Prescribed k-symmetric curvature hypersurfaces in de Sitter space

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Abstract. We prove the existence of compact spacelike hypersurfaces with prescribed k-curvature in de Sitter space, where the prescription function depends on both space and the tilt function.

1 Introduction

We consider the existence problem for embedded compact spacelike hypersurfaces Σ in de Sitter space S_1^{n+1} satisfying a prescribed curvature equation of the form

$$(1.1) \qquad H_k^{\frac{1}{k}}(\lambda[A]) = \psi.$$

Here, $1 \le k \le n$ is fixed, A is the second fundamental form of Σ, $\lambda[A] = (\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of the shape operator A^i_j, and H_k is the k-th normalised symmetric polynomial in λ, that is,

$$S_k(\lambda) := \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}, \qquad H_k := \binom{n}{k}^{-1} S_k.$$

Furthermore, we will choose our prescription function ψ to depend on both position in S_1^{n+1} and the tilt, τ, which is a measure of how spacelike Σ is at the position; see (2.7). We note that as every compact embedded spacelike surface in S_1^{n+1} can be written as a graph, (1.1) can be rewritten as a fully nonlinear elliptic second order partial differential equation in the graph function. Our main result is that, assuming some natural structural assumptions on ψ, the prescribed curvature equation (1.1) has a smooth spacelike solution Σ.

The existence of solutions of such equations was studied in [4] by L. Caffarelli, L. Nirenberg and J. Spruck. In [5], they proved the existence of star-shaped hypersurfaces in Euclidean space with prescribed k-symmetric curvature using an priori $C^{2,\alpha}$ estimate needed to carry out the continuity method. Curvature estimates for star-shaped hypersurfaces with given k-symmetric curvature have been established

for various ambient Riemannian manifolds. For hypersurfaces in the sphere, the lower order and the curvature estimate are given in [2] by M. Barbosa, L. Herbert, and V. Oliker. These were used to prove the existence result by Y. Li and V. Oliker in [16] via a degree theory argument. The curvature estimate and the existence result for prescribed curvature hypersurfaces in the hyperbolic space was proved by Q. Jin and Y. Li in [13] using similar arguments of W. Sheng, J. Urbas, and X. Wang in [17].

For spacelike hypersurfaces in Lorentz manifolds, less is known. The case of prescribed mean curvature was studied by C. Gerhardt [8, 9] and R. Bartnik and L. Simon [3]. C. Gerhardt [7] also studied more general curvature functionals, but was forced to only consider a comparatively restrictive class of curvature functionals due to a problematic sign on a term in the curvature estimates (in particular, all of these functionals are zero on the boundary of the positive cone, Γ^+). Urbas [18] obtained curvature estimates for prescribed symmetric curvature under the assumption that the mean curvature was in L^p for sufficiently large p. Y. Huang [12] noted that over compact domains Minkowski space, if the prescription function is additionally required to depend on the tilt and satisfies certain structural assumptions, then the problematic term in the curvature estimates may be cancelled and curvature estimates may be obtained. D. Ballesteros-Chávez [1] extended this result to compact domains in de Sitter space. We note that similar prescribed curvature problems with prescription function depending on the normal have also been studied in Eulidean spaces by P. Guan, J. Li, and Y. Li [10] and by P. Guan, C. Ren, and Z. Wang [11].

We will prove the following theorem.

Theorem 1.1   *Suppose that $\psi : S_1^{n+1} \times \mathbb{R}_+ \to \mathbb{R}$ is a smooth positive function satisfying structural conditions (i)–(v) (see Assumptions on ψ) and let $1 \le k \le n$. Then there exists a smooth embedded k-admissable spacelike hypersurface $\Sigma \subset S_1^{n+1}$ satisfying (1.1).*

In Section 2, we will collect all required definitions and some preliminary calculations. In Sections 3 and 4, we will prove the required C^0 and C^1 estimates, respectively. In Section 5, we extend earlier results [1, 12] to give the curvature estimates (and therefore C^2 estimates). In Section 6, we prove the existence of a solution via a regularity result of Evans and Krylov [6, 14] and the degree theory of Y. Li [15].

We now state our structural assumptions on the prescription ψ.

Assumptions on ψ *We impose the following structural assumptions on* $\psi : S_1^{n+1} \times [1.\infty) \to \mathbb{R}$:

(i)   *(Barrier conditions) There exist constants $0 < R_1 < R_2 < \infty$ such that*

$$\tanh(r) > \psi(Y(r, \xi), \cosh(r)) \quad \text{forall } \xi \in \mathbb{S}^n, r < R_1,$$
$$\tanh(r) < \psi(Y(r, \xi), \cosh(r)) \quad \text{forall } \xi \in \mathbb{S}^n, r > R_2,$$

   where Y is as in equation (2.3).

(ii)   *(Differential inequality) For all $x \in S_1^{n+1}$ and $\tau \in [1, \infty)$,*

$$\psi_\tau(x, \tau)\tau \ge \psi(x, \tau).$$

(iii)   *(Asymptotics) $\frac{\psi(x,\tau)}{\tau} \to \infty$ as $\tau \to \infty$ for all $x \in S_1^{n+1}$.*

(iv) *(C¹ bound) Taking coordinates* $(x^1, \ldots, x^{n+1}) = (r, \xi_1, \ldots, \xi_n)$ *on* S_1^{n+1} *as in* (2.3), *we have that there exists a uniform* $C > 0$ *such that for all* $x \in S_1^{n+1}$ *and* $\tau \in \mathbb{R}$,

$$\left| \frac{\partial \psi(x, \tau)}{\partial x^i} \right| \leq C \psi(x, \tau).$$

(v) *(Convexity in* τ*)* $\psi_{\tau\tau}(x, \tau) \geq 0$ *for all* $x \in S_1^{n+1}$ *and* $\tau \in [1, \infty)$.

Remark 1.2 Assumption (i) is simply to ensure the existence of suitable barriers. The relevance of the rate $\tanh(r)$ is that this is curvature of the natural foliation of totally umbilic hypersurfaces in de Sitter space.

Remark 1.3 We note that condition (ii) already implies that $\psi(x, \tau) \geq \tau \psi(x, 1)$, so (iii) can be considered as the smallest possible increase in growth on top of this assumption.

Remark 1.4 Condition (iv) is used to estimate the space derivative of ψ with respect to a multiple of ψ. This is a necessary condition in our tilt estimates, as the derivative may be vastly larger with respect to τ; for example, if in local coordinates $\psi(x, \tau) = \tau^2 + x^1 e^\tau$, then at $x^1 = 0$, $\frac{\partial \psi(x, \tau)}{\partial x^1}$ cannot be estimated by ψ.

Remark 1.5 There are an abundance of functions ψ that satisfy the structure conditions (i)–(v) (see also Lemma 3.2). Our model function is $\psi(x, \tau) = \Psi(x)\tau^p$ where $p > 1$ and Ψ is a smooth bounded function satisfying the conditions of Lemma 3.2.

2 Preliminaries

2.1 Subspace Geometry in Lorentzian Manifolds

To avoid confusion with signs, we now collect some geometric formulae for hypersurfaces in Lorentzian manifolds.

Let $\{\partial_1, \ldots, \partial_n, N\}$ be a basis for a Lorentzian manifold $(\overline{M}, \overline{g})$ and M a Lorentzian (not necessarily spacelike) hypersurface with induced metric g such that $\{\partial_i\}$ span TM, and let N be the unit normal field to M and put $\varepsilon = \overline{g}(N, N)$. When the induced metric is positive definite, then we say that M is a spacelike hypersurface; then g can be represented by the matrix $g_{ij} = g(\partial_i, \partial_j)$ with inverse denoted by g^{ij}.

The *Gauss formula* for $X, Y \in T\Sigma$ reads

$$D_X Y = \nabla_X Y + \varepsilon \, h(X, Y) N.$$

Here, D is the connection on $\overline{M}$, ∇ is the induced connection on M, and the *second fundamental form* h is the normal projection of D. In coordinate basis we write

$$h_{ij} = h(\partial_i, \partial_j).$$

The *shape operator* obtained by raising an index with the inverse of the metric

$$h_j^i = g^{ik} h_{kj}.$$

The *principal curvatures* of the hypersurface Σ are the eigenvalues of the symmetric matrix (h_j^i). The tangential projection of the covariant derivative of the normal vector field N on Σ, $\nabla_j N = (D_{\partial_j} N)^\top$, is related to the second fundamental form by the *Weingarten equation*

$$(2.1) \qquad \nabla_j N = -h_j^i \partial_i = -g^{ik} h_{kj} \partial_i.$$

The *curvature tensor* is defined for $X, Y, Z \in T\Sigma$ as

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

Contracting with the metric

$$R_{ijkl} = g\big(R(\partial_i, \partial_j)\partial_k, \partial_l\big) = g_{lm} R_{ijk}^m.$$

To relate curvature and second fundamental form, we have the *Codazzi equations*

$$\big\langle \overline{R}_{ijk}, N \big\rangle = \nabla_j h_{ik} - \nabla_i h_{jk}.$$

and the *Gauss equation*

$$\overline{R}_{ijkl} = R_{ijkl} - \varepsilon(h_{ik} h_{jl} - h_{il} h_{jk}).$$

Note that if T is a symmetric tensor, then the following *Ricci identity* holds:

$$(2.2) \qquad \nabla_k \nabla_l T_{ij} - \nabla_l \nabla_k T_{ij} = R_{kljr} T_{ir} + R_{klir} T_{rj}.$$

2.2 The Geometry of de Sitter Space

We now consider manifolds $\Sigma \subset S_1^{n+1} \subset \mathbb{R}_1^{n+2}$ where
- $\mathbb{R}_1^{n+2} = (\mathbb{R}^{n+2}, \overline{\overline{g}})$ is *Minkowski space* with metric

$$\overline{\overline{g}} = -dx_1^2 + dx_2^2 + \cdots + dx_{n+2}^2$$

  and covariant derivative $\overline{D}$;
- S_1^{n+1} is a *de Sitter space*, defined by

$$S_1^{n+1} = \big\{ x \in \mathbb{R}_1^{n+2} : -x_1^2 + x_2^2 + \cdots + x_{n+2}^2 = 1 \big\}$$

  with the induced Lorentzian metric $\overline{g}$, covariant derivative D, unit normal N and second fundamental form h;
- $\Sigma \subset S_1^{n+1}$ is a embedded spacelike hypersurface of S_1^{n+1} with induced Riemannian metric g, covariant derivative ∇, unit normal v and second fundamental form A.

Let $\mathbb{S}^n$ be the standard round sphere. Then de Sitter space can be parametrised by $Y : \mathbb{S}^n \times \mathbb{R} \to S_1^{n+1}$ given by

$$(2.3) \qquad Y(r, \xi) = \sinh(r)E_1 + \cosh(r)\xi,$$

and in these coordinates, the induced metric is

$$\overline{g} = -dr^2 + \cosh^2(r)\sigma,$$

where σ is the round metric on $\mathbb{S}^n$. We note that as $|Y|^2 = 1$, we have that $Y = N$ is a unit normal to S_1^{n+1} and (as $0 = \langle Y_{\alpha\beta}, Y \rangle + \langle Y_\alpha, Y_\beta \rangle$)

(2.4)
$$h_{\alpha\beta} = -\overline{g}_{\alpha\beta},$$

(where $1 \le \alpha, \beta \le n+1$), and so using the Gauss equation, we have

$$R^{S_1^{n+1}}_{\alpha\beta\gamma\delta} = \overline{g}_{\alpha\gamma}\overline{g}_{\beta\delta} - \overline{g}_{\alpha\delta}\overline{g}_{\beta\gamma},$$

where we used that Minkowski space is flat. This implies that on Σ (which has a timelike normal), we have that

(2.5)
$$\nabla_i h_{jk} = \nabla_j h_{ik},$$

(2.6)
$$R_{ijkl} = A_{il}A_{jk} - A_{ik}A_{jl} + g_{ik}g_{jl} - g_{il}g_{jk}.$$

We define the *tilt function* on Σ to be the function

(2.7)
$$\tau = \langle v, E_1 \rangle,$$

where v is a unit normal to Σ that has been chosen so that τ is positive.

We now represent Σ as a graph; that is, we take $u : \mathbb{S}^n \to \mathbb{R}$ so that Σ is parametrised by $X : \mathbb{S}^n \to S_1^{n+1}$ given by $X(\xi) = Y(u(\xi), \xi)$. We will use $\tilde{\nabla}$ to denote the standard covariant derivative for the metric σ on $\mathbb{S}^n$, and our indices $\partial_i, \partial_j, \dots$, etc., take values from 1 to n, except for the vector field ∂_r, which will be considered separately. The tangent space of the hypersurface at a point $Y \in \Sigma$ is spanned by the tangent vectors $Y_j = u_j \partial_r + \partial_j$; the covariant derivative ∇ corresponding to the induced metric on Σ is given by

$$g_{ij} = -u_i u_j + \cosh^2(u)\sigma_{ij}.$$

We write

$$\tilde{\tau} = \frac{\cosh^2(u)}{\sqrt{\cosh^2(u) - |\tilde{\nabla}u|^2}}$$

where $\tilde{\nabla}u = \sigma^{ij}u_j\partial_i$ and $|\tilde{\nabla}u| := \sigma^{ij}u_i u_j$ (we will see shortly that $\tilde{\tau} = \tau$). Σ is spacelike at a point if g_{ij} is invertible, which is equivalent to $\tilde{\tau}$ being finite at that point. Since Σ is spacelike, we calculate the inverse of g_{ij} to be

(2.8)
$$g^{ij} = \cosh^{-2}(u)\left[\sigma^{ij} + \tilde{\tau}^2 \frac{\sigma^{il}u_l\sigma^{jm}u_m}{\cosh^4(u)}\right].$$

A unit normal vector to Σ at the point Y can be obtained by solving the equation $\overline{g}(Y_\alpha, \hat{n}) = 0$, and then we get

$$v = -\frac{\cosh^2(u)\partial_r + \tilde{\nabla}u}{\sqrt{\cosh^4(u) - \cosh^2(u)|\tilde{\nabla}u|^2}} = \cosh^{-3}(u)\tilde{\tau}\left(\cosh^2(u)\partial_r + \widetilde{\nabla}u\right).$$

We note that

$$\cosh(u)\partial_r = E_1 + \sinh(u)Y,$$

and so we see that

$$\tau = \langle E_1, v \rangle = \tilde{\tau},$$

as claimed. The second fundamental form is the projection of the second derivatives of the parametrisation $D_{Y_\alpha} Y_\beta$ on the normal direction. Writing $\tilde{\Gamma}$ for the Christoffel symbols of the metric σ, we have

$$D_{\partial_r}\partial_r = 0, \quad D_{\partial_r}\partial_j = \tanh(r)\partial_j, \quad D_{\partial_i}\partial_j = \cosh(r)\sinh(r)\sigma_{ij}\partial_r + \tilde{\Gamma}^k_{ij}\partial_k,$$

and using these identities, we compute

$$\begin{aligned} D_{Y_i}Y_j &= D_{u_i\partial_r+\partial_i}\left(u_j\partial_r + \partial_j\right) \\ &= u_ju_jD_{\partial_r}\partial_r + u_iD_{\partial_r}\partial_j + u_{ij}\partial_r + u_jD_{\partial_i}\partial_r + D_{\partial_i}\partial_j. \end{aligned}$$

It follows that $A_{ij} = \overline{g}(D_{Y_i}Y_j, v)$ is given explicitly by

$$(2.9) \qquad A_{ij} = \cosh^{-1}(u)\tau\big(\tilde{\nabla}^2_{ij}u - 2\tanh(u)u_iu_j + \sinh(u)\cosh(u)\sigma_{ij}\big).$$

Finally, we define a notion of partial derivatives for ψ on S_1^{n+1}. Suppose we have a function $f : S_1^{n+1} \times \mathbb{R} \to \mathbb{R}$; then we define the partial derivative on S_1^{n+1} and Σ by

$$D^x f = \frac{\partial \psi}{\partial x^\alpha}\overline{g}^{\alpha\beta}\partial_\beta, \qquad \nabla^x \psi = (D^x f)^\top.$$

Similarly, we can define second derivatives of f in the usual tensorial way.

2.3 Curvature Functionals and Admissability

Throughout this section we fix $1 \le k \le n$. As described above, we will consider functions

$$F[A] := H_k^{\frac{1}{k}}(\lambda[A]),$$

where $\lambda[A] = (\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of the symmetric matrix A, and we will define $f : \mathbb{R}^n \to \mathbb{R}$ by $f = H_k^{\frac{1}{k}}$. We define the *admissable cone* of H_k to be Γ_k that is defined to be the connected component of $H_k^{-1}(\{x \in \mathbb{R} | x > 0\})$ that contains the positive cone

$$\Gamma^+ = \{\lambda \in \mathbb{R}^n \mid \lambda_i > 0, \forall i = 1, 2, \ldots, n\}.$$

We have that for all $\lambda \in \Gamma_k$, $f_{\lambda_i}(\lambda) > 0$ and f is concave in Γ_k. Since $f \in C^2(\Gamma_k) \cap C^0(\overline{\Gamma_k})$, it follows that $F[A]$ is elliptic and concave if the eigenvalues of A lie in Γ_k.

A spacelike hypersurface $\Sigma \subset S_1^{n+1}$ will be called *k-admissable* if for all $p \in \Sigma$ the eigenvalues of the shape operator $A_i^j = A_{ik}g^{kj}$ are in Γ_k.

A function $u \in C^2(\mathbb{S}^n)$ will be called *k-admissable* if the graph of u is a spacelike admissable hypersurface. We note that this implies that u is positive, as if there is a negative minimum of u at $p \in \Sigma$, the shape operator is negative definite so $\lambda[A] \notin \Gamma_k$.

As is standard, and we will write

$$F^{ij} := \frac{\partial F}{\partial A_{ij}}, \qquad F^{ij,kl} := \frac{\partial^2 F}{\partial A_{ij} \partial A_{kl}}.$$

3 A Priori C^0 Estimate

Considering Σ graphically, any solution $u \in C^2(\mathbb{S}^n)$ of (1.1), the barrier conditions (3.1) will ensure that $R_1 \le u(\xi) \le R_2$ for all $\xi \in \mathbb{S}^n$. The proof follows maximum principle arguments similar to [2, Lemma 3.1].

Lemma 3.1 *Let $1 \le k \le n$ and let $\psi : (0, \infty) \times \mathbb{S}^n \times [1, \infty) \to \mathbb{R}$ is a continuous positive function such that there exist constants $0 < R_1 < R_2 < \infty$ such that*

(3.1)
$$\tanh(r) > \psi(r, \xi, \cosh(r)), \quad \text{for } \xi \in \mathbb{S}^n, r < R_1,$$
$$\tanh(r) < \psi(r, \xi, \cosh(r)), \quad \text{for } \xi \in \mathbb{S}^n, r > R_2.$$

Then $u \in C^2(\mathbb{S}^n)$ is a solution of (1.1) then for all $\xi \in \mathbb{S}^n$, u satisfies

$$R_1 \le u(\xi) \le R_2.$$

Proof Suppose there is a point $\xi_0 \in \mathbb{S}^n$ where the maximum of u is attained, say $R_2 < r_0 = u(\xi_0)$. Then at the maximum, we have $\tilde{\nabla} u \doteq 0$ and $\tilde{\nabla}^2 u \dot{\le} 0$. Substituting these into the equations of the previous section, the inverse of the metric, the tilt, and the second fundamental form at ξ_0 are

$$g^{ij} \doteq \frac{1}{\cosh^2(r_0)} \sigma^{ij}, \quad \tau \doteq \cosh u(u) = \cosh(r_0), \quad A_{ij} \dot{\le} \sinh(r_0) \cosh(r_0) \sigma_{ij},$$

respectively. At ξ_0, the shape operator therefore satisfies

$$A^i_j \le \tanh(r_0) \delta^i_j,$$

and so $\lambda_i \le \tanh(r_0)$ for $1 \le i \le n$. Substituting into (1.1),

$$\psi(r_0, \xi_0, \cosh(r_0)) \doteq F[A^i_j] \le \tanh(r_0) < \psi(r_0, \xi_0, \cosh(r_0)),$$

which is a contradiction, and so $u \le R_2$. An analogous argument at the minimum completes the proof. ∎

Observe that we can impose a few fairly mild assumptions on ψ_r to ensure that barriers exist.

Lemma 3.2 *Let $\psi : [0, \infty) \times \mathbb{S}^n \times [1, \infty) \to \mathbb{R}$ be a uniformly bounded in C^2 that is positive for $r > 0$. We give conditions for upper and lower barriers:*
Lower Barriers: If for all $\xi \in \mathbb{S}^n$

$$\psi(0, \xi, 1) = 0 \quad \text{and} \quad \psi_r(0, \xi, 1) < 1,$$

then a lower barrier exists; that is, there exists an $0 < R_1$ such that for all $\xi \in \mathbb{S}^n$ and $r < R_1$,

$$\tanh(r) > \psi(r, \xi, \cosh(r)).$$

Upper Barriers Suppose ψ satisfies the structural condition $\psi_\tau \tau > \psi$ and let $\gamma(r)$ be any smooth monotonic function s.t. $\gamma(r) \to \infty$ as $r \to \infty$. Suppose additionally that for all $\xi \in \mathbb{S}^n$ and $r > \tilde{R}$,

$$\psi_r(r, \xi, \cosh(r)) > \left[\gamma'(r) - \tanh(r) \right] \psi \, ;$$

then an upper barrier exists; that is, there exists an $0 < R_2$ such that for all $\xi \in \mathbb{S}^n$ and $r > R_2$,

$$\tanh(r) < \psi(r, \xi, \cosh(r)).$$

Proof   We fix ξ and consider the function

$$\Psi(r, \xi) = \frac{\psi(r, \xi, \cosh(r))}{\tanh(r)}.$$

Finding a lower barrier is equivalent to showing that for all $\xi \in \mathbb{S}^n$, there exists a $R_1 > 0$ such that for all $0 < r < R_1$, $\Psi(r, \xi) < 1$. By the assumptions, Ψ is continuous up to $r = 0$, and so by compactness, there exists a $\delta > 0$ such that for all $\xi \in \mathbb{S}^n$, $\Psi(0, \xi) < 1 - \delta$. By continuity and compactness, there exists an R_1 such that $\Psi(r, \xi) < 1$ for all $0 < r < R_1$, as claimed. Finding an upper barrier is equivalent to showing that for all $\xi \in \mathbb{S}^n$, there exists a $R_2 > 0$ such that for all $r > R_2$, $\Psi(r, \xi) > 1$. We calculate that

$$\frac{d}{dr} \Psi(r, \xi) = -\frac{1}{\sinh^2(r)} \psi(r, \xi, \cosh(r)) + \coth(r) \left(\psi_r + \psi_\tau \sinh(r) \right)$$

$$\geq \left[1 - \frac{1}{\sinh^2(r)} \right] \psi + \coth(r) \psi_r$$

$$\geq \frac{\gamma'(r)}{\tanh(r)} \Psi$$

for $r > \hat{R}(\tilde{R})$ sufficiently large depending on $\tilde{R}$. Since $\tanh(r) < 1$, we see that

$$\frac{d}{dr} \Psi(r) > \gamma'(r) \Psi$$

which implies $\Psi(\xi, r) > \Psi(\hat{R}, \xi) e^{\gamma(r)}$. Since $\mathbb{S}^n$ is compact, and $\gamma(r) \to \infty$ as $r \to \infty$, this implies that the claimed upper barrier conditions are met.   ∎

Remark 3.3   A function ψ satisfying lower barrier conditions in Lemma 3.2 allows the solution to (1.1) given by $u \equiv 0$. As noted earlier, this solution is inadmissable, as we require strictly positive u for the shape operator to be in the admissible cone everywhere.

4 Tilt Estimate

We now demonstrate a strict spacelikeness estimate by estimating the tilt function τ. The height function is defined by

$$\eta := -\langle Y, E_1 \rangle,$$

and we note that in terms of the graph function, $\eta = \sinh(u)$. We now demonstrate the following identities.

Proposition 4.1 *The tilt and height functions satisfy the following identities:*

(i) $\nabla^2_{ij}\eta = \tau A_{ij} + \eta g_{ij};$

(ii) $\nabla_j \tau = A^i_j \nabla_i \eta;$

(iii) $\nabla_j \nabla_i \tau = \nabla_k A_{ij} \nabla^k \eta + \tau A^2_{ij} + A_{ij}\eta,$

where $A^2_{ij} := A_{kj}A^k_i$.

Proof Using (2.4) and the Gauss formula, we have that

$$\begin{aligned}
\nabla^2_{ij}\eta &= Y_j(Y_i\eta) - (\nabla_{Y_i}Y_j)\eta \\
&= Y_j(Y_i\eta) - (\overline{D}_{Y_i}Y_j - h_{ij}N + A_{ij}\hat{n})\eta \\
&= \overline{D}^2_{Y_jY_i}\eta + (h_{ij}N - A_{ij}v)\eta \\
&= \tau A_{ij} + \eta g_{ij},
\end{aligned}$$

where we used that $X(\eta) = -\langle X, E_1 \rangle$ and $\overline{D}^2\eta = 0$.

Using the Weingarten equation (2.1), we obtain

$$\nabla_j \tau = \langle \nabla_j v, E_1 \rangle = -g^{ik}A_{kj}\langle Y_i, E_1 \rangle = -g^{ik}A_{kj}\nabla_i\langle Y, E_1 \rangle = g^{ik}A_{kj}\nabla_i\eta,$$

and from this we have

$$\begin{aligned}
\nabla^2_{ij}\tau &= \nabla_j(g^{mn}A_{ni}\nabla_m\eta) \\
&= g^{mn}\nabla_j A_{ni}\nabla_m\eta + g^{mn}A_{ni}\nabla_{mj}\eta \\
&= g^{mn}\nabla_n A_{ij}\nabla_m\eta + \tau A_{mj}g^{mn}A_{ni} + A_{ij}\eta,
\end{aligned}$$

where we used (2.5) on the third line. ∎

Proposition 4.2 *Suppose that $\psi : S^{n+1} \times \mathbb{R} \to \mathbb{R}$ is smooth and positive and satisfies assumptions (iii) and (iv) above. Suppose that $u \in C^3(\mathbb{S}^n)$ satisfies (1.1) so that there exist $R_1, R_2 > 0$ such that,*

$$0 < R_1 < u(\xi) < R_2$$

for all $\xi \in \mathbb{S}^n$. Then there exists a constant C_τ, depending only on n, k, R_1, R_2 and ψ such that $\tau < C_\tau$.

Proof We have that

$$F^{ij}\nabla_j\nabla_i\tau = \nabla^k F\nabla_k\eta + \tau F^{ij}A_{mj}A^m_i + F\eta$$

$$= \langle \nabla^x \psi, \nabla \eta \rangle + \psi_\tau \langle \nabla \tau, \nabla \eta \rangle + \tau F^{ij} A^2_{ij} + \psi \eta.$$

Due to structural assumption (iv) on ψ, we have that

$$| \langle \nabla^x \psi, \nabla \eta \rangle | \leq C\tau^2 \psi.$$

Furthermore, following [18, equation 3.8] and using Newton's inequalities [19] (which are valid for S_k *outside* Γ_k),

(4.1) $$F^{ij} A^2_{ij} \geq H_k^{\frac{1}{k}} H_1 \geq H_k^{\frac{2}{k}} = \psi^2.$$

Substituting this into the above equation, we see that at a maximum of τ, as $\nabla \tau = 0$ and $\nabla_i \nabla_j \tau \leq 0$, we have that

$$0 \geq -C\tau^2 \psi + \psi \eta + \tau \psi^2,$$

which implies, by using the C^0 estimates, that

$$\psi \leq C\tau .$$

The C^0 estimates imply that the solution stays in a compact region of S_1^{n+1} and so, due to structural assumption (iii) on ψ, there exists a uniform τ_0 such that for all x in this region and for all $\tau > \tau_0$, $\tau < \frac{\psi}{2C}$. This yields a contradiction to $\tau > \tau_0$ and proves the lemma. ∎

5   A Priori C^2 Estimate

From [1] we have the following curvature estimates over domains $\Omega \subset S^n$.

Theorem 5.1 (Ballesteros–Chávez 2019)   *Let $\Omega \subset \mathbb{S}^n$ be a domain in the round sphere, and let $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ an admissible solution of the boundary value problem*

$$\begin{cases} F(A) = H_k^{\frac{1}{k}}(\lambda(A)) = \psi(Y, \tau) & in \quad \Omega, \\ u = \varphi & on \quad \partial\Omega, \end{cases}$$

where A is the second fundamental form of a spacelike surface Σ in de Sitter space given by (2.9) and ψ is a smooth positive function satisfying (ii) and (v). Assume additionally that there exists a $R_1, R_2, C_\tau > 0$ such that

$$R_1 < u(\xi) < R_2, \qquad \tau(\xi) < C_\tau.$$

Then

$$\sup_\Omega |A| \leq C,$$

where C depends on n, $|\varphi|_{C^1(\overline{\Omega})}$, R_1, R_2, C_τ, $|\psi|_{C^2([R_1,R_2]\times\Omega\times[1,C_\tau])}$ and $\sup_{\partial\Omega} |A|$.

We will now extend this result to all of $\mathbb{S}^n$, or, equivalently, obtain estimates on all of Σ.

Theorem 5.2   *Suppose that ψ is a smooth positive function that satisfies (ii) and (v). Suppose $u \in C^4(\mathbb{S}^n)$ is a solution of (1.1) such that there exist constants $0 < R_1 < R_2$ and*

$C_\tau > 0$ *such that for all* $\xi \in \mathbb{S}^2$,

$$R_1 < u(\xi) < R_2, \qquad \tau(\xi) < C_\tau.$$

Then there exists a constant $C_A = C_A\big(k, n, R_1, R_2, C_\tau, |\psi|_{C^2([R_1,R_2]\times\mathbb{S}^n\times[1,C_\tau])}\big)$ *so that* $|A| < C_A$.

Proof Suppose first that $k = 1$. In this case, (2.8) and (2.9) imply that if we write (1.1) in terms of the graph function u, we obtain a quasilinear equation that is uniformly elliptic if we have a uniform bound on τ. Therefore, if we have the assumed bounds, we see that this equation is uniform ellipticity with uniform C^1 estimates on the solution. We can therefore apply De Giorgi–Nash–Moser estimates and Schauder estimates to imply uniform C^2 estimates. In this case, the theorem is therefore proven.

Suppose now that $k \geq 2$. We begin by proving a Simon's-type identity for the second fundamental form. At an arbitrary point $p \in \Sigma$, we choose coordinates so that A_{ij} is diagonal and $g_{ij} = \delta_{ij}$. In these coordinates, F^{ij} is also diagonal.

The Codazzi equation (2.5) and the Ricci identity (2.2) imply that

$$
\begin{aligned}
\nabla_i \nabla_j A_{kk} &= \nabla_i \nabla_k A_{kj} \\
&= \nabla_k \nabla_i A_{kj} + R_{ikkr} A_j^r + R_{ikjr} A_k^r \\
&= \nabla_k \nabla_k A_{ij} + R_{ikkr} A_j^r + R_{ikjr} A_k^r,
\end{aligned}
$$

where we only sum over indices where one is raised and one is lowered. Choosing $i = j$ in the above, and applying the Gauss equation (2.6), we obtain

$$
\begin{aligned}
\nabla_j \nabla_j A_{kk} &= \nabla_k \nabla_k A_{jj} + R_{jkkr} A_j^r + R_{jkjr} A_k^r \\
&= \nabla_k \nabla_k A_{jj} + \big(A_{jr} A_{kk} - A_{jk} A_{kr} + g_{jk} g_{kr} - g_{jr} g_{kk}\big) A_j^r \\
&\quad + \big(A_{jr} A_{kj} - A_{jj} A_{kr} + g_{jj} g_{kr} - g_{jr} g_{kj}\big) A_k^r \\
&= \nabla_k \nabla_k A_{jj} + A_{jj}^2 A_{kk} - A_{jk} A_{kj}^2 + g_{jk} A_{jk} - A_{jj} g_{kk} \\
&\quad + A_{jk}^2 A_{kj} - A_{jj} A_{kk}^2 + g_{jj} A_{kk} - A_{kj} g_{kj} \\
&= \nabla_k \nabla_k A_{jj} + A_{jj}^2 A_{kk} - A_{jj} g_{kk} - A_{jj} A_{kk}^2 + g_{jj} A_{kk}.
\end{aligned}
$$

We therefore see that

$$
\begin{aligned}
F^{ij} \nabla_i \nabla_j A_{kk} &= \sum_{j=1}^n F^{jj} \nabla_j \nabla_j A_k k \\
&= F^{ij} \nabla_k \nabla_k A_{ij} + A_{kk} F^{ij} A_{ij}^2 - F^{ij} A_{ij} A_{kk}^2 + A_{kk} \mathrm{tr} F^{ij} - F^{ij} A_{ij} \\
&= F^{ij} \nabla_k \nabla_k A_{ij} + A_{kk} F^{ij} A_{ij}^2 - \psi A_{kk}^2 + A_{kk} \mathrm{tr} F^{ij} - \psi,
\end{aligned}
$$

and so writing $H = nH_1 = \sum_{k=1}^n A_{kk}$, we have that

$$F^{ij} \nabla_i \nabla_j H = F^{ij} \Delta A_{ij} + H F^{ij} A_{ij}^2 + H \mathrm{tr} F^{ij} - \psi\big(n + |A|^2\big),$$

where Δ is the Laplace–Beltrami operator. To estimate the first term on the right-hand side, we note that differentiating (1.1) twice gives

$$\Delta \psi = F^{ij} \Delta A_{ij} + F^{ij,kl} \nabla_t A_{ij} \nabla^t A_{kl} \leq F^{ij} \Delta A_{ij},$$

where we used the well known concavity of F.

We have that

$$\nabla_l \nabla_k \psi = \nabla_l^x \nabla_k^x \psi + \nabla_k^x \psi_\tau \nabla_l \tau + \nabla_l^x \psi_\tau \nabla_k \tau + \psi_\tau \nabla_l \nabla_k \tau + \psi_{\tau\tau} \nabla_l \tau \nabla_k \tau$$
$$= D_l^x D_k^x \psi - A_{lk} D_n^x \psi + \nabla_l^x \psi_\tau \nabla_k \tau + \psi_\tau \nabla_l \nabla_k \tau + \psi_{\tau\tau} \nabla_l \tau \nabla_k \tau.$$

Since $H^2 = 2S_2 + |A|^2$, so if $\lambda[A] \in \Gamma_k$, $|A| < H$, and so

$$\Delta \psi = g^{kl} D_l^x D_k^x \psi - H D_n^x \phi + 2 \langle \nabla^x \psi_\tau, \nabla \tau \rangle + \psi_\tau \Delta \tau + \psi_{\tau\tau} |\nabla \tau|^2$$
$$= \psi_\tau \big[\langle \nabla \eta, \nabla H \rangle + \tau |A|^2 + H\eta \big] + \psi_{\tau\tau} |\nabla \tau|^2$$
$$\quad - H D_n^x \psi + 2 \langle \nabla^x \psi_\tau, \nabla \tau \rangle + g^{kl} D_l^x D_k^x \psi$$
$$\geq \psi_\tau \langle \nabla \eta, \nabla H \rangle + \psi_\tau \tau |A|^2 - C_1 H - C_2,$$

where used structural assumption (v) of ψ and estimated using the bounds on τ and u in compactness arguments to estimate derivatives of ψ. Overall, we have that

$$F^{ij} \nabla_i \nabla_j H \geq H F^{ij} A_{ij}^2 + H \mathrm{tr} F^{ij} + (\psi_\tau \tau - \psi) |A|^2$$
$$\quad + \psi_\tau \langle \nabla \eta, \nabla H \rangle - C_1 H - C_2 - n\psi \,.$$

Using (4.1) and the structural assumption (ii) on ψ we have that

$$F^{ij} \nabla_i \nabla_j H \geq \frac{1}{n} H^2 \psi + H \mathrm{tr} F^{ij} + \psi_\tau \langle \nabla \eta, \nabla H \rangle - C_1 H - C_2 - n\psi.$$

Using the bounds on u, there exists a small constant $\delta > 0$ such that $\psi > \delta$, and so at a maximum point of H,

$$0 \geq F^{ij} \nabla_i \nabla_j H$$
$$\geq \frac{\delta}{n} H^2 - C_1 H - C_3,$$

and so H is bounded. This implies a uniform bound on H and a uniform bound on $|A|$ now follows as $k \geq 2$.   ∎

6 Proof of Existence

We now prove existence of solutions to (1.1), following the proof of V. Oliker and Y. Li [16]. Throughout this section, Σ will be considered in graphical coordinates with graph function u. Consider for $0 < \alpha < 1$, the subset of functions in $C^{4,\alpha}(\mathbb{S}^n)$ that are k-admissible, denoted by $C_{ad}^{4,\alpha}(\mathbb{S}^n)$. The idea is to consider a one parameter family of prescription functions ψ_t where

$$\psi_t(\xi, u, \tau) := t\psi(\xi, u, \tau) + (1 - t)\Psi(\xi, u, \tau),$$

where $\Psi(\xi, u, \tau)$ is to be chosen shortly. We define $\Phi : C_{ad}^{4,\alpha}(S^n) \times [0,1] \to C^{2,\alpha}$, by

$$(6.1) \qquad\qquad \Phi(u, t) := H_k^{\frac{1}{k}}(u_t) - \psi_t(\xi, u_t, \tau(u_t))$$

for all $t \in [0,1]$. We will apply degree theory to ensure that there exists at least one solution to $\Phi(u_t, t) = 0$ for all $t \in [0,1]$. As in [16], to be able to apply the beautiful degree theory of Y. Li [15], we need to verify the following three steps:

Step 1(a): Show that there exists a unique solution to

$$H_k^{\frac{1}{k}}(u_0) = \Psi(x, u_0, \tau(u_0)).$$

Step 1(b): Show that at u_0 the linearisation of Φ is invertible.

Step 2: Define a suitable set of admissible functions and show that all solutions of (6.1) stay in this set.

Step 3: Verify that we may apply degree theory to ensure that the degree of $\Phi(\cdot, 1)$ is not zero, and so a solution exists, as claimed.

We choose Ψ to be

$$\Psi(\xi, u, \tau) = \tau^p u \tanh(u)$$

for some $p > 1$.

Proof of Step 1(a)   We can easily verify that a solution exists to $\Phi(u_0, 0) = 0$, by considering constant functions. The hypersurfaces corresponding to $u_0 = \lambda$ are totally umbillic with principal curvatures $\tanh(\lambda)$, and so $H_k^{\frac{1}{k}} = \tanh(\lambda)$. We can see that on such a hypersurface, $\tau = \cosh(\lambda)$, and so if λ satisfies $\lambda \cosh^p(\lambda) = 1$, then u_0 is a solution. Clearly, such a λ exists as, writing the continuous function $\varphi(x) := x \cosh^p(x)$, $\varphi(0) = 0$ and $\varphi(1) > 1$.

Suppose that there exists another $u \in C_a^{4,\alpha}(S^n)$ satisfying $\Phi(u, 0) = 0$. Suppose, furthermore, that $\max u = u(\xi_0) = \lambda_0 > \lambda$. As in the proof of the C^0 estimates, we have that at ξ_0,

$$\psi(\xi_0, u(\xi_0), \tau(\xi_0)) = \cosh^p(\lambda_0)\lambda_0 \tanh(\lambda_0) = H_k^{\frac{1}{k}}|_{\xi_0} \le \tanh(\xi_0),$$

which is a contradiction, as $x \cosh^p(x)$ is a monotonically increasing function. Therefore, $\max u \le \lambda$. An identical argument implies $\min u \ge \lambda$, implying that $u(\xi) = u_0$. ∎

Proof of Step 1(b)   Considering τ, A_i^k as algebraic functions of $\xi, u, \widetilde{\nabla} u, \widetilde{\nabla}^2 u$, which we will write with the variables x, r, p^i, z_{ij}, respectively, then $A_i^k = A_i^k(x, r, p, z)$ and $\tau = \tau(x, r, p)$. Then the linearisation of the above in direction v is given by

$$(6.2) \qquad 0 = \frac{d}{ds}\left([H_k^{\frac{1}{k}} - \psi_t](u + sv)\right)$$

$$= F_k^i \frac{\partial A_i^k}{\partial z_{ij}} \widetilde{\nabla}_{ij} v + \left[F_k^i \frac{\partial A_i^k}{\partial p_k} - \psi_\tau \frac{\partial \tau}{\partial p_k}\right] \widetilde{\nabla}_k v + \left[F_k^i \frac{\partial A_i^k}{\partial r} - \psi_r - \psi_\tau \frac{\partial \tau}{\partial r}\right] v,$$

From equations (2.8) and (2.9), we have that

$$A_i^k = \tau\left(\sigma^{ij} + \tau^2 \frac{u_l \sigma^{li} u_m \sigma^{mj}}{\cosh^4(u)}\right) \frac{\widetilde{\nabla}_{ij} u - 2\tanh(u) u_i u_j + \sinh(u)\cosh(u)\sigma_{ij}}{\cosh^3(u)},$$

and so we see that

$$\frac{\partial A_i^k}{\partial r}\Big|_{(x, u, \widetilde{\nabla} u, \widetilde{\nabla}^2 u)} = \frac{\frac{\partial \tau}{\partial r}}{\tau} A_i^k - 3\tanh(u) A_i^k + \tau \frac{\cosh^2(u) + \sinh^2(u)}{\cosh^3(u)} \delta_i^k$$

$$+ u_l u_m[\text{bounded terms}],$$

and

$$\frac{\partial \tau}{\partial r}\Big|_{(x,u,\widetilde{\nabla}u,\widetilde{\nabla}^2 u)} = \tanh(u)\tau + |\widetilde{\nabla}u|^2 [\text{ bounded terms}].$$

We are interested in (6.2) when $t = 0$, that is, when $u = \lambda > 0$ and $\psi = \Psi = \tau^p u \tanh(u)$. In this case, we have that

$$\tau = \cosh(u), \qquad A_i^k = \tanh(u)\delta_i^k, \qquad H_k^{\frac{1}{k}} = \tanh(u), \qquad F_k^i = \frac{1}{n}\delta_k^i.$$

At such a u, the linearisation becomes

$$0 = a^{ij}\widetilde{\nabla}_{ij}v + b^i \widetilde{\nabla}_i v + cv,$$

where a is elliptic, b is bounded, and

$$c = \frac{\frac{\partial \tau}{\partial r}}{\tau}H_k^{\frac{1}{k}} - 3\tanh(u)H_k^{\frac{1}{k}} + \tau\frac{\cosh^2(u) + \sinh^2(u)}{\cosh^3(u)}\mathrm{tr}F^{ij} - \psi_r - \psi_\tau\frac{\partial \tau}{\partial z}$$

$$= \cosh^{-2}(u) - u\cosh^{p-2}(u)$$
$$\qquad - \cosh^p(u)\tanh(u) - p\cosh(u)^{p-1}u\tanh(u)\sinh(u).$$

We recall that $u = \lambda$ was chosen so that $\lambda\cosh^p\lambda = 1$, which implies that $u\cosh^{p-2}u = \cosh^{-2}(u)$, and so we see that $c < 0$. The strong maximum principle now implies that the only solution $v \in C^{4,\alpha}(S^n)$ of $\Phi_u(\cdot,0)(v) = 0$ is $v = 0$. This implies that $\ker(\Phi_u) = \{0\}$, and so, the standard theory of second order elliptic equations implies that Φ_u is invertible, as required. ∎

Proof of Step 2   By assumption, we have that $\psi(\xi,z,\cosh(z)) < \tanh(z)$ for $z < R_1$ and $\psi(\xi,z,\cosh(z)) > \tanh(z)$ for $z > R_2$. Similarly, we see directly that there exists $R_1^\Psi, R_2^\Psi > 0$ such that $\Psi(\xi,z,\cosh(z)) < \tanh(z)$ for $z < R_1^\Psi$ and $\psi(\xi,z,\cosh(z)) > \tanh(z)$ for $z > R_2^\Psi$. Setting $\overline{R}_1 = \min\{R_1, R_1^\Psi\}$ and $\overline{R}_2 = \max\{R_2, R_2^\Psi\}$, then for all $t \in [0,1]$, $\psi_t(\xi,z,\cosh(z)) < \tanh(z)$ for $z < \overline{R}_1$ and $\psi_t(\xi,z,\cosh(z)) > \tanh(z)$ for $z > \overline{R}_2$. Lemma 3.1, therefore yields uniform C^0 estimates

$$0 < \overline{R}_2 \leq u_t \leq \overline{R}_2 < \infty.$$

Proposition 4.2, in addition to giving a C^1 estimate, implies uniform spacelikeness, and so we can apply Theorem 5.2 to yield $|\lambda_i| < C_A$, which implies uniform C^2 estimates in this situation. Uniform parabolicity of the equation now follows, and so due to the classical regularity theory for uniformly elliptic equations and the Evans–Krylov theorem [6, 14], we obtain

$$\|u_t\|_{C^{4,\alpha}(\mathbb{S}^n)} < C,$$

for any admissible solution $u_t \in C_{ad}^{4,\alpha}(\mathbb{S}^n)$, where the constant C depends on k, n, R_1, R_2 and $\|\psi\|_{C^{2,\alpha}(\mathbb{S}^n)}$.

Due to compactness and the above estimates, there exists a constant $\delta > 0$ such that $\delta < \psi_t(\xi, u(\xi), \tau(u))$ for all $\xi \in S^n$. We define the bounded open set

$$V := \left\{\lambda \in \Gamma_k : H_k^{\frac{1}{k}}(\lambda) \geq \delta, |\lambda| < \sqrt{n}C_A\right\} \subset \Gamma_k,$$

and we define the bounded set

$$\mathcal{B} = \Big\{ u \in C^{4,\alpha}(\mathbb{S}^n) \mid$$

$$\frac{1}{2}\overline{R}_1 < u < 2\overline{R}_2, \|u\|_{C^{4,\alpha}(\mathbb{S}^n)} < C \text{ and } \lambda(A[u(\xi)]) \in V \ \forall \ \xi \in \mathbb{S}^n \Big\}.$$

Clearly the arguments of the previous paragraphs imply that any admissable solution $u_t \in C^{4,\alpha}_{ad}(S^n)$ is contained in $\mathcal{B}$, and $\partial \mathcal{B} \cap \Phi^{-1}(\cdot, t) = \varnothing$ for all $t \in [0, 1]$. ∎

Proof of Step 3   This step now follows exactly as in [16]. Due to Step 2 and [15, Definition 2.2, Proposition 2.2], the degree is defined and constant for $t \in [0, 1]$. By [15, Proposition 2.3, Proposition 2.2],

$$\deg\big(\Phi(\cdot, 0), \mathcal{B}, 0\big) = \deg\big(\Phi_u(\cdot, 0), B_1, 0\big),$$

where B_1 is the open unit ball in $C^{4,\alpha}(S^n)$. However, Step 1 and [15, Proposition 2.4] imply that $\deg(\Phi(\cdot, 0), \mathcal{B}, 0) = \pm 1 = \deg(\Phi(\cdot, 1), \mathcal{B}, 1)$, and we conclude that a k-admissable solution u_1 to $\Phi(u_1, 1) = 0$ exists. Standard elliptic estimates imply that u_1 is smooth, and therefore, the proof of Theorem 1.1 is complete. ∎

References

[1] D. Ballesteros-Chávez, *Curvature estimates of spacelike surfaces in deSitter space*. Preprint, 2019. https://arxiv.org/pdf/1905.09587.pdf

[2] J. L. M. Barbosa, J. H. S. Lira, and V. I. Oliker, *A priori estimates for starshaped compact hypersurfaces with prescribed mth curvature function in space forms*. In: Nonlinear problems in mathematical physics and related topics, I, Int. Math. Ser. (N. Y.), 1, Kluwer/Plenum, New York, 2002, pp. 35–52.

[3] R. Bartnik and L. Simon, *Spacelike hypersurfaces with prescribed boundary values and mean curvature*. Comm. Math. Phys. **87**(1982), 131–152.

[4] L. Caffarelli, L. Nirenberg, and J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations. III. Functions of the eigenvalues of the Hessian*. Acta Math. **155**(1985), 261–301. https://doi.org/10.1007/BF02392544

[5] L. Caffarelli, L. Nirenberg, and J. Spruck, *Nonlinear second order elliptic equations. IV. Starshaped compact Weingarten hypersurfaces*. In: Current topics in partial differential equations, Kinokuniya, Tokyo, 1986, pp. 1–26.

[6] L. C. Evans, *Classical solutions of fully nonlinear, convex, second order elliptic equations*. Comm. Pure Appl. Math. **35**(1982), 333–363.   https://doi.org/10.1002/cpa.3160350303

[7] C. Gerhardt, *Hypersurfaces of prescribed curvature in Lorentzian manifolds*. Indiana Univ. Math. J. **49**(2000), 1125–1153.   https://doi.org/10.1512/iumj.2000.49.1861

[8] C. Gerhardt, *H-surfaces in Lorentzian manifolds*. Comm. Math. Phys. **89**(1983), 523–553.

[9] C. Gerhardt, *Hypersurfaces of prescribed Weingarten curvature*. Math. Z. **224**(1997), 167–194.   https://doi.org/10.1007/PL00004580

[10] P. Guan, J. Li, and Y. Li, *Hypersurfaces of prescribed curvature measure*. Duke Math. J. **161**(2012), 1927–1942.   https://doi.org/10.1215/00127094-1645550

[11] P. Guan, C. Ren, and Z. Wang, *Global C^2 estimates for convex solutions of curvature equations*. Comm. Pure Appl. Math. **68**(2015), 1287–1325.   https://doi.org/10.1002/cpa.21528

[12] Y. Huang, *Curvature estimates of hypersurfaces in the Minkowski space*. Chin. Ann. Math. Ser. B **34**(2013), 753–764.   https://doi.org/10.1007/s11401-013-0789-5

[13] Q. Jin and Y. Li, *Starshaped compact hypersurfaces with prescribed k-th mean curvature in hyperbolic space*. Discrete Contin. Dyn. Syst. **15**(2006), 367–377.   https://doi.org/10.3934/dcds.2006.15.367

[14] N. V. Krylov, *Boundedly nonhomogeneous elliptic and parabolic equations in a domain*. Izv. Nauk. SSSR. Ser. Mat. **47**(1983), 75–108.

[15] Y. Li, *Degree theory for second order nonlinear elliptic operators and its applications*. Comm. Partial Differential Equations **14**(1989), 1541–1578.   https://doi.org/10.1080/03605308908820666

[16] Y. Li and V. I. Oliker, *Starshaped compact hypersurfaces with prescribed m-th mean curvature in elliptic space.* J. Partial Differential Equations **15**(2002), 68–80.

[17] W. Sheng, J. Urbas, and X.-J. Wang, *Interior curvature bounds for a class of curvature equations.* Duke Math. J. **123**(2004), 235–264.   https://doi.org/10.1215/S0012-7094-04-12321-8

[18] J. Urbas, *Interior curvature bounds for spacelike hypersurfaces of prescribed k-th mean curvature.* Comm. Anal. Geom.**11**(2003), 235–261.

[19] J. N. Whiteley, *On Newton's inequality for real polynomials.* Amer. Math. Monthly **76**(1969), 905–909.   https://doi.org/10.2307/2317943

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