beta}$ are such that $\sup_{(\alpha,\beta)\in\mathbb{R}^+\times\mathbb{R}^+} L(\underset{\sim}{t}; \alpha, \beta, \gamma\prime, \underline{K}) = L(\underset{\sim}{t}; \tilde{\alpha}, \tilde{\beta}, \gamma\prime, \underline{K})$ and $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ are such that $\sup_{(\alpha,\beta,\gamma)\in\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}} L(\underset{\sim}{t}; \alpha, \beta, \gamma, \underline{K}) = L(\underset{\sim}{t}; \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \underline{K})$.

The probability density function of the Arrhenius-Weibull model is assumed in this paper to describe the failure time at a fixed stress level, and its parameters are therefore maximized in the likelihood ratio test. There are in total three parameters that need to be estimated under the alternative hypothesis; α_0, β, and γ. Under the null hypothesis, for which γ is fixed while α_0 and β are estimated, the LR follows a ${\chi_1}^2$ distribution. To get $[\underline{\gamma}_i, \overline{\gamma}_i]$, for each value of γ we would have different $\hat{\alpha}_0$ and $\hat{\beta}$ but we do not use these any further in our method.

For each $i = 1, ..., m$, we find $\underline{\gamma}_i$, the smallest value for $\gamma\prime$ for which we do not reject the null hypothesis, and $\overline{\gamma}_i$, the largest value for $\gamma\prime$ for which we do not reject the null hypothesis. Then we define $\underline{\gamma} = \max \{\min \underline{\gamma}_i, 0\}$ and $\overline{\gamma} = \max \overline{\gamma}_i$. Note that, because of the physical interpretation of generally faster failures with increased stress levels, we exclude negative values which leads to some $\underline{\gamma}$ values being set at 0.

In this paper, we will always restrict $\underline{\gamma}$ to non-negative values. We find the $\underline{\gamma}_i$ and $\overline{\gamma}_i$ numerically using the statistical software R.

Note that, we do not make a confidence statement for the final NPI lower and upper survival functions, so we do not explicitly quantify the prediction accuracy. If indeed the main assumption is valid, that increased stress tends to decreased failure times, then negative $\underline{\gamma_i}$ are typically resulting from statistical variation and would disappear for larger samples.

One may wish to allow $\underline{\gamma}$ to be negative, which may e.g. be reasonable if it turned out that higher stress level could possibly improve a unit's failure time. However, in normal ALT applications there tends to be sufficient knowledge about the effect of the stress on the failure times to justify negative values for $\underline{\gamma}$ not to be considered. Hence we also do not consider such values in this paper. It should be emphasized though that our inferential method could still be used if negative values for lower $\underline{\gamma}$ were allowed.

In the second step, we apply the data transformation using $\underline{\gamma}$ for all levels $i = 1, ..., m$ to get transformed data at level 0, which are then used together with the original data at the normal stress level, to derive the NPI lower survival function $\underline{S}$. Similarly, we apply the data transformation using $\overline{\gamma}$ for all levels $i = 1, ..., m$ to get transformed data at level 0, which are then used together with the original data at the normal stress level, to derive the NPI upper survival function $\overline{S}$. Note that each observation at an increased stress level transforms into an interval-valued observation at the normal stress level 0, where the width of an interval is larger for an observation from a higher stress level.

Note that, if the model fits really well, we expect most $\underline{\gamma}_i$ values to be quite similar, as well as most $\overline{\gamma}_i$ values. If the model fits poorly, $\underline{\gamma}_i$ are most probably very different, or $\overline{\gamma}_i$ are very different, or both. Hence, in case of poor model fit, the resulting interval $[\underline{\gamma}, \overline{\gamma}]$ tends to be wider than in the case of good model fit. If the model assumed is not too far from reality, we would expect the widest interval for the parameter γ to come from the likelihood ratio test applied to levels 1 and 0.

If the model assumed is not too far from reality, we would expect the widest interval for the parameter γ to come from the likelihood ratio test applied to levels 1 and 0. If the model fits well, a level 1 observation is transformed to a smaller interval on level 0 than a level 2 interval, if the transferred intervals are close, in particular if they are overlapping. In the overlapping case, because the level 2 interval is wider, the left and right end points of these intervals from level 2 are further apart, which implies that the γ in the null hypothesis will be rejected in more cases. Hence, the interval $[\underline{\gamma}, \overline{\gamma}]$ from level 1 will be wider than the interval $[\underline{\gamma}, \overline{\gamma}]$ from level 2 (in most cases, due to variability in the samples not in all cases). If the model is worse than we would expect more often that $\underline{\gamma} = \underline{\gamma}_i$ for an $i \neq 1$, or $\overline{\gamma} = \overline{\gamma}_i$ for an $i \neq 1$. If the model assumptions are not fully correct, for example, using some misspecification cases or there is a lot of overlap between the data, then latter can happen.

Although our method involves multiple pairwise tests, we do not aim to combine these into a single test of a hypothesis involving all groups simultaneously. A multiple testing (comparison) procedure arises in many scenarios, where two groups of data are compared over time with each other [5, 24]. A well known scenario that one may be interested in testing is the comparison between two groups based on confidence level [24]. While we use pairwise tests in our approach, we do not combine these into an overall confidence level statement for the resulting inference. Instead we use NPI to derive the lower and upper predictive survival functions and we investigate the performance of our predictive method separately via simulations.

If the assumed model is fully correct then the lower and upper γ will form an interval with at least $1-\alpha$ confidence level, with α the significance level for each pairwise test. However, we explicitly develop our method for robust inference as the basic assumed model will in practice not be 'correct', and acknowledging this makes confidence statements hard to justify. This is an interesting topic for future research. The method proposed in this section is illustrated by two examples using simulated data and real data in Sections 5 and 6, and studied in more detailed by simulation in Section 7.

5. Example: simulated data

In this section we present an example consisting of two cases. In Case 1 we simulated data at all levels using the parametric model for the link function we assume for the analysis. In Case 2 we change these data such that the assumed link function will not provide a good fit anymore and we investigate the effect on the interval $[\underline{\gamma}, \overline{\gamma}]$ and the corresponding lower and upper predictive survival functions for the normal stress level. We assumed the normal temperature stress level to be $K_0 = 283$, and the increased temperatures stress levels $K_1 = 313$ and $K_2 = 353$ Kelvin. We generated ten observations from a Weibull distribution at each stress level linked by the Arrhenius link function. The Weibull distribution at level K_0 had shape parameter $\beta = 3$ and scale parameter $\alpha_0 = 7000$, and the Arrhenius parameter was set at $\gamma = 5200$. This model keeps the same shape parameter at each temperature, and the scale parameters are linked by the Arrhenius relation, which led to $\alpha_1 = 1202.942$ at level K_1 and $\alpha_2 = 183.0914$ at level K_2. The ten failure times of units were simulated at each temperature, so data of a total of 30 units is used in the study. The failure times are given in Table 1.

To illustrate the pairwise likelihood ratio tests method using these data, we first assume the Weibul distribution at each stress level and the Arrhenius link function for the data. To obtain the intervals $[\underline{\gamma}_i, \overline{\gamma}_i]$ of the values γ_i for which we do not reject the null hypothesis with regard to the well-mixed data transformation, we used the pairwise likelihood ratio test between K_i for $i = 1, 2$ and K_0. The resulting intervals $[\underline{\gamma}_i, \overline{\gamma}_i]$ for three significance levels are given in Table 2.

In the second step of our method, we transformed the data using the $[\underline{\gamma}, \overline{\gamma}]$ values. All observations at the increased stress levels were transformed to the normal stress level. Therefore, the observations at the increased stress levels K_1 and K_2

Case	$K_0 = 283$	$K_1 = 313$	$K_2 = 353$	$K_1 = 313$ (*1.4)	$K_2 = 353$ (*0.4)
1	2692.596	241.853	74.557	338.595	29.823
2	3208.336	759.562	94.983	1063.387	37.993
3	3324.788	769.321	138.003	1077.050	55.201
4	5218.419	832.807	180.090	1165.930	72.036
5	5417.057	867.770	180.670	1214.878	72.279
6	5759.910	1066.956	187.721	1493.739	75.088
7	6973.130	1185.382	200.828	1659.535	80.331
8	7690.554	1189.763	211.913	1665.668	84.765
9	8189.063	1401.084	233.529	1961.517	93.412
10	9847.477	1445.231	298.036	2023.323	119.214

Table 1: Failure times at three temperature levels in the first three columns and corresponding failure times with misspecification in the last two columns

Significance level	0.01		0.05		0.10	
Stress level	$\underline{\gamma}_i$	$\overline{\gamma}_i$	$\underline{\gamma}_i$	$\overline{\gamma}_i$	$\underline{\gamma}_i$	$\overline{\gamma}_i$
Case 1: K_1 K_0	4060.018	6605.752	4424.881	6261.168	4593.700	6100.653
K_2 K_0	4377.043	5602.321	4550.205	5434.908	4630.511	5357.037
Case 2: $K_1 * (1.4)$, K_0	3066.539	5612.273	3431.402	5267.689	3600.221	5107.174
$K_2 * (0.4)$, K_0	5684.708	6909.985	5857.870	6742.573	5938.175	6664.701

Table 2: $[\underline{\gamma_i}, \overline{\gamma_i}]$ for likelihood ratio test

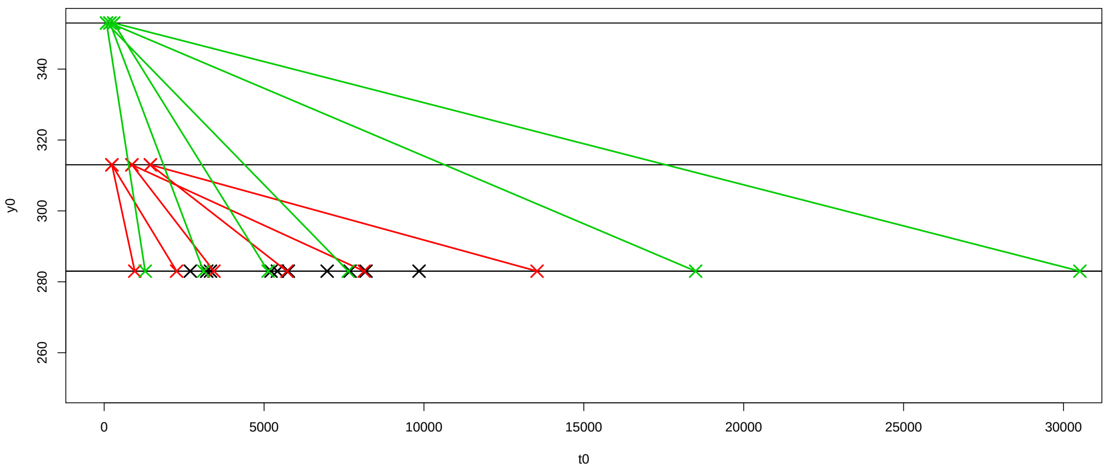


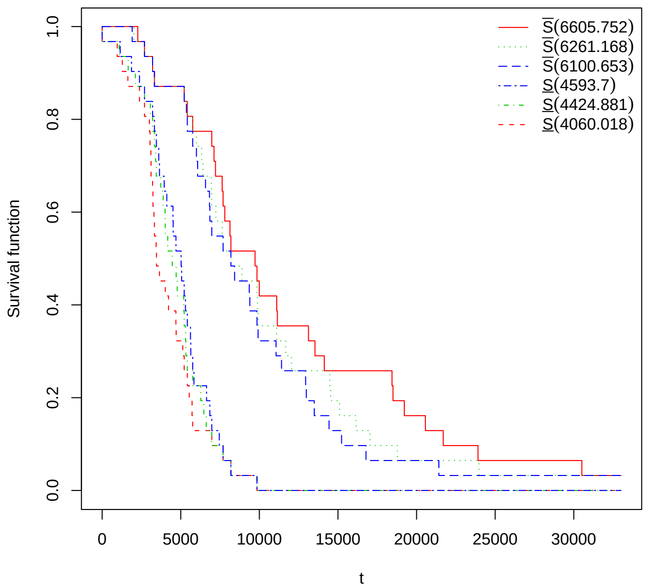
Figure 1: Some transformed data using $[4060.018, 6605.752]$

are transformed to interval-valued observations at the normal stress level K_0. We briefly illustrate this in Figure 1 using only three points of the data at each level, therefore, we have six lines going down as shown in Figure 1. We transformed the data from the higher stress levels K_1 and K_2 using $[4060.018, 6605.752]$ with 0.01 level of significance, mixed with the original samples at the normal stress level K_0. Note that, in Figure 1 the two largest transformed data points are actually the $\underline{\gamma}$ and $\overline{\gamma}$ transformations of the largest observation from level K_2. So, this illustrates the key property of our method, that data transformed from higher levels tend to be wider intervals at the normal level.

The NPI lower survival function is based on the original data at level 0 together with the transformed data from the stress levels K_1 to K_0 and K_2 to K_0 using $\underline{\gamma}$. Similarly, the NPI upper survival function is based on the original data at level 0 together with the transformed data from the stress levels K_1 to K_0 and K_2 to K_0 using $\overline{\gamma}$. The $\underline{\gamma}$ transformed the points to the smallest values and therefore is most pessimistic case which leads to the lower survival function $\underline{S}$. The $\overline{\gamma}$ transformed the points to the largest values and therefore is most optimistic case which leads to the upper survival function $\overline{S}$. In Case 1, we take $\underline{\gamma} = 4060.018$ and the $\overline{\gamma} = 6605.752$, $\underline{\gamma} = 4424.881$ and $\overline{\gamma} = 6261.168$, and $\underline{\gamma} = 4593.700$ and $\overline{\gamma} = 6100.653$ of the pairwise K_1, K_0 with 0.01, 0.05 and 0.10 significance levels, respectively. We used all the above $\underline{\gamma}$ and $\overline{\gamma}$ values to transform the data to the normal stress level 0, see Figure 2(a). In this figure, the lower survival function $\underline{S}$ is labeled as $\underline{S}(\underline{\gamma}_i)$ and the upper survival function $\overline{S}$ is labeled as $\overline{S}(\overline{\gamma}_i)$. This figure shows that higher significance levels leads to more imprecision for the NPI lower and upper survival functions.

In Case 2, we illustrate our method in case of misspecification. We multiply the data at level K_1 by 1.4 and in addition we multiply the data at level K_2 by 0.4. The simulated data values are given in the last two columns in Table 1. In this case, we take $\underline{\gamma} = 3066.539$ and $\overline{\gamma} = 6909.985$, $\underline{\gamma} = 3431.402$ and $\overline{\gamma} = 6742.573$, and $\underline{\gamma} = 3600.221$ and $\overline{\gamma} = 6664.701$ for 0.01, 0.05 and 0.10 significance levels, respectively. We used all the above $\underline{\gamma}$ and $\overline{\gamma}$ values to transform the data to the normal stress level 0, see Figure 2(b). Figure 2(b) also shows that higher significance level results in more imprecision for the NPI lower and upper survival functions. In Case 2, we can see that the $[\underline{\gamma_i}, \overline{\gamma_i}]$ intervals for the two pairwise comparisons are fully disjoint unlike in Case 1. Note that in Case 2, the observations at level K_1 have increased, leading to smaller $\underline{\gamma}_1$ and $\overline{\gamma}_1$ values, which, in turn, leads to the lower and upper survival functions to decrease in comparison to Case 1. Also, the observations at K_2 stress level in Case 2 have decreased, resulting in larger values for $\underline{\gamma}_2$ and $\overline{\gamma}_2$, and this leads to the lower and upper survival functions to increase in comparison to Case 1. Therefore, it is obvious that considering the $\underline{\gamma}$ and $\overline{\gamma}$ values give substantially more imprecision in our NPI method. In case of poor model fit, the NPI lower and upper survival functions in Figure 2(b), using our method as discussed in Section 4, have more imprecision than if the model fits well.

(a) Case 1



(b) Case 2

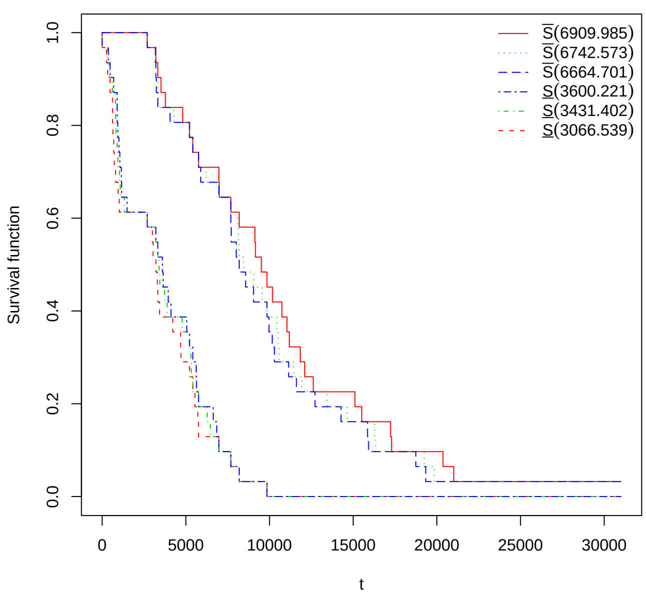


Figure 2: The NPI lower and upper survival functions: Case 1 and Case 2.

Case	$K_0 = 393$	$K_0 = 408$	$K_0 = 423$
1	3850	3300	2750
2	4340	3720	3100
3	4760	4080	3400
4	5320	4560	3800
5	5740	4920	4100
6	6160	5280	4400
7	6580	5640	4700
8	7140	6120	5100
9	7980	6840	5700
10	8960	7680	6400

Table 3: Failure times at three temperature levels

Significance level	0.01		0.05		0.1	
Stress level	$\underline{\gamma}_i$	$\overline{\gamma}_i$	$\underline{\gamma}_i$	$\overline{\gamma}_i$	$\underline{\gamma}_i$	$\overline{\gamma}_i$
K_1 K_0	0	4881.225	0	3988.558	0	3572.193
K_2 K_0	188.348	3540.639	651.091	3077.896	866.927	2862.060

Table 4: $[\underline{\gamma}_i, \overline{\gamma}_i]$ for pairwise likelihood ratio tests.

6. Example: real data

The method proposed in Section 4 is now applied to a data set from the literature [28], which result from a temperature-accelerated lifespan test. The time-to-failure data were collected at three temperatures (in Kelvin): $K_0 = 393, K_1 = 408$, and $K_2 = 423$, with 393 the normal temperature for the process of interest. Ten units were tested at each temperature, so a total of 30 units where used in the study. All of the units failed during the experiment. The failure times, in hours, are given in Table 3.

For the data in Table 3, we have assumed the Weibull failure time distributions at each stress level, with the Arrhenius link function between different stress levels. Then the pairwise likelihood ratio test is used separately between level K_i and K_0, for $i = 1, 2$ to derive the intervals $[\underline{\gamma}_i, \overline{\gamma}_i]$ for the value of the parameter γ of the Arrhenius link function, such that we do not reject the null hypothesis that two group of failure data (the transformed data from level i and the real data from level 0) come from the same underlying distribution, is not rejected for values of γ in this interval. The resulting intervals $[\underline{\gamma}_i, \overline{\gamma}_i]$ are given in Table 4, for three test significance levels.

Note that, because the data corresponding to the different stress levels already have quite some overlap, the likelihood ratio tests even did not rule out some negative values for γ. However, because of the physical interpretation of generally faster failures with increased temperature, we exclude negative values which leads to some

$\underline{\gamma}_i$ values being set at 0. Following the method presented in this paper, we transformed the data using the overall values $[\underline{\gamma}, \overline{\gamma}]$, derived as the smallest and largest corresponding values for the pairwise tests, respectively. Based on the original data at level 0 together with the data transformed from the stress levels K_1 to K_0 and K_2 to K_0 using $\underline{\gamma}$, the NPI approach provides the NPI lower survival function. Similarly, but using $\overline{\gamma}$ for the transformation, we derived the NPI upper survival function. For the significance levels $0.01, 0.05, 0.1$, the values of $\underline{\gamma}$ are always 0 in this case, and for $\overline{\gamma}$ we have, 4881.225, 3988.558 and 3572.193, respectively. The resulting lower and upper survival functions are presented in Figure 3, where of course the three different significant levels lead to the same lower survival function due to the restriction for the γ values to be non-negative. In this figure, the lower survival function $\underline{S}$ is labeled as $\underline{S}$ $(\underline{\gamma}_i)$ and the upper survival function $\overline{S}$ is labeled as $\overline{S}$ $(\overline{\gamma}_i)$. This figure shows that a smaller significance level leads to more imprecision for the NPI lower and upper survival functions, which is directly resulting from the fact that the intervals of not rejected values of γ_i in the pairwise tests will be nested, becoming larger if the significance level is decreased. These lower and upper survival functions can also be used to deduce corresponding lower and upper values for percentiles, which are found in the usual way by inverting the respective functions. These functions can also be used as inputs into decision processes, for example with regard to setting warranty policies, this is a topic of ongoing research by the authors.

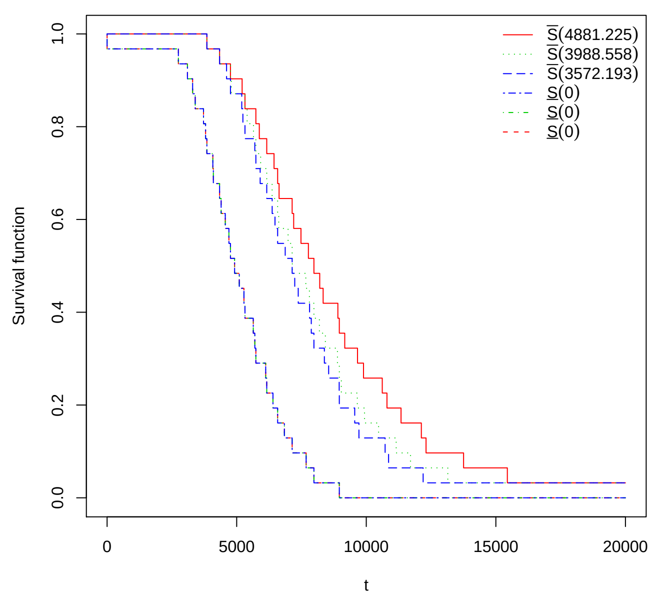


Figure 3: The NPI lower and upper survival functions. Example 6

7. Simulation studies

The imprecise predictive statistical method we propose in this paper for ALT, uses the pairwise likelihood ratio tests with the parametric Arrhenius-Weibull model. This is intuitively attractive because it shows some robustness, and demonstrates the effect of imprecision when we lack perfect information about the link between different stress levels. In this section, we first apply simulation studies to investigate the performance of our proposed method when considering the assumed Arrhenius-Weibull model as the true underlying model.

To investigate the predictive inference performance of our new method for ALT data, we conduct a simulation study. We assumed the temperature stress levels to be $K_0 = 283$, $K_1 = 313$, and $K_2 = 353$. In this analysis, we run the simulation 10,000 times with the data simulated from the Weibull distribution with $\beta = 3$, $\alpha = 7000$, and the Arrhenius link function parameter $\gamma = 2000$, where we have used $n = 10, 50, 100$ assigned to each stress level. We applied the method described in Section 4, with levels of significance 0.01, 0.05, and 0.10. For good performance of the method, we check the results by taking into account a future observation, and we consider if it mixes well among the data at level K_0, where we use both the actual data at level K_0 and the data transformed to level K_0. We examined the performance in the quartiles of NPI lower and upper survival functions for $q = 0.25, 0.50, 0.75$, and whether or not the future observation at the normal stress level has exceeded these quartiles in the right proportion. One can use different quartiles but these quartiles provided a good indicator of the performance of our method. We conducted simulation with assumed Arrhenius-Weibull model, hence with data generated from the same model as used for our method, hence we expect a good performance.

For good performance of the method, we require that the future observation for each run at the normal stress level has exceeded the first, second, and third quartiles of the NPI lower survival functions just over proportions $0.75, 0.50$ and 0.25 of all runs, respectively, and for the NPI upper survival functions just under proportions $0.75, 0.50$ and 0.25. Table 5 presents the results of the performance of our method for this simulation. All cases in Table 5 show an overall good performance of the proposed method, with levels of significance $0.01, 0.05$, and 0.10, and with sample size $n = 10, 50, 100$. Note that the quartiles in this table are denoted by qL and qU, corresponding to the NPI lower and upper survival functions, respectively. We note that for corresponding proportion with larger values of n, the differences between lower and upper survival functions tend to decrease. That means that when have more data, the NPI lower and upper survival functions allow more precise inference. As mentioned in Section 4, if the model fits very well or perfectly, then in most cases $\underline{\gamma} = \underline{\gamma}_1$, and $\overline{\gamma} = \overline{\gamma}_1$. Table 6 shows that when we simulate from the assumed Arrhenius-Weibull model out of 10,000 runs with 0.05 level of significance, hence with data generated from the same model, that most of the widest intervals come from level K_1. From these simulations, we conclude that the proposed approach using the Arrhenius-Weibull model provides suitable predictive inference if the model assumptions are fully correct. We will investigate robustness in case of model misspecification in the next three simulation cases.

In Case 1, we are dealing with a particular misspecification of the model for statistical inference for ALT data [20, 21]. Table 7 presents the results of the predictive performance of our method. We used the same scenario as in the first simulation but we assumed shape parameter $\beta = 2$ for each level, in the analysis. Now only the scale parameter α and the parameter γ of the link function are explicitly maximize in the likelihood ratio. In comparison to the first simulation where the shape parameter $\beta = 3$ was estimated, there is a slight effect on the quartiles in Table 7. From Table 5 and Table 7, the effect of fixing the shape parameter $\beta = 2$ in the analysis can also be seen on the quartiles of the NPI lower and upper survival functions for $q = 0.25, 0.50, 0.75$, which is reflected by more imprecision for this case. Also, we checked whether or not the future observation at the normal stress level has exceeded these quartiles in the right proportion. Further note that, in Table 7, there seems to be more imprecision than in Table 5. Throughout this simulation, when we fixed the shape parameter $\beta = 2$ in the fitted model, our novel method provided sufficient robustness against the misspecification.

In Case 2, we investigate the predictive performance of our method for ALT data we change these data such that the assumed link function will not provide a good fit and we show the change to the resulting quartiles of NPI lower and upper survival functions for $q = 0.25, 0.50, 0.75$, and whether or not the future observation at the normal stress level has exceeded these quartiles in the right proportion.

To illustrate this, we generated data sets as before with $n = 10, 50, 100$ observation at each stress level using the Arrhenius-Weibull model. But now all the data at stress level K_1 are multiplied by 1.2 and in addition we multiply all the data at level K_2 by 0.8. We run the simulation 10,000 times. Using these generated data, we again applied our method as described in Section 4, with levels of significance 0.01, 0.05, and 0.10. We compute the likelihood ratio test within the statistical software R to the simulated data set separately between the stress levels K_1 to K_0 then K_2 to K_0 for finding the values $\underline{\gamma}$ and $\overline{\gamma}$. Then we used these $\underline{\gamma}$ and $\overline{\gamma}$ values to transform the data to the normal stress level K_0. All the results in Case 2 provide an insight into whether or not the presented method shows sufficient robustness against the misspecification case considered. Table 8 presents the results of these simulations with $n = 10, 50, 100$.

For $n = 50, 100$ there are a few cases for which the future observation for each run at the normal stress level has exceeded the first, second, and third quartile ($qU0.25$, $qU0.50$, $qU0.75$) corresponding to the NPI upper survival functions just over $0.75, 0.50, 0.25$ of the pairwise level $K_1 * (1.2)$ to K_0, receptively, see Table 8. They are highlighted by bold font in Table 8. Exceeding the first, second, and third quartile ($qU0.25$, $qU0.50$, $qU0.75$) of the NPI upper survival function just over $0.75, 0.50, 0.25$ corresponds to the use of this misspecification case where the data at stress level K_1 are multiplied by 1.2 and in addition we multiply all the data at level

K_2 by 0.8. There are slight increase in $qU0.25$, $qU0.50$, $qU0.75$, which means that we have too many observations passing these quartiles from the upper $0.75, 0.50, 0.25$. They seem that the points $0.75, 0.50, 0.25$ occurred a bit earlier and they should be related to the effect of multiplying the data by 1.2 and 0.8 for the stress levels K_1 to K_0 and K_2 to K_0, respectively. As mentioned, this is in line with expectation, which is mainly due to the misspecification case we assumed as well as increasing n that caused the imprecision in the method to become small. Note that, the lower significance level leads to less imprecision for the NPI lower and upper survival functions. Further note that in this simulation, the observations at the stress level K_1 have increased, resulting in smaller $\underline{\gamma_1}$ and $\overline{\gamma_1}$, and hence possibly smaller $\underline{\gamma}$ and $\overline{\gamma}$ compared to the earlier simulations, when we assume the model assumptions are fully correct. Also, the data at level $K_2 * (0.8)$ have decreased, resulting in larger $\underline{\gamma_2}$ and $\overline{\gamma_2}$ and hence possibly larger $\underline{\gamma}$ and $\overline{\gamma}$, in comparison to the first simulation results in Table 5.

Table 9 shows the numbers of the simulation runs with $\underline{\gamma} = \underline{\gamma_1}$ or $\overline{\gamma} = \overline{\gamma_1}$ or both for this case considered, out of 10,000 simulation runs with 0.05 level of significance. It shows that when we use the simulated data for this case, that most of the $\underline{\gamma} = \underline{\gamma_1}$ intervals come from level K_1 and few of the $\overline{\gamma} = \overline{\gamma_1}$ intervals come from level K_1 in comparison to the Table 6 when we assume the correct model. Note that, the resulting intervals at level $K_2 * (0.8)$ become larger, that why most cases of the $\overline{\gamma}$ come from $\overline{\gamma}_2$ in comparison to the first simulation. Therefore, in case of worse model fit, the NPI lower and upper survival functions for $q = 0.25, 0.50, 0.75$, using our proposed method discussed in Section 4, have more imprecision.

We next investigate the robustness and the performance of our predictive inference against the necessary assumptions. To conduct this, we simulated the data from the Eyring-Weibull model with the parameters $\alpha_0 = 7000, \beta = 3$ and $\lambda = 2000$. The Weibull distributions for different stress levels are assumed to have different scale parameters $\alpha_i > 0$ for level i, but the same shape parameter β. The Eyring link function for the Weibull scale parameters is

$$\alpha_i = \alpha_0 * (K_0/K_i) * \exp\Big[(\lambda/K_i - \lambda/K_0)\Big] \tag{7}$$

Where using this model with temperatures as stress levels, K_0 is the normal temperature (Kelvin) at stress level 0, K_i is the higher temperature (Kelvin) at stress level i, and $\lambda > 0$ is the parameter of the Eyring link function model. In this simulation we used the assumed Arrhenius link function model between different stress levels. We applied the method described in Section 4, with levels of significance 0.01, 0.05, and 0.10, with $10,000$ simulation runs. Table 8 shows the results of the predictive performance of our method. We use the same scenario as in the first simulation but with data generated from the Eyring-Weibull model. In comparison with the simulation where the model assumptions are fully correct, the results in Table 8 are very similar to those in Table 7, which means that from the preceding investigation that the Eyring model and the Arrhenius model lead to similar conclusions. These simulations show that the proposed approach provides suitable robustness in predictive inference against the model assumptions in case of this specific model misspecification. One can similarly investigate other cases of model misspecification.

The main findings drawn from the above simulations are: the future observation at the normal stress level K_0 has exceeded the quartiles that we considered in the right proportions. Both with the Arrhenius-Weibull model and the power-Weibull model, our method achieves suitable predictive inference if the model assumptions are fully correct, and the intervals $[\underline{\gamma}, \overline{\gamma}]$ have reasonable imprecision. However, in case of model misspecification, the intervals $[\underline{\gamma}, \overline{\gamma}]$ have wider imprecision compared to if the model assumptions are correct. One can similarly investigate other cases of model misspecification. Of course, in the case of large misspecification, no method would give meaningful inferences; in our model it would lead to large imprecision, which would reflect that there is a problem of model fit. We have seen that when the number of observations at each stress level is $n = 100$, the imprecision between the NPI lower and upper survival functions tends to decrease compared to when the number of observations at each stress level $n = 10$.

K_1K_0		$n=10$		$n=50$		$n=100$	
α	q	qL	qU	qL	qU	qL	qU
0.01	0.25	0.9386	0.4960	0.8565	0.6287	0.8293	0.6717
	0.50	0.8227	0.1407	0.6726	0.3208	0.6349	0.3684
	0.75	0.5670	0.0197	0.4314	0.0900	0.3818	0.1245
0.05	0.25	0.9058	0.5531	0.8326	0.6585	0.8131	0.6938
	0.50	0.7557	0.2192	0.6322	0.3660	0.6028	0.4000
	0.75	0.5049	0.0470	0.3966	0.1238	0.3511	0.1531
0.1	0.25	0.8871	0.5804	0.8214	0.6730	0.8048	0.7030
	0.50	0.7143	0.2604	0.6137	0.3866	0.5880	0.4155
	0.75	0.4664	0.0667	0.3755	0.1415	0.3351	0.1681
K_2K_0		$n=10$		$n=50$		$n=100$	
α	q	qL	qU	qL	qU	qL	qU
0.01	0.25	0.8795	0.5054	0.8088	0.6923	0.7974	0.7132
	0.50	0.6974	0.2869	0.5919	0.4072	0.5712	0.4361
	0.75	0.4588	0.0649	0.3525	0.1599	0.3162	0.1838
0.05	0.25	0.8538	0.6418	0.7945	0.7073	0.7881	0.7220
	0.50	0.6524	0.3392	0.5699	0.4287	0.5552	0.4518
	0.75	0.4073	0.1035	0.3287	0.1792	0.2998	0.2005
0.1	0.25	0.8398	0.6589	0.7881	0.7146	0.7835	0.7260
	0.50	0.6285	0.3644	0.5594	0.4411	0.5470	0.4605
	0.75	0.3818	0.1239	0.3175	0.1900	0.2917	0.2079
min $\underline{\gamma}$ and the max $\overline{\gamma}$		$n=10$		$n=50$		$n=100$	
α	q	qL	qU	qL	qU	qL	qU
0.01	0.25	0.9427	0.4925	0.8577	0.6273	0.8299	0.6708
	0.50	0.8277	0.1352	0.6749	0.3189	0.6363	0.3672
	0.75	0.5732	0.0149	0.4333	0.0886	0.3834	0.1235
0.05	0.25	0.9122	0.5459	0.8347	0.6562	0.8144	0.6920
	0.50	0.7664	0.2087	0.6362	0.3625	0.6058	0.3982
	0.75	0.5150	0.0376	0.4003	0.1203	0.3538	0.1500
0.1	0.25	0.8957	0.5714	0.8238	0.6700	0.8074	0.7002
	0.50	0.7299	0.2485	0.6202	0.3818	0.5920	0.4122
	0.75	0.4792	0.0546	0.3812	0.1366	0.3384	0.1642

Table 5: Proportion of runs with future observation greater than the quartiles, Arrhenius model.

min $\underline{\gamma}$ and the max $\overline{\gamma}$	$n=10$	$n=50$	$n=100$
$\underline{\gamma}=\underline{\gamma}_1$	8699	8963	9022
$\overline{\gamma}=\overline{\gamma}_1$	8675	9003	9087
$\underline{\gamma}=\underline{\gamma}_1$ and $\overline{\gamma}=\overline{\gamma}_1$	7374	7966	8109

Table 6: Number of the simulation runs with $\underline{\gamma}=\underline{\gamma_1}$ or $\overline{\gamma}=\overline{\gamma_1}$ or both out of 10,000 simulation runs with 0.05 level of significance.

K_1K_0		$n = 10$		$n = 50$		$n = 100$	
α	q	qL	qU	qL	qU	qL	qU
0.01	0.25	0.9524	0.4640	0.8727	0.6028	0.8599	0.6357
	0.50	0.8432	0.0915	0.7047	0.2859	0.6913	0.3120
	0.75	0.5807	0.0028	0.4629	0.0795	0.4400	0.0831
0.05	0.25	0.9169	0.5211	0.8396	0.6336	0.8339	0.6669
	0.50	0.7686	0.1542	0.6447	0.3336	0.6450	0.3625
	0.75	0.5056	0.0112	0.4019	0.0995	0.3910	0.1193
0.1	0.25	0.8941	0.5413	0.8193	0.6518	0.8218	0.6945
	0.50	0.7230	0.1878	0.6076	0.3521	0.6217	0.4096
	0.75	0.4600	0.0221	0.3752	0.1151	0.3663	0.1618
K_2K_0		$n = 10$		$n = 50$		$n = 100$	
α	q	qL	qU	qL	qU	qL	qU
0.01	0.25	0.8828	0.5851	0.8196	0.6791	0.8118	0.6982
	0.50	0.7026	0.2563	0.6096	0.3971	0.6012	0.4088
	0.75	0.4514	0.0447	0.3700	0.1518	0.3472	0.1565
0.05	0.25	0.8524	0.6135	0.7998	0.6960	0.7976	0.7135
	0.50	0.6491	0.3076	0.5780	0.4161	0.5740	0.4368
	0.75	0.3939	0.0862	0.3338	0.1683	0.3186	0.1834
0.1	0.25	0.8356	0.6029	0.7869	0.7038	0.7910	0.7240
	0.50	0.6169	0.3132	0.5587	0.4258	0.5622	0.4584
	0.75	0.3601	0.1266	0.3165	0.1781	0.3068	0.2042
min $\underline{\gamma}$ and the max $\overline{\gamma}$		$n = 10$		$n = 50$		$n = 100$	
α	q	qL	qU	qL	qU	qL	qU
0.01	0.25	0.9539	0.4623	0.8748	0.5958	0.8602	0.6356
	0.50	0.8468	0.0889	0.7081	0.2757	0.6915	0.3118
	0.75	0.5844	0.0027	0.4667	0.0656	0.4402	0.0826
0.05	0.25	0.9216	0.5089	0.8450	0.6303	0.8357	0.6663
	0.50	0.7779	0.1484	0.6522	0.3267	0.6481	0.3614
	0.75	0.5184	0.0222	0.4085	0.0948	0.3938	0.1177
0.1	0.25	0.9017	0.5064	0.8258	0.6476	0.8243	0.6896
	0.50	0.7369	0.1727	0.6182	0.3474	0.6258	0.4008
	0.75	0.4782	0.0580	0.3824	0.1108	0.3714	0.1519

Table 7: Proportion of runs with future observation greater than the quartiles, Arrhenius model, $\beta = 2$ in the fitted model. Case 1.

$K_1 * (1.2), K_0$		$n = 10$		$n = 50$		$n = 100$	
α	q	qL	qU	qL	qU	qL	qU
0.01	0.25	0.9684	0.5460	0.9179	0.7178	0.8995	0.7575
	0.50	0.8544	0.2169	0.7662	0.4495	0.7398	0.5044
	0.75	0.5674	0.0501	0.4701	0.1909	0.4467	0.2421
0.05	0.25	0.9487	0.6200	0.9013	0.7468	0.8857	0.7770
	0.50	0.8145	0.3172	0.7370	0.4931	0.7183	0.5373
	0.75	0.5236	0.1020	0.4466	0.2316	0.4271	0.2689
0.1	0.25	0.9353	0.6551	0.8933	0.7608	0.8785	0.7883
	0.50	0.7872	0.3607	0.7208	0.5141	0.7072	0.5530
	0.75	0.4972	0.1333	0.4350	0.2499	0.4169	0.2815
$K_2 * (0.8), K_0$		$n = 10$		$n = 50$		$n = 100$	
α	q	qL	qU	qL	qU	qL	qU
0.01	0.25	0.8679	0.5767	0.7896	0.6642	0.7778	0.6918
	0.50	0.6690	0.2470	0.5596	0.3710	0.5368	0.3960
	0.75	0.4135	0.0481	0.3106	0.1303	0.2788	0.1479
0.05	0.25	0.8379	0.6135	0.7756	0.6793	0.7648	0.7011
	0.50	0.6209	0.3024	0.5377	0.3928	0.5204	0.4171
	0.75	0.3675	0.0793	0.2864	0.1470	0.2637	0.1621
0.1	0.25	0.8232	0.6307	0.7690	0.6869	0.7594	0.7055
	0.50	0.5976	0.3282	0.5260	0.4041	0.5130	0.4233
	0.75	0.3465	0.0971	0.2747	0.1563	0.2557	0.1701
min $\underline{\gamma}$ and the max $\overline{\gamma}$		$n = 10$		$n = 50$		$n = 100$	
α	q	qL	qU	qL	qU	qL	qU
0.01	0.25	0.9807	0.5149	0.9483	0.6253	0.9368	0.6541
	0.50	0.8619	0.1722	0.7977	0.3107	0.7756	0.3328
	0.75	0.5681	0.0182	0.4721	0.0780	0.4498	0.0952
0.05	0.25	0.9675	0.5637	0.9358	0.6430	0.9261	0.6666
	0.50	0.8309	0.2346	0.7764	0.3369	0.7619	0.3503
	0.75	0.5263	0.0393	0.4506	0.0973	0.4332	0.1063
0.1	0.25	0.9585	0.5868	0.9283	0.6505	0.9198	0.673
	0.50	0.8125	0.2648	0.7635	0.3478	0.7547	0.3592
	0.75	0.5020	0.0536	0.4398	0.1042	0.4239	0.1131

Table 8: Proportion of runs with future observation greater than the quartiles, Arrhenius model. $K_1 * (1.2)$ and $K_2 * (0.8)$. Case 2.

min $\underline{\gamma}$ and the max $\overline{\gamma}$	$n=10$	$n=50$	$n=100$
$\underline{\gamma}=\underline{\gamma}_1$	9993	10000	10000
$\overline{\gamma}=\overline{\gamma}_1$	1821	12	5
$\underline{\gamma}=\underline{\gamma}_1$ and $\overline{\gamma}=\overline{\gamma}_1$	1814	12	5

Table 9: Number of the simulation runs with $\underline{\gamma}=\underline{\gamma_1}$ or $\overline{\gamma}=\overline{\gamma_1}$ or both out of 10,000 simulation runs with 0.05 level of significance, Arrhenius model.$K_1*(1.2)$ and $K_2*(0.8)$. Case 2.

K_1K_0		$n=10$		$n=50$		$n=100$	
α	q	qL	qU	qL	qU	qL	qU
0.01	0.25	0.9396	0.4970	0.8583	0.6306	0.8332	0.6718
	0.50	0.8238	0.1426	0.6762	0.3254	0.6426	0.3706
	0.75	0.5694	0.0205	0.4338	0.0938	0.3889	0.1251
0.05	0.25	0.9068	0.5547	0.8355	0.6612	0.8162	0.6942
	0.50	0.7573	0.2230	0.6350	0.3700	0.6089	0.4007
	0.75	0.5051	0.0477	0.3988	0.1264	0.3570	0.1544
0.1	0.25	0.8887	0.5822	0.8244	0.6763	0.8084	0.7035
	0.50	0.7184	0.2630	0.6176	0.3908	0.5944	0.4165
	0.75	0.4692	0.0694	0.3785	0.1457	0.3418	0.1686
K_2K_0		$n=10$		$n=50$		$n=100$	
α	q	qL	qU	qL	qU	qL	qU
0.01	0.25	0.8782	0.6038	0.8072	0.6911	0.7959	0.7116
	0.50	0.6965	0.2848	0.5905	0.4053	0.5704	0.4336
	0.75	0.4571	0.0648	0.3486	0.1576	0.3140	0.1820
0.05	0.25	0.8529	0.6407	0.7943	0.7054	0.7875	0.7204
	0.50	0.6497	0.3381	0.5673	0.4271	0.5540	0.4501
	0.75	0.4047	0.1020	0.3265	0.1775	0.2970	0.1971
0.1	0.25	0.8379	0.6558	0.7871	0.7131	0.7822	0.7243
	0.50	0.6261	0.3618	0.5571	0.4393	0.5455	0.4578
	0.75	0.3795	0.1222	0.3153	0.1883	0.2902	0.2056
min $\underline{\gamma}$ and the max $\overline{\gamma}$		$n=10$		$n=50$		$n=100$	
α	q	qL	qU	qL	qU	qL	qU
0.01	0.25	0.9433	0.4934	0.8583	0.6286	0.8338	0.6705
	0.50	0.8281	0.1366	0.6777	0.3229	0.6435	0.3690
	0.75	0.5749	0.0153	0.4349	0.0916	0.3902	0.1238
0.05	0.25	0.9133	0.5472	0.8376	0.6583	0.8176	0.6915
	0.50	0.7671	0.2116	0.6384	0.3650	0.6117	0.3975
	0.75	0.5142	0.0373	0.4016	0.1222	0.3592	0.1503
0.1	0.25	0.8966	0.5723	0.8261	0.6727	0.8103	0.6999
	0.50	0.7334	0.2495	0.6229	0.3848	0.5980	0.4116
	0.75	0.4800	0.0566	0.3831	0.1386	0.3445	0.1634

Table 10: Proportion of runs with future observation greater than the quartiles, Eyring model. Case 3.

8. Concluding remarks

This paper has presented statistical methods for ALT using the Arrhenius-Weibull model under constant stress testing, supported by the theory of imprecise probability [3, 4], where the imprecision results from the use of the likelihood ratio test [26]. The proposed method applies the use of the likelihood ratio test to compare the survival distribution of pairwise stress levels, in combination with the Arrhenius model finding the interval of γ values according to the null hypothesis. The observations at the increased stress levels were transformed to interval-valued observations at the normal stress level by developing imprecision in the link function of the Arrhenius model via a classical test between the pairwise stress levels. We found that using an interval of values for the parameter in the link function between different stress levels enabled us to achieve a greater level of robustness than if we were to use a single point for the parameter. Using the Arrhenius model, we linked the data at different stress levels to the normal stress level, after which NPI can be used at normal stress level. The most pessimistic case, which leads to the lower survival function $\underline{S}$, uses $\underline{\gamma}$ to transform the data points to the smallest values at the normal stress level. Unlike the most optimistic case, which uses $\overline{\gamma}$ to transform the data points to the largest values at the normal stress level.

We discussed the use of pairwise tests instead of one test on all stress levels simultaneously, elsewhere [1]. If the model fits poorly, a single test on all stress levels would result in less imprecision while our proposed method, combining pairwise tests, tends to result in more imprecision. It should be mentioned that we were quite surprised to see that the classical literature in this field mostly presents methods for parameter estimation, with no explicit attention to prediction at the normal stress level. Hence, we did not find classical methods that were suitable for direct comparison with our method. Using the proposed approach, the intervals $[\underline{\gamma}, \overline{\gamma}]$ for the parameter for the link function have adequate imprecision if the model fits well. However, the intervals $[\underline{\gamma}, \overline{\gamma}]$ for the parameter for the link function get wider if the model fits poorly. The latter can happen if the model assumptions are not fully correct. However, if we have large imprecision, the remaining inferences are probably of no use at all. Therefore, it will be a strong recommendation to do more detailed modelling or to sample more data in such cases. Regarding the choice of the values of the factors for assessing robustness of the methods, we only show that any suggested form of misspecification can be included in simulations to then study the level of robustness. We have shown how the use of different significant levels for the pairwise likelihood ratio tests lead logically to varying imprecision in the final inferences. An interesting topic for future research concerns the choice of the significance level, in particular related to theoretical coverage properties for the final predictive inference resulting from the NPI-based inference in the form of lower and upper survival functions for a future observation at the normal stress level. It should be emphasized here, however, that our aim differs substantially from the usual hypothesis testing in frequentist statistics, as we do not aim at a good test performance in order to decide between two hypotheses, but we want to provide a method which can also be used if the assumed basic model does not fit well, which will then be indicated by large imprecision.

The application of our novel method in this paper assumed that we have failure times observed at the normal stress level K_0. The assumption of having failure data at the normal stress level K_0 may not be realistic in real world applications. In this case, we can apply our method with basic function to a higher stress level that is above the normal stress level K_0. The combined data at that level are then transformed all together to the normal stress level K_0. Investigating this is a topic for future research.

Usually, events of interest in reliability and survival analysis are failure times [8, 9]. However, such data often includes right-censored observations. The $A_{(n)}$ assumption cannot handle right-censored observations, and demands fully observed data. Coolen and Yan [10] presented a generalization of $A_{(n)}$, called rc-$A_{(n)}$, which is suitable for NPI with right-censored data. This method can be used at the second step in our approach if there are right-censored data, and also the likelihood ratio test can be applied. So generalizing this method to data including right-censored observations is straightforward. Moreover, a transformation link function, generalizing Equation (3), can also be derived if we allow different shape parameters β_i for each level i, so our method can be generalized in this way as well. Investigating these generalizations, for example to consider the effect of such changes on the imprecision in the resulting lower and upper survival functions is an important topic for future research. Of course, one may want to use a different parametric model at each stress level. As long as the model enables the Arrhenius link function (or a similar link function) to be used to link scale parameters, and transformations of data at higher stress levels to the normal stress levels can be written explicitly as function of a parameter of the link function, then the method as presented in this paper can straightforwardly be adapted. It is also possible to apply the basic idea of our new method without assuming a parametric model at each stress level, but instead using the Arrhenius link function directly to transform observations to the normal stress level. In that case, the likelihood ratio test used in this paper must be replaced by a nonparametric test to determine a range of values for the parameter of the Arrhenius link function for which the transformed data mix well with actual data at the normal stress level. As this approach requires substantially different aspects to be explained compared to the method presented in this paper, it will be presented elsewhere.

As mentioned in the introduction section, there is an increasing literature on more detailed modelling of degradation processes related to accelerated test scenarios. It will be of interest to investigate if the method presented in this paper can also be developed for inference based on such more detailed models. While this will pose substantial challenges, the main idea to avoid very detailed or complicated models through the use of simpler models in combination with imprecision suggests that there may be strong advantages possible for similar methods related to degradation processes.

Since we began this research, we have not found ALT methods with imprecision in the literature. The classical methods as presented in the literature seem effectively to stop at parameter estimates, so no predictions and certainly not predictions explicitly at the normal stress level are considered. Of course, there are more published imprecise statistical methods for other reliability inferences [8, 9]. The main approach presented in this paper, that is taking a simple model and adding imprecision to a parameter, then using corresponding transformed data for imprecision at the level of interest, has not been presented before for any inference problem. One important aspect of ALT applications is the decision of the design of the experiment, so which stress levels to use and how many items to use at each level. There is substantial attention to this in the literature on ALT, and we have not addressed it yet in combination with the inferential method presented in this paper, hence this is left as an important topic for future research.

As with any novel statistical method developed for real-world applications, the real value of our method should be shown in practical applications. To implement the method, not more is needed on the modelling side then for the classic inference methods with the same model assumptions, but the main questions are how one can use the resulting lower and upper survival functions to support real-world decisions. In continuing research, we are investigating this important aspect in the context of warranty contracts, while it is also interesting to consider the use of our method for other decision support scenarios. Finally, we are starting research into different ALT scenarios, where we aim to develop a similar approach, e.g. for the case of step-stress testing.

Acknowledgements

The authors thank two anonymous reviewers for helpful suggestions that have led to improved presentation. Abdullah A.H. Ahmadini gratefully acknowledges the financial support from Jazan University in Saudi Arabia.

References

[1] Ahmadini AAH, Coolen FPA (2018, September). Imprecise statistical inference for accelerated life testing data: imprecision related to log-rank test. *In International Conference Series on Soft Methods in Probability and Statistics*, Advances in Intelligent Systems and Computing (AISC). **823**, pp. 1–8. Springer.

[2] Alam K, Sudhir P (2015). Testing equality of scale parameters of two Weibull distributions in the presence of unequal shape parameters. *Acta et Commentationes Universitatis Tartuensis de Mathematica*, **19**, 11–26.

[3] Augustin T, Coolen FPA (2004). Nonparametric predictive inference and interval probability. *Journal of Statistical Planning and Inference*, **124**, 251–272.

[4] Augustin T, Coolen FPA, de Cooman G, Troffaes MCM (2014). *Introduction to Imprecise Probabilities*. Wiley, Chichester.

[5] Bretz, F., Genz, A., A. Hothorn, L. (2001). On the numerical availability of multiple comparison procedures. *Biometrical Journal: Journal of Mathematical Methods in Biosciences*, **43**, 645-656.

[6] Coolen FPA (2006). On nonparametric predictive inference and objective Bayesianism. *Journal of Logic, Language and Information*, **15**, 21–47.

[7] Coolen FPA (2011). Nonparametric predictive inference. *International Encyclopedia of Statistical Science*, pp. 968–970. Springer, Berlin.

[8] Coolen FPA, Coolen-Schrijner P, Yan KJ (2002). Nonparametric predictive inference in reliability. *Reliability Engineering and System Safety*, **78**, 185–193.

[9] Coolen FPA, Utkin LV (2011). *Imprecise reliability.* In: Lovric, M. (ed.), *International Encyclopedia of Statistical Science*, pp. 649–650. Springer, Berlin.

[10] Coolen FPA, Yan KJ (2004). Nonparametric predictive inference in reliability. *Reliability Engineering and System Safety*, **78**, 185–193.

[11] De Finetti B (1974). *Theory of Probability.* Wiley, Chichester.

[12] Duan F, Wang G (2018). Bivariate constant-stress accelerated degradation model and inference based on the inverse Gaussian process. *Journal of Shanghai Jiaotong University (Science)*, **23**, 784–790.

[13] Elsayed EA, Zhang H (2007). Design of PH-based accelerated life testing plans under multiple-stress-type. *Reliability Engineering and System Safety*, **92**, 286–292.

[14] Fan T, Wang W, Balakrishnan N (2008). Exponential progressive step-stress life-testing with link function based on Box–Cox transformation. *Journal of Statistical Planning and Inference*, **138**, 2340–2354.

[15] Fard N, Li C (2009). Optimal simple step stress accelerated life test design for reliability prediction. *Journal of statistical planning and inference*, **139**(5), 1799–1802.

[16] Han D (2015). Time and cost constrained optimal designs of constant-stress and step-stress accelerated life tests. *Reliability Engineering and System Safety*, **140**, 1-14.

[17] Hill BM (1968). Posterior distribution of percentiles: Bayes' theorem for sampling from a population. *Journal of the American Statistical Association*, **63**, 677–691.

[18] Hill BM (1988). De Finetti's theorem, induction, and Bayesian nonparametric predictive inference (with discussion). *Bayesian Statistics 3*, (Editors) J.M. Bernando, M.H. DeGroot, D.V. Lindley and A. Smith, pp. 211–241. Oxford University Press.

[19] Liao CM, Tseng ST (2006). Optimal design for step-stress accelerated degradation tests. *IEEE Transactions on Reliability*, **55**, 59–66.

[20] Meeker WQ, Escobar LA (1998). *Statistical Methods for Reliability Data*. Wiley, New York.

[21] Meeker WQ, Sarakakis G, Gerokostopoulos A (2013). More pitfalls in conducting and interpreting the results of accelerated tests. *Journal of Quality Technology*, **45**, 213–222.

[22] Nasira EA, Pan R (2015). Simulation-based Bayesian optimal ALT designs for model discrimination. *Reliability Engineering and System Safety*, **134**, 1–9.

[23] Nelson WB (1990). *Accelerated Testing: Statistical Models, Test Plans, and Data Analysis*. Wiley, Hoboken, New Jersey.

[24] O'Brien, P. C., Fleming, T. R. (1979). A multiple testing procedure for clinical trials. *Biometrics*, 549-556.

[25] Pan Z, Balakrishnan N, Sun Q (2011). Bivariate constant-stress accelerated degradation model and inference. *Communications in Statistics - Simulation and Computation*, **40**, 247–257.

[26] Pawitan Y (2001). *In all likelihood: Statistical Modelling and Inference using Likelihood*. Oxford University Press.

[27] Sha N, Pan R (2014). Bayesian analysis for step-stress accelerated life testing using Weibull proportional hazard model. *Statistical Papers*, **55**, 715–726.

[28] Vassilious A, Mettas A (2001). Understanding accelerated life-testing analysis. In *Annual Reliability and Maintainability symposium, Tutorial Notes*, 1–21.

[29] Yin Y, Coolen FPA, Coolen-Maturi TA (2017). An imprecise statistical method for accelerated life testing using the power-Weibull model. *Reliability Engineering and System Safety*, **167**, 158-167.