The continuous-time lace expansion

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Abstract

We derive a continuous-time lace expansion for a broad class of self-interacting continuous-time random walks. Our expansion applies when the self-interaction is a sufficiently nice function of the local time of a continuous-time random walk. As a special case we obtain a continuous-time lace expansion for a class of spin systems that admit continuous-time random walk representations.

We apply our lace expansion to the n-component $g|\varphi|^4$ model on $\mathbb{Z}^d$ when $n = 1, 2$, and prove that the critical Green's function $G_{\nu_c}(x)$ is asymptotically a multiple of $|x|^{2-d}$ when $d \geq 5$ at weak coupling. As another application of our method we establish the analogous result for the lattice Edwards model at weak coupling.

1 Introduction

Many lattice spin systems are expected to exhibit mean-field behaviour on $\mathbb{Z}^d$ when $d > d_c = 4$. Results of this type have been proven by making use of *random walk representations* [1, 2, 18, 17] and recently [3] these methods have been extended to $d = 4$ by taking into account logarithmic corrections to mean field theory. In this paper we extend this program by analyzing the critical Green's function of n-component lattice spin models for $n = 1, 2$ at weak coupling for $d > 4$. We use the random walk representation originating in the work of Symanzik [47] and developed in [10, 14]. For recent developments regarding this representation see [5, 48], and for alternative random walk representations see [2, 26].

To be more precise, we determine the asymptotics of the infinite volume critical two-point function $\langle \varphi_a \cdot \varphi_b \rangle$. Here $\langle \cdot \rangle$ denotes the expectation of an $O(n)$-invariant $g|\varphi|^4$ spin model; the spins φ take values in $\mathbb{R}^n$. The definitions of these models and what it means to be critical are given in Section 3. Let $|x|$ and $|\varphi|$ denote the Euclidean norms of $x \in \mathbb{Z}^d$ and $\varphi \in \mathbb{R}^n$.

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Theorem 1.1. *Let $d > d_c = 4$ and $n \in \{1,2\}$. Let $\langle \cdot \rangle$ denote expectation with respect to the critical n-component $g|\varphi|^4$ model. For $g > 0$ sufficiently small there is a constant $C > 0$ such that*

$$\langle \varphi_a \cdot \varphi_b \rangle \sim \frac{C}{|b-a|^{d-2}}, \qquad \text{as } |b-a| \to \infty. \tag{1.1}$$

The relation $\sim$ in (1.1) means the ratio of the left-hand and right-hand sides tends to one in the designated limit. Our theorem exhibits mean-field behaviour in the sense that the exponent $d-2$ in (1.1) is the exponent predicted by Landau's extension of mean-field theory [32, Chapter 2]. The right hand side of (1.1) is Euclidean invariant, so for weak coupling the conclusion strengthens the triviality results [1, 18] by showing that the scaling limit of the two-point function of this model is Euclidean invariant and equals the massless free field two-point function. When $n = 1$ Theorem 1.1 was already proven by Sakai [40]. The case $n = 2$ is new. For $d = d_c = 4$ the asymptotics in (1.1) have been established by a rigorous renormalization group technique for the n-component $g|\varphi|^4$ model for all $n \in \mathbb{N}$ [45]. For $n = 1$ and $d = 4$ the analysis of the Green's function by rigorous renormalization group techniques began with [20, (8.32)] and [16, Theorem I.2].

Sakai's proof of the $n = 1$ case of Theorem 1.1 made use of the *lace expansion*, a technique originally introduced to prove mean-field behaviour for discrete-time weakly self-avoiding walk [11]. The lace expansion has since been reformulated in many different settings: unoriented and oriented percolation [23, 37, 55], the contact process [54], lattice trees and animals [24], Ising and $g|\varphi|^4$ models [39, 40], the random connection model [28], and various self-interacting random walk models [52, 21, 27, 50]. Within these settings the lace expansion has been applied to a variety of problems, ranging from proofs of weak convergence on path space for branching particle systems [12, 53, 30] to proofs of monotonicity properties of self-interacting random walks [51, 29, 31]. In each case the expansion is based on a discrete parameter that plays the role of time.

To prove Theorem 1.1 we introduce a lace expansion in continuous time. Our methods naturally apply to a broader class of problems than $g|\varphi|^4$ models, and to illustrate this we also analyze the lattice Edwards model. A precise formulation of our main results is given in Section 3, after the introduction of the basic objects of our paper.

2 Random walk and local times

To fix notation and assumptions, we define continuous-time random walk started at a point a in $\mathbb{Z}^d$ and killed outside of a finite subset Λ of $\mathbb{Z}^d$. These stochastic processes are central to the rest of the paper.

2.1 Infinite volume

We begin by defining the class of jump distributions that we will allow. Recall that a one-to-one map T from the vertex set of $\mathbb{Z}^d$ onto itself such that edges $\{x,y\}$

of $\mathbb{Z}^d$ are mapped to edges $\{Tx, Ty\}$ of $\mathbb{Z}^d$ is called an *automorphism*. Let $\mathrm{Aut}_0(\mathbb{Z}^d)$ denote the subgroup of automorphisms that fix the origin 0. For example, reflections in lattice planes containing the origin are in $\mathrm{Aut}_0(\mathbb{Z}^d)$. An automorphism in $\mathrm{Aut}_0(\mathbb{Z}^d)$ permutes the nearest neighbors in $\mathbb{Z}^d$ of the origin and this permutation determines the automorphism. A function f on $\mathbb{Z}^d$ is $\mathbb{Z}^d$*-symmetric* if $f(Tx) = f(x)$ for $x \in \mathbb{Z}^d$ and $T \in \mathrm{Aut}_0(\mathbb{Z}^d)$. A function $f(x,y)$ of two variables is $\mathbb{Z}^d$*-symmetric* if $f(Tx,Ty) = f(x,y)$ for all automorphisms T of $\mathbb{Z}^d$. The condition of being $\mathbb{Z}^d$-symmetric includes translation invariance ($f(x,y) = f(0,y-x)$) and hence is equivalent to the function $g(x) := f(0,x)$ of one variable being $\mathbb{Z}^d$-symmetric.

Assumptions 2.1. Assume $J\colon \mathbb{Z}^d \to \mathbb{R}$ satisfies

(J1) $J(x) \geq 0$ for $x \neq 0$, and $J(0) := -\sum_{x\neq 0} J(x)$ is finite,
(J2) the set $\{x \in \mathbb{Z}^d \mid J(x) > 0\}$ is a generating set for $\mathbb{Z}^d$,
(J3) J is $\mathbb{Z}^d$-symmetric,
(J4) J has finite range $R > 0$, i.e., $J(x) = 0$ if $|x| \geq R$.

A condition that J decays like $|x|^{-3(d-2)}$ might serve instead of (J4); to avoid having to extend standard cited results such as [41, 35, 38] we work with a fixed choice of J satisfying (J1)–(J4). Let $\Delta^{(\infty)}\colon \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ be the infinite matrix with entries

$$\Delta^{(\infty)}_{x,y} := J(y-x). \tag{2.1}$$

By (J3), $\Delta^{(\infty)}$ is symmetric, and (J1) implies that $\Delta^{(\infty)}$ has non-negative off-diagonal elements and that its row sums are all equal to zero. This implies $\Delta^{(\infty)}$ is the generator of a continuous-time random walk $X^{(\infty)}$ on $\mathbb{Z}^d$. The assumption (J2) ensures this walk is irreducible. Let

$$\hat{J} := -\Delta^{(\infty)}_{x,x} = -J(0), \qquad J_+(y) := J(y)\mathbb{1}_{\{y\neq 0\}}. \tag{2.2}$$

By (J1), $\hat{J}$ is finite. The walk $X^{(\infty)}$ has a mean $\hat{J}^{-1}$ exponential holding time at each x, and jumps from x to $y \neq x$ with probability $J_+(y-x)/\hat{J}$. We write P_a for a probability measure under which $X^{(\infty)}$ is a continuous-time random walk on $\mathbb{Z}^d$ started at $a \in \mathbb{Z}^d$, and E_a for the corresponding expectation.

Example 2.2. The most important example is when

$$J(x) = \mathbb{1}_{\{|x|=1\}} - 2d\mathbb{1}_{\{x=0\}}.$$

In this case $\Delta^{(\infty)}$ is called the *lattice Laplacian*, and $X^{(\infty)}$ is a continuous-time nearest-neighbour random walk on $\mathbb{Z}^d$.

2.2 Finite volume

Let Λ be a finite subset of $\mathbb{Z}^d$ and let $T^{(\Lambda)}$ be the first time $X^{(\infty)}$ exits Λ:

$$T^{(\Lambda)} := \inf\{t \geq 0 : X^{(\infty)}_t \notin \Lambda\}. \tag{2.3}$$

Let $* \notin \mathbb{Z}^d$ be an additional "cemetery" state, and define $X_t^{(\Lambda)}$ by

$$X_t^{(\Lambda)} = \begin{cases} X_t^{(\infty)}, & t < T^{(\Lambda)}, \\ *, & t \geq T^{(\Lambda)}. \end{cases} \tag{2.4}$$

For each finite Λ the process $X^{(\Lambda)} = (X_t^{(\Lambda)})_{t\geq 0}$ is a continuous-time Markov chain on $\Lambda \cup \{*\}$ with absorbing state $*$. Note that this construction defines the processes $X^{(\Lambda)}$ on the same probability space for all finite $\Lambda \subset \mathbb{Z}^d$.

The generator $\Delta_*^{(\Lambda)}$ of $X^{(\Lambda)}$ is a $(|\Lambda|+1) \times (|\Lambda|+1)$ matrix with a row of zeros since $*$ is an absorbing state. We will let $\Delta^{(\Lambda)} \colon \Lambda \times \Lambda \to \mathbb{R}$ denote the matrix obtained by removing the row and column corresponding to transition rates to and from $*$, i.e., $(\Delta^{(\Lambda)})_{x,y} := (\Delta^{(\infty)})_{x,y}$ for $x, y \in \Lambda$.

2.3 Local time and free Green's functions

For $x \in \Lambda$ and a Borel set $I \subset [0,\infty)$ the *local time of $X^{(\Lambda)}$ at x during I* is

$$\tau_{I,x}^{(\Lambda)} := \int_I d\ell\, \mathbb{1}_{\{X_\ell^{(\Lambda)} = x\}}, \tag{2.5}$$

where $d\ell$ is Lebesgue measure. Let $\boldsymbol{\tau}_I^{(\Lambda)} := (\tau_{I,x}^{(\Lambda)})_{x\in\Lambda}$ denote the vector of all local times, and let $\tau_x^{(\Lambda)}$ denote $\tau_{[0,\infty),x}^{(\Lambda)}$. We will often omit the superscript Λ when there is no risk of confusion. For $a, b \in \Lambda$ we define the *free Green's function in* Λ by

$$S^{(\Lambda)}(a,b) := E_a[\tau_b^{(\Lambda)}] = \int_0^\infty d\ell\, P_a(X_\ell^{(\Lambda)} = b). \tag{2.6}$$

Note that $S^{(\Lambda)}(a,b) < \infty$ since the expected time for $X^{(\Lambda)}$ to exit Λ is finite. The next lemma, proved in Appendix A.1, explains why $S^{(\Lambda)}(a,b)$ is called the free Green's function.

Lemma 2.3. $\Delta^{(\Lambda)}$ *is invertible, and for* $a, b \in \Lambda$

$$S^{(\Lambda)}(a,b) = (-\Delta^{(\Lambda)})^{-1}_{a,b}. \tag{2.7}$$

There is also an infinite volume version of Lemma 2.3, and it is nicer because the infinite volume limit restores translation invariance. For $d \geq 3$, define $S(a,b)$ to be the expected time spent at b by the random walk $X^{(\infty)}$ started from a, i.e.,

$$S(a,b) := E_a[\tau_b^{(\infty)}], \tag{2.8}$$

where $\tau_b^{(\infty)}$ is defined as in (2.5) but with $X_\ell^{(\Lambda)}$ replaced by $X_\ell^{(\infty)}$. Since $T^{(\Lambda)} \uparrow \infty$ as $\Lambda \uparrow \mathbb{Z}^d$ a.s., monotone convergence implies that

$$S^{(\Lambda)}(a,b) = E_a[\tau^{(\infty)}_{[0,T^{(\Lambda)}],b}] \uparrow E_a[\tau^{(\infty)}_{[0,\infty),b}] = S(a,b) < \infty. \tag{2.9}$$

where the finiteness holds by transience. The translation invariance of the infinite volume random walk implies

$$S(x) := S(a, a+x), \qquad x, a \in \mathbb{Z}^d \tag{2.10}$$

is well-defined, i.e., independent of $a \in \mathbb{Z}^d$. Recall, see [33, Theorem 4.3.5], that there is a C_J depending on J such that

$$S(x) \sim \frac{C_J}{|x|^{d-2}}. \tag{2.11}$$

Since $S(x)$ is positive for all x this implies that there is a constant c_J depending on J such that $S(x) \geq \frac{c_J}{|x|^{d-2}}$ for $x \neq 0$.

Recall that the discrete convolution $f * h$ of functions f and h on $\mathbb{Z}^d$ is defined by $f * h(x) := \sum_{y \in \mathbb{Z}^d} f(y)h(x-y)$. For $n \in \mathbb{N}$ let f^{*n} denote the n-fold convolution of f with itself, and let $f^{*0}(x) = \mathbb{1}_{\{x=0\}}$. The next lemma is proved in Appendix A.1.

Lemma 2.4. *Suppose $d \geq 3$. For S defined by* (2.10) *and $x \in \mathbb{Z}^d$,*

$$J * S(x) = S * J(x) = -\mathbb{1}_{\{x=0\}}. \tag{2.12}$$

Moreover, recalling $\hat{J}$ and J_+ from (2.2)*,*

$$S(x) = \sum_{n \geq 0} \hat{J}^{-(n+1)} J_+^{*n}(x), \tag{2.13}$$

where the right-hand side is a convergent sum of positive terms.

2.4 A convenient technical choice

In this section we make a specific choice for the measurable space $(\Omega_1, \mathscr{F})$ on which $X^{(\infty)}$ is defined so that the paths of $X^{(\infty)}$ have desirable regularity properties. This reduces the number of statements that have to be qualified as holding almost surely (a.s.). Let

$$\Omega_1 = \left\{ X^{(\infty)} \colon [0, \infty) \to \mathbb{Z}^d \,\middle|\, X^{(\infty)} \text{ is càdlàg} \right\}, \tag{2.14}$$

where we recall a function is càdlàg if it is right continuous with left limits. Let $(\mathscr{F}_t)_{t \geq 0}$ denote the natural filtration of $X^{(\infty)}$, i.e., $\mathscr{F}_t$ is the smallest σ-algebra on Ω_1 such that $\{X^{(\infty)} \,|\, X_s^{(\infty)} = y\} \in \mathscr{F}_t$ for each $s \in [0,t]$ and $y \in \mathbb{Z}^d$, and let $\mathscr{F}$ denote the smallest σ-algebra on Ω_1 containing $\cup_{t \geq 0} \mathscr{F}_t$. Henceforth we let P_a denote the probability measure on $(\Omega_1, \mathscr{F})$ under which $X^{(\infty)}$ is a continuous-time random walk on $\mathbb{Z}^d$ started at $a \in \mathbb{Z}^d$.

3 The Green's function

In this section we define the object $G_{\boldsymbol{t}}^{(\Lambda)}(a,b)$ at the center of our results and we refer to it as the *Green's function*. It involves a self-interacting walk starting at a and ending at b. We have included a parameter t that specifies the additional interaction that arises when conditioning on an initial segment of the walk. It would be more standard to reserve the name "Green's function" for the case $t = 0$.

We have two motivations for studying this Green's function. The first is that two-point correlations of lattice spin models such as the n-component $g|\varphi|^4$ model have representations in terms of $G_{\boldsymbol{t}}^{(\Lambda)}(a,b)$; see Definition 3.2 and Theorem 3.3. The second motivation is that it is a point of departure for the study of random walks with self-interactions that are functions of local time. Such models are of interest in chemistry, physics, and probability; they include the lattice Edwards model which we define in Definition 3.1. This is a canonical model of self-avoiding walk.

Fix a finite set $\Lambda \subset \mathbb{Z}^d$. For a set A, A^Λ denotes the set of sequences $(x_v)_{v\in\Lambda}$ with each component x_v in A. Let $Z\colon [0,\infty)^\Lambda \to (0,\infty)$, $\boldsymbol{t} \mapsto Z_{\boldsymbol{t}}$ be a bounded continuous positive function. For a random variable $\boldsymbol{\sigma}$ taking values in $[0,\infty)^\Lambda$, $Z_{\boldsymbol{\sigma}}$ denotes Z evaluated at the random point $\boldsymbol{\sigma}$. For $\boldsymbol{t},\boldsymbol{s} \in [0,\infty)^\Lambda$ let

$$Y_{\boldsymbol{t},\boldsymbol{s}} := \frac{Z_{\boldsymbol{t}+\boldsymbol{s}}}{Z_{\boldsymbol{t}}}. \tag{3.1}$$

For $a,b \in \Lambda$ and $\boldsymbol{t} \in [0,\infty)^\Lambda$ define the *Green's function*

$$G^{(\Lambda)}_{\boldsymbol{t}}(a,b) := \int_{[0,\infty)} d\ell\; E_a\left[Y_{\boldsymbol{t},\boldsymbol{\tau}^{(\Lambda)}_{[0,\ell]}} \mathbb{1}_{\{X^{(\Lambda)}_\ell = b\}}\right]. \tag{3.2}$$

Note that $G^{(\Lambda)}_{\boldsymbol{t}}(a,b) > 0$ since $Z_{\boldsymbol{t}}$ is continuous and positive. We extend the definition (3.2) by setting $G^{(\Lambda)}_{\boldsymbol{t}}(a,b) = 0$ if a or b is the cemetery state $*$.

The free Green's function $S^{(\Lambda)}(a,b)$ is the special case of $G^{(\Lambda)}_{\boldsymbol{t}}$ when $Z := 1$, see (2.6). For each $\boldsymbol{t}$ the function $Y_{\boldsymbol{t},\boldsymbol{\tau}^{(\Lambda)}_{[0,\ell]}}$ is bounded as a function of $\omega \in \Omega_1$ and $\ell \in [0,\infty)$ because $Z_{\boldsymbol{t}}$ is bounded and positive. By (2.9) this implies

$$G^{(\Lambda)}_{\boldsymbol{t}}(a,b) < \infty, \qquad \boldsymbol{t} \in [0,\infty)^\Lambda. \tag{3.3}$$

Our primary interest in this paper is $G^{(\Lambda)}_{\boldsymbol{0}}(a,b)$ given by (3.2) when $\boldsymbol{t} \mapsto Z_{\boldsymbol{t}}$ is one of the choices described in the next two sections. Both choices involve parameters $g > 0$ and $\nu \in \mathbb{R}$ called *coupling constants*.

3.1 The Edwards model

Definition 3.1. Fix $g > 0$ and $\nu \in \mathbb{R}$. The *Green's function* $G^{(\Lambda)}_{\boldsymbol{t}}(a,b)$ *of the (lattice) Edwards model* is given by (3.2) and (3.1) with the choice

$$Z_{\boldsymbol{t}} := \exp\left\{-g\sum_{x\in\Lambda} t_x^2 - \nu \sum_{x\in\Lambda} t_x\right\}. \tag{3.4}$$

To explain Definition 3.1 note that

$$\sum_x \tau^2_{[0,\ell],x} = \iint_{[0,\ell]^2} ds\,dr\, \mathbb{1}_{\{X^{(\Lambda)}_s = X^{(\Lambda)}_r\}}$$

is the time $X^{(\Lambda)}$ spends intersecting itself up to time ℓ. Since $g > 0$, the choice of $Z_{\boldsymbol{t}}$ in Definition 3.1 weights a walk in (3.2) by the exponential of minus its self-intersection time: self-intersection is discouraged. The parameter $\nu \in \mathbb{R}$ is called the *chemical potential*, and it controls the expected length of a walk. Thus the Edwards model is a continuous time self-avoiding walk. See [6] for further details and background on this model.

3.2 The $g|\varphi|^4$ models

Our second choice of $Z_{\boldsymbol{t}}$ requires some preparation. Let $\mathbb{R}^{n\Lambda} := (\mathbb{R}^n)^\Lambda$, $\varphi := (\varphi_x)_{x\in\Lambda}$ be a point in $\mathbb{R}^{n\Lambda}$, and let $\varphi_x^{[i]}$ denote the ith component of $\varphi_x \in \mathbb{R}^n$. Define a centered Gaussian measure P on the Borel sets of $\mathbb{R}^{n\Lambda}$ in terms of a density with respect to Lebesgue measure $d\varphi$ on $\mathbb{R}^{n\Lambda}$ by

$$dP(\varphi) := Ce^{\frac{1}{2}(\varphi,\Delta^{(\Lambda)}\varphi)}\,d\varphi, \tag{3.5}$$

where C normalises the measure to have total mass one and the quadratic form $(\varphi,\Delta^{(\Lambda)}\varphi)$ is defined by:

$$(\Delta^{(\Lambda)}\varphi)_x^{[i]} := \sum_{y\in\Lambda}\Delta^{(\Lambda)}_{x,y}\varphi_y^{[i]}, \qquad (f,h) := \sum_{x\in\Lambda}\sum_{i=1}^{n} f_x^{[i]}h_x^{[i]}. \tag{3.6}$$

The covariance of φ under P is the $n|\Lambda|\times n|\Lambda|$ positive definite matrix $(-\Delta^{(\Lambda)})^{-1}_{x,y}\delta_{i,j}$; positive definiteness follows from (A.3) in Appendix A.1. By Lemma 2.3,

$$\int_{\mathbb{R}^{n\Lambda}} dP(\varphi)\,\varphi_x^{[i]}\varphi_y^{[j]} = S^{(\Lambda)}(x,y)\delta_{i,j}. \tag{3.7}$$

Definition 3.2. Fix $g>0$, $\nu\in\mathbb{R}$, and $n\in\mathbb{N}_{\geq 1}$. The *Green's function* $G^{(\Lambda)}_{\boldsymbol{t}}(a,b)$ *of the n-component* $g|\varphi|^4$ *model* is given by (3.2) and (3.1) with the choice

$$Z_{\boldsymbol{t}} := \int_{\mathbb{R}^{n\Lambda}} dP(\varphi)\,\exp\left\{-\sum_{x\in\Lambda}\left(g(\frac{1}{2}|\varphi_x|^2+t_x)^2+\nu(\frac{1}{2}|\varphi_x|^2+t_x)\right)\right\}. \tag{3.8}$$

The justification for Definition 3.2 is given by the next theorem. To state the theorem, define an expectation operator $\langle\cdot\rangle^{(\Lambda)}_{g,\nu,\boldsymbol{t}}$ by

$$\langle F\rangle^{(\Lambda)}_{g,\nu,\boldsymbol{t}} := \frac{1}{Z_{\boldsymbol{t}}}\int_{\mathbb{R}^{n\Lambda}} F(\varphi)e^{-\frac{1}{2}(\varphi,-\Delta^{(\Lambda)}\varphi)}\prod_{x\in\Lambda}\left(e^{-V_{t_x}(\frac{1}{2}|\varphi_x|^2)}d\varphi_x\right), \tag{3.9}$$

where for $\psi,s\in\mathbb{R}$, $V_s(\psi)=g(\psi+s)^2+\nu(\psi+s)$. We abbreviate $\langle F\rangle^{(\Lambda)}_{g,\nu,\boldsymbol{0}}$ to $\langle F\rangle^{(\Lambda)}_{g,\nu}$.

Theorem 3.3. *Let* $G^{(\Lambda)}_{\boldsymbol{t}}(a,b)$ *be given by Definition 3.2. Then*

$$G^{(\Lambda)}_{\boldsymbol{t}}(a,b) = \frac{1}{n}\langle\varphi_a\cdot\varphi_b\rangle^{(\Lambda)}_{g,\nu,\boldsymbol{t}}. \tag{3.10}$$

When $\boldsymbol{t}=\boldsymbol{0}$ the right hand side $\frac{1}{n}\langle\varphi_a\cdot\varphi_b\rangle^{(\Lambda)}_{g,\nu}$ in (3.10) is the standard definition of the n-component $g|\varphi|^4$ two-point function, see, e.g., [4, Section 1.6]. Note this reference writes $\left\langle\varphi_a^{[1]}\varphi_b^{[1]}\right\rangle^{(\Lambda)}_{g,\nu}$ in place of $\frac{1}{n}\langle\varphi_a\cdot\varphi_b\rangle^{(\Lambda)}_{g,\nu}$ which is the same by $O(n)$ invariance.

Proof of Theorem 3.3. We will use the BFS-Dynkin isomorphism as formulated in [4, Theorem 11.2.3]. To translate between the notation of the present article and [4] note that in the latter $\tau_x=\frac{1}{2}|\varphi_x|^2$ and L_T is the vector of local times of the walk up to time T. For $x\neq y\in\Lambda$ let $\beta_{x,y}=J(y-x)$ and for $x\in\Lambda$ let $\gamma_x=\sum_{y\notin\Lambda}J(y-x)$. By

[4, (11.1.9)] the Laplacian Δ_β in [4, Theorem 11.2.3] is a $\Lambda \times \Lambda$ matrix whose rows sum to zero with matrix elements $(\Delta_\beta)_{x,y} = J(y-x) + \mathbb{1}_{x=y}\gamma_x$. By comparison with the matrix $\Delta^{(\Lambda)}$ defined by the last line of Section 2.2 and (2.1) we obtain $(-\Delta_\beta)_{x,y} = (-\Delta^{(\Lambda)})_{x,y} - \mathbb{1}_{x=y}\gamma_x$. Therefore, by [4, Theorem 11.2.3] with

$$F(\boldsymbol{s}) = \exp\Big[\sum_{x\in\Lambda}\big(\gamma_x s_x - V(s_x)\big)\Big], \tag{3.11}$$

we obtain $\left\langle \varphi_a^{[1]}\varphi_b^{[1]}\right\rangle_{g,\nu,\mathbf{0}}^{(\Lambda)} = G_{\mathbf{0}}^{(\Lambda)}(a,b)$ which is the same as (3.10) with $\boldsymbol{t}=0$. The desired (3.10) with $\boldsymbol{t}$ not necessarily zero is obtained by replacing $F(\boldsymbol{s})$ by $F(\boldsymbol{s}+\boldsymbol{t})$. ■

3.3 Main result

Our main result Theorem 3.6 concerns the infinite volume limit of the Green's function for the examples in the previous sections.

Lemma 3.4 (Proof in Section 11)**.** *Let $a,b\in\Lambda$. $G_{\mathbf{0}}^{(\Lambda)}(a,b)$ is non-decreasing in Λ for the $n=1,2$-component $g|\varphi|^4$ and Edwards models.*

By inspecting Definitions 3.1 and 3.2 we observe that our examples satisfy $Z_{\boldsymbol{t}+\boldsymbol{s}} \le e^{-\nu s}Z_{\boldsymbol{t}}$ for all Λ. Therefore, by (3.1) and (3.2), $\sum_b G_{\mathbf{0}}^{(\Lambda)}(a,b) \le \int d\ell\, E_a\left[e^{-\nu\ell}1\right] \le \frac{1}{\nu}$ for $\nu>0$. Accordingly, Lemma 3.4 implies that $\lim_{\Lambda\uparrow\mathbb{Z}^d} G_{\mathbf{0}}^{(\Lambda)}(a,b)$ exists for our examples, and is finite if $\nu>0$. By a standard monotonicity argument (Lemma 9.1) the limit is $\mathbb{Z}^d$ invariant. We define

$$G_{g,\nu}(x) = G_{g,\nu}^{(\infty)}(x) := \lim_{\Lambda\uparrow\mathbb{Z}^d} G_{\mathbf{0}}^{(\Lambda)}(a,a+x). \tag{3.12}$$

A related monotonicity property of our models is

Lemma 3.5 (Proof in Section 11)**.** *For each $x\in\mathbb{Z}^d$, $G_{g,\nu}(x)$ is non-increasing in ν for the $n=1,2$-component $g|\varphi|^4$ and Edwards models.*

This lemma motivates defining the *critical value* of ν by

$$\nu_c := \inf\Big\{\nu\in\mathbb{R} \mid \sum_{x\in\mathbb{Z}^d} G_{g,\nu}(x) < \infty\Big\}. \tag{3.13}$$

Up to this point all we know is that $\nu_c\in[-\infty,0]$. We will prove that $\nu_c\neq-\infty$; this is known [19], [36, Corollary 3.2.6] for essentially the same models. Since our models depend on g, ν_c is a function of g and, when necessary, we write $\nu_c=\nu_c(g)$. When $\nu=\nu_c$ we say that the Green's function is *critical*.

Our main result is the following precise version of Theorem 1.1 which now also includes the Edwards model.

Theorem 3.6. *Suppose $d\ge5$, J satisfies (J1)–(J4), and consider the Edwards and the $g|\varphi|^4$ models given by Definition 3.1 and Definition 3.2 with $n=1,2$. For both* *models there exists* $g_0 = g_0(J) > 0$ *such that, for* $g \in (0, g_0)$, $\nu_c(g)$ *is finite,* $G_{g,\nu_c}(x)$ *is finite for all* $x \in \mathbb{Z}^d$*, and there exists* $C = C(g,J) > 0$ *such that*

$$G_{g,\nu_c}(x) \sim \frac{C}{|x|^{d-2}}, \qquad \text{as } |x| \to \infty. \tag{3.14}$$

Theorem 3.6 describes mean field asymptotics of the infinite volume Green's function at the critical point, c.f. (2.11). The restriction to $n = 1,2$ for the $g|\varphi|^4$ models is necessary because our proof uses the Griffiths II inequality, which is not known to hold for $n > 2$.

The proof of Theorem 3.6 occupies Sections 4 through 11. Section 4 serves as an overview of lace expansion methods and reduces a key step of our argument to some auxiliary lemmas. The remainder of the argument is comprised of three parts: the derivation of a lace expansion in finite volume (Sections 5 and 6), establishing an infinite volume expansion (Section 7 through 9), the analysis of this expansion (Section 10), and the application of this analysis to our examples (Section 11). The contents of individual sections will be discussed locally.

3.4 Related lace expansion results

In [39] Sakai proved a similar result for the Green's function of the Ising model. He applied the lace expansion for percolation to the random current representation of the Ising model. For his expansion to converge he required the dimension d of the lattice to be large or alternatively the range of the Ising coupling to be large. In a second paper [40] he extended his results to the scalar $g|\varphi|^4$ model; he approximated the scalar $g|\varphi|^4$ model by Ising models using the Griffiths-Simon trick [42] and thereby derived a lace expansion for $g|\varphi|^4$ that converges for weak coupling. His breakthrough inspired us to find the expansion used in this paper.

4 Infrared bound and overview

A key step in the proof of Theorem 3.6 is to obtain the upper bound on $G_{g,\nu_c}(x)$ provided by Theorem 4.1 below. This section begins the proof of Theorem 4.1 by reducing it to lemmas which will be proved in later sections. Our reduction reviews the guiding ideas of proofs by lace expansion, which are explained in more detail and attribution in [44]. See also [25, 7].

Recall the definitions of the Edwards model and the $g|\varphi|^4$ model from Sections 3.1 and 3.2. The infinite volume Green's functions $G_{g,\nu}(x)$ for these models are given by (3.12). We are mainly interested in the case where $\nu = \nu_c$, the critical value given by (3.13). *The hypotheses for results in this section include Assumptions 2.1.*

Theorem 4.1. *Suppose* $d \geq 5$*. For the* $n = 1,2$*-component* $g|\varphi|^4$ *and Edwards models there are* $g_0 = g_0(d,J) > 0$ *such that if* $0 < g < g_0$ *then* $\nu_c(g)$ *is finite and*

$$G_{g,\nu_c}(x) \leq 2S(x), \qquad x \in \mathbb{Z}^d. \tag{4.1}$$

Equation (4.1) is called an *infrared bound*. Infrared bounds in Fourier space for nearest neighbour models (i.e., J as in Example 2.2) were first proved for $n \geq 1$ and $d > 2$ in [19] with the 2 replaced by a 1. The relation between Fourier infrared bounds and (4.1) is not trivial, see [46, Appendix A] and [36, Example 1.6.2].

While Theorem 4.1 only stated ν_c is finite, a more precise estimate holds:

Proposition 4.2 (Proof in Section 11)**.** *For the $n = 1,2$-component $g|\varphi|^4$ and Edwards models, $\nu_c = -O(g)$ as $g \downarrow 0$.*

In Proposition 4.2 and in what follows, for functions f, r, the notation $f(x) = O(r(x))$ as $x \to a$ has its standard meaning, i.e., that there exists a $C > 0$ such that $|f(x)| \leq Cr(x)$ if x is sufficiently close to a.

4.1 The infrared bound

Recall that S is the free Green's function from (2.8). The heart of the proof of Theorem 4.1 is establishing the next proposition.

Proposition 4.3. *Suppose $d \geq 5$. For the $n = 1,2$-component $g|\varphi|^4$ and Edwards models, there are $g_0 = g_0(d,J) > 0$ such that if $0 < g < g_0$ then*

$$G_{g,\nu} \leq 2S, \qquad \text{for } \nu > \nu_c. \tag{4.2}$$

The possibility $\nu_c(g) = -\infty$ is included.

Before giving the proof we review the strategy and state some preparatory results. Lace expansion arguments have been reduced to three schematic steps, all for $\nu > \nu_c$. As we discuss these steps it will be helpful to recall (2.12), i.e., $J * S = -\mathbb{1}_{\{x=0\}}$. This is equivalent to

$$\hat{J}S(x) = \mathbb{1}_{\{x=0\}} + J_+ * S(x), \tag{4.3}$$

where $\hat{J}$ and J_+ were defined in (2.2). Define $L_{g,\nu} \in \mathbb{R}$ by

$$L^{(\Lambda)}_{g,\nu,x} := \lim_{\boldsymbol{t}\downarrow\boldsymbol{0}} \partial_{t_x} \log Z^{(\Lambda)}_{\boldsymbol{t}}, \qquad L_{g,\nu} = L^{(\infty)}_{g,\nu,x} := \lim_{\Lambda\uparrow\mathbb{Z}} L^{(\Lambda)}_{g,\nu,x}, \tag{4.4}$$

where x is any point in $\mathbb{Z}^d$ and $Z^{(\Lambda)}_{\boldsymbol{t}}$ depends on the model: for the Edwards model $Z^{(\Lambda)}_{\boldsymbol{t}} = Z_{\boldsymbol{t}}$ in (3.4); for the $g|\varphi|^4$ model $Z^{(\Lambda)}_{\boldsymbol{t}} = Z_{\boldsymbol{t}}$ in (3.8). We say that $L_{g,\nu}$ is well-defined if the limits exist and $L_{g,\nu}$ does not depend on x. For the Edwards model (3.4) implies that $L^{(\Lambda)}_{g,\nu,x} = -\nu\mathbb{1}_{\{x\in\Lambda\}}$ and therefore $L_{g,\nu} = -\nu$.

Step one. We will call a bound of the form

$$G_{g,\nu} \leq KS \tag{4.5}$$

a *K-infrared bound* or *K-IRB*. Step one assumes a 3-IRB and uses the assumption that g is sufficiently small to prove that there exists an $O(g)$ integrable function $\Psi_{g,\nu} : \mathbb{Z}^d \to \mathbb{R}$ such that for all x

$$(\hat{J} - L_{g,\nu})G_{g,\nu}(x) = \mathbb{1}_{\{x=0\}} + J_+ * G_{g,\nu}(x) + \Psi_{g,\nu} * G_{g,\nu}(x). \tag{4.6}$$

This is a generalization of (4.3). The proof of this step is accomplished by a formula for $\Psi_{g,\nu}$ called the *lace expansion*.

Step two. Step two assumes $\Psi_{g,\nu}$ is small relative to J and shows that (4.6) implies that $G_{g,\nu}(x)$ satisfies a 2-IRB. Thus steps one and two combined show that a 3-IRB implies a 2-IRB.

Step three. Step three removes the 3-IRB assumption of step one so that (4.2) holds unconditionally. The removal of the 3-IRB assumption uses continuity of $G_{g,\nu}(x)$ in ν together with an auxiliary result that $G_{g,\nu}(x)$ satisfies a 2-IRB if ν is large enough. The 2-IRB cannot be lost as ν is decreased towards ν_c because step two implies that $G_{g,\nu}(x)$ cannot continuously become greater than $3S(x)$.

4.2 Proof of Proposition 4.3

The three steps outlined in the previous section become the proof of Proposition 4.3 given at the end of this section, but first we state the lemmas which are the precise versions of these steps. This requires two auxiliary functions. For $|z| \leq \hat{J}^{-1}$, define $\tilde{S}_z$ and $D_z^{\tilde{S}}$ on $\mathbb{Z}^d$ by

(4.7) $$\tilde{S}_z(x) := \sum_{n\geq 0} (zJ_+)^{*n}(x), \quad D_z^{\tilde{S}}(x) := -\mathbb{1}_{\{x=0\}} + zJ_+(x).$$

$D_z^{\tilde{S}}$ is a variant of J such that when $z = \hat{J}^{-1}$ the jump rates are normalised by z to probabilities and when $z \in [0, \hat{J}^{-1})$ the walk has a positive killing rate. By (2.13) the series defining $\tilde{S}_z(x)$ in (4.7) is absolutely convergent, and it is straightforward to check that $D_z^{\tilde{S}} * \tilde{S}_z(x) = -\mathbb{1}_{\{x=0\}}$. Let

(4.8) $$\tilde{S}(x) := \tilde{S}_{\hat{J}^{-1}}(x).$$

By comparing the definition of $\tilde{S}_z$ with Equation (2.13) and using (2.11)

(4.9) $$\tilde{S}_z(x) \leq \tilde{S}(x) = \hat{J}S(x) \leq \frac{\tilde{C}_J}{|||x|||^{d-2}},$$

for some $\tilde{C}_J > 0$, where $|||x||| := \max\{|x|, 1\}$.

Step one is Lemma 4.4 (i), (ii) and Lemma 4.5 (ii) below.

Lemma 4.4 (Proof in Section 11)**.** *Let $d \geq 5$, and consider the lattice Edwards model or the $g|\varphi|^4$ model with $n = 1, 2$. There exist $\alpha > 0$, $g_0 > 0$, $\Psi_{g,\nu}$ such that for all $g \in (0, g_0)$ if $G_{g,\nu} \leq 3S$ then $L_{g,\nu} = O(g)$ is well-defined and*

- *(i)* Ψ *is a $\mathbb{Z}^d$-symmetric function.*
- *(ii)* $|\Psi_{g,\nu}(x)| \leq g\alpha |||x|||^{-3(d-2)}$.
- *(iii)* (4.6) *holds.*
- *(iv)* $\hat{J} - L_{g,\nu} \geq \frac{1+O(g)}{3S(0)}$.

By the lower bound of item (iv) we can rewrite equation (4.6) as

(4.10) $$D_{g,\nu} * \tilde{G}_{g,\nu}(x) = -\mathbb{1}_{\{x=0\}},$$

with the definitions

$$w(g,\nu) := (\hat{J} - L_{g,\nu})^{-1}, \qquad \tilde{G}_{g,\nu}(x) := w(g,\nu)^{-1} G_{g,\nu}(x),$$
$$D_{g,\nu} := D^{\tilde{S}}_{w(g,\nu)} + \tilde{\Psi}_{g,\nu}, \qquad \tilde{\Psi}_{g,\nu}(x) := w(g,\nu)\Psi_{g,\nu}(x). \tag{4.11}$$

For $C > 0$ let $\mathscr{D}_C$ be the class of all functions $D\colon \mathbb{Z}^d \to \mathbb{R}$ with the properties

(i) D is $\mathbb{Z}^d$-symmetric,
(ii) $\sum_{x\in\mathbb{Z}^d} D(x) \leq 0$,
(iii) there exists $z = z(g,D) \in \left[0, \hat{J}^{-1}\right]$ such that $\left|D(x) - D^{\tilde{S}}_z(x)\right| \leq Cg|||x|||^{-(d+4)}$.

Lemma 4.5 (Proof in Section 11)**.** *With the same hypotheses as Lemma 4.4, for $\nu \in (\nu_c, g]$, there exists $C_0 > 0$ such that*

(i) $D_{g,\nu} \in \mathscr{D}_{C_0}$,
(ii) $|L_{g,\nu}| \leq C_0 g$.

Step two rests on Lemma 4.6 below, which is a generalization of [7, Lemma 2] to more general step distributions.

Lemma 4.6 (Proof in Section 10.2)**.** *Let $d \geq 5$, and $C > 0$. There exist $g_0 = g_0(d,J,C) > 0$ and $C' > 0$ such that for $g \in (0, g_0)$ and $D \in \mathscr{D}_C$ there exists $H\colon \mathbb{Z}^d \to \mathbb{R}$ such that*

$$D * H = -\mathbb{1}_{\{x=0\}} \tag{4.12}$$

$$|H(x) - \tilde{S}_\mu(x)| \leq C' g|||x|||^{-(d-2)}, \qquad x \in \mathbb{Z}^d, \tag{4.13}$$

where $\mu := \hat{J}^{-1}\left(1 + \sum_{x\in\mathbb{Z}^d} D(x)\right) \in \left[-(2\hat{J})^{-1}, \hat{J}^{-1}\right]$.

By (J3) and Lemma 4.4, $D_{g,\nu}$ given by (4.11) is in $\mathscr{D}_C$. For details refer to the proof of Lemma 10.5. We will apply Lemma 4.6 with $D = D_{g,\nu}$, and the μ created by the lemma will be denoted by $\mu(g,\nu)$.

Step three uses the fact that the continuous image of a connected interval is connected. We state this in the form of the next lemma and apply it to the function F defined in (4.14) below. The use of this lemma to extend lace expansion estimates up to the critical point originated in [43]; a related application is in [9].

Lemma 4.7 ([25, Lemma 2.1])**.** *For $\nu_1 > \nu_c$ let $F\colon (\nu_c, \nu_1] \to \mathbb{R}$. If*

(i) $F(\nu_1) \leq 2$,
(ii) F is continuous on $(\nu_c, \nu_1]$,
(iii) for $\nu \in (\nu_c, \nu_1]$ the inequality $F(\nu) \leq 3$ implies the inequality $F(\nu) \leq 2$,

then $F(\nu) \leq 2$ for $\nu \in (\nu_c, \nu_1]$.

The next two lemmas provide hypotheses (i) and (ii) of Lemma 4.7 with $\nu_1 = g$ for the function $F\colon (\nu_c, \infty) \to \mathbb{R}$ defined by

$$F(\nu) := \sup_{x\in\mathbb{Z}^d} \frac{G_{g,\nu}(x)}{S(x)}. \tag{4.14}$$

Lemma 4.8 (Proof in Section 11)**.** *For the lattice Edwards model and the $n = 1,2$-component $g|\varphi|^4$ model, with F as in* (4.14)

$$F(\nu) \le 2 \text{ when } \nu = g. \tag{4.15}$$

Lemma 4.9 (Proof in Section 11)**.** *For the lattice Edwards model and the $n = 1,2$-component $g|\varphi|^4$ model the function F in* (4.14) *is continuous on* $(\nu_c, g]$.

For $U,V\colon \Lambda\times\Lambda \to \mathbb{R}$ we write $UV(x,y) = \sum_{u\in\Lambda} U(x,u)V(u,y)$. The following lemma is the well-known algebraic fact that left and right inverses coincide for algebraic structures with an associative product. We will use it in the proof of Proposition 4.3 and in Section 9.2.

Lemma 4.10. *Let $U,V,W\colon \mathbb{Z}^d\times\mathbb{Z}^d \to \mathbb{R}$ satisfy $UV(x,y) = WU(x,y) = \mathbb{1}_{\{x=y\}}$ and $\sum_{u,v\in\mathbb{Z}^d} |W(x,u)|\,|U(u,v)|\,|V(v,y)| < \infty$ for all $x,y \in \mathbb{Z}^d$. Then $V = W$.*

Proof. The absolute convergence of the sum over u and v implies associativity, $W(UV) = (WU)V$. Therefore $W = W(UV) = (WU)V = V$. ∎

Let $\ell^p = \ell^p(\mathbb{Z}^d)$ denote the set of $f\colon \mathbb{Z}^d \to \mathbb{R}$ with $\sum_{x\in\mathbb{Z}^d} |f(x)|^p$ finite.

Proof of Proposition 4.3. By hypothesis, $d \ge 5$ and g is small enough such that the results we have stated above in this section are applicable. By the argument below (3.13) $\nu_c \le 0$ so $(\nu_c, g]$ is not empty. In terms of definition (4.14) we will prove that

$$F(\nu) \le 2 \qquad \text{for } \nu \in (\nu_c, g], \tag{4.16}$$

and the desired conclusion (4.2) then follows by Lemma 3.5. In Lemma 4.7 set $\nu_1 = g$ so that hypotheses (i) and (ii) of Lemma 4.7 are supplied by Lemmas 4.8 and 4.9. Since (4.16) is the conclusion of Lemma 4.7, it is enough to prove hypothesis (iii) which is: for $\nu \in (\nu_c, g]$, $F(\nu) \le 3$ implies $F(\nu) \le 2$. To this end, assume $F(\nu) \le 3$. Then the conclusions of Lemma 4.4 and Lemma 4.5 hold and provide the hypotheses of Lemma 4.6 for $D = D_{g,\nu}$ given by (4.11). By part (4.12) of Lemma 4.6, H is the right-convolution inverse of $-D_{g,\nu}$. By (4.10) $\tilde{G}_{g,\nu}$ is also a right-convolution inverse of $-D_{g,\nu}$. We will show that this implies that $\tilde{G}_{g,\nu} = H$ after demonstrating that the result follows from this claim.

By the lower bound for $S(x)$ below (2.11), the equality $\tilde{S}(x) = \hat{J}S(x)$ in (4.9) and (4.13) we have $|\tilde{G}_{g,\nu}(x) - \tilde{S}_\mu(x)| \le O(g)\tilde{S}(x)$. Therefore $\tilde{G}_{g,\nu}(x) \le (1+O(g))\tilde{S}(x)$ since $|\mu| = |\mu(g,\nu)| \le \hat{J}^{-1}$. By (4.11), Lemma 4.5 and $\tilde{S}(x) = \hat{J}S(x)$ this is the same as

$$G_{g,\nu}(x) \le \frac{\hat{J}}{\hat{J} - O(g)}\big(1+O(g)\big)S(x) = \big(1+O(g)\big)S(x), \tag{4.17}$$

for $\nu \in (\nu_c, g]$ and $x \in \mathbb{Z}^d$. By taking g smaller if necessary we have $G_{g,\nu}(x) \le 2S(x)$ as desired.

It only remains to prove our claim that $\tilde{G}_{g,\nu} = H$ given that $F(\nu) \le 3$ and $\nu \in (\nu_c, g]$. By (4.13), $|H(x)| \le C|x|^{-(d-2)}$. The function $\tilde{G}_{g,\nu}(x)$ also decays like

$|x|^{-(d-2)}$ by $F(\nu) \leq 3$. By Lemma 4.4 part (ii), $D_{g,\nu}(x)$ decays like $|x|^{-3(d-2)}$. By the decay of $\tilde{G}_{g,\nu}$ and $D_{g,\nu}$ and $d > 4$ the sum that defines the convolution in (4.10) is absolutely convergent, see Lemma 8.2. Therefore this convolution is commutative and $\tilde{G}_{g,\nu}$ is a two-sided convolution inverse to $-D_{g,\nu}$. Furthermore, by Lemma 8.2, $\sum_{u,v\in\mathbb{Z}^d} |\tilde{G}_{g,\nu}(x-u)|\,|D_{g,\nu}(u-v)|\,|H(v-y)| < \infty$. By Lemma 4.10 with $W = \tilde{G}_{g,\nu}$, $U = D_{g,\nu}$ and $V = H$, we have $\tilde{G}_{g,\nu} = H$ as claimed, and hence the proof is complete. ∎

4.3 Proof of Theorem 4.1

Lemma 4.11 (Proof in Section 11)**.** *For the lattice Edwards model and the $n = 1,2$-component $g|\varphi|^4$ models, $L_{g,\nu} \to \infty$ as $\nu \to -\infty$.*

We use this result to prove that ν_c is finite as claimed in Theorem 4.1: since $\nu_c \leq 0$ it is enough to rule out $\nu_c = -\infty$. Towards a contradiction, suppose $\nu_c = -\infty$. Then Proposition 4.3 implies a 2-IRB holds for all $\nu \leq g$, and hence Lemma 4.5 (ii) implies $|L_{g,\nu}| \leq C_0 g$ for all $\nu \leq g$. This contradicts Lemma 4.11.

Lemma 4.12. *For the lattice Edwards model and the $n = 1,2$-component $g|\varphi|^4$ models, the finite volume $G_{\mathbf{0}}^{(\Lambda)}(0,x)$ is continuous in ν at ν_c.*

Proof. For the Edwards model, observe from (3.4) that $Z_{\boldsymbol{t}}$ is continuous in ν pointwise in $\boldsymbol{t}$ and uniformly bounded in $\boldsymbol{t}$ for each ν. By dominated convergence it follows from (3.2) that the *finite* volume $G_{\mathbf{0}}^{(\Lambda)}(0,x)$ is continuous in ν at ν_c. A similar argument applies to the $g|\varphi|^4$ model. ∎

Proof of Theorem 4.1. For future reference, we note that the remainder of this proof deduces the desired (4.1) from (4.2), (i) $G^{(\Lambda)}$ is monotone in Λ and (ii) the *finite volume* $G_{\mathbf{0}}^{(\Lambda)}(0,x)$ is continuous in ν at ν_c. Claim (i) is Lemma 3.4. Claim (ii) is Lemma 4.12.

By (i) and (4.2), for $\nu > \nu_c$,

$$G_{\mathbf{0}}^{(\Lambda)}(0,x) \leq G_{g,\nu}(x) \leq 2S(x) \tag{4.18}$$

By (ii) $G_{\mathbf{0}}^{(\Lambda)}(0,x)$ is bounded above by $2S(x)$ when $\nu = \nu_c$. By taking the infinite volume limit with $\nu = \nu_c$ we obtain $G_{g,\nu_c}(x) \leq 2S(x)$. ∎

Remark 4.13. By reviewing Sections 4.2 and 4.3 we find that the proofs of Theorem 4.1 and Proposition 4.3 have been reduced to Lemmas 3.4, 3.5 and 4.4 to 4.11 excluding Lemmas 4.7 and 4.10. Note that the proof of Lemma 4.12 is valid for any $Z_{\boldsymbol{t}}$ that is continuous in ν pointwise in $\boldsymbol{t}$ and uniformly bounded in $\boldsymbol{t}$ for each ν. We classify Lemmas 4.4 to 4.6 and 4.9 as *model-independent*: although the hypotheses of Lemmas 4.4 and 4.5 mention our specific models, they apply, with understood variations in the $|||x|||^{-3(d-2)}$ decay, to all models with convergent lace expansions. We classify Lemmas 3.4, 3.5, 4.8 and 4.11 as *model dependent*.

4.4 Outline of the remainder of the paper

Our analysis is done in a general context that includes the Edwards and the $g|\varphi|^4$ models with $n = 1,2$ as special cases. The general context is a set of hypotheses on the function $\boldsymbol{t} \mapsto Z_{\boldsymbol{t}}$ that enters into the definition (3.2) of the Green's function; see Section 10.1 for a full list of hypotheses. In the course of the paper we introduce these hypotheses on $Z_{\boldsymbol{t}}$ as they are needed. In some initial sections we use hypotheses that will eventually be superseded; these are indicated by ending in a 0, e.g., (G0) below. We verify that the Edwards and the $g|\varphi|^4$ models with $n = 1,2$ satisfy the hypotheses in Section 11. We have based our proof on hypotheses on Z to facilitate extending the continuous-time lace expansion to other models: isolating properties that currently play a role should help the search for more appealing hypotheses.

In Section 5 and Section 6 we develop a lace expansion for Green's functions as in (3.2) in *finite volumes* $\Lambda \subset \mathbb{Z}^d$. Working in a finite volume is essential, as we have only defined the $g|\varphi|^4$ model as the infinite volume limit of finite volume models.

The next part of the paper, Sections 7 through 9, develops estimates on our finite volume lace expansion, under the hypothesis that the Green's function satisfies a 3-IRB. These estimates establish the infinite-volume lace expansion equation (4.6) under the general hypotheses on $Z_{\boldsymbol{t}}$ and provide the key inputs for the proofs of Lemma 4.4 and Lemma 4.5.

In Section 10.2 and 10.3 we complete the proofs of the lemmas we have used in the last two sections, and thus establish the conclusions of Theorem 4.1 for any $Z_{\boldsymbol{t}}$ satisfying our hypotheses. We then make use of this result, in conjunction with a theorem of Hara [22], to obtain the Gaussian asymptotics of Theorem 3.6.

5 Functions on a set of intervals

In this section we begin to derive the lace expansion needed for step one of Section 4.1. The main result of this section is Theorem 5.2, which is an expansion for a function $\mathscr{Y}: \mathscr{D} \to \mathbb{R}$, $(s,t) \mapsto \mathscr{Y}_{s,t}$, where

$$\mathscr{D} := \{(s,t) : 0 \le s \le t < \infty\} \subset [0,\infty)^2. \tag{5.1}$$

Here (s,t) denotes an ordered pair, but the same notation will be used for an open interval. Theorem 5.2 will be used in the next section to derive our lace expansion for Green's functions of the form (3.2). We begin with notation and minimal assumptions on $\mathscr{Y}$ needed for the main result of the section.

For a function $\mathscr{Y}$ on $\mathscr{D}$ and $t \in (0,\infty)$, we denote by $\mathscr{Y}_{\cdot,t}\colon [0,t] \to \mathbb{R}$ the function $s \mapsto \mathscr{Y}_{s,t}$, and for each $s \in [0,\infty)$, we denote by $\mathscr{Y}_{s,\cdot}\colon [s,\infty) \to \mathbb{R}$ the function $t \mapsto \mathscr{Y}_{s,t}$. We will write ∂_1 and ∂_2 to denote partial differentiation with respect to the first and second coordinates, respectively.

We will assume $\mathscr{Y}$ satisfies the following assumptions. The *almost every* (a.e.) statements in the assumptions are with respect to Lebesgue measure.

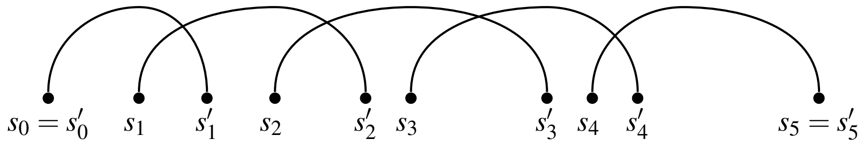


Figure 5.1. A lace with $m = 5$ intervals

Assumptions 5.1.

(1) $\mathscr{Y}$ is continuous and strictly positive on $\mathscr{D}$, and $\mathscr{Y}_{s,s} = 1$ for all $s \geq 0$.
(2) For each $t \in (0,\infty)$, $\mathscr{Y}_{\cdot,t}$ is absolutely continuous. For each $s \in [0,\infty)$, $\mathscr{Y}_{s,\cdot}$ is absolutely continuous on bounded subintervals of $[s,\infty)$.
(3) For a.e. $t \in (0,\infty)$, the function $(\partial_2 \mathscr{Y})_{\cdot,t}$ is absolutely continuous on $(0,t)$. For a.e. $s \in [0,\infty)$, the function $(\partial_1 \mathscr{Y})_{s,\cdot}$ is absolutely continuous on bounded subintervals of (s,∞).
(4) $\partial_1 \partial_2 \mathscr{Y} = \partial_2 \partial_1 \mathscr{Y}$ a.e. on the interior of $\mathscr{D}$.

We will see in Section 6.1 that the absolute continuity statements are properties of the random function $(s,t) \mapsto \tau_{[s,t],x}$ defined by (2.5) with $I = [s,t]$ and x fixed. The derivatives in Item 3 have open intervals of definition, while the standard definition of absolute continuity concerns closed intervals. In this paper we say a function f defined on a bounded open interval $I \subset \mathbb{R}$ is absolutely continuous on I if for $\varepsilon > 0$ there exists $\delta > 0$ such that for finitely many disjoint subintervals (a_i, b_i) of I with endpoints in I, $\sum |b_i - a_i| < \delta$ implies $\sum |f(b_i) - f(a_i)| < \varepsilon$. Such functions are uniformly continuous on bounded intervals and therefore the derivative $\partial_2 \mathscr{Y}_{\cdot,t}$ in Item 3 extends to an absolutely continuous function on the closed interval $[0,t]$ and similarly $\partial_1 \mathscr{Y}_{s,\cdot}$ extends to $[s,\infty)$. When we write derivatives on the boundaries of their domains we mean these extensions by continuity.

5.1 The expansion

In this section we introduce the objects that enter into our expansion, and state the expansion in Theorem 5.2 below.

Define *vertex functions* by

$$(5.2) \qquad \begin{aligned} r_s &:= -\lim_{t \downarrow s} \partial_1 \log \mathscr{Y}_{s,t}, \qquad && 0 \leq s < \infty, \\ r_{s,t} &:= -\partial_1 \partial_2 \log \mathscr{Y}_{s,t}, \qquad && 0 \leq s < t < \infty. \end{aligned}$$

These are a.e. equalities. By Assumptions 5.1 and the paragraph that follows them, the chain rule, which applies to a smooth function composed with an absolutely continuous function, proves these derivatives exist a.e. and provides formulas for them. The limit defining r_s exists by the discussion under Assumptions 5.1.

For $m \in \mathbb{N}$ a *lace* L is a sequence

$$(5.3) \qquad L = \big((s_i, s'_{i+1})\big)_{i=0,\dots,m-1}$$

of m open intervals (s_i, s'_{i+1}) with $s_0 := s'_0$, $s_m := s'_m$, and

$$0 \le s'_0 < s_1 < s'_1 < s_2 < s'_2 < \cdots < s'_{m-1} < s_m < \infty. \tag{5.4}$$

The meaning of (5.4) is illustrated by Figure 5.1. The union of all of the intervals of a lace is (s_0, s'_m), and if any single interval is excluded from the union the resulting set does not cover (s_0, s'_m).

Let $\mathscr{L}_m$ be the region in $\mathbb{R}^{2m}$ defined by the inequalities in (5.4). We identify $\mathscr{L}_m$ with the collection of all laces containing m intervals. Let

$$\mathscr{L}_0 := \{s_0 \mid 0 \le s_0 < \infty\}. \tag{5.5}$$

We associate to a lace L a product

$$r(L) := \begin{cases} r_{s_0}, & \text{if } L = \{s_0\} \in \mathscr{L}_0, \\ \prod_{i=0}^{m-1} r_{s_i, s'_{i+1}}, & \text{if } L = \big((s_i, s'_{i+1})\big)_{i=0,\dots,m-1} \in \mathscr{L}_m,\ m \ge 1, \end{cases} \tag{5.6}$$

of vertex functions. The *weight* $L \mapsto w(L)$ of a lace is defined to be

$$w(L) := r(L) \times \begin{cases} 1, & L \in \mathscr{L}_0, \\ P(L), & L \in \mathscr{L}_m,\ m \ge 1, \end{cases} \tag{5.7}$$

where for $L = \big((s_i, s'_{i+1})\big)_{i=0,\dots,m-1}$

$$P(L) := \mathscr{Y}_{s'_0, s'_1} \prod_{i=0}^{m-2} \frac{\mathscr{Y}_{s'_i, s'_{i+2}}}{\mathscr{Y}_{s'_i, s'_{i+1}}}. \tag{5.8}$$

For $m = 1$ the empty product in (5.8) is defined to be one by convention.

For $m \ge 0$ we define integration $\int_{\mathscr{L}_m} dL$ over $\mathscr{L}_m$ to be integration with respect to Lebesgue measure on $\mathscr{L}_m$. For example, if $m = 0$ then $\int_{\mathscr{L}_m} dL = \int_{[0,\infty)} ds_0$. Let $\mathscr{L}_{m,\ell}$ be the subset of $\mathscr{L}_m$ such that $s_m \le \ell$, and let $\mathscr{D}_\ell$ be the subset of $\mathscr{D}$ with $t \le \ell$.

Theorem 5.2. *Let $\mathscr{Y}$ be such that (i) $\mathscr{Y}$ satisfies Assumption 5.1, (ii) the function $r_{s,t}$ defined in* (5.2) *is Lebesgue a.e. bounded on $\mathscr{D}_\ell$. Then for $\ell > 0$*

$$\mathscr{Y}_{0,\ell} = 1 + \sum_{m \ge 0} \int_{\mathscr{L}_{m,\ell}} dL\, w(L)\, \mathscr{Y}_{s'_m, \ell}\,, \tag{5.9}$$

and the sum is absolutely convergent.

The proof of this theorem is given in Section 5.4 and is based on the two identities given in Lemmas 5.4 and 5.5. We will need the following fact from real analysis, whose proof we omit.

Lemma 5.3. *Let $I \subset \mathbb{R}$ be bounded. If $f\colon I \to \mathbb{R}$ is Lipschitz continuous and $g\colon I \to I$ is absolutely continuous, then $f \circ g$ is absolutely continuous.*

5.2 The identity that starts the expansion

Lemma 5.4. *Under Assumptions 5.1,*

$$\mathscr{Y}_{0,\ell} = 1 + \int_{(0,\ell)} ds_0 \, r_{s_0} \, \mathscr{Y}_{s_0,\ell} + \int_{(0,\ell)} ds_0 \int_{(s_0,\ell)} ds'_1 \, r_{s_0,s'_1} \, \mathscr{Y}_{s_0,\ell}. \tag{5.10}$$

Proof. By Assumption 5.1(2),

$$\mathscr{Y}_{0,\ell} = \mathscr{Y}_{\ell,\ell} - \int_{(0,\ell)} ds_0 \, \partial_1 \mathscr{Y}_{s_0,\ell} \, . \tag{5.11}$$

$\mathscr{Y}$ is bounded below by a positive constant on $\mathscr{D}_\ell$ by Assumption 5.1(1) and the compactness of $\mathscr{D}_\ell$. Hence, by $\mathscr{Y}_{\ell,\ell} = 1$, (5.11) can be rewritten as

$$\mathscr{Y}_{0,\ell} = 1 - \int_{(0,\ell)} ds_0 \, \frac{\partial_1 \mathscr{Y}_{s_0,\ell}}{\mathscr{Y}_{s_0,\ell}} \, \mathscr{Y}_{s_0,\ell}. \tag{5.12}$$

By Assumption 5.1(1) the range of $t \mapsto \mathscr{Y}_{s_0,t}$ for $t \in [s_0,\ell]$ is the continuous image of a compact set, and the range does not contain zero. As $z \mapsto 1/z$ is Lipschitz on compact subsets of $(0,\infty)$, it follows from Lemma 5.3 that $t \mapsto \mathscr{Y}^{-1}_{s_0,t}$ is absolutely continuous in t for t in bounded subintervals of (s_0,∞). Combined with Assumption 5.1(3) this shows $\mathscr{Y}^{-1}_{s_0,t} \, \partial_1 \mathscr{Y}_{s_0,t}$ is absolutely continuous in $t \in (s_0,\ell)$ for a.e. s_0, hence for a.e. $s_0 > 0$

$$\partial_1 \log \mathscr{Y}_{s_0,\ell} = (\partial_1 \log \mathscr{Y})_{s_0,s_0} + \int_{(s_0,\ell)} ds'_1 \, \partial_2 \partial_1 \log \mathscr{Y}_{s_0,s'_1}. \tag{5.13}$$

In this equation $(\partial_1 \log \mathscr{Y})_{s_0,s_0}$ is by definition the continuous extension of $(\partial_1 \log \mathscr{Y})_{s_0,t} = \mathscr{Y}^{-1}_{s_0,t} \, \partial_1 \mathscr{Y}_{s_0,t}$ as a function of $t > 0$ to the boundary $t = s_0$ of its domain. This is an instance of the convention we declared below Assumption 5.1. By inserting (5.13) into (5.12) we obtain

$$\begin{aligned} \mathscr{Y}_{0,\ell} = 1 - &\int_{(0,\ell)} ds_0 \, \mathscr{Y}_{s_0,\ell} \, (\partial_1 \log \mathscr{Y})_{s_0,s_0} \\ &- \int_{(0,\ell)} ds_0 \int_{(s_0,\ell)} ds'_1 \, \mathscr{Y}_{s_0,\ell} \, \partial_2 \partial_1 \log \mathscr{Y}_{s_0,s'_1}. \end{aligned} \tag{5.14}$$

The proof is completed by substituting in the definitions of r_{s_0} and r_{s_0,s'_1}. For the last term the interchange of derivatives is justified by Assumption 5.1(4). ■

5.3 The identity that generates the expansion

Lemma 5.5. *Suppose $\mathscr{Y}$ satisfies Assumptions 5.1. If the point $(u,v) \in \mathscr{D}_\ell$ then*

$$\mathscr{Y}_{u,\ell} = \mathscr{Y}_{u,v} \, \mathscr{Y}_{v,\ell} + \int_{(u,v)} ds_+ \int_{(v,\ell)} ds'_+ \, r_{s_+,s'_+} \, \mathscr{Y}_{u,s'_+} \, \frac{\mathscr{Y}_{v,\ell}}{\mathscr{Y}_{v,s'_+}}. \tag{5.15}$$

Proof. As $\mathscr{Y}$ is bounded below by a positive constant on $\mathscr{D}_\ell$, we can rewrite the left-hand side:

$$\mathscr{Y}_{u,\ell} = \mathscr{Y}_{u,v} \mathscr{Y}_{v,\ell} + \left(\frac{\mathscr{Y}_{u,\ell}}{\mathscr{Y}_{v,\ell}} - \mathscr{Y}_{u,v} \right) \mathscr{Y}_{v,\ell}. \tag{5.16}$$

By Assumption 5.1(2), $\mathscr{Y}_{s,s}=1$, and the absolute continuity of $\mathscr{Y}_{v,\ell}^{-1}$ in ℓ noted after (5.12), this can be rewritten as

$$\mathscr{Y}_{u,\ell}=\mathscr{Y}_{u,v}\mathscr{Y}_{v,\ell}+\left(\int_{(v,\ell)} ds'_+\,\partial_2\frac{\mathscr{Y}_{u,s'_+}}{\mathscr{Y}_{v,s'_+}}\right)\mathscr{Y}_{v,\ell}. \tag{5.17}$$

Since exp and log are Lipschitz on compact subsets of their open domains we can compute the derivative in (5.17) using $f(x)=\exp\log f(x)$:

$$\mathscr{Y}_{u,\ell}=\mathscr{Y}_{u,v}\mathscr{Y}_{v,\ell}+\left(\int_{(v,\ell)} ds'_+\left(\partial_2\log\left(\frac{\mathscr{Y}_{u,s'_+}}{\mathscr{Y}_{v,s'_+}}\right)\right)\frac{\mathscr{Y}_{u,s'_+}}{\mathscr{Y}_{v,s'_+}}\right)\mathscr{Y}_{v,\ell} \tag{5.18}$$

$$=\mathscr{Y}_{u,v}\mathscr{Y}_{v,\ell}-\left(\int_{(v,\ell)} ds'_+\left(\int_{(u,v)} ds_+\,\partial_1\partial_2\log\mathscr{Y}_{s_+,s'_+}\right)\frac{\mathscr{Y}_{u,s'_+}}{\mathscr{Y}_{v,s'_+}}\right)\mathscr{Y}_{v,\ell}, \tag{5.19}$$

where we have used Assumption 5.1(3) in the second step. By Fubini's theorem this can be rewritten as the desired result. ■

5.4 Proof of Theorem 5.2

Recall the definitions of $w(L)$, $r(L)$ and $P(L)$ in (5.6)–(5.8) and define

$$R_n:=\int_{\mathscr{L}_{n,\ell}} dL\,w(L)\,\frac{\mathscr{Y}_{s'_{n-1},\ell}}{\mathscr{Y}_{s'_{n-1},s'_n}},\qquad n\geq 1. \tag{5.20}$$

Lemma 5.6. *Suppose $\mathscr{Y}$ satisfies Assumptions 5.1. Then*

$$\mathscr{Y}_{0,\ell}=1+\sum_{m=0}^{n-1}\int_{\mathscr{L}_{m,\ell}} dL\,w(L)\,\mathscr{Y}_{s'_m,\ell}+R_n,\qquad n\geq 1. \tag{5.21}$$

Proof. We first prove (5.21) with $n=1$. By Lemma 5.4

$$\mathscr{Y}_{0,\ell}=1+\int_{(0,\ell)} ds_0\,r_{s_0}\,\mathscr{Y}_{s_0,\ell}+\int_{(0,\ell)} ds_0\int_{(s_0,\ell)} ds'_1\,r_{s_0,s'_1}\,\mathscr{Y}_{s_0,\ell}. \tag{5.22}$$

By the definition (5.7) of $w(L)$ and the definition of integration over $\mathscr{L}_{m,\ell}$ given below (5.8) (recall also that $s'_0:=s_0$), this can be rewritten as

$$\mathscr{Y}_{0,\ell}=1+\int_{\mathscr{L}_{0,\ell}} dL\,w(L)\,\mathscr{Y}_{s'_0,\ell}+\int_{\mathscr{L}_{1,\ell}} dL\,r_{s'_0,s'_1}\,\mathscr{Y}_{s'_0,\ell}. \tag{5.23}$$

This establishes (5.21) when $n=1$ as the final term is R_1.

We now prove (5.21) holds for $n\geq 1$ by induction, using (5.21) as the inductive hypothesis. By Lemma 5.5 with (u,v) replaced by (s'_{n-1},s'_n),

$$\begin{aligned}\mathscr{Y}_{s'_{n-1},\ell}&=\mathscr{Y}_{s'_{n-1},s'_n}\,\mathscr{Y}_{s'_n,\ell}\\&\quad+\int_{(s'_{n-1},s'_n)} ds_+\int_{(s'_n,\ell)} ds'_+\,r_{s_+,s'_+}\,\mathscr{Y}_{s'_{n-1},s'_+}\,\frac{\mathscr{Y}_{s'_n,\ell}}{\mathscr{Y}_{s'_n,s'_+}}.\end{aligned} \tag{5.24}$$

We insert (5.24) into the definition (5.20) of R_n and use the definition (5.7) of $w(L)$ for the contribution from the first term $\mathscr{Y}_{s'_{n-1},s'_n}\,\mathscr{Y}_{s'_n,\ell}$:

$$R_n = \int_{\mathscr{L}_{n,\ell}} dL\, w(L)\mathscr{Y}_{s'_n,\ell} + \Bigg(\int_{\mathscr{L}_{n,\ell}} dL\, r(L)\, P(L)$$

$$\times \int_{(s'_{n-1},s'_n)} ds_+ \int_{(s'_n,\ell)} ds'_+ \, r_{s_+,s'_+} \frac{\mathscr{Y}_{s'_{n-1},s'_+}}{\mathscr{Y}_{s'_{n-1},s'_n}} \frac{\mathscr{Y}_{s'_n,\ell}}{\mathscr{Y}_{s'_n,s'_+}} \Bigg). \tag{5.25}$$

Renaming s_+, s'_+ as s_n, s'_{n+1} and combining the integrals in the second term into an integral over $\mathscr{L}_{n+1,\ell}$ yields

$$R_n = \int_{\mathscr{L}_{n,\ell}} dL\, w(L)\, \mathscr{Y}_{s'_n,\ell} + \int_{\mathscr{L}_{n+1,\ell}} dL\, r(L)\, P(L) \frac{\mathscr{Y}_{s'_n,\ell}}{\mathscr{Y}_{s'_n,s'_{n+1}}} \tag{5.26}$$

For the second term on the right of (5.25), $r_{s_+,s'_+} := r_{s_n,s'_{n+1}}$ became part of $r(L)$ and the ratio of $\mathscr{Y}$'s became part of $P(L)$ when the range of integration became $\mathscr{L}_{n+1,\ell}$. By the definition (5.20) of R_{n+1}, this is

$$R_n = \int_{\mathscr{L}_{n,\ell}} dL\, w(L)\, \mathscr{Y}_{s'_n,\ell} + R_{n+1}. \tag{5.27}$$

Inserting (5.27) into the inductive hypothesis completes the proof. ■

Proof of Theorem 5.2. We justify taking the $n \to \infty$ limit of Lemma 5.6.

The factors $\mathscr{Y}_{s,t}$ under the integrals in (5.21) are bounded above and below because they are strictly positive and continuous functions on the compact domain $\mathscr{D}_\ell$. Together with the assumption that $r_{s,t}$ is uniformly bounded this proves that there is a constant $C = C(\ell)$ such that $\left| w(L)\mathscr{Y}_{s'_n,\ell} \right| \le C^{n+1}$ for $L \in \mathscr{L}_{n,\ell}$ and $n \ge 1$, where w was defined in (5.7). Similarly the integrand in R_n is bounded by C^{n+1} for $L \in \mathscr{L}_{n,\ell}$. Because $\mathscr{Y}_{s,t}$ is bounded above and below we also have that $\left| w(L)\mathscr{Y}_{s'_0,\ell} \right|$ is bounded by a constant $C' = C'(\ell)$ when $L \in \mathscr{L}_{0,\ell}$.

The Lebesgue measure of $\mathscr{L}_{n,\ell}$ is the Lebesgue measure of all $2n$-tuples of ordered points in $(0,\ell)$, which is $\frac{1}{(2n)!}\ell^{2n}$. Therefore $|R_n| \le C^{n+1}\frac{1}{(2n)!}\ell^{2n}$ and the mth term in the sum over m in (5.21) is bounded by $C^{m+1}\frac{1}{(2m)!}\ell^{2m}$. Therefore the series in the right hand side of (5.9) is absolutely convergent and equals $\mathscr{Y}_{0,\ell}$ as claimed because $\lim_{n\to\infty} R_n = 0$ in (5.21). ■

6 The lace expansion in finite volume

In this section we continue with step one of Section 4.1. Throughout this section $\Lambda \subset \mathbb{Z}^d$ denotes a fixed finite set. The main result is Proposition 6.2, which provides the finite-volume version of (4.6). This proposition involves a function $\Pi^{(\Lambda)}(x,y)$, and the formula (6.3) for $\Pi^{(\Lambda)}(x,y)$ is called a lace expansion for reasons to be explained following Proposition 7.4. We begin by introducing some further definitions and assumptions.

Given $Z^{(\Lambda)}\colon [0,\infty)^{\Lambda} \to (0,\infty)$, $\boldsymbol{u} \mapsto Z^{(\Lambda)}_{\boldsymbol{u}}$, we choose the function $\mathscr{Y}\colon \mathscr{D} \to \mathbb{R}$ of Section 5 to be the random function

$$\mathscr{Y}_{s,t} = \mathscr{Y}^{(\Lambda)}_{s,t} := \left(\frac{Z^{(\Lambda)}_{\cdot}}{Z^{(\Lambda)}_{\mathbf{0}}}\right) \circ \boldsymbol{\tau}^{(\Lambda)}_{[s,t]} = \frac{1}{Z^{(\Lambda)}_{\mathbf{0}}} Z^{(\Lambda)}_{\boldsymbol{\tau}^{(\Lambda)}_{[s,t]}}. \tag{6.1}$$

Recall (3.1) and note that $\mathscr{Y}_{s,t} = Y_{\mathbf{0},\boldsymbol{\tau}^{(\Lambda)}_{[s,t]}}$. Henceforth $\mathscr{Y}$ is given by (6.1). Let $G^{(\Lambda)}$ be the Green's function determined by (3.2) with this choice of $Z^{(\Lambda)}$. Recall that the weight $w(L)$ of a lace is defined in terms of $\mathscr{Y}_{s,t}$ in (5.7). We write $w^{(\Lambda)}(L) := w(L)$ for the weight with the choice (6.1).

Let $\mathscr{L}_m(s) \subset \mathscr{L}_m$ be the hypersurface defined by $s'_0 = s$. For $m \geq 1$ we write $\int_{\mathscr{L}_m(s)} dL$ for integration with respect to Lebesgue measure on $\mathscr{L}_m(s)$. Then, by (5.4), we have

$$\int_{\mathscr{L}_m} dL = \int_{[0,\infty)} ds \int_{\mathscr{L}_m(s)} dL. \tag{6.2}$$

For $m = 0$, since $\mathscr{L}_0(s)$ consists of the single point $s'_0 = s$ we let dL in the inner integral denote a unit Dirac mass at s.

In the following assumptions, and hereafter, we write Z in place of $Z^{(\Lambda)}$ when there is no ambiguity.

Assumptions 6.1. For all $a \in \Lambda$,

- (Z1) $\boldsymbol{t} \mapsto Z_{\boldsymbol{t}}$ is strictly positive and continuous on $[0,\infty)^{\Lambda}$.
- (Z2) $\boldsymbol{t} \mapsto Z_{\boldsymbol{t}}$ is $\mathscr{C}^2$ on $[0,\infty)^{\Lambda}$.
- (G0) $\int_{[0,\infty)} d\ell\, E_a \left[\mathscr{Y}^{(\Lambda)}_{0,\ell}\right] < \infty$.
- (F0) $\sum_{m\geq 0} \int_{\mathscr{L}_m(0)} dL\, E_a \left[|w^{(\Lambda)}(L)|\,\right] < \infty$.

In (Z2), $\mathscr{C}^2$ at the boundary means that the derivatives have continuous extensions to the boundary. Assumption (F0) enables us to define $\Pi^{(\Lambda)}\colon \Lambda \times \Lambda \to \mathbb{R}$ by

$$\Pi^{(\Lambda)}(x,y) = \sum_{m\geq 0} \int_{\mathscr{L}_m(0)} dL\, E_x \left[w^{(\Lambda)}(L) \mathbb{1}_{\{X^{(\Lambda)}_{s'_m} = y\}}\right]. \tag{6.3}$$

and (G0) is equivalent to the Greens function $G^{(\Lambda)}(x)$ being finite for all $x \in \Lambda$.

Proposition 6.2. *Under Assumptions 6.1,*

$$G^{(\Lambda)}_{\mathbf{0}}(a,b) = S^{(\Lambda)}(a,b) + \sum_{x,y\in\Lambda} S^{(\Lambda)}(a,x)\Pi^{(\Lambda)}(x,y)G^{(\Lambda)}_{\mathbf{0}}(y,b). \tag{6.4}$$

In the literature of theoretical physics this relation between $G^{(\Lambda)}_{\mathbf{0}}$, $S^{(\Lambda)}$ and $\Pi^{(\Lambda)}$ is called a Dyson equation [15, Equation (92)]. The proof of the above proposition occupies the rest of this section.

6.1 Derivatives of local time

For each $x \in \Lambda$, (2.5) defines a local time $\boldsymbol{\tau}_{[s,s'],x}$ that is absolutely continuous in s for $s \leq s'$ with s' fixed, and similarly is absolutely continuous in s' for fixed s when s' is restricted to a bounded interval. Note that

$$\partial_2 \tau_{[s,s'],x} = \partial_2 \int_{[s,s']} \mathbb{1}_{\{X_r^{(\Lambda)}=x\}}\, dr = \mathbb{1}_{\{X_{s'}^{(\Lambda)}=x\}}, \tag{6.5}$$

$$\partial_1 \tau_{[s,s'],x} = -\mathbb{1}_{\{X_s^{(\Lambda)}=x\}}, \tag{6.6}$$

$$\partial_2 \partial_1 \tau_{[s,s'],x} = \partial_1 \partial_2 \tau_{[s,s'],x} = 0. \tag{6.7}$$

The first equation holds a.e. in s' for $s \leq s'$. The derivative does not depend on s, so it is absolutely continuous in s. By similar reasoning (6.6) holds a.e. in s for $s \leq s'$, and as a consequence (6.7) holds a.e. in $\{s \leq s'\}$.

6.2 Proof of Proposition 6.2

Lemma 6.3. *If $Z_{\boldsymbol{t}}$ satisfies (Z1) and (Z2) of Assumptions 6.1 then $w^{(\Lambda)}(L)$ is well-defined and $\mathscr{Y}_{s,t}^{(\Lambda)}$ defined by* (6.1) *satisfies the hypotheses of Theorem 5.2.*

Proof. The hypothesis that $\mathscr{Y}_{s,s}^{(\Lambda)} = 1$ holds as $\boldsymbol{\tau}_{[s,s]} = \mathbf{0}$. By (Z2) and the compactness of $[0,\ell]^\Lambda$ the function $Z_{\boldsymbol{t}}$ is Lipschitz in $\boldsymbol{t}$. For each s, $\boldsymbol{\tau}_{[s,t]}$ is absolutely continuous as a function of t when t is restricted to a bounded interval, and vice-versa by Section 6.1. By Lemma 5.3 this implies that for each s, $Z_{\boldsymbol{\tau}_{[s,t]}}$ is absolutely continuous as a function of t when t is restricted to a bounded interval and vice-versa. Combined with (Z1) this proves the first of Assumptions 5.1. Furthermore, by the chain rule, the composition $Z_{\boldsymbol{\tau}_{[s,t]}}$ is differentiable in t at points (s,t) where $\boldsymbol{\tau}_{[s,t]}$ has this property. Therefore for each s, $Z_{\boldsymbol{\tau}_{[s,t]}}$ is differentiable in t at all but a countable number of points (recall that simple random walk takes only finitely many jumps in any finite time interval), and hence is absolutely continuous in t, and vice-versa. This verifies the second item of Assumptions 5.1, and an analogous argument verifies the third item. The fourth follows by (Z2) and (6.7).

Lastly, we must prove that $s,t \mapsto r_{s,t}$ is a.e. bounded on $\mathscr{D}_\ell$. By (5.2) we must show that $\partial_1\partial_2 \log \mathscr{Y}_{s,t}$ is a.e. bounded on $\mathscr{D}_\ell$. By (Z2), $\boldsymbol{u} \mapsto Z_{\boldsymbol{u}}$ is $\mathscr{C}^2$ on $[0,\ell]^\Lambda$. By (Z1) and the compactness of $\mathscr{D}_\ell$ the range of $\boldsymbol{u} \mapsto Z_{\boldsymbol{u}}$ is bounded away from zero. Therefore $F\colon \boldsymbol{u} \mapsto \log Z_{\boldsymbol{u}}$ is $\mathscr{C}^2$ on $[0,\ell]^\Lambda$ and $\log \mathscr{Y}_{s,t} = F(\boldsymbol{\tau}_{[s,t]})$. Let $F_x^{(1)}(u)$ be the partial derivative of $F(\boldsymbol{u})$ with respect to u_x and let $F_{xy}^{(2)}(u)$ be the second partial derivative of $F(\boldsymbol{u})$ with respect to u_x and u_y. By the chain rule and (6.5) $\partial_1\partial_2 \log \mathscr{Y}_{s,t} = \partial_1\big(\sum_x F_x^{(1)}(\boldsymbol{\tau}_{[s,t]})\mathbb{1}_{\{X_t=x\}}\big)$. The sum over $x \in \Lambda$ is finite and $\mathbb{1}_{\{X_t=x\}}$ does not depend on s so it is sufficient to prove that $\partial_1 F_x^{(1)}(\boldsymbol{\tau}_{[s,t]})$ is a.e. bounded on $\mathscr{D}_\ell$. By the chain rule and (6.6) $\partial_1 F_x^{(1)}(\boldsymbol{\tau}_{[s,t]})$ is a finite sum over y of $F_{xy}^{(2)}(\boldsymbol{\tau}_{[s,t]})\mathbb{1}_{\{X_s=y\}}$. Therefore it is sufficient to prove that $F_{xy}^{(2)}(\boldsymbol{\tau}_{[s,t]})$ is bounded. By (Z2) $F_{xy}^{(2)}$ is continuous on $[0,\ell]^\Lambda$ and by Section 6.1 $\boldsymbol{\tau}_{[s,t]}$ is jointly continuous on

$\mathscr{D}_\ell$. Therefore the composition $F^{(2)}_{xy}(\boldsymbol{\tau}_{[s,t]})$ is continuous on the compact set $\mathscr{D}_\ell$ and hence bounded as desired. ■

Proof of Proposition 6.2. We omit the superscript Λ on Green's functions, etc., since Λ is fixed. At two points in the proof we will use the Markov property; the justifications for these applications are given in Appendix A.2.

By definition (3.2) and (6.1),

$$G_{\mathbf{0}}(a,b) = \int_{[0,\infty)} d\ell \, E_a \left[\mathscr{Y}_{0,\ell} \mathbb{1}_{\{X_\ell = b\}} \right]. \tag{6.8}$$

Lemma 6.3 implies we can expand $\mathscr{Y}_{0,\ell}$ by Theorem 5.2. Using the definition (2.6) of $S(a,b)$ this yields

$$G_{\mathbf{0}}(a,b) = S(a,b) + \int_{[0,\infty)} d\ell \, E_a \Big[\sum_{m \geq 0} \int_{\mathscr{L}_{m,\ell}} dL \, w(L) \, \mathscr{Y}_{s'_m,\ell} \, \mathbb{1}_{\{X_\ell = b\}} \Big], \tag{6.9}$$

where s'_m is defined by the lace L as explained in (5.4). For convenience, define $U_{a,b} := G_{\mathbf{0}}(a,b) - S(a,b)$. Using (6.2) and (F0) we obtain

$$U_{a,b} = \int_{[0,\infty)} ds \sum_{m \geq 0} \int_{\mathscr{L}_m(s)} dL \int_{[s'_m,\infty)} d\ell \, E_a \left[w(L) \, \mathscr{Y}_{s'_m,\ell} \, \mathbb{1}_{\{X_\ell = b\}} \right]. \tag{6.10}$$

By the change of variable $\ell \mapsto s'_m + \ell$ in the integral with respect to ℓ

$$U_{a,b} = \int_{[0,\infty)} ds \sum_{m \geq 0} \int_{\mathscr{L}_m(s)} dL \int_{[0,\infty)} d\ell \, E_a \left[w(L) \, \mathscr{Y}_{s'_m,s'_m+\ell} \, \mathbb{1}_{\{X_{s'_m+\ell} = b\}} \right]. \tag{6.11}$$

For $\ell > 0$, let $h(\ell,y,b) := E_y \left[\mathscr{Y}_{0,\ell} \, \mathbb{1}_{\{X_\ell = b\}} \right]$. By conditioning on $\mathscr{F}_{s'_m}$ in the last expectation in (6.11), using $w(L) \in \mathscr{F}_{s'_m}$, integrability by (G0), and the Markov property for $E_a \left[\mathscr{Y}_{s'_m,s'_m+\ell} \, \mathbb{1}_{\{X_{s'_m+\ell} = b\}} \middle| \mathscr{F}_{s'_m} \right]$,

$$U_{a,b} = \int_{[0,\infty)} ds \sum_{m \geq 0} \int_{\mathscr{L}_m(s)} dL \int_{[0,\infty)} d\ell \, E_a \left[w(L) \, h(\ell, X_{s'_m}, b) \right]. \tag{6.12}$$

By (G0) and (F0) the right-hand side converges absolutely and likewise for the following equations. We bring the integral with respect to ℓ inside the expectation and rewrite $\int_{[0,\infty)} d\ell \, h(\ell, X_{s'_m}, b)$ using the definition (6.8) of $G_{\mathbf{0}}(a,b)$:

$$U_{a,b} = \int_{[0,\infty)} ds \sum_{m \geq 0} \int_{\mathscr{L}_m(s)} dL \, E_a \left[w(L) \, G_{\mathbf{0}}(X_{s'_m}, b) \right]. \tag{6.13}$$

By changing variables in the integral over $\mathscr{L}_m(s)$ so that it becomes an integral over $\mathscr{L}_m(0)$ we rewrite this as

$$U_{a,b} = \int_{[0,\infty)} ds \sum_{m \geq 0} \int_{\mathscr{L}_m(0)} dL \, E_a \left[w(L+s) \, G_{\mathbf{0}}(X_{s'_m+s}, b) \right], \tag{6.14}$$

where for $L \in \mathscr{L}_m(0)$, $L + s$ is defined to be the lace in $\mathscr{L}_m(s)$ obtained from L by adding s to each s_i, s'_i. For $L \in \mathscr{L}_m(0)$ define $f(x,b,L) := E_x \Big[w(L) \, G_{\mathbf{0}}(X_{s'_m}, b) \Big]$.

By conditioning on $\mathscr{F}_s$ inside the expectation in (6.14) and applying the Markov property to $E_a\left[w(L+s)\, G_{\mathbf{0}}(X_{s'_m+s},b)\middle|\mathscr{F}_s\right]$ we obtain

$$U_{a,b} = \int_{[0,\infty)} ds \sum_{m\geq 0} \int_{\mathscr{L}_m(0)} dL\, E_a\left[f(X_s,b,L)\right]. \tag{6.15}$$

The expectation is equal to

$$\begin{aligned}
&\sum_{x,y\in\Lambda} E_a\left[E_{X_s}\left[w(L)G_{\mathbf{0}}(X_{s'_m},b)\mathbb{1}_{\{X_{s'_m}=y\}}\right]\mathbb{1}_{\{X_s=x\}}\right]\\
&\qquad = \sum_{x,y\in\Lambda} G_{\mathbf{0}}(y,b)\, E_a\left[\mathbb{1}_{\{X_s=x\}}\right] E_x[w(L)\mathbb{1}_{\{X_{s'_m}=y\}}],
\end{aligned} \tag{6.16}$$

where we have used the fact that the sums over $x,y\in\Lambda$ are finite to take them outside the expectation in the first line. Recalling the definition (6.3) we see that (6.15) can be written as

$$U_{a,b} = \int_{[0,\infty)} ds \sum_{x,y\in\Lambda} G_{\mathbf{0}}(y,b)\, E_a\left[\mathbb{1}_{\{X_s=x\}}\right]\Pi(x,y). \tag{6.17}$$

By the definitions of $S(x,y)$ and $U_{a,b}$ this is the same as

$$G_{\mathbf{0}}(a,b) = S(a,b) + \sum_{x,y\in\Lambda} S(a,x)\Pi(x,y)G_{\mathbf{0}}(y,b). \qquad \blacksquare$$

7 The terms $\Pi_m^{(\Lambda)}$ of the lace expansion

Throughout this section $\Lambda\subset\mathbb{Z}^d$ is a fixed finite set. Recall the definitions below (6.1) and define

$$\Pi_m^{(\Lambda)}(x,y) := \int_{\mathscr{L}_m(0)} dL\, E_x\left[w^{(\Lambda)}(L)\mathbb{1}_{\{X^{(\Lambda)}_{s'_m}=y\}}\right], \qquad m\geq 0. \tag{7.1}$$

Thus $\Pi_m^{(\Lambda)}$ is the m^{th} term in the series (6.3) that defines $\Pi^{(\Lambda)}(x,y)$. This section has two parts. The first provides formulas for the weights $w^{(\Lambda)}$, and the second derives bounds on $\Pi_m^{(\Lambda)}$ for $m\geq 1$. The main result is Proposition 7.4, which bounds $\Pi_m^{(\Lambda)}$ in terms of $G_{\mathbf{0}}^{(\Lambda)}$; these bounds are used in implementing step one of Section 4.1.

7.1 Formulas for weights

We give formulas for $\Pi_0^{(\Lambda)}$ and the factors $r^{(\Lambda)}$ that enter into $w^{(\Lambda)}$. Both computations are applications of the chain rule to our choice (6.1) of $\mathscr{Y}$ together with the formulas (6.5)–(6.7) for derivatives of the local time. The formulas of this section are valid under Assumptions 6.1.

The term $\Pi_0^{(\Lambda)}$

Recall the definition (4.4) of $L_{g,\nu,x}^{(\Lambda)}$ where $Z^{(\Lambda)}$ is now the function entering in the definition (6.1) of $\mathscr{Y}$. At this level of generality $Z^{(\Lambda)}$ need not depend on g,ν, but we retain them in our notation. The limit $L_{g,\nu,x}^{(\Lambda)}$ exists and is finite by (Z1) and (Z2); see below (5.2). The next result shows that $L_{g,\nu,x}^{(\Lambda)}$ is essentially the first (zeroth) term in the finite volume lace expansion.

Lemma 7.1. *For all finite* Λ*,* $\Pi_0^{(\Lambda)}(x,y) = L_{g,\nu,x}^{(\Lambda)} \mathbb{1}_{\{x=y\}}$.

Proof. From (7.1),

$$\Pi_0^{(\Lambda)}(x,y) = \int_{\mathscr{L}_0(0)} dL E_x \left[r_{s_0}^{(\Lambda)} \mathbb{1}_{\{X_{s_0'}^{(\Lambda)}=y\}} \right]. \tag{7.2}$$

By definition, dL for $L \in \mathscr{L}_0(0)$ is a unit mass at $s_0 = 0$, and by definition $s_0' = s_0$. Moreover, $X_0^{(\Lambda)} = x$ under the measure E_x. Hence (7.2) becomes

$$\Pi_0^{(\Lambda)}(x,y) = \mathbb{1}_{\{x=y\}} E_x \left[r_0^{(\Lambda)} \right] \tag{7.3}$$

By the definition (5.2) of $r_0^{(\Lambda)}$, (6.1), and $\partial_{t_x} Z_{\mathbf{0}}^{(\Lambda)} = 0$,

$$r_0^{(\Lambda)} = -\lim_{s'\downarrow 0} \sum_{z\in\Lambda} \partial_{t_z} \log Z_{\boldsymbol{t}}^{(\Lambda)}\big|_{\boldsymbol{t}=\boldsymbol{\tau}_{[0,s']}^{(\Lambda)}} \partial_1 \tau_{[0,s'],z}^{(\Lambda)} \tag{7.4}$$

$$= \sum_{z\in\Lambda} \partial_{t_z} \log Z_{\boldsymbol{t}}^{(\Lambda)}\Big|_{\boldsymbol{t}=\mathbf{0}} \mathbb{1}_{\{X_0^{(\Lambda)}=z\}}, \qquad \text{a.s.} \tag{7.5}$$

In obtaining (7.5) we used (6.6) and the right-continuity of the random walk. Since $X_0^{(\Lambda)} = x$ a.s. under E_x, the lemma follows by inserting the definition (4.4) of $L_{g,\nu,x}^{(\Lambda)}$. ■

The vertex weight $r^{(\Lambda)}$

Recall from (5.2) that $r_{u,v}^{(\Lambda)} = r_{u,v} := -\partial_1\partial_2 \log \mathscr{Y}_{u,v}$. Define, for $x,y \in \Lambda$ and $u < v$,

$$r_{u,v}^{(\Lambda)}(x,y) := \partial_{t_x}\partial_{t_y} \log Z_{\boldsymbol{t}}^{(\Lambda)}\big|_{\boldsymbol{t}=\boldsymbol{\tau}_{[u,v]}^{(\Lambda)}}. \tag{7.6}$$

Lemma 7.2. *For all finite* Λ *and all* $u < v$*,*

$$r_{u,v}^{(\Lambda)} = \sum_{x,y\in\Lambda} r_{u,v}^{(\Lambda)}(x,y) \mathbb{1}_{\{X_u^{(\Lambda)}=x\}} \mathbb{1}_{\{X_v^{(\Lambda)}=y\}}. \tag{7.7}$$

Proof. This follows from a calculation similar to the proof of Lemma 7.1. ■

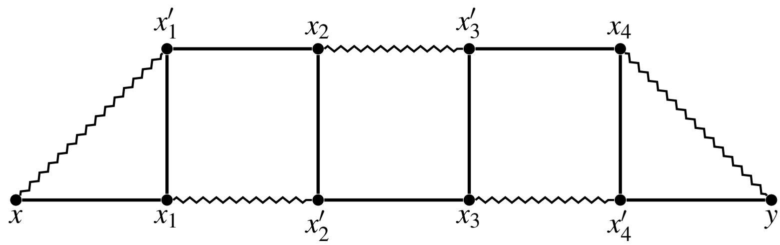


Figure 7.1. The upper bound on $\Pi_5^{(\Lambda)}(x,y)$ from Proposition 7.4. All vertices except x and y are summed over Λ. Lines connecting vertices represent functions: wavy lines represent $\bar{r}^{(\Lambda)}$ and straight lines represent $G_{\mathbf{0}}^{(\Lambda)}$.

7.2 Bounds on $\Pi_m^{(\Lambda)}$, $m \geq 1$

Our bounds on $\Pi_m^{(\Lambda)}$ for $m \geq 1$ will rely on two assumptions. Recall the definition (7.6) of $r_{u,v}^{(\Lambda)}(x,y)$.

Assumptions 7.3.

- (G1) For all $\boldsymbol{t} \in [0,\infty)^\Lambda$, $G_{\boldsymbol{t}}^{(\Lambda)} \leq G_{\mathbf{0}}^{(\Lambda)}$.
- (R0) There exists $\bar{r}^{(\Lambda)}\colon \Lambda \times \Lambda \to \mathbb{R}$ such that $\left|r_{u,v}^{(\Lambda)}(x,y)\right| \leq \bar{r}^{(\Lambda)}(x,y)$ for all $x,y \in \Lambda$ and $0 \leq u < v < \infty$.

Given vertices x and y in Λ, and $m \geq 1$, define

$$\Lambda_{x,y}^{2m-2} := \{((x_i,x_i'))_{i=0,\dots,m} \in \Lambda^{2(m+1)} \,|\, x_0 = x_0' = x \text{ and } x_m = x_m' = y\}. \tag{7.8}$$

Generic elements of $\Lambda_{x,y}^{2m-2}$ will be denoted by $\boldsymbol{x} = ((x_i,x_i'))_{i=0,\dots,m}$.

Proposition 7.4. *Suppose (Z1)–(Z2), (G0), and Assumptions 7.3 hold. For $m \geq 1$ and $x,y \in \Lambda$,*

$$\begin{aligned}\left|\Pi_m^{(\Lambda)}(x,y)\right| \leq \sum_{\boldsymbol{x} \in \Lambda_{x,y}^{2m-2}} & G_{\mathbf{0}}^{(\Lambda)}(x,x_1)\bar{r}^{(\Lambda)}(x,x_1') \\ & \times \prod_{j=1}^{m-1} G_{\mathbf{0}}^{(\Lambda)}(x_j,x_j')G_{\mathbf{0}}^{(\Lambda)}(x_j',x_{j+1})\bar{r}^{(\Lambda)}(x_j,x_{j+1}'). \end{aligned} \tag{7.9}$$

See Figure 7.1 for a diagrammatic representation of the upper bound, which explains our use of the term *lace expansion*: the upper bound is of exactly the form that occurs in discrete-time lace expansion analyses of self-avoiding walk [11, 44]. In more detail, for the Edwards model, a computation (see (11.1)) shows that we can choose $\bar{r}^{(\Lambda)}$ to be a constant times $\mathbb{1}_{\{x=y\}}$. This amounts to shrinking the wavy edges in Figure 7.1 to points, and these are the diagrams occurring in [11, 44].

The next two subsections prove Proposition 7.4. As Λ is fixed it will be omitted from the notation.

A preparatory lemma

Recall the definition (6.1) of $\mathscr{Y}_{s,t}$ and define

$$\bar{\mathscr{Y}}_{u,v}(w) := \frac{\mathscr{Y}_{u,w}}{\mathscr{Y}_{u,v}}, \qquad u \leq v \leq w. \tag{7.10}$$

By the definitions (5.7) and (5.8) of $w(L)$ and $P(L)$ for $L \in \mathscr{L}_m$ with $m \geq 1$,

$$\begin{aligned} w(L) &= r(L)P(L), \\ P(L) &= \prod_{i=-1}^{m-2} \bar{\mathscr{Y}}_{s'_i, s'_{i+1}}(s'_{i+2}), \qquad s'_{-1} := s'_0. \end{aligned} \tag{7.11}$$

The term $\bar{\mathscr{Y}}_{s'_{-1},s'_0}(s'_1)$ in the product is the factor $\mathscr{Y}_{s'_0,s'_1}$ in (5.8) (recall that $\mathscr{Y}_{s'_0,s'_0} = 1$). For $k = 1,2$ define $P_{-k}(L)$ by replacing the upper limit $m-2$ in (7.11) by $m-2-k$. By convention empty products are defined to be one.

Lemma 7.5. *Let $0 \leq u_1 \leq u_2 \leq u_3$, and let $H \geq 0$ be $\mathscr{F}_{u_3}$-measurable. Then almost surely*

$$\int_{[u_3,\infty)} d\ell\, E_x\left[H\, \bar{\mathscr{Y}}_{u_1,u_2}(\ell)\, \mathbb{1}_{\{X_\ell = y\}} \,\middle|\, \mathscr{F}_{u_3}\right] = H\, \bar{\mathscr{Y}}_{u_1,u_2}(u_3)\, G_{\boldsymbol{\tau}_{[u_1,u_3]}}(X_{u_3}, y) \tag{7.12}$$

Proof. By the definition (7.10) of $\bar{\mathscr{Y}}_{u_1,u_2}(u_3)$

$$\bar{\mathscr{Y}}_{u_1,u_2}(\ell) = \frac{\mathscr{Y}_{u_1,\ell}}{\mathscr{Y}_{u_1,u_2}} = \frac{\mathscr{Y}_{u_1,u_3}}{\mathscr{Y}_{u_1,u_2}} \frac{\mathscr{Y}_{u_1,\ell}}{\mathscr{Y}_{u_1,u_3}} = \bar{\mathscr{Y}}_{u_1,u_2}(u_3) \bar{\mathscr{Y}}_{u_1,u_3}(\ell). \tag{7.13}$$

Insert (7.13) into the left-hand side of (7.12). Using the nonnegativity of H, $\bar{\mathscr{Y}}_{u_1,u_2}$ and $\bar{\mathscr{Y}}_{u_1,u_3}$ we take the $\mathscr{F}_{u_3}$-measurable factor $H\,\bar{\mathscr{Y}}_{u_1,u_2}(u_3)$ outside the conditional expectation and the integral of what remains is

$$\int_{[u_3,\infty)} d\ell\, E_a\left[\bar{\mathscr{Y}}_{u_1,u_3}(\ell)\, \mathbb{1}_{\{X_\ell = y\}} \middle| \mathscr{F}_{u_3}\right] = G_{\boldsymbol{\tau}_{[u_1,u_3]}}(X_{u_3}, y) \qquad \text{a.s.,} \tag{7.14}$$

by the Markov property as stated in Lemma A.2. ■

Proof of Proposition 7.4

Before giving the proof of Proposition 7.4, we recall the following consequence of the Fubini–Tonelli theorem that will be used in the proof. If X_u is a real-valued stochastic process satisfying $E[\int_{[a,b]} du\, |X_u|] = \int_{[a,b]} du\, E[|X_u|] < \infty$, then integration and conditional expectation can be interchanged:

$$\int_{[a,b]} du\, E\left[X_u \mid \mathscr{G}\right] = E\left[\int_{[a,b]} du X_u \mid \mathscr{G}\right], \quad \text{a.s.} \tag{7.15}$$

Proof of Proposition 7.4. Let $L \in \mathscr{L}_m(0)$. Given a sequence $\boldsymbol{x} \in \Lambda_{x,y}^{2m-2}$ as in (7.8) and a time $u \in [0,\infty)$ define the indicator function

$$\mathscr{I}_{L,\boldsymbol{x},u} := \prod_{j: s_j \leq u} \mathbb{1}_{\{X_{s_j} = x_j\}} \prod_{j': s'_{j'+1} \leq u} \mathbb{1}_{\{X_{s'_{j'+1}} = x'_{j'+1}\}} \tag{7.16}$$

of the event that the path X is at the points (x_i, x'_{i+1}) at the times (s_i, s'_{i+1}) up to u in $L = ((s_i, s'_{i+1}))_{i=0,\dots,m-1}$. See Figures 5.1 and 7.1 and think of the solid lines in the latter figure as a representation of paths X with $X_0 = x$, $X_{s_1} = x_1$, $X_{s'_1} = x'_1$, etc. For $y \in \Lambda$ we have

$$\mathbb{1}_{\{X_0=x\}}\mathbb{1}_{\{X_{s'_m}=y\}} = \sum_{\boldsymbol{x}\in\Lambda^{2m-2}_{x,y}} \mathscr{I}_{L,\boldsymbol{x},s'_m} \tag{7.17}$$

since X cannot be at the absorbing state $*$ at times earlier than s'_m on the event $\{X_{s'_m} = y\}$. We have also used that $\{X_0 = x\} = \{X_{s_0} = x\}$ since $L \in \mathscr{L}_m(0)$. Define

$$\Gamma_{m,\boldsymbol{x}} := \int_{\mathscr{L}_m(0)} dL\, E_x\Big[\mathscr{I}_{L,\boldsymbol{x},s'_m} P(L)\Big]. \tag{7.18}$$

Since $\mathbb{1}_{\{X_0=x\}} = 1$ a.s. under E_x, we can insert (7.17) into the definition (7.1) of Π_m to obtain

$$|\Pi_m(x,y)| = \Big|\sum_{\boldsymbol{x}\in\Lambda^{2m-2}_{x,y}} \int_{\mathscr{L}_m(0)} dL\, E_x\Big[\mathscr{I}_{L,\boldsymbol{x},s'_m} r(L)P(L)\Big]\Big| \tag{7.19}$$

$$\leq \sum_{\boldsymbol{x}\in\Lambda^{2m-2}_{x,y}} \prod_{j=0}^{m-1} \bar{r}(x_j, x'_{j+1})\Gamma_{m,\boldsymbol{x}}, \tag{7.20}$$

where we have used the triangle inequality and (R0) to bound the vertex functions in $r(L)$, and $P(L) > 0$ to remove absolute values. This reduces Proposition 7.4 to proving

$$\Gamma_{m,\boldsymbol{x}} \leq G_{\mathbf{0}}(x_0,x_1)\prod_{j=1}^{m-1} G_{\mathbf{0}}(x_j,x'_j)G_{\mathbf{0}}(x'_j,x_{j+1}), \tag{7.21}$$

for $m \geq 1$ and $\boldsymbol{x} \in \Lambda^{2m-2}_{x,y}$, which we will do by induction on m. The base case $m = 1$ follows by noting that $\mathscr{I}_{L,\boldsymbol{x},s'_1} = \mathbb{1}_{\{X_{s'_1}=y\}}$ under E_x and recalling that $\int_{\mathscr{L}_1(0)} dL = \int_0^\infty ds'_1$, so $\Gamma_{1,\boldsymbol{x}} = G_{\mathbf{0}}(x,y)$.

Suppose (7.21) holds when $m = n$ for some $n \geq 1$. By (5.4) with $s'_0 = 0$, for $L \in \mathscr{L}_{n+1}(0)$ the measure dL factorizes as $dL'\, ds'_n\, ds_{n+1}$, where dL' is Lebesgue measure on

$$\mathscr{L}'_n(0) = \{(s_1, s'_1, s_2, \dots, s_{n-1}, s'_{n-1}, s_n) \mid 0 < s_1 < s'_1 < \cdots < s'_{n-1} < s_n\} \tag{7.22}$$

and $ds'_n\, ds_{n+1}$ is Lebesgue measure on the set of (s'_n, s_{n+1}) such that $s_n < s'_n < s_{n+1}$. Rewriting $\Gamma_{n+1,\boldsymbol{x}}$ using this factorization yields

$$\Gamma_{n+1,\boldsymbol{x}} = \int_{\mathscr{L}'_n(0)} dL' \int_{[s_n,\infty)} ds'_n \int_{[s'_n,\infty)} ds_{n+1}\, E_x\Big[\mathscr{I}_{L,\boldsymbol{x},s_{n+1}} P(L)\Big]. \tag{7.23}$$

The induction step involves estimating the integrals over s_{n+1} and s'_n by Lemma 7.5 and (G1). To bound the s_{n+1} integral note the range of integration starts at s'_n and accordingly insert a conditional expectation with respect to $\mathscr{F}_{s'_n}$ under the expectation E_x. Bringing the s_{n+1} integral inside the expectation yields

$$\Gamma_{n+1,\boldsymbol{x}} = \int_{\mathscr{L}'_n(0)} dL' \int_{[s_n,\infty)} ds'_n \, E_x \Big[J_{L,\boldsymbol{x},s'_n} \Big], \quad \text{where} \tag{7.24}$$

$$J_{L,\boldsymbol{x},s'_n} := \int_{[s'_n,\infty)} ds_{n+1} \, E_x \Big[\mathscr{I}_{L,\boldsymbol{x},s_{n+1}} P(L) \Big| \mathscr{F}_{s'_n} \Big]. \tag{7.25}$$

Recall that $P_{-1}(L)$ was defined below (7.11), and note that

$$\mathscr{I}_{L,\boldsymbol{x},s_{n+1}} = \mathscr{I}_{L,\boldsymbol{x},s'_n} \mathbb{1}_{\{X_{s_{n+1}} = x_{n+1}\}}, \qquad P(L) = P_{-1}(L) \bar{\mathscr{Y}}_{s'_{n-1},s'_n}(s_{n+1}). \tag{7.26}$$

We insert these identities into $J_{L,\boldsymbol{x},s'_n}$ and apply Lemma 7.5 with $(u_1,u_2,u_3) = (s'_{n-1}, s'_n, s'_n)$ and $H = \mathscr{I}_{L,\boldsymbol{x},s'_n} P_{-1}(L)$. Since $H \geq 0$, after using (G1) with $\boldsymbol{t} = \boldsymbol{\tau}_{[s'_{n-1},s'_n]}$ we obtain

$$J_{L,\boldsymbol{x},s'_n} \leq \mathscr{I}_{L,\boldsymbol{x},s'_n} P_{-1}(L) \, G_{\mathbf{0}}(x'_n, x_{n+1}) \tag{7.27}$$

because $\bar{\mathscr{Y}}_{s'_{n-1},s'_n}(s'_n) = 1$. Hence by (7.24) and that $x'_{n+1} = x_{n+1}$ by (7.8),

$$\Gamma_{n+1,\boldsymbol{x}} \leq \int_{\mathscr{L}'_n(0)} dL' \int_{[s_n,\infty)} ds'_n \, E_x \Big[\mathscr{I}_{L,\boldsymbol{x},s'_n} P_{-1}(L) \Big] \, G_{\mathbf{0}}(x'_n, x_{n+1}). \tag{7.28}$$

For the s'_n integral in (7.28) the procedure is similar so we will be brief. Insert a conditional expectation with respect to $\mathscr{F}_{s_n}$ under the expectation in (7.28), bring the integral over s'_n inside E_x, and then insert

$$\mathscr{I}_{L,\boldsymbol{x},s'_n} = \mathscr{I}_{L,\boldsymbol{x},s_n} \mathbb{1}_{\{X_{s'_n} = x'_n\}}, \qquad P_{-1}(L) = P_{-2}(L) \bar{\mathscr{Y}}_{s'_{n-2},s'_{n-1}}(s'_n). \tag{7.29}$$

We apply Lemma 7.5 with $(u_1,u_2,u_3) = (s'_{n-2}, s'_{n-1}, s_n)$ and $H = \mathscr{I}_{L,\boldsymbol{x},s_n}$, with the result, again after using (G1),

$$\begin{aligned} \Gamma_{n+1,\boldsymbol{x}} \leq \int_{\mathscr{L}'_n(0)} dL' \, E_x \Big[\mathscr{I}_{L,\boldsymbol{x},s_n} P_{-2}(L) \, \bar{\mathscr{Y}}_{s'_{n-2},s'_{n-1}}(s_n) \Big] \\ \times G_{\mathbf{0}}(x_n, x'_n) G_{\mathbf{0}}(x'_n, x_{n+1}). \end{aligned} \tag{7.30}$$

By (7.11) $P_{-2}(L) \bar{\mathscr{Y}}_{s'_{n-2},s'_{n-1}}(s_n)$ equals $P(L') \mid_{s'_n = s_n, x'_n = x_n}$. By (7.22) the measure spaces $(\mathscr{L}'_n(0), dL')$ and $(\mathscr{L}_n(0), dL)$ are the same. Therefore

$$\Gamma_{n+1,\boldsymbol{x}} \leq \left(\int_{\mathscr{L}_n(0)} dL E_x \Big[\mathscr{I}_{L,\boldsymbol{x},s_n} P(L) \Big]_{s'_n = s_n, x'_n = x_n} \right) G_{\mathbf{0}}(x_n, x'_n) G_{\mathbf{0}}(x'_n, x_{n+1}). \tag{7.31}$$

By (7.18) the quantity in brackets is $\Gamma_{m,\boldsymbol{x}}$ with $m = n$. Applying the inductive hypothesis (7.21) to this term completes the proof. ■

8 Preparation for the infinite volume limit

This section continues with step one of Section 4.1. The main result is Corollary 8.6, which is a bound on $\Psi^{(\Lambda)}$, which is the finite volume version of the term Ψ in (4.6). A crucial aspect of the bound given by Corollary 8.6 is that it is uniform

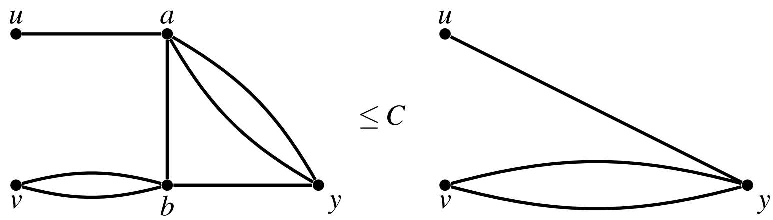


Figure 8.1. A diagrammatic depiction of Lemma 8.3. Solid lines represents factors $|||x_2 - x_1|||^{2-d}$. The vertices u, v, y are fixed, but a and b are summed over $\mathbb{Z}^d$.

in Λ. The bound relies on Assumptions 8.4, which play a continuing role in the remainder of the paper.

8.1 Convolution estimates

Recall that $|||x||| = \max\{|x|, 1\}$. The next lemma says that when the sum over $w \in \mathbb{Z}^d$ is sufficiently convergent there is a bound as if $w = 0$.

Lemma 8.1 (Equation (4.17) of [25])**.** *Let $d \geq 5$, $u, v \in \mathbb{Z}^d$. There exists a $C > 0$ such that*

$$\sum_{w \in \mathbb{Z}^d} |||w|||^{4-2d} |||w - v|||^{2-d} |||w - u|||^{2-d} \leq C |||v|||^{2-d} |||u|||^{2-d}.$$

The next estimate says the convolution of two functions decays according to whichever has the weaker decay, provided at least one of them is integrable.

Lemma 8.2 (Proposition 1.7(i) of [25])**.** *Let $f, g \colon \mathbb{Z}^d \to \mathbb{R}$ be such that $|f(x)| \leq |||x|||^{-a}$, $|g(x)| \leq |||x|||^{-b}$, $a \geq b > 0$. There exists a $C > 0$ such that*

$$|(f * g)(x)| \leq \begin{cases} C|||x|||^{-b}, & a > d, \\ C|||x|||^{d-(a+b)}, & a < d \text{ and } a + b > d. \end{cases}$$

Figure 8.1 gives a diagrammatic representation of the next lemma.

Lemma 8.3. *Fix $u, v, y \in \mathbb{Z}^d$, $d \geq 5$. There exists a $C > 0$ such that*

$$\sum_{a,b \in \mathbb{Z}^d} |||u - a|||^{2-d} |||y - a|||^{4-2d} |||b - a|||^{2-d} |||v - b|||^{4-2d} |||y - b|||^{2-d} \leq C |||y - u|||^{2-d} |||y - v|||^{4-2d}. \tag{8.1}$$

Proof. Lemma 8.1 can be applied to the sum over a to upper bound the left-hand side of (8.1) as if $a = y$, that is, by a constant $C > 0$ times

$$|||y - u|||^{2-d} \sum_{b \in \mathbb{Z}^d} |||y - b|||^{2-d} |||v - b|||^{4-2d} |||y - b|||^{2-d}.$$

The sum over b is a convolution of two functions that decay at rate $\gamma = 2d-4$. As γ exceeds d when $d \geq 5$, Lemma 8.2 implies the claimed upper bound. ■

8.2 $\Psi^{(\Lambda)}$ and uniform bounds on $\Psi^{(\Lambda)}$

Assumptions 8.4. For all $\Lambda \subset \mathbb{Z}^d$ finite,

(G2) For $\Lambda' \subset \Lambda$ and $x,y \in \Lambda'$, $G_{\mathbf{0}}^{(\Lambda')}(x,y) \leq G_{\mathbf{0}}^{(\Lambda)}(x,y)$;
(R1) There exists $\eta > 0$ independent of Λ, such that for $0 \leq u < v < \infty$

$$\left| r_{u,v}^{(\Lambda)}(x,y) \right| \leq \eta \left(\mathbb{1}_{\{x=y\}} + G_{\mathbf{0}}^{(\Lambda)}(x,y)^2 \right), \qquad x,y \in \Lambda. \tag{8.2}$$

The assumption (R1) supersedes Assumptions 7.3(R0) by stipulating the specific form

$$\bar{r}^{(\Lambda)} = \eta \left(\mathbb{1}_{\{x=y\}} + G_{\mathbf{0}}^{(\Lambda)}(x,y)^2 \right) \tag{8.3}$$

for the bound $\bar{r}^{(\Lambda)}$ of (R0). This form is motivated by our applications, as will become clear in Section 11. Note that (G2) implies $\lim_{\Lambda \uparrow \mathbb{Z}^d} G_{\mathbf{0}}^{(\Lambda)} = \sup_{\Lambda \uparrow \mathbb{Z}^d} G_{\mathbf{0}}^{(\Lambda)}$. The propositions of this section will be made under the assumption that this limit satisfies a K-IRB, i.e., that

$$G_{\mathbf{0}}^{(\infty)}(x,y) := \sup_{\Lambda \uparrow \mathbb{Z}^d} G_{\mathbf{0}}^{(\Lambda)}(x,y) \leq KS(y-x), \qquad x,y \in \mathbb{Z}^d. \tag{8.4}$$

The key aspect of the next proposition is that the bound is independent of Λ and proportional to $(c\eta)^m$. Recall that $\Pi_m^{(\Lambda)}$ is defined by (7.1).

Proposition 8.5. *Suppose $d \geq 5$, and that (Z1)–(Z2), (G1)–(G2), and (R1) hold, and that a K-IRB* (8.4) *holds. Then there are constants $c_1, c_2 > 0$ depending only on d, J and K such that for each $m \geq 1$*

$$\left| \Pi_m^{(\Lambda)}(x,y) \right| \leq c_1 (c_2 \eta)^m |||y-x|||^{-3(d-2)}. \tag{8.5}$$

Proof. The basic input in our estimates is that by (G1) and (G2), $G_{\boldsymbol{t}}^{(\Lambda)}(x,y) \leq G_{\mathbf{0}}^{(\infty)}(x,y)$, so the K-IRB and (4.9) imply

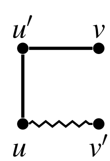
$$G_{\boldsymbol{t}}^{(\Lambda)}(x,y) \leq K\tilde{C}_J |||y-x|||^{2-d}, \tag{8.6}$$

and hence, letting $K_1 = \max\{K\tilde{C}_J, 1\}$, by (R1) in the form (8.3) and (8.6),

$$\bar{r}^{(\Lambda)}(x,y) \leq \eta \left(\mathbb{1}_{\{x=y\}} + K_1^2 \right) |||y-x|||^{4-2d}. \tag{8.7}$$

For $u, u', v, v' \in \mathbb{Z}^d$, define

$$A^{(\Lambda)}(u,u';v,v') := G_{\mathbf{0}}^{(\Lambda)}(u,u') \bar{r}^{(\Lambda)}(u,v') G_{\mathbf{0}}^{(\Lambda)}(u',v) =: \tag{8.8}$$



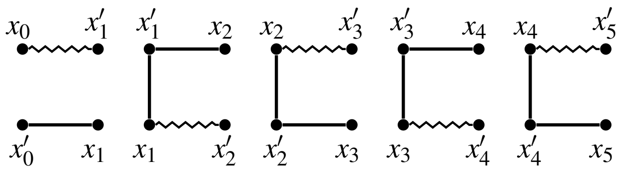
,

where the right-hand side follows the diagrammatic notation of Figure 7.1. Recall the notation $\boldsymbol{x}$ defined in (7.8), and note that (G2) combined with a K-IRB implies (G0) holds. Hence we can apply Proposition 7.4, and this proposition can be rewritten as

$$\left|\Pi_m^{(\Lambda)}(x,y)\right| \leq \sum_{\boldsymbol{x}\in\Lambda_{x,y}^{2m-2}} G_{\mathbf{0}}^{(\Lambda)}(x_0',x_1)\bar{r}^{(\Lambda)}(x_0,x_1')\prod_{j=1}^{m-1} A^{(\Lambda)}(x_j,x_j';x_{j+1},x_{j+1}'). \tag{8.9}$$

To check this claim compare Figure 7.1 with

(8.10)



and recall that $x_0 = x_0' = x$ and $x_m = x_m' = y$ and there is a sum over the remaining x_i, x_i'. To estimate the summands in (8.9) we introduce

$$\bar{A}(u,u';v,v') := \frac{K_1^2(1+K_1^2)}{|||u-u'|||^{d-2}|||u-v'|||^{2d-4}|||u'-v|||^{d-2}}. \tag{8.11}$$

Inserting the bounds (8.6) (with $\boldsymbol{t}=\mathbf{0}$) and (8.7) into $A^{(\Lambda)}(u,u';v,v')$ we obtain

$$A^{(\Lambda)}(u,u';v,v') \leq \eta\bar{A}(u,u';v,v'), \tag{8.12}$$

$$G_{\mathbf{0}}^{(\Lambda)}(x_0',x_1)\bar{r}^{(\Lambda)}(x_0,x_1') \leq \eta K_1^{-1}\bar{A}(x_0,x_0';x_1,x_1'), \tag{8.13}$$

where (8.13) holds because $x_0' = x_0$. Inserting (8.12)–(8.13) into (8.9) yields

$$\left|\Pi_m^{(\Lambda)}(x,y)\right| \leq K_1^{-1}\eta^m U_m(x,y) \tag{8.14}$$

where

$$U_m(x,y) := \sum_{\boldsymbol{x}\in\mathbb{Z}_{x,y}^{d(2m-2)}}\prod_{j=0}^{m-1}\bar{A}(x_j,x_j';x_{j+1},x_{j+1}'). \tag{8.15}$$

In obtaining (8.14) sums over vertices in Λ have been extended to sums over $\mathbb{Z}^d$, which gives an upper bound as all terms are non-negative. To prove (8.5) holds for $m \geq 1$ it therefore suffices to establish the upper bound

$$U_m(x,y) \leq c_1 c_2^m |||y-x|||^{-3(d-2)}. \tag{8.16}$$

We prove (8.16) with $c_1 = K_1^2(1+K_1^2)$ and $c_2 = \max\{1, C(K_1)\}$ (where C is a constant defined in (8.17) below) by induction. For $m=1$ there is no sum in (8.15), so the bound follows from $x_0 = x_0' = x$, $x_1 = x_1' = y$, and (8.11).

Suppose the upper bound has been established for some $n-1 \geq 1$. By multiplying both sides of Lemma 8.3 by $|||u-v|||^{2-d}$ and inserting (8.11) there is a

$C = C(K_1) > 0$ such that for $u, v, y \in \mathbb{Z}^d$,

$$\sum_{a,b\in\mathbb{Z}^d} \bar{A}(v,u;a,b)\bar{A}(a,b;y,y) \le C\bar{A}(v,u;y,y). \tag{8.17}$$

Let $m = n$. By using (8.17) to estimate the sum over x_{n-1}, x'_{n-1} in the definition of U_n and then using the induction hypotheses we have

$$U_n(x,y) = \sum_{\boldsymbol{x}\in\mathbb{Z}^{d(2n-2)}_{x,y}} \prod_{j=0}^{n-1} \bar{A}(x_j,x'_j;x_{j+1},x'_{j+1}) \tag{8.18}$$

$$\le C \sum_{\boldsymbol{x}\in\mathbb{Z}^{d(2n-4)}_{x,y}} \prod_{j=0}^{n-2} \bar{A}(x_j,x'_j;x_{j+1},x'_{j+1}) \tag{8.19}$$

$$\le c_1 c_2^n |||y-x|||^{-3(d-2)},$$

where in the second line we have redefined $x_{n-1} := x_n$ and $x'_{n-1} := x'_n$. The final line follows by recalling that $x_n = x'_n = y$. ■

Recall that $\Pi^{(\Lambda)} = \sum_{m\ge 0} \Pi^{(\Lambda)}_m$. Define $\Psi^{(\Lambda)}$, the finite-volume precursor to Ψ from Section 4.1, by

$$\Psi^{(\Lambda)}(x,y) := \sum_{m\ge 1} \Pi^{(\Lambda)}_m(x,y) = \Pi^{(\Lambda)}(x,y) - \Pi^{(\Lambda)}_0(x,y). \tag{8.20}$$

Summing (8.5) over $m \ge 1$ immediately gives the following.

Corollary 8.6. *Under the hypotheses of Proposition 8.5, if $c_2\eta < 1$ then*

$$|\Psi^{(\Lambda)}(x,y)| \le \frac{c_1 c_2 \eta}{1 - c_2\eta} |||y-x|||^{-3(d-2)}, \qquad x,y \in \mathbb{Z}^d. \tag{8.21}$$

9 The lace expansion in infinite volume

The main result of this section is Proposition 9.8, which constructs $L_{g,\nu}$ and $\Psi_{g,\nu}$ such that (4.6) holds. This completes a key part of step one of Section 4.1. The proof uses Corollary 8.6 and the algebraic structure of Proposition 6.2 to take the infinite volume limit of Proposition 6.2. In particular we prove the existence of the infinite volume limit $\Pi^{(\infty)}$ of $\Pi^{(\Lambda)}$.

9.1 The infinite volume limit of $G^{(\Lambda)}$

We begin by establishing some properties of $G^{(\infty)}_{\mathbf{0}}$ which was defined in (8.4).

Lemma 9.1. *If (Z1), (G2) and a K-IRB* (8.4) *holds then $G^{(\infty)}_{\mathbf{0}}(a,b)$ is non-negative and $\mathbb{Z}^d$-symmetric.*

Proof. Non-negativity is clear from (Z1) and the definition (3.2) of $G^{(\Lambda)}$.

By the K-IRB and monotone convergence provided by (G2) $\lim_{\Lambda\uparrow\mathbb{Z}^d} G^{(\Lambda)}_{\mathbf{0}}(a,b)$ exists for any choice of exhaustion $\Lambda_n \uparrow \mathbb{Z}^d$. Given $\Lambda_n \uparrow \mathbb{Z}^d$ and $\Lambda'_n \uparrow \mathbb{Z}^d$ there exists $\tilde{\Lambda}_n \uparrow \mathbb{Z}^d$ such that $\tilde{\Lambda}_{n=1,3,5,\ldots}$ is a subsequence of Λ_n and $\tilde{\Lambda}_{n=2,4,6,\ldots}$ is a subsequence of Λ_n'. Since these three sequences have the same limit, the limit is independent of the exhaustion. Independence of the exhaustion implies $G_{\mathbf{0}}^{(\infty)}(a,b)$ is translation invariant: the limit of $G_{\mathbf{0}}^{(\Lambda)}(a,b)$ through Λ_n equals the limit of $G_{\mathbf{0}}^{(\Lambda)}(a',b')$ through (Λ_n+e), where $a'=a+e$, $b'=b+e$, and e a unit vector in $\mathbb{Z}^d$. Simultaneously, this latter limit is the same as the limit of $G_{\mathbf{0}}^{(\Lambda)}(a',b')$ through (Λ_n). This implies that $G_{\mathbf{0}}^{(\infty)}(a,b)=G_{\mathbf{0}}^{(\infty)}(0,b-a)$. A similar argument shows $G_{\mathbf{0}}^{(\infty)}(x):=G_{\mathbf{0}}^{(\infty)}(0,x)$ is $\mathbb{Z}^d$-symmetric. Therefore $G_{\mathbf{0}}^{(\infty)}(x,y)$ is $\mathbb{Z}^d$-symmetric; see the discussion above Assumptions 2.1. ∎

Since $G_{\mathbf{0}}^{(\infty)}(x,y)$ is translation invariant, a K-IRB of the form (8.4) implies $G_{\mathbf{0}}^{(\infty)}(x)$ satisfies a K-IRB of the form (4.5). Thus in the sequel there is no ambiguity when we say $G_{\mathbf{0}}^{(\infty)}$ satisfies a K-IRB without further specification.

9.2 The infinite volume limit of $\Pi^{(\Lambda)}$

In this section we prove the existence of the infinite volume limit of $\Pi^{(\Lambda)}$. Recall that $\Pi^{(\Lambda)}=\sum_{m\geq 0}\Pi_m^{(\Lambda)}$ and $\Pi_m^{(\Lambda)}$ is defined by (7.1). By Lemma 7.1 the $m=0$ term is $\Pi_0^{(\Lambda)}(x,y)=L_{g,\nu,x}^{(\Lambda)}\mathbb{1}_{\{x=y\}}$ where $L_{g,\nu,x}^{(\Lambda)}$ is defined in (4.4). The next assumption postpones proving that $L_{g,\nu,x}^{(\Lambda)}$ has an infinite volume limit to the next section.

Assumptions 9.2.

(Z3) If a K-IRB holds, then $L_{g,\nu,x}^{(\Lambda)}$ is bounded uniformly in x and Λ, and the limit $L_{g,\nu}=L_{g,\nu,x}^{(\infty)}:=\lim_{\Lambda\uparrow\mathbb{Z}^d}L_{g,\nu,x}^{(\Lambda)}$ in (4.4) exists and is independent of x.

Lemma 9.3. *Assume the hypotheses of Proposition 8.5 and (Z3). If η is sufficiently small, then for $x,y\in\mathbb{Z}^d$*

$$\left|\Pi^{(\Lambda)}(x,y)\right|=O(\|\!|y-x|\!\|^{-3(d-2)}), \tag{9.1}$$

uniformly in x,y and Λ.

Proof. This is immediate from (8.20), Corollary 8.6, Lemma 7.1 and (Z3). ∎

Lemma 9.4. *Assume the hypotheses of Proposition 8.5 and (Z3) and that η is sufficiently small. Then for any sequence of volumes $\Lambda_n\uparrow\mathbb{Z}^d$ there exists a subsequence Λ_{n_k} such that $\Pi(x,y):=\lim_{k\to\infty}\Pi^{(\Lambda_{n_k})}(x,y)$ exists pointwise in $x,y\in\mathbb{Z}^d$.*

Proof. Extend the definition of $\Pi^{(\Lambda)}:\Lambda\times\Lambda\to\mathbb{R}$ to $\mathbb{Z}^d\times\mathbb{Z}^d$ by letting $\Pi^{(\Lambda)}(x,y)=0$ if $x\notin\Lambda$ or $y\notin\Lambda$. By Lemma 9.3, $|\Pi^{(\Lambda)}(x,y)|$ is $O(\|\!|y-x|\!\|^{-3(d-2)})$ uniformly in Λ. Thus for any $x,y\in\mathbb{Z}^d$ and any increasing sequence of volumes $\Lambda_n\uparrow\mathbb{Z}^d$, there exists a subsequence $\Lambda_{n_k(x,y)}$ such that $\Pi^{(\Lambda_{n_k(x,y)})}(x,y)$ converges as $k\to\infty$. By a diagonal argument we can refine this sequence such that the limit exists for all $x,y\in\mathbb{Z}^d$. ∎

Lemma 9.5. *Assume the hypotheses of Proposition 8.5 and (Z3) and that η is sufficiently small. For a sequence $\Lambda_n\uparrow\mathbb{Z}^d$ for which $\Pi^{(\Lambda_n)}$ converges pointwise to Π,*

$$S^{(\Lambda_n)}\Pi^{(\Lambda_n)}G_{\mathbf{0}}^{(\Lambda_n)}(x,y)\to S\Pi G_{\mathbf{0}}^{(\infty)}(x,y)\qquad x,y\in\mathbb{Z}^d, \tag{9.2}$$

and the product on the right-hand side is absolutely convergent, so there is no ambiguity in the order of the products.

Proof. By Lemma 9.3, $\Pi^{(\Lambda)}(x,y)$ is uniformly bounded above by a multiple of $U(x,y) := |||y-x|||^{-3(d-2)}$, $S^{(\Lambda)}$ is bounded above by S and, by (G2), $G_{\mathbf{0}}^{(\Lambda)}$ is bounded above by $G_{\mathbf{0}}^{(\infty)}$. Both $S(x,y)$ and $G_{\mathbf{0}}^{(\infty)}(x,y)$ are non-negative and bounded above by a multiple of $|||y-x|||^{-d+2}$. Hence the products $SU(x,y)$ and $UG_{\mathbf{0}}^{(\infty)}(x,y)$ are both absolutely convergent by Lemma 8.2, and decay at least as fast as a multiple of $|||y-x|||^{-d+2}$. Applying Lemma 8.2 once more with $d \geq 5$ shows $SUG_{\mathbf{0}}^{(\infty)}(x,y)$ is given by an absolutely convergent double sum.

Since $\Pi^{(\Lambda_n)} \to \Pi$ pointwise by hypothesis, $S^{(\Lambda_n)} \to S$ pointwise (see (2.9)), and $G_{\mathbf{0}}^{(\Lambda_n)} \to G_{\mathbf{0}}^{(\infty)}$ pointwise by Lemma 9.1, (9.2) follows by the dominated convergence theorem. ■

Recall the definition of $\Delta^{(\infty)}$ from (2.1).

Lemma 9.6. *Assume the hypotheses of Proposition 8.5 and (Z3), that η is sufficiently small, and that $\Lambda_n \uparrow \mathbb{Z}^d$ is such that $\Pi^{(\Lambda_n)} \to \Pi$ pointwise. Then $-(\Delta^{(\infty)}+\Pi)$ is a two-sided inverse of $G_{\mathbf{0}}^{(\infty)}$ and $\Pi(x,y) = \Pi(y,x)$ for $x,y \in \mathbb{Z}^d$.*

Proof. By (6.4), Lemma 9.1 and Lemma 9.5,

$$G_{\mathbf{0}}^{(\infty)} = S + S\Pi G_{\mathbf{0}}^{(\infty)}. \tag{9.3}$$

Multiplying (9.3) on the left by $-\Delta^{(\infty)}$ and using (2.12) yields

$$-(\Delta^{(\infty)}+\Pi)G_{\mathbf{0}}^{(\infty)}(x,y) = \mathbb{1}_{\{x=y\}}. \tag{9.4}$$

In applying (2.12) we have used that $\Delta^{(\infty)}(S\Pi G_{\mathbf{0}}^{(\infty)}) = (\Delta^{(\infty)}S)(\Pi G_{\mathbf{0}}^{(\infty)})$, which holds as $\Delta^{(\infty)}(x,\cdot)$ is finite range by (J4). Thus $-(\Delta^{(\infty)}+\Pi)$ is a left-inverse of $G_{\mathbf{0}}^{(\infty)}$.

Letting A^t denote the transpose of a matrix A, note that

$$-\mathbb{1}_{\{x=y\}} = (G_{\mathbf{0}}^{(\infty)})^t(\Delta^{(\infty)}+\Pi)^t(x,y) = G_{\mathbf{0}}^{(\infty)}(\Delta^{(\infty)}+\Pi)^t(x,y), \tag{9.5}$$

as $(G_{\mathbf{0}}^{(\infty)})^t = G_{\mathbf{0}}^{(\infty)}$ by Lemma 9.1. Note $\Pi(x,y) = O(|||y-x|||^{-3(d-2)})$, as Π is a pointwise limit of functions satisfying this uniform bound by Lemma 9.3. Since $\Delta^{(\infty)}(x,\cdot)$ is finite-range by (J4), this implies $(\Delta^{(\infty)}+\Pi)(x,y)$ is $O(|||y-x|||^{-3(d-2)})$, and hence $(\Delta^{(\infty)}+\Pi)^t(x,y)$ is also $O(|||y-x|||^{-3(d-2)})$. Thus $(\Delta^{(\infty)}+\Pi)G_{\mathbf{0}}^{(\infty)}(\Delta^{(\infty)}+\Pi)^t$ is absolutely convergent and unambiguously defined by Lemma 8.2 and $-(\Delta^{(\infty)}+\Pi)^t$ is a right-inverse of $G_{\mathbf{0}}^{(\infty)}$. By Lemma 4.10 $-(\Delta^{(\infty)}+\Pi)$ is two-sided-inverse to $G_{\mathbf{0}}^{(\infty)}$ and

$$\Delta^{(\infty)}+\Pi = (\Delta^{(\infty)}+\Pi)^t,$$

and consequently $\Pi(x,y) = \Pi(y,x)$ as desired. ■

Proposition 9.7. *Assume the hypotheses of Proposition 8.5 and (Z3) and that η is sufficiently small. The limit $\Pi^{(\infty)}(x,y) := \lim_{\Lambda\uparrow\mathbb{Z}^d}\Pi^{(\Lambda)}(x,y)$ exists and is $\mathbb{Z}^d$-symmetric.*

Proof. Let Π and $\tilde{\Pi}$ be pointwise limit points of exhaustions $\Lambda_n \uparrow \mathbb{Z}^d$ and $\tilde{\Lambda}_n \uparrow \mathbb{Z}^d$. Let $A = -(\Delta^{(\infty)} + \Pi)$ and $\tilde{A} = -(\Delta^{(\infty)} + \tilde{\Pi})$. By Lemma 9.3 A and $\tilde{A}$ are $O(|||y-x|||^{-3(d-2)})$. Then $\sum_{u,v\in\mathbb{Z}^d} |A(x,u| \, |G_{\mathbf{0}}^{(\infty)}(u,v)| \, |\tilde{A}(v,y)| < \infty$ by Lemma 8.2. By Lemma 9.6 A and $\tilde{A}$ are two-sided inverses of $G_{\mathbf{0}}^{(\infty)}$. By Lemma 4.10 $A = \tilde{A}$. By Lemma 9.4 limit points exist for every exhaustion and we have just shown that the limit point is unique so $\Pi^{(\infty)}(x,y) := \lim_{\Lambda\uparrow\mathbb{Z}^d} \Pi^{(\Lambda)}(x,y)$ exists.

Let T be an automorphism of $\mathbb{Z}^d$. By Lemma 9.1 and the definition of $\mathbb{Z}^d$-symmetry above Assumptions 2.1 $G_{\mathbf{0}}^{(\infty)}(Tx,Ty) = G_{\mathbf{0}}^{(\infty)}(x,y)$ for all x,y. Let $A'(x,y) = A(Tx,Ty)$ for A as above. By changing the summation variable from y to $T^{-1}y$, $\sum_y G_{\mathbf{0}}^{(\infty)}(x,y)A'(y,z)$ equals $\sum_y G_{\mathbf{0}}^{(\infty)}(x,T^{-1}y)A(y,Tz)$ which equals $\sum_y G_{\mathbf{0}}^{(\infty)}(Tx,y)$ $A(y,Tz)$ by the $\mathbb{Z}^d$-symmetry of $G_{\mathbf{0}}^{(\infty)}$. Since A is a right inverse this sum simplifies to $\mathbb{1}_{\{Tx=Tz\}} = \mathbb{1}_{\{x=z\}}$. We conclude that A' is a right-inverse to $G_{\mathbf{0}}^{(\infty)}$. By a similar calculation A' is a left-inverse so A' is a two-sided inverse to $G_{\mathbf{0}}^{(\infty)}$. Repeating the argument in the first paragraph we have $A' = A$ so $A = -(\Delta^{(\infty)} + \Pi^{(\infty)})$ is $\mathbb{Z}^d$-symmetric as claimed. ■

9.3 The infinite-volume lace expansion equation

In the next proposition $\Psi^{(\infty)}(x) := \Psi^{(\infty)}(0,x)$ is the infinite volume limit of $\Psi^{(\Lambda)}$ defined by (8.20) and $L_{g,\nu}^{(\infty)}$ is as defined in (4.4), in which the limit exists by Assumptions 9.2. Recall $G_{\mathbf{0}}^{(\infty)}(x) = G_{\mathbf{0}}^{(\infty)}(0,y-x)$. In Item (iii) of the Proposition below we prove (4.6).

Proposition 9.8. *Assume the hypotheses of Proposition 8.5 and (Z3) hold. Then there exist $\alpha > 0$, $\eta_0 > 0$, and $\Psi^{(\infty)}$ such that for all $\eta \in (0,\eta_0)$,*

- *(i)* $\Psi^{(\infty)}(x)$ *exists and is a $\mathbb{Z}^d$-symmetric function of x.*
- *(ii)* $|\Psi^{(\infty)}(x)| \leq \alpha\eta\, |||x|||^{-3(d-2)}$.
- *(iii)* $(\hat{J} - L_{g,\nu}^{(\infty)})G_{\mathbf{0}}^{(\infty)}(x) = \mathbb{1}_{\{x=0\}} + J_+ * G_{\mathbf{0}}^{(\infty)}(x) + \Psi^{(\infty)} * G_{\mathbf{0}}^{(\infty)}(x)$.
- *(iv)* $\hat{J} - L_{g,\nu}^{(\infty)} \geq \frac{1+O(\eta)}{KS(0)}$.

Proof. Item (i): Note $\Psi^{(\Lambda)}(x,y) = \Pi^{(\Lambda)}(x,y) - \Pi_0^{(\Lambda)}(x,y)$. Both terms on the right-hand side have $\mathbb{Z}^d$-symmetric infinite volume limits by Proposition 9.7, Lemma 7.1 and (Z3). Therefore $\Psi^{(\infty)}(x)$ exists and is a $\mathbb{Z}^d$-symmetric function.

Item (ii) follows from the finite-volume estimate given by Corollary 8.6.

Item (iii): By Lemma 9.6 and Proposition 9.8 $-(\Delta^{(\infty)} + \Pi^{(\infty)})G_{\mathbf{0}}^{(\infty)} = \mathbb{1}$. In terms of the one variable functions $\Pi^{(\infty)}(x) := \Pi^{(\infty)}(0,x)$ and $G_{\mathbf{0}}^{(\infty)}(x) := G_{\mathbf{0}}^{(\infty)}(0,x)$ with convolution replacing matrix products this is $(J - \Pi^{(\infty)}) * G_{\mathbf{0}}^{(\infty)}(x) = \mathbb{1}_{\{x=0\}}$. We insert $J(x) = -\hat{J}\mathbb{1}_{\{x=0\}} + J_+(x)$ from (2.2) and $\Pi^{(\infty)}(x) = L_{g,\nu}^{(\infty)}\mathbb{1}_{\{x=0\}} + \Psi^{(\infty)}(x)$ from Assumptions 9.2. After rearranging we obtain (iii).

Item (iv): We evaluate (iii) at $x=0$, insert item (ii) using $d \geq 5$ to obtain $|G_{\mathbf{0}}^{(\infty)} * \Psi^{(\infty)}(0)| = O(\eta)$ and insert $J_+ * G_{\mathbf{0}}^{(\infty)}(0) \geq 0$. The result is the desired bound

$$\hat{J} - L_{g,\nu}^{(\infty)} \geq \frac{1+O(\eta)}{G_{\mathbf{0}}^{(\infty)}(0)} \geq \frac{1+O(\eta)}{KS(0)}, \tag{9.6}$$

where the second inequality is implied by the K-IRB hypothesis. ■

10 Final hypotheses and proof of asymptotic behaviour

In Remark 4.13 we listed the lemmas that collectively prove the infrared bound of Theorem 4.1. As outlined in Section 4.4 we now revise the hypotheses of these lemmas to replace their specialisation to lattice Edwards model and $g|\varphi|^4$ model by Assumptions 10.1 and 10.2 and we prove these revised lemmas. The main result Theorem 10.11 is that these assumptions imply the desired asymptotic law for the Green's function $G_{g,\nu_c}(x)$. In Section 11 we will verify that the Edwards model and the $n=1,2$-component $g|\varphi|^4$ models satisfy these assumptions.

10.1 Final hypotheses

This subsection summarizes the hypotheses under which we will draw conclusions about the asymptotic behaviour of the Green's function. For each finite $\Lambda \subset \mathbb{Z}^d$ and parameters $g>0$ and $\nu \in \mathbb{R}$ let $Z_{g,\nu}^{(\Lambda)} \colon [0,\infty)^\Lambda \to (0,\infty)$, $t \mapsto Z_{g,\nu,\boldsymbol{t}}^{(\Lambda)}$ satisfy

Assumptions 10.1. Assume J is such that (J1)–(J4) hold, and

(i) for each $g>0, \nu \in \mathbb{R}$, (Z1), (Z2), (Z3), (G1)–(G2) hold for $Z_{g,\nu}^{(\Lambda)}$;
(ii) there exists $\nu_1 > 0$ such that for all $g \geq 0$ $G_{g,\nu_1}^{(\Lambda)}(x)$ is summable in x uniformly in Λ;
(iii) there exists $c_* > 0$ such that for each $g>0, \nu \in \mathbb{R}$, (R1) holds for $Z_{g,\nu}^{(\Lambda)}$ with $\eta = c_* g$.

Throughout this section we write $G_{g,\nu} = G_{g,\nu}^{(\infty)}$, which exists as a possibly infinite monotone limit by (G2). The *susceptibility* is defined by

$$\chi_g(\nu) := \sum_{x\in\mathbb{Z}^d} G_{g,\nu}(x), \tag{10.1}$$

and the *critical value* $\nu_c(g)$ of ν is defined by

$$\nu_c(g) := \inf\{\nu \in \mathbb{R} \mid \chi_g(\nu) < \infty\}. \tag{10.2}$$

In particular, $\nu_c \leq \nu_1$ with ν_1 given in Assumptions 10.1.

Assumptions 10.2. Assume that $g>0$, that $\nu_c(g) \leq 0$, and that

(G3) $G_{g,\nu}(x)$ is non-increasing in $\nu \in \mathbb{R}$. Moreover,
  (a) For $\nu \in (\nu_c(g), \infty)$ and $x \in \mathbb{Z}^d$, $G_{g,\nu}(x)$ is continuous in ν.
  (b) If $G_{g,\nu} \leq 3S$ for some $\nu \in [\nu_c, \infty)$, then $\{G_{g,\nu'}(x)\}_{x\in\mathbb{Z}^d}$ is a uniformly equicontinuous family of functions of $\nu' \in [\nu, \infty)$.

(G4) For $\nu \in (\nu_c(g),\infty)$, $\sup_{x\in\mathbb{Z}^d:|x|\geq r} G_{g,\nu}(x)/S(x) \to 0$ as $r\to\infty$.
(G5) $G_{g,g} \leq 2S$.
(Z4) For each finite Λ and $g>0$, $Z^{(\Lambda)}_{g,\nu}\colon [0,\infty)^\Lambda \to (0,\infty)$ is continuous in $\nu\in\mathbb{R}$ pointwise in $\boldsymbol{t}\in[0,\infty)^\Lambda$ and uniformly bounded in $\boldsymbol{t}$ for each ν.
(Z5) (a) For $g>0$, $L_{g,\nu}$ is continuous in $\nu\in(\nu_c(g),\infty)$.
(b) If $G_{g,\nu}\leq 3S$ for some ν, then $L_{g,\nu'}$ is continuous for $\nu'\in[\nu,\infty)$. If, additionally, $L_{g,\nu}\leq 0$ and $\nu\in(\nu_c(g),g]$, then $L_{g,\nu}=O(g)$.
(Z6) $L_{g,\nu}\to\infty$ as $\nu\to-\infty$.

These assumptions suffice for our applications, and are intended to be a starting point on the road to better assumptions. The next lemma is a step in this direction.

Lemma 10.3. *(Z1), (G1), (G2) and $\sum_{x\in\mathbb{Z}^d} G^{(\infty)}(0,x)<\infty$ implies $G^{(\infty)}(0,x)$ decays exponentially in x as $x\to\infty$. In particular (Z1), (G1), (G2) imply (G4).*

To prove this we establish a Simon inequality.

Proposition 10.4 (Simon inequality)**.** *Let $G^{(\infty)}$ be the infinite volume limit of the Green's function of a model that satisfies (Z1), (G1), (G2). Assume $G^{(\infty)}(0,x)$ is summable in $x\in\mathbb{Z}^d$. For $a,b\in\mathbb{Z}^d$ and $\Lambda'\subset\mathbb{Z}^d$ such that Λ' contains a but not b,*

$$G^{(\infty)}(a,b) \leq \sum_{x'\in\Lambda',x\in\mathbb{Z}^d\setminus\Lambda'} G^{(\infty)}(a,x')J^{(\partial\Lambda')}(x',x)G^{(\infty)}(x,b), \tag{10.3}$$

where $J^{(\partial\Lambda')}(x',x)=J(x-x')\mathbb{1}_{\{x'\in\Lambda'\}}\mathbb{1}_{\{x\in\mathbb{Z}^d\setminus\Lambda'\}}$, i.e., $J^{(\partial\Lambda')}(x',x)$ is non-zero only for jumps out of Λ'.

Simon [41] proved the progenitor for this inequality for the Ising model, and showed that models that satisfy Simon inequalities are such that whenever the two-point correlation is summable the two-point function decays exponentially. Lieb [35] gave an important improvement in the Simon inequality, which was extended by Rivasseau [38] to two-component models. The improvement was to replace $G^{(\infty)}(a,x')$ by $G^{(\Lambda)}(a,x')$. We are unable to obtain this improvement in the generality of Proposition 10.4; see [10, Theorem 6.1] for a random walk proof of this improvement for $g|\varphi|^4$ models.

The next proof paraphrases [41] in the notation of this paper. It explains why we assumed the finite range condition (J4); without this (G4) would have to be established by an alternative argument.

Proof of Lemma 10.3. Recall from Assumptions 2.1 (J4) that there exists a range R such that $J^{(\partial\Lambda')}(x',x)=0$ for $|x'-x|\geq R$. For $r\geq 1$ choose Λ' to be the ball $\{x:|x|\leq rR\}$ in $\mathbb{Z}^d$, let $F(r)=\sum_{x'\in\Lambda'}\sum_{x\notin\Lambda'} G^{(\infty)}(0,x')J^{(\partial\Lambda')}(x',x)$, and let $f(s)=\sum_{x:|x|\geq sR} G^{(\infty)}(0,x)$. The choice of Λ' and the range R of $J^{(\partial\Lambda')}$ imply that we can upper bound $F(r)$ by replacing $x\notin\Lambda'$ by $|x|>rR$ and $x'\in\Lambda'$ by $|x'|\geq(r-1)R$. This yields $F(r)\leq \hat{J}f(r-1)$, where $\hat{J}$ is given by (2.2). Note that for $x'\in\Lambda'$ and $|x|\geq(r+1)R$, $G^{(\infty)}(0,x')J^{(\partial\Lambda')}(x',x)=0$, and hence by summing (10.3) over $|b|\geq nrR$ with $a=0$ we have $f(nr)\leq F(r)f(nr-(r+1))\leq \hat{J}f(r-1)f((n-2)r)$.

By iteration we obtain $f(nr) \leq (\hat{J} f(r-1))^{n/2} f(0)$ for n even. By the summability hypothesis $f(0)$ is finite and $f(r-1) \downarrow 0$ as $r \uparrow \infty$. Therefore, for r sufficiently large, $\hat{J} f(r-1) \leq \frac{1}{2}$. With this choice of r and n even $f(nr) \leq 2^{-n/2} f(0)$ which implies $G^{(\infty)}(0,x)$ decays exponentially in x as desired and, by (2.11), (G4) is an immediate consequence. ■

Proof of Proposition 10.4. Let $\Lambda \subset \mathbb{Z}^d$ such that $a \in \Lambda' \subset \Lambda$. Let $\mathscr{S} = \inf\{t \geq 0 : X_t^{(\Lambda)} \notin \Lambda'\}$ be the time of first exit from Λ', and note that $X_{\mathscr{S}}^{(\Lambda)}$ is the position of $X^{(\Lambda)}$ immediately after it jumps for the first time out of Λ'. Let

$$P(a,x) := E_a \left[\frac{Z_{\boldsymbol{\tau}[0,\mathscr{S}]}}{Z_{\mathbf{0}}} \mathbb{1}_{\{X_{\mathscr{S}}^{(\Lambda)}=x\}} \right]. \tag{10.4}$$

Recall that the finite volume Greens function $G^{(\Lambda)}(a,b)$ is $G_{\boldsymbol{t}}^{(\Lambda)}(a,b)$ as defined by (3.2) with $\boldsymbol{t}$ set to $\mathbf{0}$. Using (Z1) we interchange the integral with the expectation in (3.2) and obtain

$$G^{(\Lambda)}(a,b) = E_a \left[\int_{[0,\infty)} d\ell \, \frac{Z_{\boldsymbol{\tau}[0,\ell]}}{Z_{\mathbf{0}}} \mathbb{1}_{\{X_\ell^{(\Lambda)}=b\}} \right]. \tag{10.5}$$

The hypothesis on a,b implies that $\ell > \mathscr{S}$. By conditioning on $\mathscr{F}_{\mathscr{S}}$

$$G^{(\Lambda)}(a,b) = E_a \left[\frac{Z_{\boldsymbol{\tau}[0,\mathscr{S}]}}{Z_{\mathbf{0}}} \int_{[\mathscr{S},\infty)} d\ell \, E_a \left[\frac{Z_{\boldsymbol{\tau}[0,\mathscr{S}]+\boldsymbol{\tau}[\mathscr{S},\ell]}}{Z_{\boldsymbol{\tau}[0,\mathscr{S}]}} \mathbb{1}_{\{X_\ell^{(\Lambda)}=b\}} \middle| \mathscr{F}_{\mathscr{S}} \right] \right]. \tag{10.6}$$

By Lemma A.2 with $I = [0,\mathscr{S}]$ and $G_{\boldsymbol{\tau}_I}^{(\Lambda)}(X_{\mathscr{S}}^{(\Lambda)},b) \leq G^{(\Lambda)}(X_{\mathscr{S}}^{(\Lambda)},b)$ from (G1) we have $G^{(\Lambda)}(a,b) \leq E_a \left[\frac{Z_{\boldsymbol{\tau}[0,\mathscr{S}]}}{Z_{\mathbf{0}}} G^{(\Lambda)}(X_{\mathscr{S}}^{(\Lambda)},b) \right]$ which is the same as

$$G^{(\Lambda)}(a,b) \leq \sum_{x \in \Lambda} P(a,x) G^{(\Lambda)}(x,b). \tag{10.7}$$

To complete the proof it suffices to show that

$$P(a,x) \leq \sum_{x' \in \Lambda'} G^{(\Lambda)}(a,x') J^{(\partial\Lambda')}(x',x) \tag{10.8}$$

because inserting (10.8) into (10.7) and using (G2) to take the infinite volume limit gives (10.3). *For the remainder of this proof we write $X = X^{(\Lambda)}$.* By summing over the possible values of X_ℓ, and the (Poisson) number of jumps in $(\ell,\ell+\delta]$, one can easily show that for $x \in \Lambda \setminus \Lambda'$, $\ell > 0$ and $\delta > 0$

$$E_a \left[\mathbb{1}_{\{X_{\ell+\delta}=x\}} | \mathscr{F}_\ell \right] \mathbb{1}_{\{\mathscr{S}>\ell\}} = J(x-X_\ell)\delta \mathbb{1}_{\{\mathscr{S}>\ell\}} + O(\delta^2), \tag{10.9}$$

where the $O(\delta^2)$ term is uniform in ℓ,x, a.s. ω. We let $Y_\ell = \frac{Z_{\boldsymbol{\tau}[0,\ell]}}{Z_{\mathbf{0}}}$ and use this to obtain

$$\begin{aligned} E_a \left[\mathbb{1}_{\{X_{\ell+\delta}=x\}} \mathbb{1}_{\{\mathscr{S}>\ell\}} Y_\ell \right] &= E_a \left[E_a \left[\mathbb{1}_{\{X_{\ell+\delta}=x\}} | \mathscr{F}_\ell \right] \mathbb{1}_{\{\mathscr{S}>\ell\}} Y_\ell \right] \\ &= \delta E_a \left[J(x-X_\ell) \mathbb{1}_{\{\mathscr{S}>\ell\}} Y_\ell \right] + O(\delta^2). \end{aligned} \tag{10.10}$$

Since for ℓ in a compact set $[0,M]$ the vector of local times ranges within a compact subset $[0,M]^\Lambda$, we have by (Z1) that Y_ℓ is bounded by a constant for $\ell \in [0,M]$.

Therefore $O(\delta^2)$ is uniform in x and $\ell \in [0,M]$. Note that Y_ℓ is pathwise continuous in ℓ by (Z1) and (2.5) and since also the probability that the walker jumps in a small interval goes to zero as the length of the interval goes to 0, the expectation is continuous in ℓ. We insert $\ell = \ell_m = m\delta$, sum over $m = 0,1,...,\lfloor M/\delta \rfloor$ and take the limit as $\delta \downarrow 0$ to obtain

(10.11)
$$\lim_{\delta\downarrow 0} \sum_{m=0}^{\lfloor M/\delta \rfloor} E_a\left[\mathbb{1}_{\{X_{\ell_{m+1}}=x\}}\mathbb{1}_{\{\mathscr{S}>\ell_m\}}Y_{\ell_m}\right] = \int_{[0,M]} E_a\left[J(x-X_\ell)\mathbb{1}_{\{\mathscr{S}>\ell\}}Y_\ell\right] d\ell.$$

On the other hand, taking the sum inside the expectation, and using Dominated convergence we see that the left hand side converges to $E_a\left[\mathbb{1}_{\{X_{\mathscr{S}}=x\}}\mathbb{1}_{\{\mathscr{S}\le M\}}Y_{\mathscr{S}}\right]$ because it partitions the event $\{\mathscr{S} \le \delta(\lfloor M/\delta \rfloor + 1)\}$ into events $\{\mathscr{S} \in (m\delta,(m+1)\delta]\}$, Y_ℓ is pathwise continuous, and X_ℓ is right-continuous. We let $M \uparrow \infty$ to obtain

$$E_a\left[\mathbb{1}_{\{X_{\mathscr{S}}=x\}}Y_{\mathscr{S}}\right] = \int_{[0,\infty)} E_a\left[J(x-X_\ell)\mathbb{1}_{\{\mathscr{S}>\ell\}}Y_\ell\right] d\ell. \tag{10.12}$$

Recalling that $Y_\ell = \frac{Z_{\tau_{[0,\ell]}}}{Z_0}$ the left hand side is $P(a,x)$ by definition (10.4). By inserting $J(x-X_\ell) = \sum_{x'\in\Lambda'} \mathbb{1}_{\{X_\ell=x'\}}J(x-x')$ we have

$$P(a,x) = \sum_{x'\in\Lambda'} \int_{[0,\infty)} d\ell\, E_a\left[\frac{Z_{\tau_{[0,\ell]}}}{Z_0}\mathbb{1}_{\{X_\ell=x'\}}\mathbb{1}_{\{\mathscr{S}>\ell\}}\right] J^{(\partial\Lambda')}(x',x). \tag{10.13}$$

We insert $\mathbb{1}_{\{\mathscr{S}>\ell\}} \le 1$ and (3.2) with $\boldsymbol{t} = 0$ to obtain (10.8) and thereby complete the proof. ∎

10.2 Model independent lemmas

In this section we prove and revise hypotheses for the model independent Lemmas 4.4 to 4.6 and 4.9 in the list of Remark 4.13.

Lemma 10.5. *Lemmas 4.4 and 4.5 revised by replacing lattice Edwards model and $g|\varphi|^4$ model with Assumptions 10.1 and (Z5).*

Proof. Recall that $D_{g,\nu}, \tilde{G}_{g,\nu}(x), \tilde{\Psi}_{g,\nu}(x)$ are defined in (4.11) in terms of $L_{g,\nu}$ and $\Psi_{g,\nu}(x) := \Psi^{(\infty)}_{g,\nu}(x)$, and in (4.7) we defined $D_z^{\tilde{S}}(x) = -\mathbb{1}_{\{x=0\}} + zJ_+(x)$.

Proof of revised Lemma 4.4. By Proposition 9.8, Assumptions 10.1(iii) and hypothesis $K = 3$ we immediately obtain Lemma 4.4 with the desired revision. Note that this revised lemma implies there is a constant c_d such that for $d \ge 5$

$$0 \le w(g,\nu) \le c_d, \quad \text{and } |\Psi_{g,\nu}(x)| \le g\alpha c_* |||x|||^{-3(d-2)} \le g\alpha c_* |||x|||^{-(d+4)}. \tag{10.14}$$

Proof of revised Lemma 4.5. Part (i): recall the definition of $\mathscr{D}_C$ above Lemma 4.5 and that we have chosen $D = D_{g,\nu}$. We have to prove item (i): $D_{g,\nu}$ is $\mathbb{Z}^d$-symmetric; item (ii): $\sum_{x\in\mathbb{Z}^d} D_{g,\nu}(x) \le 0$; item (iii): there exists C_0 and there exists $z = z(g,D_{g,\nu}) \in \left[0,\hat{J}^{-1}\right]$ such that

$$\left|D_{g,\nu}(x) - D_z^{\tilde{S}}(x)\right| \le C_0 g |||x|||^{-(d+4)}. \tag{10.15}$$

Item (i) holds by (J3) and Proposition 9.8. To obtain item (ii), we sum (4.10) over x and interchange the sum over x with the sum in the convolution in (4.10). Since $\nu > \nu_c$ the sums are absolutely convergent and the interchange is valid. The result is

$$\sum_{x\in\mathbb{Z}^d} D_{g,\nu}(x) = -\left(\sum_{x\in\mathbb{Z}^d} \tilde{G}_{g,\nu}(x)\right)^{-1} < 0, \tag{10.16}$$

as desired. The inequality follows from $G_{g,\nu}(x) > 0$ and $w(g,\nu) > 0$.

Item (iii).

$$\left|D(x) - D_z^{\tilde{S}}(x)\right| \leq \left|\left(w(g,\nu) - z\right)\right| J_+(x) + w(g,\nu)\left|\Psi_{g,\nu}(x)\right| \tag{10.17}$$

For $w(g,\nu) \leq \hat{J}^{-1}$, the choice $z = w(g,\nu)$ satisfies (10.15) with $C_0 \geq c_d\alpha c_*$ by (10.14). Otherwise, by (10.14), $w(g,\nu) > \hat{J}^{-1}$ and we will now prove that $z = \hat{J}^{-1}$ satisfies (10.15). Accordingly set $z = \hat{J}^{-1}$ until the end of the proof of this part. Item (ii) bounds how much $w(g,\nu)$ can exceed z as in the final inequality of

$$\begin{aligned} 0 &\leq w(g,\nu) - z = \hat{J}^{-1}\sum_x D^{\tilde{S}}_{w(g,\nu)}(x) \\ &= \hat{J}^{-1}\sum_x \Big(D_{g,\nu}(x) - w(g,\nu)\Psi_{g,\nu}(x)\Big) \leq \hat{J}^{-1}\sum_x -w(g,\nu)\Psi_{g,\nu}(x), \end{aligned} \tag{10.18}$$

where equalities follow from $\sum_x J_+(x) = \hat{J}$, the definition of $D_z^{\tilde{S}}(x)$ with z replaced by $w(g,\nu)$ and the definition of $D_{g,\nu}(x)$. This together with (10.17) implies

$$\left|D(x) - D_z^{\tilde{S}}(x)\right| \leq \Big(\sum_{x'} w(g,\nu)\left|\Psi_{g,\nu}(x')\right|\Big)\frac{J_+(x)}{\hat{J}} + w(g,\nu)\Psi_{g,\nu}(x). \tag{10.19}$$

We insert (10.14). By (J4) there is a constant c_{J_+} such that $\big(\sum_{x'} |||x'|||^{-3(d-2)}\big)J_+(x)\hat{J}^{-1} \leq c_{J_+}|||x|||^{-(d+4)}$. Therefore the right hand side is bounded by $C_0 g|||x|||^{-(d+4)}$ for $C_0 \geq c_d\alpha c_*(c_{J_+}+1)$. The proof of item (iii) and therefore of part (i) is complete.

Lemma 4.5, part (ii): We must show that $L_{g,\nu} = O(g)$. By (Z5) and the hypothesis $\nu \in [\nu_c, g]$ this holds if $L_{g,\nu} \leq 0$. If $L_{g,\nu} \geq 0$ this follows by inserting $w(g,\nu) = (\hat{J} - L_{g,\nu})^{-1}$ into (10.18) and solving the inequality for $L_{g,\nu}$. ■

Next we prove Lemma 4.6. The hypotheses need no revision because they do not reference our models. This lemma extends [7, Lemma 2], where the Laplacian is nearest neighbour, to the finite range context of Assumptions 2.1 (J4). The proof is, *mutatis mutandis*, that of [7, Lemma 2], so we discuss only part that needed care.

Proof of Lemma 4.6. Note this reference uses $-\Delta$ for what we denote by D, and that the formula for μ and its range is stated in the body of the proof of Lemma 2 in [7, below (28)].

The most significant step to check is the Edgeworth expansion (24) in the proof of [7, Lemma 4]. According to [7] this is equation (1.5b) of [49, Theorem 2] with

$m = 4$. This is misleading even for the nearest neighbour Laplacian because (1.5b) is not the same as (24), but the proof of [7, Lemma 4] remains valid with (1.5b) in place of (24) so we momentarily set this aside. The equation (1.5b) of [49, Theorem 2] continues to hold under our Assumptions 2.1. In particular, (J3) implies that in (1.5b) the norm $\|\cdot\|$ is the Euclidean norm $|\cdot|$ and the odd Edgeworth coefficients U_1, U_3 vanish. By the discussion below [49, Theorem 2] and (2.4) of [49, Theorem 2] the Edgeworth coefficients $U_2(\tilde{\omega}^x)$ and $U_4(\tilde{\omega}^x)$ in (1.5b) are continuous functions of the unit vector $\tilde{\omega}^x$ and are therefore bounded.

We return to the problem with [7, equation (24)]. The coefficients in (1.5b) of [49, Theorem 2] depend on the direction $\tilde{\omega}^x$ by which x approaches infinity, whereas [7, equation (24)] has no directional dependence. However the boundedness of this directional dependence is all that is used in the proof of [7, Lemma 2, Lemma 4]. ■

Lemma 10.6. *Lemma 4.9 revised by replacing lattice Edwards model and $g|\varphi|^4$ model with Assumptions 10.1 and (G3), (G4) of Assumptions 10.2.*

Proof. Recall from (4.14) that $F(\nu) := \sup_{x\in\mathbb{Z}^d} f(x,\nu)$ where $f(x,\nu) = G_{g,\nu}(x)/S(x)$. We have to prove that F is continuous on $(\nu_c, g]$. It suffices to prove that F is continuous on $(\tilde{\nu}_c, g]$ for $\tilde{\nu}_c > \nu_c$. As in step (i) in the proof of [25, Proposition 2.2] we reduce the supremum defining F to a finite set. By (G4) there exists r such that $\sup_{|x|>r} f(x,\tilde{\nu}_c) \leq \frac{1}{2} f(0,g)$. For $\nu \in (\tilde{\nu}_c, g]$ we have that $F(\nu) = \sup_{|x|\leq r} G_{g,\nu}(x)/S(x)$ because (G3)

$$\sup_{|x|>r} f(x,\nu) \leq \sup_{|x|>r} f(x,\tilde{\nu}_c) \leq \frac{1}{2} f(0,g) \leq \frac{1}{2} f(0,\nu). \tag{10.20}$$

For fixed x, $f(x,\nu)$ is continuous in ν by (G3). Since the supremum of finitely many continuous functions is continuous, F is continuous as desired. ■

10.3 Model-dependent lemmas and proof of Theorem 4.1

By the previous section the list in Remark 4.13 has been reduced to the model-dependent Lemmas 3.4, 3.5, 4.8 and 4.11. The conclusions of Lemmas 3.4, 3.5, 4.8 and 4.11 are contained in assumptions (G2), (G3), (G5), (Z6). Remark 4.13 also records continuity properties that are used in the proof of Lemma 4.12. These continuity properties are provided in Assumption (Z4). Therefore all the lemmas listed in Remark 4.13 hold with revised hypotheses. As explained in Remark 4.13 this proves:

Theorem 10.7. *Theorem 4.1 and Proposition 4.3 revised by replacing lattice Edwards model and $g|\varphi|^4$ model with Assumptions 10.1, 10.2: in particular, under these assumptions, there is $g_0 = g_0(d,J) > 0$ such that if $0 < g < g_0$ then $\nu_c(g)$ is finite and*

$$G_{g,\nu_c}(x) \leq 2S(x), \qquad x \in \mathbb{Z}^d. \tag{10.21}$$

10.4 Proof of asymptotic behaviour

We begin with two lemmas. The first, Lemma 10.8, is an extension of a lemma from [7], and hence we only describe where care must be taken in obtaining this extension.

Lemma 10.8. *Let D and H be as in Lemma 4.6. If* $\sum_{x\in\mathbb{Z}^d} D(x) < 0$ *then* $H(x) \in \ell^1(\mathbb{Z}^d)$.

Proof. By the definition (4.7) of $\tilde{S}_z$ as a series

$$\sum_{x\in\mathbb{Z}^d} \left|\tilde{S}_z(x)\right| \leq \sum_{x\in\mathbb{Z}^d}\sum_{n\geq 0} (|z|J_+)^{*n}(x) = \sum_{n\geq 0} |z|^n \sum_{x\in\mathbb{Z}^d} J_+^{*n}(x) = \sum_{n\geq 0} |z|^n \hat{J}^n,$$

where $\hat{J} = \sum_x J_+(x)$ by (J1) and (2.2). The interchange of sums is justified because all terms on the right hand side are positive. The right hand side is absolutely convergent iff $|z| < \hat{J}^{-1}$ and when it is absolutely convergent $\left|\tilde{S}_z(x)\right|$ decays exponentially like $(|z|\hat{J})^{O(|||x|||)}$ as $|||x||| \to \infty$ by (J4).

From $D * H = -\mathbb{1}$ we can generate a series similar to $\tilde{S}_z$ but it is inadequate because, unlike $J_+(x)$, $D(x) \not\geq 0$ for $x \neq 0$. However, this sign problem was solved in [7], where in the proof of [7, Lemma 2], it is shown that

$$H = (-D * \tilde{S}_\mu)^{-1} * \tilde{S}_\mu,$$

where we have expressed the equation preceding [7, Equation (32)] in the notation of the present paper. In [7] it is shown that the first term $(-D*\tilde{S}_\mu)^{-1}$ is an element of the Banach algebra B defined at the beginning of [7, Section 4], i.e., the set of functions f on $\mathbb{Z}^d$ that are ℓ^1 and have $\sup_x |f(x)||x|^d$ finite. Since $\sum_{x\in\mathbb{Z}^d} D(x) < 0$, μ defined in Lemma 4.6 satisfies $|\mu| < \hat{J}^{-1}$. Therefore, by the preceding paragraph, $\tilde{S}_\mu$ is an element of B since it decays exponentially in $|||x|||$. Hence the convolution defining H is an element of B; in particular it is ℓ^1 as desired. ■

Lemma 10.9. *Under Assumptions 10.1 and 10.2, if* $d \geq 5$ *and g is sufficiently small, then* $\{D_{g,\nu}(x)\}_{x\in\mathbb{Z}^d}$ *is an equicontinuous family of functions for* $\nu \in [\nu_c, \infty)$. *Moreover* $D_{g,\nu}$ *is continuous in* $\nu \in [\nu_c,\infty)$ *as an* ℓ^1*-valued function.*

Proof. To prove that $D_{g,\nu}(x)$ is defined and continuous in $\nu \in [\nu_c,\infty)$ we now discuss the definitions in (4.11) for $\nu = \nu_c$ as well as $\nu > \nu_c$. By Theorem 10.7 and (G3) the infrared bound $G_{g,\nu_c} \leq 2S$ holds for $\nu \geq \nu_c$. This implies the hypotheses of Proposition 9.8 hold with $\eta = c_* g$ for $\nu \geq \nu_c$. By item (iv) of Proposition 9.8 we conclude that $D_{g,\nu} := D^{\tilde{S}}_{w(g,\nu)} + \tilde{\Psi}_{g,\nu}$ exists for $\nu \geq \nu_c$ as desired. This definition together with item (iii) of Proposition 9.8 asserts that

$$D_{g,\nu} * \tilde{G}_{g,\nu}(x) = -\mathbb{1}_{\{x=0\}}. \tag{10.22}$$

Moreover, by (J4), items (ii) and (iv) of Proposition 9.8, and the lower bound on $w(g,\nu)$ following (10.14), there is a $c_1 > 0$ such that for $\nu \geq \nu_c$,

$$|D_{g,\nu}(x)| \leq c_1 |||x|||^{-d-4}. \tag{10.23}$$

For $\nu_1, \nu_2 \in [\nu_c, \infty)$, (10.22) implies that

$$D_{\nu_2} * (\tilde{G}_{\nu_1} - \tilde{G}_{\nu_2}) * D_{\nu_1} + (D_{\nu_1} - D_{\nu_2}) = 0, \tag{10.24}$$

where we have omitted the subscript g. Note that the omission of the order of the convolutions in this equation is valid as the iterated convolutions are absolutely convergent by (10.23), the infrared bound $G_{g,\nu_c} \leq 2S$, and Lemma 8.2. Therefore

$$\begin{aligned} |D_{\nu_2}(x) - D_{\nu_1}(x)| &\leq \sup_{y\in\mathbb{Z}^d} \left|\tilde{G}_{\nu_1}(y) - \tilde{G}_{\nu_2}(y)\right| \|D_{\nu_1}\|_1 \|D_{\nu_2}\|_1 \\ &\leq C \sup_{y\in\mathbb{Z}^d} \left|\tilde{G}_{\nu_1}(y) - \tilde{G}_{\nu_2}(y)\right| \end{aligned} \tag{10.25}$$

for some $C > 0$ by (10.23). By (Z5) part (b), $w(g,\nu)$ is continuous in ν for $\nu \geq \nu_c$. Therefore the functions $\tilde{G}_{g,\nu}(x)$ are equicontinuous on $[\nu_c, \infty)$ by (G3) part (b). This proves that the functions $D_{g,\nu}(x)$ are equicontinuous in $\nu \in [\nu_c, \infty)$ as desired.

The second claim, that $\nu \mapsto D_{g,\nu}$ is continuous in ℓ^1, follows from the first. This is so because $\sum_{x\in\mathbb{Z}^d} |D_{g,\nu}(x)|$ converges uniformly in ν by item (ii) and item (iv) of Proposition 9.8. ■

Lemma 10.10. *Consider a model satisfying Assumptions 10.1 and 10.2. If $d \geq 5$ and g is sufficiently small then $\sum_{x\in\mathbb{Z}^d} D_{g,\nu_c}(x) = 0$.*

Proof. This follows from Lemmas 10.8 and 10.9 and the definition of ν_c. (Recall that we showed $H = G$ in the proof of Theorem 10.7). ■

Theorem 10.11. *For models satisfying Assumptions 10.1 and 10.2, if $d \geq 5$ there exists $g_0 = g_0(d,J)$ such that if $0 < g < g_0$, then there are constants $C > 0$, $\varepsilon > 0$ such that*

$$G_{g,\nu_c}(x) \sim \frac{C}{|||x|||^{d-2}} + O\left(\frac{1}{|||x|||^{d-2+\varepsilon}}\right). \tag{10.26}$$

Proof. By Theorem 10.7, (Z5) part (b) $L_{g,\nu}$ is (right-)continuous in ν at ν_c and by the revised Lemma 4.5 part of Lemma 10.5, $L_{g,\nu} = O(g)$ for $\nu \in [\nu_c, g]$. Therefore $w(g,\nu_c) = \hat{J}^{-1}(1+O(g))$ and it suffices to prove (10.26) for $\tilde{G}_{g,\nu_c}$.

Let $Q(x) := w(g,\nu_c)\big(J_+(x) + \Psi_{g,\nu_c}(x)\big)$, and note that

$$D_{g,\nu}(x) = -\mathbb{1}_{\{x=0\}} + Q(x), \tag{10.27}$$

by the definition of $D_{g,\nu}$, see (4.11). Let $\hat{Q}(k) = \sum_{x\in\mathbb{Z}^d} Q(x) e^{ik\cdot x}$ be the Fourier transform of Q. By [22, Theorem 1.4], (10.26) holds for $\tilde{G}_{g,\nu_c}$ if there is a $\rho > 0$ such that

(H1) $\hat{Q}(0) = 1$,

(H2) $|Q(x)| \leq K_1 |||x|||^{-(d+2+\rho)}$, some $K_1 > 0$,

(H3) $\sum_{x\in\mathbb{Z}^d} \|x\|_2^{2+\rho} |Q(x)| \le K_2$, some $K_2 > 0$,
(H4) there is a $K_0 > 0$ such that $\hat{Q}(0) - \hat{Q}(k) \ge K_0 \|k\|_2^2$, $k \in [-\pi,\pi]^d$.

By Theorem 10.7, G_{g,ν_c} satisfies an infrared bound. Hence (H2)–(H3) follow from Proposition 9.8, (J4), and the assumption $d \ge 5$. Furthermore (H1) follows from $\sum_{x\in\mathbb{Z}^d} D_{g,\nu_c} = 0$, i.e., Lemma 10.10. Thus the proof of (10.26) is reduced to proving (H4).

To prove (H4) let $\hat{J}_+(k) = \sum_{x\in\mathbb{Z}^d} J_+(x)e^{ik\cdot x}$ be the Fourier transform of J_+ and note that $\hat{J}$ defined by (2.2) equals the Fourier transform $\hat{J}_+(k)$ evaluated at $k=0$ by (J1). By absorbing $w(g,\nu_c)$ into K_0 and $\hat{J} = \hat{J}_+(0)$, (H4) can be re-expressed as

$$\hat{J} - \hat{J}_+(k) + (\hat{\Psi}_{g,\nu_c}(0) - \hat{\Psi}_{g,\nu_c}(k)) \ge K_0 \|k\|_2^2.$$

By item (ii) of Proposition 9.8, the $\mathbb{Z}^d$-symmetry of Ψ_{g,ν_c}, $1 - \cos(k\cdot x) \le c_1 (k\cdot x)^2$ and $\sum_x |||x|||^{-3(d-2)} |x|^2 < \infty$ for $d \ge 5$,

$$\left|\hat{\Psi}_{g,\nu_c}(0) - \hat{\Psi}_{g,\nu_c}(k)\right| \le \sum_{x\in\mathbb{Z}^d} |\Psi_{g,\nu_c}(x)| \left(1 - \cos(k\cdot x)\right) = O(g)\,|k|^2$$

Thus to prove (H4) it suffices to show $\hat{J} - \hat{J}_+(k) \ge c' \|k\|_2^2$ for some $c' > 0$; the desired bound then follows by taking g small enough. The stated lower bound on $\hat{J} - \hat{J}_+(k)$ follows from [33, Lemma 2.3.2], whose hypotheses are provided by (J4) and the irreducibility assumption (J2). ∎

11 Verification of hypotheses

Recall that in Sections 4.2 and 4.3 we reduced the proof of Theorems 3.6 and 4.1 and Proposition 4.3 to the lemmas listed in Remark 4.13. Then, in Section 10 we revised these lemmas by replacing the lattice Edwards model and $n = 1,2$ $g|\varphi|^4$ models by Assumptions 10.1 and 10.2 in the hypotheses and we proved these revised lemmas. Therefore the next Lemma 11.1 completes the proof of Theorems 3.6 and 4.1 and Proposition 4.3.

Lemma 11.1. *The Edwards and $n = 1,2$ $g|\varphi|^4$ models defined in Sections 3.1 and 3.2 satisfy Assumptions 10.1 and 10.2.*

The remainder of this section proves this lemma, first for the Edwards model and then for the $g|\varphi|^4$ model. For each model we also prove Proposition 4.2.

11.1 Edwards model

By the argument below Lemma 3.4 Assumptions 10.1(ii) holds and $\nu_c \le 0$ as required by Assumptions 10.2. By the Definition 3.1 of the Edwards model it is clear that (Z1), (Z2), and (Z4) hold.

Short calculations starting from the definition (4.4) of $L^{(\Lambda)}_{g,\nu,x}$ and the definition (7.6) of $r^{(\Lambda)}_{s,s'}(x,y)$ show that

$$L^{(\Lambda)}_{g,\nu,x} = -\nu \mathbb{1}_{\{x\in\Lambda\}}, \qquad r^{(\Lambda)}_{s,s'}(x,y) = -2g\mathbb{1}_{\{x=y\in\Lambda\}} \tag{11.1}$$

and therefore the infinite volume limits are $L_{g,\nu} = -\nu$ and $r_{s,s'}(x,y) = -2g\mathbb{1}_{\{x=y\}}$. From (11.1) we immediately obtain (Z3), (Z6), and Assumptions 10.1(iii). The continuity statements in (Z5) are clear from (11.1), and the claim $L_{g,\nu} = O(g)$ in part (b) of (Z5) follows as $L_{g,\nu} \le 0$ implies $\nu \ge 0$, and (b) includes the hypothesis $\nu \in (\nu_c(g), g]$.

By (3.4) and (3.1) it follows that

$$Y_{\boldsymbol{t},\boldsymbol{s}} := \exp\left\{-g\sum_{x\in\Lambda}(2t_x s_x + s_x^2) - \nu\sum_{x\in\Lambda}s_x\right\} \tag{11.2}$$

which is decreasing in t_x for each x, so (G1) holds. To verify (G2), note that $\{X_\ell^{(\Lambda)} = b\} = \{T^{(\Lambda)} > \ell, X_\ell^{(\infty)} = b\}$ and on this event $\boldsymbol{\tau}_{[0,\ell]}{}^{(\Lambda)} = \boldsymbol{\tau}_{[0,\ell]}{}^{(\infty)}$. Hence

$$Y_{\boldsymbol{0},\boldsymbol{\tau}^{(\Lambda)}_{[0,\ell]}}\mathbb{1}_{\{X_\ell^{(\Lambda)}=b\}} = Y_{\boldsymbol{0},\boldsymbol{\tau}^{(\infty)}_{[0,\ell]}}\mathbb{1}_{\{X_\ell^{(\infty)}=b\}}\mathbb{1}_{\{T^{(\Lambda)}>\ell\}}, \tag{11.3}$$

which is non-negative and increasing in Λ since $T^{(\Lambda)}$ is.

(G3). Monotonicity in ν is clear from (3.4). We defer the proofs of (G3) parts (a) and (b) until after Lemma 11.4 below, as they are very similar to the detailed proof of Lemma 11.4.

Lemma 10.3 shows that (G4) is a consequence of (Z1), (G1), (G2) which we have already established. (G5) is clear since $G_{g,g}^{(\infty)} \le S$ by (3.4).

We complete the proof of Assumptions 10.2 by showing $\nu_c \le 0$. This is immediate from Definition 3.1: if $\nu > 0$ the Green's function is dominated by the Green's function of a simple random walk with non-zero killing.

Proof of Proposition 4.2 for Edwards model. By $L_{g,\nu} = -\nu$ this is immediate from $L_{g,\nu_c} = O(g)$ by (Z5) part (b) and Lemma 4.5 part (ii). ∎

11.2 $g|\varphi|^4$ theory with $n = 1,2$

Recall the definition of $\langle\cdot\rangle_{g,\nu,\boldsymbol{t}}^{(\Lambda)}$ from (3.9). In this section we abbreviate this to $\langle\cdot\rangle_{\boldsymbol{t}}^{(\Lambda)}$. By the argument below Lemma 3.4 Assumptions 10.1(ii) holds and $\nu_c \le 0$ as required by Assumptions 10.2. From Definition 3.2 it is clear that (Z1), (Z2) and (Z4) hold. By the definition (4.4) of $L_{g,\nu,x}^{(\Lambda)}$ and the definition (7.6) of $r_{s,s'}^{(\Lambda)}(x,y)$ straightforward calculations show that

$$L_{g,\nu,x}^{(\Lambda)} = \left(-\nu - g\left\langle|\varphi_x|^2\right\rangle_{\boldsymbol{0}}^{(\Lambda)}\right)\mathbb{1}_{\{x\in\Lambda\}} \tag{11.4}$$

$$r_{s,s'}^{(\Lambda)}(x,y) = \left(-2g\mathbb{1}_{\{x=y\}} + g^2\left\langle|\varphi_x|^2;|\varphi_y|^2\right\rangle_{\boldsymbol{\tau}^{(\Lambda)}_{[s,s']}}^{(\Lambda)}\right)\mathbb{1}_{\{x,y\in\Lambda\}}, \tag{11.5}$$

where $\langle A;B\rangle^{(\Lambda)} := \langle AB\rangle^{(\Lambda)} - \langle A\rangle^{(\Lambda)}\langle B\rangle^{(\Lambda)}$. Assumption (Z3) follows by (11.4) and Lemma 9.1. The continuity statements in (Z5) follow from (G3), which we will verify below. The $L_{g,\nu} = O(g)$ part of (Z5) follows from (11.4) as the 3-IRB implies the expectation is finite. Assumption (Z6) follows from (11.4).

Lemma 11.2. *For the n-component $g|\varphi|^4$-model with $n=1,2$, $a,b\in\Lambda$ and $\boldsymbol{t}\in[0,\infty)^\Lambda$, $G^{(\Lambda)}_{g,\nu,\boldsymbol{t}}(a,b)$ is non-decreasing in ν, in each component of $\boldsymbol{t}$, and in Λ.*

Proof. These statements follow by writing $G^{(\Lambda)}_{g,\nu,\boldsymbol{t}}(a,b)$ in terms of $\langle\varphi_a\cdot\varphi_b\rangle^{(\Lambda)}_{g,\nu,\boldsymbol{t}}$ using Theorem 3.3. We prove that this expectation has the claimed monotonicity in ν by showing that the derivative with respect to ν is nonpositive. Up to a positive constant of proportionality this derivative is *minus* the sum over $y\in\Lambda$ of $\langle\varphi_a\cdot\varphi_b;\varphi_y\cdot\varphi_y\rangle^{(\Lambda)}_{g,\nu,\boldsymbol{t}}$. As desired this correlation is nonnegative by the GKS inequality for $n=1$ and the Ginibre inequality for $n=2$. See [17, Lemmas 11.3 and 11.4]. Monotonicity in $\boldsymbol{t}$ is proved in a similar way by differentiating $\langle\varphi_a\cdot\varphi_b\rangle^{(\Lambda)}_{g,\nu,\boldsymbol{t}}$ with respect to a component t_y of $\boldsymbol{t}$. Monotonicity in Λ follows from monotonicity in $\boldsymbol{t}$ because letting $t_y\uparrow\infty$ is the same as adding the point y to Λ. ■

Proposition 11.3 (Lebowitz Inequality)**.** *Consider the $g|\varphi|^4$ model with $n=1,2$ components. Then for all Λ, $x,y,u\in\Lambda$ and $\boldsymbol{t}\in[0,\infty)^\Lambda$,*

$$0\le\langle\varphi_x\cdot\varphi_y;\varphi_u\cdot\varphi_u\rangle^{(\Lambda)}_{\boldsymbol{t}}\le 2\,\langle\varphi_x\cdot\varphi_u\rangle^{(\Lambda)}_{\boldsymbol{t}}\,\langle\varphi_y\cdot\varphi_u\rangle^{(\Lambda)}_{\boldsymbol{t}}.$$

Proof. The lower bound is the GKS inequality for $n=1$ and the Ginibre inequality for $n=2$. See [17, Lemmas 11.3 and 11.4]. The upper bound for $n=1$ is the Lebowitz inequality [34] and for $n=2$ was proved by Bricmont [8, Theorem 2.1]. See also [17, Theorem 12.1]. ■

Both (G1) and (G2) follow immediately from Lemma 11.2. By Lemma 10.3 property (G4) is a consequence of (Z1), (G1) and (G2). Property (G5) is clear since $G^{(\infty)}_{g,g}\le S$ by (3.4). The next lemma establishes (G3) parts (a) and (b); that $G^{(\Lambda)}_{g,\nu}(x)$ is non-increasing in ν for $\nu\in\mathbb{R}$ was already established in Lemma 11.2.

Lemma 11.4.

(1) $G_{g,\nu}(x)$ is Lipschitz as a function of $\nu\in(\nu_c,\infty)$.
(2) If $d\ge5$ and G_{g,ν_c} satisfies a K-IRB for some K then $G_{g,\nu}(x)$ is uniformly Lipschitz as a function of $\nu\in[\nu_c,\infty)$, and hence $\{G_{g,\nu}(x)\}_{x\in\mathbb{Z}^d}$ is uniformly equicontinuous in $\nu\in[\nu_c,\infty)$.

Proof. (1) For any finite volume Λ and any ν, by Proposition 11.3, (3.10) and (G2)

$$\begin{aligned}\big\langle\varphi_0\cdot\varphi_x;|\varphi_y|^2\big\rangle^{(\Lambda)}_{\boldsymbol{0}}&\le 2\,\langle\varphi_0\cdot\varphi_y\rangle^{(\Lambda)}_{\boldsymbol{0}}\,\langle\varphi_x\cdot\varphi_y\rangle^{(\Lambda)}_{\boldsymbol{0}}\le 2n^2G_{g,\nu}(y)G_{g,\nu}(y-x)\\&\le n^2G^2_{g,\nu}(y)+n^2G^2_{g,\nu}(y-x),\end{aligned}\tag{11.6}$$

and the final inequality is the elementary inequality $2uv\le u^2+v^2$ for $u,v\in\mathbb{R}$. Since

$$-\frac{\partial}{\partial\nu}G^{(\Lambda)}_{g,\nu}(x)=\frac{1}{2}\sum_{y\in\Lambda}\big\langle\varphi_0\cdot\varphi_x;\varphi_y^2\big\rangle^{(\Lambda)}_{\boldsymbol{0}}\tag{11.7}$$

we have, for ν and a such that $\nu \geq a \geq \nu_c$,

$$|\frac{\partial}{\partial \nu} G^{(\Lambda)}_{g,\nu}(x)| \leq n^2 \sum_{y\in\mathbb{Z}^d} G^2_{g,\nu}(y) = c_a, \tag{11.8}$$

where $c_a = n^2 \sum_{y\in\mathbb{Z}^d} G^2_{g,a}(y)$ and c_a is finite for $a > \nu_c$ because $G_{g,a}(y)$ is summable by (10.2). For $a > \nu_c$ and $\nu, \nu' \in [a,\infty)$, by writing $G^{(\Lambda)}_{g,\nu'}(x) - G^{(\Lambda)}_{g,\nu}(x)$ as the integral of its derivative we have $|G^{(\Lambda)}_{g,\nu'}(x) - G^{(\Lambda)}_{g,\nu}(x)| \leq c_a|\nu' - \nu|$. Taking $\Lambda \uparrow \mathbb{Z}^d$ by (G2), we obtain $|G_{g,\nu'}(x) - G_{g,\nu}(x)| \leq c_a|\nu' - \nu|$ and therefore $G_{g,\nu}(x)$ is Lipschitz as claimed.

(2) We repeat part (1) with $a = \nu_c$. Since $d \geq 5$, c_{ν_c} is finite by the K-IRB, so $G_{g,\nu}(x)$ is Lipschitz on $[\nu_c,\infty)$ with uniform constant c_{ν_c}. ■

Proof of (G3) parts (a) and (b) for the Edwards model. We first claim that for any finite volume Λ and any $\nu \in \mathbb{R}$,

$$-\frac{d}{d\nu} G^{(\Lambda)}_{g,\nu}(x) \leq G^{(\Lambda)}_{g,\nu} * G^{(\Lambda)}_{g,\nu}(x) \leq \frac{1}{2}\sum_{y\in\mathbb{Z}^d} ((G^{(\Lambda)}_{g,\nu}(y-x))^2 + (G^{(\Lambda)}_{g,\nu}(y))^2), \tag{11.9}$$

where the second inequality is the elementary $2ab \leq a^2 + b^2$. Granting the claim, note that by (G2) and translation invariance this proves (11.8), and the remainder of the proof is essentially identical to the proof above.

We now prove the claimed first inequality in (11.9). By the definitions (3.2) and (2.5), the left-hand side of (11.9) is

$$\sum_{x'\in\Lambda} \int_{[0,\infty)} d\ell \int_{[0,\ell]} d\ell' \; E_a \left[\mathscr{Y}_{0,\ell} \mathbb{1}_{\{X^{(\Lambda)}_{\ell'}=x'\}} \mathbb{1}_{\{X^{(\Lambda)}_{\ell}=x\}} \right]. \tag{11.10}$$

where $\mathscr{Y}_{s,t} = Z^{(\Lambda)}_{\tau^{(\Lambda)}_{[s,t]}} / Z^{(\Lambda)}_{\mathbf{0}}$ as in (6.1), and, for the Edwards model, $Z^{(\Lambda)}_{\mathbf{0}} = 1$. We reverse the order of integration over ℓ, ℓ' and insert $\mathscr{Y}_{0,\ell} = \mathscr{Y}_{0,\ell'} \bar{\mathscr{Y}}_{0,\ell'}(\ell)$, where $\bar{\mathscr{Y}}_{0,\ell'}(\ell) = \left(\mathscr{Y}_{0,\ell}/\mathscr{Y}_{0,\ell'}\right)$ as in (7.10). By Lemma 7.5 with $H = \mathscr{Y}_{0,\ell'}$ and $(u_1,u_2,u_3) = (0,\ell',\ell')$ the result is

$$\sum_{x'\in\Lambda} \int_{[0,\infty)} d\ell' \; E_a \left[\mathscr{Y}_{0,\ell'} G^{(\Lambda)}_{\tau^{(\Lambda)}_{[0,\ell']}} (X^{(\Lambda)}_{\ell'}, x) \mathbb{1}_{\{X^{(\Lambda)}_{\ell'}=x'\}} \right], \tag{11.11}$$

By (G1) and (3.2) read from right to left we obtain the first inequality in (11.9) as desired. ■

For $0 < u < v$ define

$$\bar{r}^{(\Lambda)}_{u,v}(x,y) := 2g \left[\mathbb{1}_{\{x=y\}} + n^2 g \Big(G^{(\Lambda)}_{\tau^{(\Lambda)}_{[u,v]}}(x-y) \Big)^2 \right], \tag{11.12}$$

$$\bar{r}^{(\Lambda)}(x,y) := \bar{r}^{(\Lambda)}_{0,0}(x,y). \tag{11.13}$$

The next lemma verifies Assumptions 10.1(iii).

Lemma 11.5. *Suppose* $0 < u < v$*,* $x, y \in \Lambda$*. Then*

$$\left| r^{(\Lambda)}_{u,v}(x,y) \right| \le \bar{r}^{(\Lambda)}_{u,v}(x,y) \le \bar{r}^{(\Lambda)}(x,y). \tag{11.14}$$

Proof. As Λ is fixed we will omit it from the notation. Applying the triangle inequality to (11.5) and using Proposition 11.3 and (3.10),

$$r_{u,v}(x,y) \le \bar{r}_{u,v}(x,y), \tag{11.15}$$

The remaining inequality $\bar{r}_{u,v}(x,y) \le \bar{r}(x,y)$ follows by Lemma 11.2. ∎

Proof of Proposition 4.2 for the $g|\varphi|^4$ *model.* The same argument as for the Edwards model shows L_{g,ν_c} is $O(g)$. Since $L_{g,\nu} = -g\left\langle \varphi_x^2 \right\rangle_0 - \nu$, and $g\left\langle \varphi_x^2 \right\rangle_0$ is $O(g)$ at ν_c since an infrared bound holds, the claim follows. ∎

Appendix: Random walk and the Markov property

A.1 Properties of continuous-time random walk

Proof of Lemma 2.3. We first prove that $\Delta^{(\Lambda)}$ is invertible. Let $f, h\colon \Lambda \to \mathbb{R}$. The quadratic form associated to $\Delta^{(\Lambda)}$ is given by

$$(f, -\Delta^{(\Lambda)} h) := \sum_{x \in \Lambda} f_x (-\Delta^{(\Lambda)} h)_x. \tag{A.1}$$

For $f\colon \Lambda \to \mathbb{R}$ define the extension by zero: $\tilde{f} = f$ on Λ and $\tilde{f}_x = 0$ for $x \notin \Lambda$. We *claim* that

$$(f, -\Delta^{(\Lambda)} h) = \frac{1}{2} \sum_{x,y \in \mathbb{Z}^d} J(x-y)(\tilde{f}_x - \tilde{f}_y)(\tilde{h}_x - \tilde{h}_y). \tag{A.2}$$

By choosing $h = f$ we obtain

$$(f, -\Delta^{(\Lambda)} f) = \frac{1}{2} \sum_{x \neq y \in \mathbb{Z}^d} J(x-y)|\tilde{f}_x - \tilde{f}_y|^2 > 0, \quad f \neq 0. \tag{A.3}$$

The strict inequality holds because $\tilde{f}_y = 0$ for $y \notin \Lambda$ and for every point $v \in \Lambda$ there is a walk with transitions of nonzero rate that starts at v and reaches a point not in Λ. This positivity implies that the eigenvalues of $-\Delta^{(\Lambda)}$ are strictly positive and therefore $-\Delta^{(\Lambda)}$ is invertible as desired. Thus it suffices to prove the claim (A.2).

To prove (A.2) we start with the right-hand side which contains

$$(\tilde{f}_x - \tilde{f}_y)(\tilde{h}_x - \tilde{h}_y) = \tilde{f}_x(\tilde{h}_x - \tilde{h}_y) + \tilde{f}_y(\tilde{h}_y - \tilde{h}_x), \tag{A.4}$$

so by the symmetry under exchanging x and y we can rewrite the right-hand side of (A.2) as

$$\begin{aligned} \frac{1}{2} \sum_{x,y \in \mathbb{Z}^d} J(x-y)(\tilde{f}_x - \tilde{f}_y)(\tilde{h}_x - \tilde{h}_y) &= \sum_{x,y \in \mathbb{Z}^d} J(x-y)\tilde{f}_x(\tilde{h}_x - \tilde{h}_y) \\ &= \sum_{x,y \in \mathbb{Z}^d} J(x-y)\tilde{f}_x(-\tilde{h}_y). \end{aligned} \tag{A.5}$$

For the final equality we used the zero row sum property $\sum_y J(x-y)=0$. Recall from (2.1) that $J(x-y)=\Delta^{(\infty)}_{x,y}$ and that $\Delta^{(\Lambda)}_{x,y}$ is the restriction of $\Delta^{(\infty)}_{x,y}$ to Λ. Therefore, in (A.5) we insert

$$\sum_{y\in\mathbb{Z}^d} J(x-y)(-\tilde{h}_y)=\sum_{y\in\mathbb{Z}^d}(-\Delta^{(\infty)}_{x,y})\tilde{h}_y=\sum_{y\in\Lambda}(-\Delta^{(\infty)}_{x,y})h_y=(-\Delta^{(\Lambda)}h)_x \tag{A.6}$$

which proves (A.2) and hence completes the proof that $\Delta^{(\Lambda)}$ is invertible.

Next we prove (2.7). By definition

$$S^{(\Lambda)}(a,b)=\int_0^\infty dt\,\mathbb{P}_a(X_t=b)=\int_0^\infty dt\,(e^{t\Delta_*})_{a,b}=\int_0^\infty dt\,(e^{t\Delta^{(\Lambda)}})_{a,b}. \tag{A.7}$$

where the last equality holds since $(\Delta_*^k)_{a,b}=((\Delta^{(\Lambda)})^k)_{a,b}$, and where for a square matrix A, e^{tA} denotes the matrix exponential $\sum_{k=0}^\infty \frac{t^k}{k!}A^k$. The right hand side of (A.7) is $(-\Delta^{(\Lambda)})^{-1}_{a,b}$ as desired: since $-\Delta^{(\Lambda)}$ is real symmetric with positive eigenvalues, this follows by diagonalizing and integrating. ■

Proof of Lemma 2.4. Recall the definition of $\tilde{S}_z(x)$ from (4.7), and let $\tilde{S}(x)=\tilde{S}_{\hat{J}^{-1}}(x)$. Let T_n denote the time of the nth jump of $X^{(\infty)}$ and $T_0=0$. By (2.8)

$$\begin{aligned} S(x)&=E_0\Big[\sum_{n=0}^\infty (T_{n+1}-T_n)\mathbb{1}_{\{X^{(\infty)}_{T_n}=x\}}\Big]\\ &=\sum_{n=0}^\infty E_0[T_{n+1}-T_n]E_0\Big[\mathbb{1}_{\{X^{(\infty)}_{T_n}=x\}}\Big] &\text{(A.8)}\\ &=\hat{J}^{-1}\sum_{n=0}^\infty E_0\Big[\mathbb{1}_{\{X^{(\infty)}_{T_n}=x\}}\Big]=\hat{J}^{-1}\tilde{S}(x). &\text{(A.9)}\end{aligned}$$

By using the strong Markov property to restart at time T_1 (see [33, Section 4.3]), we have

$$\tilde{S}(x)=\mathbb{1}_{\{x=0\}}+\sum_y \hat{J}^{-1}J_+(y)\tilde{S}(x-y), \tag{A.10}$$

and by (A.9) this can be rewritten as

$$\hat{J}S(x)=\mathbb{1}_{\{x=0\}}+\sum_y J_+(y)S(x-y). \tag{A.11}$$

Collecting terms gives the first claim. To verify (2.13), we use (A.8):

$$S(x)=\sum_{n=0}^\infty E_0[T_{n+1}-T_n]P_0(X^{(\infty)}_{T_n}=x)=\sum_{n=0}^\infty \hat{J}^{-1}(\hat{J}^{-1}J_+)^{*n}(x). \qquad ■$$

A.2 The Markov Property

Recall that Ω_1 is the space of paths defined in Section 2.4 and $(\Omega_1,\mathscr{F},P_x)$ is the probability space for random walk X_t such that $X_0=x$. For $s\geq 0$ define the $\mathscr{F}$-measurable map $\theta_s\colon \Omega_1\to\Omega_1$ by $\theta_s((x_t)_{t\geq 0})=(x_{s+t})_{t\geq 0}$. The following is a standard formulation of the Markov property.

Proposition A.1. *Let $H\colon \Omega_1 \to \mathbb{R}$ be $\mathscr{F}$-measurable and integrable with respect to E_a for each $a \in \mathbb{Z}^d$, and let $h(x) = E_x\big[H\big]$. Then for every $x \in \mathbb{Z}^d$ and $s \geq 0$,*

$$E_x\big[H \circ \theta_s \big| \mathscr{F}_s\big] = h(X_s), \quad P_x\text{-a.s.} \tag{A.12}$$

The Markov property as used to obtain (6.12)

To justify this application of the Markov property, for $\ell > 0$ and $b \in \mathbb{Z}^d$ let $H_{\ell,b} := \mathscr{Y}_{0,\ell}\, \mathbb{1}_{\{X_\ell = b\}}$ and $h(\ell,y,b) := E_y\,[H_{\ell,b}]$. Then, by Proposition A.1,

$$E_a\big[\mathscr{Y}_{s,s+\ell}\, \mathbb{1}_{\{X_{s+\ell}=b\}} \big| \mathscr{F}_s\big] = h(\ell, X_s, b), \quad P_a\text{-a.s.} \tag{A.13}$$

The Markov property as used to obtain (6.15)

To justify this application of the Markov property, for $L \in \mathscr{L}_m(0)$ let $H_{L,b} := w(L) G_{\mathbf{0}}(X_{s'_m}, b)$, and $f(x,b,L) := E_x\,[H_{L,b}]$. Then, by Proposition A.1,

$$E_a\big[w(L+s)\; G_{\mathbf{0}}(X_{s'_m+s}, b) \big| \mathscr{F}_s\big] = f(X_s, b, L). \tag{A.14}$$

The Markov property in the proof of Lemma 7.5

Fix $b \in \Lambda$. We assume that the function $G^{(\Lambda)} : (\boldsymbol{r}, x) \mapsto G^{(\Lambda)}_{\boldsymbol{r}}(x,b)$ is defined for $(\boldsymbol{r}, x) \in [0,\infty)^\Lambda \times \Lambda$ and is bounded on this domain. Let $\mathscr{S}$ be a $(\mathscr{F}_t)_{t\geq 0}$-stopping time which is P_a-a.s. finite. By definition the random variable $G^{(\Lambda)}_{\boldsymbol{\tau}_I}(X_{\mathscr{S}}, b)$ is the composition of $G^{(\Lambda)}$ with $(\boldsymbol{\tau}_I, X_{\mathscr{S}})$.

Lemma A.2. *For $b \in \Lambda$, and any $\mathscr{F}_{\mathscr{S}}$-measureable Borel set $I \subset [0,\mathscr{S}]$,*

$$G^{(\Lambda)}_{\boldsymbol{\tau}_I}(X_{\mathscr{S}}, b) = E_a\Big[\int_{[\mathscr{S},\infty)} d\ell\; \frac{Z_{\boldsymbol{\tau}_I + \boldsymbol{\tau}_{[\mathscr{S},\ell]}}}{Z_{\boldsymbol{\tau}_I}}\, \mathbb{1}_{\{X_\ell = b\}} \Big| \mathscr{F}_{\mathscr{S}}\Big] \quad P_a\text{-a.s.} \tag{A.15}$$

The proof of Lemma A.2 requires two preparatory ingredients stated for a general probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Lemma A.3. *Let $W : \Omega \to \mathbb{R}$ be integrable with respect to $\mathbb{P}$, $A \in \mathscr{F}$ with $\mathbb{P}(A) > 0$ and $\mathbb{E}_A = \mathbb{E}[\cdot|A]$. Then*

$$\mathbb{1}_A \mathbb{E}[W|\mathscr{F}] = \mathbb{1}_A \mathbb{E}_A[W|\mathscr{F}], \qquad \mathbb{P}\text{-a.s.} \tag{A.16}$$

Proof. Let L and R denote the left- and right-hand sides, respectively. Then both L and R are $\mathscr{F}$-measurable. Let $B \in \mathscr{F}$. Then $\mathbb{E}[\mathbb{1}_B(L-R)] = 0$ since

$$\mathbb{E}[\mathbb{1}_B \mathbb{1}_A \mathbb{E}_A[W|\mathscr{F}]] = \mathbb{E}_A[\mathbb{1}_B \mathbb{1}_A \mathbb{E}_A[W|\mathscr{F}]]\mathbb{P}(A) \tag{A.17}$$

$$= \mathbb{E}_A[\mathbb{1}_B \mathbb{1}_A W]\mathbb{P}(A) \tag{A.18}$$

$$= \mathbb{E}[\mathbb{1}_B \mathbb{1}_A W] = \mathbb{E}[\mathbb{1}_B \mathbb{1}_A \mathbb{E}[W|\mathscr{F}]]. \tag{A.19}$$

Taking $B_1 = \{L > R\} \in \mathscr{F}$ and $B_2 = \{L < R\} \in \mathscr{F}$ completes the proof. ∎

The following lemma is a standard result in the case that $\mathscr{G} = \sigma(W_1)$, see [13, Example 5.1.5]. Since we have been unable to find this particular formulation in the literature, we give a proof below.

Lemma A.4. *For measurable spaces $(S_1,\mathscr{S}_1)$ and $(S_2,\mathscr{S}_2)$ let $W_1\colon \Omega\to S_1$ and $W_2\colon \Omega\to S_2$ be measurable, and let $f\colon S_1\times S_2\to\mathbb{R}$ be Borel-measurable on the corresponding product space $(S_1\times S_2,\mathscr{S})$ and either bounded or non-negative and such that $\mathbb{E}[f(W_1,W_2)]$ is finite. Define $h\colon S_1\to\mathbb{R}$ by*

$$h(w_1)=\mathbb{E}[f(w_1,W_2)]. \tag{A.20}$$

If W_2 is independent of $\mathscr{G}\subset\mathscr{F}$ and W_1 is $\mathscr{G}$-measurable then

$$h(W_1)=\mathbb{E}[f(W_1,W_2)|\mathscr{G}], \quad a.s. \tag{A.21}$$

Proof. If $f(w_1,w_2)=f_1(w_1)f_2(w_2)$, where f_1 and f_2 are bounded and $\mathscr{B}(\mathbb{R})$-measurable then $h(w_1)=f_1(w_1)\mathbb{E}[f_2(W_2)]$. Then $h(W_1)=f_1(W_1)\mathbb{E}[f_2(W_2)]$ (is $\mathscr{G}$ measurable) and by independence, almost surely

$$f_1(W_1)\mathbb{E}[f_2(W_2)]=\mathbb{E}[f_1(W_1)f_2(W_2)|\mathscr{G}]. \tag{A.22}$$

In particular this holds for any f of the form $f(w_1,w_2)=\mathbb{1}_{A_1\times A_2}\equiv\mathbb{1}_{\{w_1\in A_1\}}\mathbb{1}_{\{w_2\in A_2\}}$, where $A_i\in\mathscr{S}_i$. Therefore by linearity of expectation it also holds for indicators of finite disjoint unions of events of the form $A_1\times A_2$.

Let $\mathscr{A}\subset\mathscr{S}$ denote the collection of events for which the claim of the lemma holds with $f=\mathbb{1}_A$. Then $\mathscr{A}$ contains the field of finite disjoint unions of events of the form $A_1\times A_2$, and by dominated convergence $\mathscr{A}$ is a monotone class. Thus by the Monotone Class Theorem $\mathscr{A}=\mathscr{S}$, and hence by linearity the claim holds for all simple functions f.

For non-negative f such that $f(W_1,W_2)$ is integrable we can take non-negative simple functions f_n increasing to f pointwise. Let $h_n(w_1)=\mathbb{E}[f_n(w_1,W_2)]$. Then $h_n(w_1)\uparrow\mathbb{E}[f(w_1,W_2)]=:h(w_1)$ pointwise by monotone convergence. Next, by the result for simple functions we have for each n

$$h_n(W_1)=\mathbb{E}[f_n(W_1,W_2)|\mathscr{G}]. \tag{A.23}$$

The right hand side increases to $\mathbb{E}[f(W_1,W_2)|\mathscr{G}]$ by monotone convergence and the left hand side increases to $h(W_1)$ by the above pointwise convergence. This proves the result for non-negative f such that $f(W_1,W_2)$ is integrable.

The claim for bounded measurable f follows by considering the positive and negative parts of f. ■

Proof of Lemma A.2. For $\boldsymbol{r}\in[0,\infty)^\Lambda$ and a path $\tilde{y}$ in Ω_1 let

$$f(\boldsymbol{r},\tilde{y}):=\int_{[0,\infty)}d\ell'\,\frac{Z_{\boldsymbol{r}+\tilde{\boldsymbol{\tau}}_{[0,\ell']}(\tilde{y})}}{Z_{\boldsymbol{r}}}\mathbb{1}_{\{\tilde{y}_{\ell'}=b\}}, \tag{A.24}$$

where $\tilde{\boldsymbol{\tau}}_{[u,v]}(\tilde{y})$ denotes the vector of local times of the path $\tilde{y}$ in the interval $[u,v]$.

Let $\theta_{\mathscr{S}}$ be the time shift $\theta_s:\Omega_1\to\Omega_1$ defined at the beginning of this subsection with $s=\mathscr{S}$. Since $\mathscr{S}$ is finite $\theta_{\mathscr{S}}$ is defined for P_a almost all paths. Let $\tilde{X}=\theta_{\mathscr{S}}(X)$. Then $\boldsymbol{\tau}_{[\mathscr{S},\ell]}=\tilde{\boldsymbol{\tau}}_{[0,\ell-\mathscr{S}]}$, where $\tilde{\boldsymbol{\tau}}$ is the local time of $\tilde{X}$, and $\mathbb{1}_{\{X_\ell=b\}}=\mathbb{1}_{\{\tilde{X}_{\ell-\mathscr{S}}=b\}}$. Equalities of random variables in this proof are P_a-a.s. Integrals and expectations are applied only to nonnegative functions and integrability is eventually implied by the assumed boundedness of $G^{(\Lambda)}_{\boldsymbol{\tau}_I}(X_{\mathscr{S}},b)$. Let R be the right hand side of (A.15). Then

$$\begin{aligned} R &= E_a\Big[\int_{[\mathscr{S},\infty)} d\ell\ \frac{Z_{\boldsymbol{\tau}_I+\tilde{\boldsymbol{\tau}}_{[0,\ell-\mathscr{S}]}}}{Z_{\boldsymbol{\tau}_I}}\mathbb{1}_{\{\tilde{X}_{\ell-\mathscr{S}}=b\}}\big|\mathscr{F}_{\mathscr{S}}\Big] \\ &= E_a\Big[f(\boldsymbol{\tau}_I,\tilde{X})\big|\mathscr{F}_{\mathscr{S}}\Big] = \sum_{x\in\Lambda}\mathbb{1}_{\{X_{\mathscr{S}}=x\}}E_a\Big[f(\boldsymbol{\tau}_I,\tilde{X})\big|\mathscr{F}_{\mathscr{S}}\Big]. \end{aligned} \tag{A.25}$$

The first equality is obtained by the change of variables $\ell = \ell' + \mathscr{S}$ followed by inserting the definition of $f(\boldsymbol{r},\tilde{y})$ and the second equality is obtained by inserting $1 = \sum_{x\in\Lambda}\mathbb{1}_{\{X_{\mathscr{S}}=x\}}$ under the conditional expectation. The indicator function can be moved outside since the indicator function is $\mathscr{F}_{\mathscr{S}}$ measurable. If for some x, $P_a(X_{\mathscr{S}} = x) = 0$, then the corresponding contribution to the above sum is 0 P_a-a.s. Otherwise, let $\tilde{P}_x(\cdot) := P_a(\cdot|\tilde{X}_0 = x)$. By Lemma A.3 (A.25) becomes

$$R = \sum_{x\in\Lambda}\mathbb{1}_{\{X_{\mathscr{S}}=x\}}\tilde{E}_x\Big[f(\boldsymbol{\tau}_I,\tilde{X})\big|\mathscr{F}_{\mathscr{S}}\Big]. \tag{A.26}$$

Let $h_x(\boldsymbol{r}) = \tilde{E}_x\Big[f(\boldsymbol{r},\tilde{X})\Big]$. By the strong Markov property $\tilde{X}$ is a random walk with $\tilde{X}_0 = x$ that is independent of $\mathscr{F}_{\mathscr{S}}$. By Lemma A.4 $\tilde{E}_x\big[f(\boldsymbol{\tau}_I,\tilde{X})\big|\mathscr{F}_{\mathscr{S}}\big] = h_x(\boldsymbol{\tau}_I)$, but this is a $\tilde{P}_x$-a.s.equality. However it also holds P_a-a.s. because $P_a = \sum_{x\in\Lambda}P_a(\tilde{X}_0 = x)\tilde{P}_x(\cdot)$ is a countable sum. Therefore (A.26) becomes

$$R = \sum_{x\in\Lambda}\mathbb{1}_{\{X_{\mathscr{S}}=x\}}h_x(\boldsymbol{\tau}_I) = \sum_{x\in\Lambda}\mathbb{1}_{\{X_{\mathscr{S}}=x\}}G^{(\Lambda)}_{\boldsymbol{\tau}_I}(x,b) = G^{(\Lambda)}_{\boldsymbol{\tau}_I}(X_{\mathscr{S}},b) \qquad P_a\text{-a.s.} \tag{A.27}$$

as desired. The second equality is obtained from (3.2) by interchanging the integral over ℓ' in the definition of $f(\boldsymbol{\tau}_I,\tilde{X})$ with the expectation. ■

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