COHOMOGENEITY ONE TOPOLOGICAL MANIFOLDS REVISITED

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Abstract. We prove a structure theorem for closed topological manifolds of cohomogeneity one; this result corrects an oversight in the literature. We complete the equivariant classification of closed, simply-connected cohomogeneity one topological manifolds in dimensions 5, 6, and 7 and obtain topological characterizations of these spaces. In these dimensions, these manifolds are homeomorphic to smooth manifolds.

1. Main Results

A topological manifold with an (effective) topological action of a compact Lie group is of *cohomogeneity one* if its orbit space is one-dimensional. These manifolds were introduced by Mostert [42] in 1957 and their topology and geometry have been extensively studied in the smooth category (see, for example, [15, 16, 19, 28, 29, 30, 45, 48] and [3, 9, 11, 20, 21, 22, 23, 52, 51, 61]). Much less attention has been given to these spaces in the topological category, probably because of the assertion in [42] that every topological manifold of cohomogeneity one is equivariantly homeomorphic to a smooth manifold. This statement originates in the claim that an integral homology sphere that is also a homogeneous space for a compact Lie group must be a sphere (see [42, Section 2, Corollary]). This, however, is not the case. Indeed, the Poincaré homology sphere $\mathbf{P}^3$ is a homogeneous space for the Lie groups $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ and it can be written as $\mathbf{P}^3 \approx \mathrm{SU}(2)/I^* \approx \mathrm{SO}(3)/I$, where I^* is the binary icosahedral group, I is the icosahedral group (see [34] or [56, p. 89]), and the symbol "$\approx$" denotes homeomorphism between topological spaces. One can combine the fact that $\mathbf{P}^3$ is a homogeneous space with the Double Suspension Theorem of Edwards and Cannon [8, 13] to construct topological manifolds with cohomogeneity one actions that are not equivariantly homeomorphic to smooth actions (see Example 2.4). We point out that, by work of Bredon [4], the Poincaré homology sphere is the only integral homology sphere that is also a homogeneous space, besides the usual spheres. In the present article we fix the gap in [42] and explore some of its consequences.

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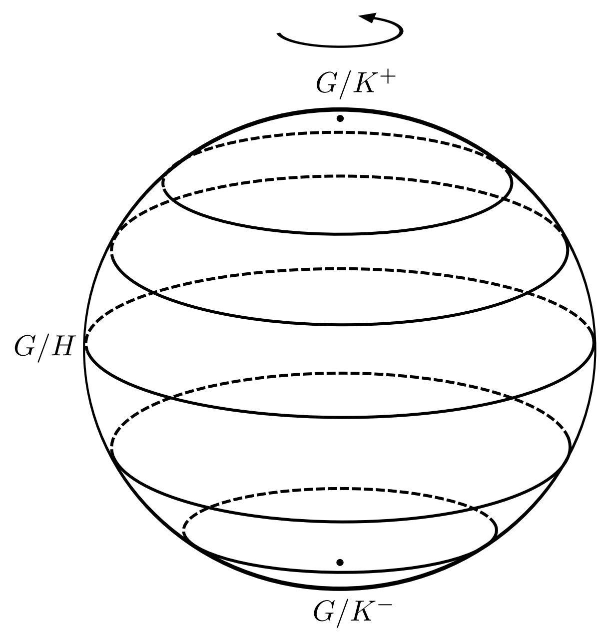

FIGURE 1. Cohomogeneity one action of SO(2) on a round 2-sphere.

Our first result is a complete structure theorem for closed cohomogeneity one topological manifolds (cf. Theorem 2.5, which corresponds to [42, Theorem 4]). As is customary, we say that a manifold is *closed* if it is compact and has no boundary. We briefly discuss the non-compact case in Section 2.

Before stating the structure theorem, let us recall that when the orbit space of a cohomogeneity one G-action on a closed topological manifold is homeomorphic to a closed interval $[-1, 1]$, then there exist compact Lie subgroups H and $K^{\pm}$ of G such that $H \subseteq K^{\pm} \subseteq G$. The group H is the principal isotropy group of the action and the groups $K^{\pm}$ are isotropy groups of points in the orbits corresponding to the boundary points ± 1 of the orbit space. The groups $K^{\pm}$ are called *non-principal isotropy groups* and the orbits $G/K^{\pm}$ are called *non-principal orbits*. We collect these groups in the quadruple (G, H, K^-, K^+), called the *group diagram* of the action (see Section 2.2 for more details). Figure 1 illustrates the orbit space structure in the simple case of the cohomogeneity one action of SO(2) by rotations on a round 2-sphere.

Theorem A. *Let M be a closed topological manifold with an (almost) effective topological G-action of cohomogeneity one with principal isotropy H. Then the orbit space is homeomorphic to either a closed interval or to a circle, and the following hold:*

(1) *If the orbit space of the action is an interval* $[-1, 1]$*, then the orbits corresponding to points in* $(-1, 1)$ *are principal orbits* G/H*, the orbits corresponding to the boundary points* ± 1 *are the non-principal orbits* $G/K^{\pm}$ *and* M *is the union of two fiber bundles over the two non-principal orbits whose fibers are cones over spheres or the Poincaré homology sphere, that is,*

$$M = G \times_{K^-} C(K^-/H) \cup_{G/H} G \times_{K^+} C(K^+/H).$$

In particular, M *is the union of the two mapping cylinders of* $G/H \to G/K^{\pm}$*. The group diagram of the action is given by* (G, H, K^-, K^+)*, where* $K^{\pm}/H$ *are spheres or the Poincaré homology sphere. Conversely, a group diagram* (G, H, K^-, K^+)*, where* $K^{\pm}/H$ *are homeomorphic to a sphere, or to the Poincaré homology sphere with* $\dim G/H \geq 4$*, determines a cohomogeneity one topological manifold.*

(2) *If the orbit space of the action is a circle, then* M *is equivariantly homeomorphic to a* G/H*-bundle over a circle with structure group* $N(H)/H$*. Conversely, every such bundle admits a cohomogeneity one* G*-action with principal isotropy* H*.*

This theorem stands in contrast with the corresponding statement in the smooth category, where the fibers of the bundle decomposition are cones over spheres, i.e. disks, and the manifold decomposes as a union of two disk bundles. We prove Theorem A in Section 2.

Remark. In part (1) of Theorem A, when considering M as a union of the two mapping cylinders of $G/H \to G/K^{\pm}$, one must also take into account the presence of a gluing map $\varphi\colon G/H \to G/H$, which may not be the identity. It is possible, however, to choose the isotropy groups $K^{\pm}$ so that M is equivariantly homeomorphic to the union of the two mapping cylinders of $G/H \to G/K^{\pm}$ with gluing map the identity (see [5, Ch. IV, Theorem 8.2]). Thus, with this choice of $K^{\pm}$, one can write M as the so-called *homotopy pushout* of the diagram $G/K^- \leftarrow G/H \to G/K^+$.

In part (2) of Theorem A, where the orbit space is a circle, M is the quotient of $[-1, 1] \times G/H$ by the identification induced by right multiplication by a single element a of $N_G(H)/H$ (see [5, Ch. 1, Corollary 4.3]). In this case, the structure group can be reduced to the cyclic group generated by this element.

It is well known that a closed smooth cohomogeneity one G-manifold admits a G-invariant Riemannian metric with a lower sectional curvature bound. Alexandrov spaces are synthetic generalizations of Riemannian manifolds with curvature bounded below (see [6, 7]). Theorem A, in combination with work of Galaz-García and Searle [18], implies:

Corollary B. *A closed topological manifold of cohomogeneity one admits an invariant Alexandrov metric.*

We prove this corollary in Section 3. It is an open question whether every topological manifold admits an Alexandrov metric. Corollary B shows that this is true if M admits a cohomogeneity one action.

A topological manifold M is *smoothable* if it is homeomorphic to a smooth manifold. A topological G-action on a smoothable topological manifold M is *smoothable* if it is equivariantly homeomorphic to a smooth G-action on M. When the orbit space of a closed cohomogeneity one topological manifold M is a circle, part (2) of Theorem A implies that both M and the action are smoothable. When the orbit space is an interval, we have the following results, which are simple consequences of Theorem A.

Corollary C. *A closed cohomogeneity one topological manifold with diagram (G, H, K^-, K^+) is equivariantly homeomorphic to a smooth manifold if and only if $K^{\pm}/H$ are homeomorphic to spheres.*

Corollary D. *Let M be a closed cohomogeneity one topological manifold with diagram (G, H, K^-, K^+).*

(1) If the codimension of the non-principal orbits $G/K^{\pm}$ is not 4, then the action is smoothable.
(2) If the codimension of some non-principal orbit is 4, then the action is
  - *(a) smoothable if and only if both $K^{\pm}/H$ are homeomorphic to spheres.*
  - *(b) non-smoothable if and only if K^+/H or K^-/H is homeomorphic to the Poincaré homology sphere.*

Corollary E. *Every cohomogeneity one action on a topological n-manifold, $n \leq 4$, is smoothable.*

Corollary C follows from the fact that, if a closed cohomogeneity one topological manifold with diagram (G, H, K^-, K^+) is equivariantly homeomorphic to a smooth manifold, then the Slice Theorem for smooth actions implies that $K^{\pm}/H$ are homeomorphic to spheres (see also [5, Ch. IV, Theorem 8.2]). Conversely, if $K^{\pm}/H$ are spheres, then Theorem A implies that the action is equivariantly homeomorphic to a smooth action, since one can construct a smooth cohomogeneity one smooth manifold with the same group diagram.

Closed, smooth manifolds of cohomogeneity one have been classified equivariantly by Mostert [42] and Neumann [45] in dimensions 2 and 3, by Parker [48] in dimension 4, and, assuming simply-connectedness, by Hoelscher [29] in dimensions 5, 6 and 7. Mostert, Neumann and Parker gave canonical representatives for the classes in the equivariant classification in dimension

$n \leq 4$. This was also done by Hoelscher [28] in dimensions 5 and 6 in the simply-connected case. In dimension 7, Hoelscher [30], Escher and Ultman [15] computed the homology groups of the manifolds appearing in the equivariant classification. By Corollary E, the classification of closed, cohomogeneity one topological manifolds is complete in dimension $n \leq 4$. In dimensions 5, 6 and 7, however, it follows from Corollary D that Hoelscher's results do not yield a classification in the topological category. Our second theorem completes the equivariant classification of closed, cohomogeneity one topological manifolds in these dimensions.

Theorem F. *Let M be a closed, simply-connected topological n-manifold, $n \leq 7$, with an (almost) effective cohomogeneity one action of a compact connected Lie group G. If the action is non-smoothable, then it is given by one of the diagrams in Table 1 and M can be exhibited as one of the manifolds in this table.*

Dimension	Diagram (G, H, K^-, K^+)	Manifold
5	$(S^3 \times S^1, I^* \times \mathbb{Z}_k, I^* \times S^1, S^3 \times \mathbb{Z}_k)$	$\mathbf{P}^3 * \mathbf{S}^1 \approx \mathbf{S}^5$
6	$(S^3 \times S^3, I^* \times S^1, S^3 \times S^1, S^3 \times S^1)$	$\Sigma(\mathbf{P}^3) \times \mathbf{S}^2$ $\approx \mathbf{S}^4 \times \mathbf{S}^2$
	$(S^3 \times S^3, S^1 \times I^*, S^3 \times I^*, S^1 \times S^3)$	$\mathbf{P}^3 * \mathbf{S}^2 \approx \mathbf{S}^6$
7	$(S^3 \times S^3, I^* \times 1, I^* \times S^3, S^3 \times 1)$	$\mathbf{P}^3 * \mathbf{S}^3 \approx \mathbf{S}^7$
	$(S^3 \times S^3, \Delta I^*, I^* \times S^3, \Delta S^3)$	$\mathbf{P}^3 * \mathbf{S}^3 \approx \mathbf{S}^7$
	$(S^3 \times S^3, I^* \times \mathbb{Z}_k, I^* \times S^1, S^3 \times \mathbb{Z}_k)$	$\mathbf{S}^5$-bundle over $\mathbf{S}^2$
	$(S^3 \times S^3, I^* \times I^*, S^3 \times I^*, I^* \times S^3)$	$\mathbf{P}^3 * \mathbf{P}^3 \approx \mathbf{S}^7$
	$(S^3 \times S^3, I^* \times 1, S^3 \times 1, S^3 \times 1)$	$\Sigma(\mathbf{P}^3) \times \mathbf{S}^3$ $\approx \mathbf{S}^4 \times \mathbf{S}^3$
	$(S^3 \times S^3, \Delta I^*, \Delta S^3, \Delta S^3)$	$\Sigma(\mathbf{P}^3) \times \mathbf{S}^3$ $\approx \mathbf{S}^4 \times \mathbf{S}^3$
	$(S^3 \times S^3 \times S^1, I^* \times S^1 \times \mathbb{Z}_k, I^* \times T^2, S^3 \times S^1 \times \mathbb{Z}_k)$	$\mathbf{S}^5$-bundle over $\mathbf{S}^2$

TABLE 1. Non-smoothable cohomogeneity one actions in dimensions 5, 6 and 7

The proof of Theorem F follows the outline of the proofs of the classification in the smooth case (see, for example, [29]). After determining the admissible group diagrams, we can write the manifolds as joins, products or bundles in terms of familiar spaces. The problem still remains whether the topological manifolds in Table 1 are smoothable. One can quickly settle this question for the joins in Table 1 since, by the Double Suspension Theorem, these manifolds are homeomorphic to spheres and therefore they are smoothable. The situation for the products $\Sigma(\mathbf{P}^3)\times\mathbf{S}^2$ and $\Sigma(\mathbf{P}^3)\times\mathbf{S}^3$ in Table 1, where $\Sigma(\mathbf{P}^3)$ denotes the suspension of $\mathbf{P}^3$, is more delicate. For example, the 6-dimensional product $\Sigma(\mathbf{P}^3)\times\mathbf{S}^2$ is homotopy equivalent to $\mathbf{S}^4\times\mathbf{S}^2$. By the classification of closed, oriented, simply-connected topological 6-manifolds with torsion-free homology, carried out by Wall [54], Jupp [31] and Zhubr [60], there exist infinitely many homeomorphism types for a homotopy $\mathbf{S}^4\times\mathbf{S}^2$, parametrized by a nonnegative integer k. For k even, the corresponding homeomorphism type is smoothable; for k odd, the corresponding homeomorphism type is non-smoothable (see Section 6). Our third theorem settles the smoothability of $\Sigma(\mathbf{P}^3)\times\mathbf{S}^2$.

Theorem G. *The manifold $\Sigma(\mathbf{P}^3)\times\mathbf{S}^2$ is homeomorphic to $\mathbf{S}^4\times\mathbf{S}^2$.*

The proof of Theorem G is an application of the classification of closed, oriented, simply-connected topological 6-manifolds with torsion-free homology, and essentially reduces to computing the first Pontryagin class of $\Sigma(\mathbf{P}^3)\times\mathbf{S}^2$ (see Section 6). To do this, we use results of Zagier [59].

Observe that $\Sigma(\mathbf{P}^3)\times\mathbf{S}^3$ is the total space of a principal S^1-bundle over $\Sigma(\mathbf{P}^3)\times\mathbf{S}^2$. By Theorem G, $\Sigma(\mathbf{P}^3)\times\mathbf{S}^2\approx\mathbf{S}^4\times\mathbf{S}^2$. Hence, $\Sigma(\mathbf{P}^3)\times\mathbf{S}^3$ is smoothable and, since the Euler class of the bundle is a generator of $H^2(\mathbf{S}^4\times\mathbf{S}^2)$, we obtain the following result.

Corollary H. *The manifold $\Sigma(\mathbf{P}^3)\times\mathbf{S}^3$ is homeomorphic to $\mathbf{S}^4\times\mathbf{S}^3$.*

Let M be the total space of a topological $\mathbf{S}^5$-bundle over $\mathbf{S}^2$. Since $H^4(M,\mathbb{Z}_2)=0$, the Kirby–Siebenmann class of M vanishes, so M admits a PL structure (see [35]). Since, in dimensions $n\leq 7$, every PL n-manifold admits at least one compatible smooth structure (see [24, 33, 38] or [55, p. 66]), it follows that M is smoothable. Thus, the homeomorphisms in the third column of Table 1, combined with Corollary E, yield the following result.

Corollary I. *A closed, simply-connected topological n-manifold of cohomogeneity one is homeomorphic to a smooth manifold, provided $n\leq 7$.*

Our paper is organized as follows. In Section 2 we discuss Mostert's article [42] and prove Theorem A. We prove Corollary B in Section 3. In Section 4 we collect some results on cohomogeneity one topological manifolds that we will use in the proof of Theorem F. Sections 5 and 6 contain, respectively, the proofs of Theorems F and G.

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2. Setup and proof of Theorem A

2.1. **Notation.** Let M be a topological manifold and let x be a point in M. Given a topological (left) action $G \times M \to M$ of a Lie group G, we let $G(x) = \{ gx \mid g \in G \}$ be the *orbit* of x under the action of G. The *isotropy group* of x is the subgroup $G_x = \{ g \in G \mid gx = x \}$. Observe that $G(x) \approx G/G_x$. We will denote the orbit space of the action by M/G and let $\pi \colon M \to M/G$ be the orbit projection map. The *(ineffective) kernel* of the action is the subgroup $K = \bigcap_{x \in M} G_x$. The action is *effective* if K is the trivial subgroup $\{e\}$ of G; the action is *almost effective* if K is finite. We will also denote the normalizer $N_G(L)$ of a subgroup L of G by $N(L)$.

We will say that two G-manifolds are *equivalent* if they are equivariantly homeomorphic. From now on, we will suppose that G is compact and assume that the reader is familiar with the basic notions of compact transformation groups (see, for example, Bredon [5]). We will assume all manifolds to be connected.

As for locally smooth actions (see [5, Ch. IV, Section 3]), for a topological action of G on M there also exists a maximum orbit type G/H, i.e. H is conjugate to a subgroup of each isotropy group. One sees this as follows. Let M_0 be the set of points with isotropy group of smallest dimension and least number of components. By work of Montgomery and Yang [40], M_0 is an open, dense and connected subset of M. Moreover, by work of Montgomery and Zippin [41], for every $x \in M$ there is a neighborhood V such that G_y is conjugate to a subgroup of G_x for $y \in V$. It then follows from the connectedness of M_0, that the isotropy groups G_y, $y \in M_0$ are conjugate to each other. By the density of M_0 and the existence of the neighborhood V, each group G_y, for $y \in M_0$, is conjugate to a subgroup of every isotropy group. Therefore, the orbit type G/G_y, for $y \in M_0$, is maximal. We call this orbit type the *principal orbit type* and orbits of this type *principal orbits.* A non-principal orbit is *exceptional* if it has the same dimension as a principal orbit and is called *singular* if its dimension is less than the dimension of a principal orbit.

A *homology sphere* is a closed topological n-manifold M^n such that $H_*(M^n, \mathbb{Z}) \cong H_*(\mathbf{S}^n, \mathbb{Z})$. We will denote the suspension of a topological space X by $\Sigma(X)$ and the join of X with a topological space Y by $X * Y$. Recall that $\Sigma(X) \approx X * \mathbf{S}^0$ and, in general, $\Sigma^n(X) \approx X * \mathbf{S}^{n-1}$ for $n \geq 1$. If two compact Lie groups G and H act on X and Y, respectively, then the product $G \times H$ acts on the join $X * Y$ in a natural way, via $(g, h) \cdot [(x, y, t)] = [(gx, hy, t)]$, where $(g, h) \in G \times H$ and $[(x, y, t)] \in X * Y$. We call this action the *join action* of $G \times H$ on $X * Y$.

We will denote the Poincaré homology sphere by $\mathbf{P}^3$; it is homeomorphic to the homogeneous spaces $\mathrm{SU}(2)/I^*$ and $\mathrm{SO}(3)/I$, where I^* is the binary icosahedral group and I is the icosahedral group. We will use some basic concepts of piecewise-linear topology in the proof of Theorem A. We refer the reader to [50] for the relevant definitions.

2.2. **Cohomogeneity one topological manifolds.** In this subsection we collect basic facts on cohomogeneity one topological manifolds, discuss the omission in Mostert's work [42] that gave rise to the present article, and prove some preliminary results that we will use in the proof of Theorems A and F.

Definition 2.1. Let M be a connected topological n-manifold with a topological action of a compact connected Lie group G. The action is of *cohomogeneity one* if the orbit space is one-dimensional or, equivalently, if there exists an orbit of dimension $n-1$. A topological manifold with a topological action of cohomogeneity one is a *cohomogeneity one manifold*.

By [42, Theorem 1], the orbit space of a cohomogeneity one manifold is homeomorphic to a connected 1-manifold (possibly with boundary). Orbits that project to interior points of the orbit space all have the same isotropy group H (up to conjugacy). The subgroup H is called the *principal* isotropy group and these orbits are *principal orbits*. We refer to orbits which map to boundary points of the orbit space as *non-principal*. We call the isotropy groups of points in these orbits *non-principal isotropy groups*. When the orbit space is homeomorphic to $[-1,1]$, we denote a non-principal isotropy group corresponding to a point in the orbit mapped to ± 1 by $K^{\pm}$.

Remark 2.2. In [42] non-principal orbits are called "singular" and principal orbits are called "regular". We have departed from the original notation in [42] in order to follow the current standard terminology in the theory of transformation groups (see, for example, [5, Ch. IV]).

As indicated in the introduction, the oversight in [42] stems from the claim that a homology sphere that is a homogeneous space must be a sphere. More precisely, in [42, Sections 2 and 4] Mostert shows that K/H, where K is a non-principal isotropy group, must be a homology sphere and a homogeneous space (see [42, Lemma 2 and proof of Theorem 2]) and concludes, erroneously, that K/H must be a sphere (see [42, Section 2, Corollary]). This, as explained in the introduction, is not the case. The following result of Bredon [4, Theorem 1.1, Corollary 1.2] implies that the Poincaré homology sphere $\mathbf{P}^3$ and spheres are the only possibilities for K/H.

Theorem 2.3 (Bredon)**.** *Let G be a compact Lie group and H a closed subgroup of G.*

1. *If G/H is a homology k-sphere, then G/H is homeomorphic to either $\mathbf{S}^k$ or to the Poincaré homology sphere $\mathbf{P}^3$.*
2. *If G acts almost effectively and transitively on $\mathbf{P}^3$, then G is isomorphic to $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$, with I^* or I as the isotropy group, respectively.*

The following example shows that there are cohomogeneity one topological manifolds with $K/H \approx \mathbf{P}^3$.

Example 2.4. Let $S^3 \times \mathrm{SO}(n+1)$, $n \geq 1$, act on $\mathbf{P}^3 * \mathbf{S}^n$ as the join action of the standard transitive actions of $S^3 \cong \mathrm{SU}(2)$ on $\mathbf{P}^3$ and $\mathrm{SO}(n+1)$ on $\mathbf{S}^n$. The orbit space is homeomorphic to $[-1,1]$ and $K^+ = S^3 \times \mathrm{SO}(n)$, $K^- = I^* \times \mathrm{SO}(n+1)$ and $H = I^* \times \mathrm{SO}(n)$. Thus $K^+/H = \mathbf{P}^3$. By the Double Suspension Theorem, $\Sigma^2(\mathbf{P}^3) \approx \mathbf{S}^5$ and it follows that $\mathbf{P}^3 * \mathbf{S}^n \approx \Sigma^{n+1}(\mathbf{P}^3)$ is homeomorphic to $\mathbf{S}^{n+4}$.

Let us now recall the statement of Theorem 4 in [42] (considering the corrections in the Errata to [42]). Note that the statement to follow is still incorrect. We have highlighted in bold the problematic claims.

Theorem 2.5 (see [42, Theorem 4])**.** *Let M be a topological manifold with a cohomogeneity one G-action. Then the following hold:*

(i) *The orbit space M/G is homeomorphic to one of (a) the circle, (b) the line, (c) the half-open interval, or (d) the closed interval.*

(ii) *In case (a), M is a fiber bundle over M/G with fiber G/G_x and structure group a (finite) cyclic subgroup of $N(G_x)/N(G_x)^0$, where $N(G_x)$ is the normalizer of G_x in G, and $N(G_x)^0$ is the identity component of $N(G_x)$. In particular, M is homeomorphic to $(M/G) \times (G/G_x)$ if $N(G_x)$ is connected.*

(ii′) *In case (b), M is homeomorphic to $(M/G) \times (G/G_x)$ for $x \in M$.*

(iii) *In case (c),* ***there exist subgroups*** $H \subset K$ ***of*** G ***such that*** K/H ***is an*** r***-sphere for some*** $r \geq 0$ *and, if p denotes the boundary point of M/G, then M is homeomorphic to the quotient of $(M/G)\times(G/H)$ by the relation $(p, gH) \equiv (p, hH)$ if $g^{-1}h \in K$, which identifies $\{p\} \times (G/H)$ to $\{p\} \times (G/K)$.*

(iv) *In case (d),* ***there exist subgroups*** $H \subset K^- \cap K^+$ ***of*** G ***such that*** $K^{\pm}/H$ ***is an*** $r_{\pm}$***-sphere*** *and, if $p_{\pm}$ are the boundary points of M/G, then M is homeomorphic to the quotient of $(M/G) \times (G/H)$ by the relation $(p_{\pm}, gH) \equiv (p_{\pm}, hH)$ if $g^{-1}h \in K^{\pm}$, which identifies $\{p_{\pm}\} \times (G/H)$ to $\{p_{\pm}\} \times (G/K^{\pm})$.*

(v) ***The action of the group*** G ***on a space with structure as in items (ii)–(iv) is equivalent to a cohomogeneity one*** G***-action on the manifold*** M ***and, conversely, a space*** M ***constructed in such a way is a topological manifold with a cohomogeneity one action of*** G***.***

Taking into account Theorem 2.3, the comments preceding it, and Example 2.4, we conclude that one must amend Theorem 2.5 by adding $\mathbf{P}^3$ as a second possibility for K/H in items (iii) and (iv).

Item (v) in Theorem 2.5 is true when all the $K^{\pm}/H$ are spheres and follows as in [42]. In the case where at least one of the $K^{\pm}/H$ is homeomorphic to $\mathbf{P}^3$, one must reprove this claim. We do this at the end of this section, in the case where M is closed (i.e. compact and without boundary). This yields Theorem A. The remaining cases, where the orbit space is not compact, can be dealt with in an analogous way, and we leave this task to the interested reader.

By item (iv) in Theorem 2.5, a cohomogeneity one G-action on a closed topological manifold with orbit space an interval determines a group diagram

$$\begin{tikzcd} & G & \\ K^- \arrow[ur, "j_-"] & & K^+ \arrow[ul, "j_+"'] \\ & H \arrow[ul, "i_-"] \arrow[ur, "i_+"'] & \end{tikzcd}$$

where $i_\pm$ and $j_\pm$ are the inclusion maps, $K^\pm$ are the isotropy groups of the non-principal orbits at the boundary points of the interval, and H is the principal isotropy group of the action. We will denote this diagram by the quadruple (G, H, K^-, K^+). The inclusion maps are an important element in the group diagram, as illustrated by the following simple example: $(T^2, \{e\}, T^1, T^1)$ determines both $\mathbf{S}^3$ and $\mathbf{S}^2 \times \mathbf{S}^1$, where in the first case the inclusion maps are to the first and second factors of T^2, respectively, and in the second case, both inclusion maps are the same (see [45]). Now we prove Theorem A.

2.3. **Proof of Theorem A.** Let M^n be a closed topological n-manifold with an (almost) effective topological G-action of cohomogeneity one with principal isotropy H. By item (i) in Theorem 2.5, the orbit space is homeomorphic to either a closed interval or to a circle. Part (2) of Theorem A follows from item (ii) in Theorem 2.5. Therefore, we need only prove part (1) of Theorem A, where the orbit space M/G is homeomorphic to a closed interval $[-1, 1]$. The "if" statement in this case corresponds to part (iv) of Theorem 2.5 (keeping in mind that one must add $\mathbf{P}^3$ as a possibility for $K^\pm/H$). Now we prove the "only if" statement.

Let (G, H, K^-, K^+) be a group diagram satisfying the hypotheses of part (1) of Theorem A. By the work of Mostert, we need only consider the case where at least one of $K^\pm/H$ is the Poincaré sphere $\mathbf{P}^3$. In this case, $n \geq 5$.

Suppose, without loss of generality, that $K^+/H = \mathbf{P}^3$. Since $n \geq 5$, the non-principal orbit G/K^+ is at least one-dimensional. Observe now that the space

$$X = G \times_{K^-} C(K^-/H) \cup_{G/H} G \times_{K^+} C(K^+/H) \tag{2.1}$$

is a finite polyhedron. A sufficiently small neighborhood of a point in the non-principal orbit G/K^+ is the PL product of an $(n-4)$-ball and a cone over $\mathbf{P}^3$. Therefore, since the link of a vertex (a,b) in a PL product $A \times B$ is the join of the link of a in A and of the link of b in B (see Exercise 3 in [50, Ch. 2, Section 24]), the link of every point in the non-principal orbit G/K^+ is $\mathbf{S}^{n-5} * \mathbf{P}^3$. The following result (see [14, Section 8] or [57, p. 742]) implies now that X is a topological manifold:

Theorem 2.6 (Edwards)**.** *A finite polyhedron P is a closed topological n-manifold if and only if the link of every vertex of P is simply-connected if $n \geq 3$, and the link of every point of P has the homology of the $(n-1)$-sphere.*

□

3. Existence of Invariant Alexandrov metrics

In this section we point out that every closed cohomogeneity one topological manifold admits an invariant Alexandrov metric. Let us first recall some basic facts about Alexandrov spaces, all of which can be found in [6].

A finite dimensional length space (X,d) has curvature bounded from below by k if every point $x \in X$ has a neighborhood U such that for any collection of four different points (x_0, x_1, x_2, x_3) in U, the following condition holds:

$$\angle_{x_1,x_2}(k) + \angle_{x_2,x_3}(k) + \angle_{x_3,x_1}(k) \leq 2\pi.$$

Here, $\angle_{x_i,x_j}(k)$, called the *comparison angle*, is the angle at $x_0(k)$ in the geodesic triangle in M_k^2, the simply-connected 2-manifold with constant curvature k, with vertices $(x_0(k), x_i(k), x_j(k))$, which are the isometric images of (x_0, x_i, x_j). An *Alexandrov space* is a complete length space of curvature bounded below by k, for some $k \in \mathbb{R}$. The isometry group $\mathrm{Isom}(X)$ of an Alexandrov space X is a Lie group (see [17, Theorem 1.1]) and $\mathrm{Isom}(X)$ is compact if X is compact and connected (see [10, p. 370, Satz I] or [37, Corollary 4.10 and its proof in pp. 46–50]). Alexandrov spaces of cohomogeneity one have been studied in [18]. The following result will play an essential role in the proof of Corollary B:

Proposition 3.1 ([18, Proposition 5])**.** *If X is a closed, cohomogeneity one Alexandrov space with orbit space an interval, then X is the union of two bundles whose fibers are cones over positively curved homogeneous spaces, that is*

$$G \times_{K_-} C(K_-/H) \cup_{G/H} G \times_{K_+} C(K_+/H),$$

where the group diagram of the action is given by (G, H, K_-, K_+) and $K_\pm/H$ are positively curved homogeneous spaces. Conversely, any diagram (G, H, K_-, K_+), with $K_\pm/H$ positively curved homogeneous spaces, gives rise to a cohomogeneity one Alexandrov space.

3.1. **Proof of Corollary B.** Let M be a closed topological manifold with a cohomogeneity one action of a compact Lie group G. By Theorem A(1), we have a group diagram (G, H, K^-, K^+), where the spaces $K^\pm/H$ are spheres or P^3. In either case, $K^\pm/H$ are positively curved and homogeneous. Therefore, by Proposition 3.1, M admits an invariant Alexandrov metric. $\square$

Remark 3.2. Corollary B follows trivially in the case when the cohomogeneity one G-action on M is equivalent to a smooth action. Indeed, it is well known that one can construct a G-invariant Riemannian metric on M. Since, M is compact, this Riemannian metric has a lower sectional curvature bound and hence M is an Alexandrov space.

4. Tools and further definitions

In this section we review some standard results for cohomogeneity one smooth manifolds in the context of topological manifolds. We will use these tools in the proof of Theorem F.

We first point out that all the propositions and lemmas used by Hoelscher in [29] to determine both the groups G that may act by cohomogeneity one on a smooth closed manifold M and the fundamental group of M also hold for topological manifolds. Indeed, the fact that M is a union of two mapping cylinders is a key point in the proofs of most statements in [29]. By Mostert's work [42], this is also the case for a cohomogeneity one topological manifold (see also Theorem A). We collect the relevant results here for easy reference, focusing our attention on the cases where at least one of $K^\pm/H$ is the Poincaré sphere.

The following proposition determines when two different group diagrams yield the same manifold. Its proof follows as in [5, Ch. IV, Theorem 8.2], after observing that a cohomogeneity one topological manifold decomposes as the union of two mapping cylinders.

Proposition 4.1. *If a cohomogeneity one topological manifold is given by a group diagram (G, H, K^-, K^+), then any of the following operations on the group diagram will result in a G-equivariantly homeomorphic topological manifold:*

(1) Switching K^- and K^+,
(2) Conjugating each group in the diagram by the same element of G,
(3) Replacing K^- with gK^-g^{-1} for $g \in N(H)_0$.

Conversely, the group diagrams for two G-equivariantly homeomorphic cohomogeneity one, closed topological manifolds must be mapped to each other by some combination of these three operations.

The following result of Parker [29, Proposition 1.8] is stated for smooth cohomogeneity one manifolds but the proof carries over to the topological category.

Proposition 4.2 (Parker)**.** *A closed simply-connected cohomogeneity one topological manifold has no exceptional orbits.*

The van Kampen Theorem applied to a closed cohomogeneity one manifold written as a union of two mapping cylinders yields the following result (see [29, Proposition 1.8]).

Proposition 4.3 (Corollary of the van Kampen Theorem). *Let M be the closed cohomogeneity one topological manifold given by the group diagram (G,H,K^-,K^+) with $\dim(K^\pm/H)\geq 1$. Then $\pi_1(M)\cong\pi_1(G/H)/N^-N^+$, where*

$$N^\pm=\ker\{\pi_1(G/H)\to\pi_1(G/K^\pm)\}=\mathrm{Im}\{\pi_1(K^\pm/H)\to\pi_1(G/H)\}.$$

In particular M is simply-connected if and only if the images of $K^\pm/H$ generate $\pi_1(G/H)$ under the natural inclusions.

Corollary 4.4. *Let M be the closed simply-connected cohomogeneity one topological manifold given by the group diagram (G,H,K^-,K^+), with $\dim(K^\pm/H)\geq 1$, and $K^-/H=\mathbf{S}^l$, for $l\geq 2$. Then G/K^+ is simply-connected and, if G is connected, then K^+ is also connected.*

Proof. Consider the fiber bundle

$$K^\pm/H\to G/H\to G/K^\pm.$$

This gives a long exact sequence of homotopy groups:

$$\cdots\to\pi_1(K^\pm/H)\xrightarrow{i_*^\pm}\pi_1(G/H)\xrightarrow{\rho_*^\pm}\pi_1(G/K^\pm)\xrightarrow{\partial_*}\pi_0(K^\pm/H).$$

Since $\dim(K^-/H)\geq 1$ and $K^-/H=\mathbf{S}^l$ is simply-connected for $l\geq 2$, we get the following sequences:

$$0\to\pi_1(G/H)\xrightarrow{\rho_*^-}\pi_1(G/K^-)\to 0,$$

$$\cdots\to\pi_1(K^+/H)\xrightarrow{i_*^+}\pi_1(G/H)\xrightarrow{\rho_*^+}\pi_1(G/K^+)\to 0.$$

In particular, $N^-=\ker\rho_*^-=0$ and $\pi_1(G/K^+)=\pi_1(G/H)/N^+$. Further, by Proposition 4.3, since M is simply-connected, $\pi_1(G/H)=N^-N^+$. Hence, $\pi_1(G/H)=N^+$, and $\pi_1(G/K^+)$ is trivial.

Now assume that G is connected. Similarly, the following fiber bundle

$$K^+\to G\to G/K^+,$$

gives a long exact sequence of homotopy groups:

$$\cdots\to\pi_1(K^+)\to\pi_1(G)\to\pi_1(G/K^+)\to\pi_0(K^+)\to\pi_0(G).$$

As G is connected and $\pi_1(G/K^+)$ is trivial by the above argument, $\pi_0(K^+)=0$, i.e. K^+ is connected. □

As the homogeneous spaces $K^\pm/H$ are homeomorphic to either spheres or the Poincaré homology sphere, their fundamental groups are $\mathbb{Z}$, the identity or the binary icosahedral group. Since these groups are finitely generated, the next lemma follows as in the proof of [29, Lemma 1.10].

Lemma 4.5. *Let M be the cohomogeneity one topological manifold given by the group diagram (G, H, K^-, K^+) with at least one of $K^\pm/H$ homeomorphic to $\mathbf{P}^3$. Denote $H^\pm = H \cap K_0^\pm$, and let $\alpha_\pm^i\colon [0,1] \to K_0^\pm$ be curves that generate $\pi_1(K^\pm/H)$, with $\alpha_\pm^i(0) = 1 \in G$. The manifold M is simply-connected if and only if*

(1) H is generated as a subgroup by H^- and H^+, and
(2) α_-^i and α_+^i generate $\pi_1(G/H_0)$.

Recall that a cohomogeneity one action on a closed manifold M is *non-primitive* if for some diagram (G, H, K^-, K^+) for M the isotropy groups $K^\pm$ and H are contained in some proper subgroup L of G. Such a non-primitive action is well known to be equivalent to the usual G-action on $G \times_L M_L$, where M_L is the cohomogeneity one manifold given by the group diagram (L, H, K^-, K^+).

A cohomogeneity one action of G on a closed topological manifold M is *reducible* if there is a proper normal subgroup of G that still acts by cohomogeneity one with the same orbits. Conversely, there is a natural way of extending an arbitrary cohomogeneity one action to an action by a possibly larger group. Such extensions, called *normal extensions*, are described as follows (see [18, Propositions 11–13] and [29, Section 1.11]). Let M be a cohomogeneity one topological manifold with group diagram (G_1, H_1, K_1^-, K_1^+) and let L be a compact connected subgroup of $N(H_1) \cap N(K_1^-) \cap N(K_1^+)$. Notice that $L \cap H_1$ is normal in L and let $G_2 = L/(L \cap H_1)$. We then define an action of $G_1 \times G_2$ on M orbitwise by

$$(\hat{g}_1, [l]) \cdot g_1(G_1)_x = \hat{g}_1 g_1 l^{-1}(G_1)_x$$

on each orbit $G_1/(G_1)_x$ for $(G_1)_x = H_1$ or $K_1^\pm$. This action is of cohomogeneity one, has the same orbits as the action of G_1 and has group diagram

$$(G_1 \times G_2, (H_1 \times 1)\Delta L, (K^- \times 1)\Delta L, (K^+ \times 1)\Delta L),$$

where $\Delta L = \{(l, [l]) \mid l \in L\}$.

Notice that every reducible action is a normal extension of its restricted action (see [29, Proposition 1.15]). Therefore it is natural to consider non-reducible actions in the classification. We will use the following result on reducible actions (see [18, Proposition 11] and [29, Proposition 1.12]) in the proof of Theorem F.

Proposition 4.6. *Let M be a cohomogeneity one manifold given by the group diagram (G, H, K^-, K^+) and suppose that $G = G_1 \times G_2$ with $\mathrm{Proj}_2(H) = G_2$. Then the subaction of $G_1 \times 1$ on M is also by cohomogeneity one, with the same orbits, and with isotropy groups $K_1^\pm = K^\pm \cap (G_1 \times 1)$ and $H_1 = H \cap (G_1 \times 1)$.*

The following two results give restrictions on the groups that may act by cohomogeneity one on a closed topological manifold. The next proposition can be found in [29, Proposition 1.19] for smooth actions. Here we prove it in the slightly more general case of topological actions on topological manifolds.

Proposition 4.7. *If a compact connected Lie group G acts (almost) effectively on a topological manifold with principal orbits of dimension k, then $k \leq \dim G \leq k(k+1)/2$.*

Proof. Let G/H be a principal orbit. Since $\dim G/H = k$, the left inequality is immediate. To verify the right inequality, it suffices to show that G acts almost effectively on principal orbits, since then we can equip G/H with a G-invariant Riemannian metric and obtain a homomorphism $\varphi\colon G \to \mathrm{Isom}(G/H)$ with finite kernel L. It then follows that $G/L \cong \varphi(G) \leq \mathrm{Isom}(G/H)$. Since L is finite,

$$\begin{aligned}\dim G = \dim G/L &\leq \dim \mathrm{Isom}(G/H)\\ &\leq \frac{k(k+1)}{2},\end{aligned}$$

where the last inequality follows from a well-known theorem of Myers and Steenrod [44, Theorem 7] (see also [36, Ch. II, Theorem 3.1]). To finish the proof, let us show that G acts almost effectively on principal orbits. As mentioned in Section 2, all principal isotropy groups are conjugate to each other and conjugate to subgroups of the non-principal isotropy groups. As a result, G acts almost effectively on the principal orbits. Observe that every conjugate of the principal isotropy group occurs as an isotropy group within each principal orbit. $\square$

An argument as in the proof of [29, Proposition 1.18] yields the following lemma:

Lemma 4.8. *Let M be a closed, simply-connected topological manifold with an (almost) effective cohomogeneity one action of a compact Lie group G. Suppose that the following conditions hold:*

- *$G = G_1 \times T^m$ and G_1 is semisimple;*
- *G acts non-reducibly;*
- *at least one of the homogeneous spaces $K^\pm/H$ is the Poincaré sphere.*

Then, $G_1 \neq 1$ and $m \leq 1$. Moreover, if $m = 1$, then one of the homogeneous spaces $K^\pm/H$ is a circle and $K^+ \subset G_1 \times 1$.

It is well known that every compact connected Lie group has a finite cover of the form $G_{ss} \times T^k$, where G_{ss} is semisimple and simply connected and T^k is a torus. The classification of compact simply-connected semisimple Lie groups is also well known. By Proposition 4.7, every compact connected Lie group G acting (almost) effectively with cohomogeneity one on a topological n-manifold, $n \leq 7$, must have dimension at most 21. A list of compact simply-connected semisimple Lie groups of dimension at most 21, along with their subgroups, can be found in [29, Section 1.24] (see also Tables 2.2.1 and 2.2.2 in [27]). We have collected the relevant groups in Tables 2 and 3. If an arbitrary compact group G acts on a topological manifold M, then every cover $\widetilde{G}$ of G still acts on M, although less effectively. Hence, if we consider almost effective actions, and since in our case G has dimension 21 or less, we can assume that G is a product of groups from Table 2 with a torus T^k.

The following proposition (see [29, Proposition 1.25]) gives further restriction on the groups.

Proposition 4.9. *Let M be the cohomogeneity one topological manifold given by the group diagram (G, H, K^-, K^+), where G acts non-reducibly on M. Suppose that G is the product of groups*

$$G = \mathrm{SU}(4)^i \times (G_2)^j \times \mathrm{Sp}(2)^k \times \mathrm{SU}(3)^l \times (S^3)^m \times (S^1)^n.$$

Then

$$\dim H \leq 10i + 8j + 6k + 4l + m.$$

We will also use the following result on transitive actions.

Proposition 4.10 ([47, Ch. 1, §5, Proposition 7])**.** *Let a Lie group G act transitively on a manifold M. Then G_0 acts transitively on any connected component of M. In particular, if M is connected, then G_0 acts transitively on M, and $(G_0)_x = G_0 \cap G_x$, $G = G_0 G_x$, for all $x \in M$.*

We conclude this section with an observation on groups acting on the Poincaré homology sphere.

Lemma 4.11. *Let G be a compact Lie group of dimension at most 2. If $S^3 \times G$ acts transitively on $\mathbf{P}^3$, then G acts trivially on $\mathbf{P}^3$ and the isotropy group of the $(S^3 \times G)$-action is $I^* \times G$.*

Proof. Assume first that G is connected and let L be the kernel of the action. By Theorem 2.3, $(S^3 \times G)/L$ is isomorphic to $\mathbf{S}^3$. Hence, $\dim G = \dim L$. Since L is a normal and connected subgroup of $S^3 \times G$, $\mathrm{Proj}_1(L)$ is a normal connected subgroup of S^3. Thus $\mathrm{Proj}_1(L)$ is trivial, as $\dim G \leq 2$. As a result, $L = 1 \times G$ and $H = I^* \times G$, where H is the principal isotropy group.

Suppose now that G is not connected. In this case, $S^3 \times G_0$ is connected and acts transitively on $\mathbf{P}^3$ as a restriction of the action of $S^3 \times G$. Since $S^3 \times G_0$ is connected, we may apply the argument in the preceding paragraph. Therefore, $I^* \times G_0 \subseteq H \subseteq I^* \times G$. Connectedness of the quotient $\mathbf{P}^3 = (S^3 \times G)/H$ gives $H = I^* \times G$. $\square$

5. Proof of Theorem F

Let G be a compact, connected Lie group acting almost effectively, non-reducibly and with cohomogeneity one on a closed, simply-connected topological n-manifold M^n, $5 \leq n \leq 7$. We assume that the action is non-smoothable. Hence, by Corollary C, at least one of $K^{\pm}/H$, say K^+/H, is homeomorphic to $\mathbf{P}^3$, the Poincaré homology sphere. We analyze each dimension separately.

Group	Dimension	Rank
$S^3 \cong \mathrm{SU}(2) \cong \mathrm{Sp}(1) \cong \mathrm{Spin}(3)$	3	1
$\mathrm{SU}(3)$	8	2
$\mathrm{Sp}(2) \cong \mathrm{Spin}(5)$	10	2
G_2	14	2
$\mathrm{SU}(4) \cong \mathrm{Spin}(6)$	15	3
$\mathrm{Sp}(3)$	21	3
$\mathrm{Spin}(7)$	21	3

Table 2. Simple compact simply-connected Lie groups in dimensions 21 and less.

Group	Subgroups
T^2	$\{(e^{ip\theta}, e^{iq\theta})\}$
S^3	$\{e^{x\theta} = \cos\theta + x\sin\theta\}$, where $x \in \mathrm{Im}(S^3)$.
$\mathrm{SU}(3)$	$S^1 \subset T^2$, T^2, $\mathrm{SO}(3)$, $\mathrm{SU}(2)$ and $\mathrm{U}(2)$
$\mathrm{Sp}(2)$	$\mathrm{U}(2)$, $\mathrm{Sp}(1)\mathrm{SO}(2)$ and $\mathrm{Sp}(1)\mathrm{Sp}(1)$, in dimensions 4 and higher.
G_2	$\mathrm{SU}(3)$, in dimensions 8 and higher.
$\mathrm{SU}(4)$	$\mathrm{U}(3)$ and $\mathrm{Sp}(2)$ in dimensions 9 and higher.

Table 3. Groups and their subgroups playing a role in the classification.

Dimension 5. By Proposition 4.7, we have $4 \leq \dim G \leq 10$. Hence, by Table 2, G is, up to a finite cover, one of $(S^3)^m \times T^n$, $\mathrm{SU}(3) \times T^n$ or $\mathrm{Spin}(5)$ (see also [29, 1.24]). From Lemma 4.8, we see that $n \leq 1$. Since $\dim H = \dim G - 4$, Proposition 4.9 gives the possible groups. These are, up to a finite cover, $S^3 \times S^1$, $S^3 \times S^3$, $\mathrm{SU}(3)$ and $\mathrm{Spin}(5)$. On the other hand, since $K^+/H = \mathbf{P}^3$, $\dim K^+ = 3 + \dim H = \dim G - 1$. Therefore, $G = S^3 \times S^1$ is the only possibility, since the other groups do not have a subgroup of codimension 1.

Now we determine the group diagrams for $S^3 \times S^1$. Here H_0 is trivial because $\dim H = \dim(S^3 \times S^1) - 4 = 0$. From Lemma 4.8 we have $K^-/H = S^1$, and $K_0^+ \subseteq S^3 \times 1$. Therefore, $K_0^- = \{(e^{ip\theta}, e^{iq\theta})\}$ and $K_0^+ = S^3 \times 1$, for $\dim K^+ = 3 + \dim H = 3$. By Proposition 4.10, $K_0^+ = S^3 \times 1$ acts transitively on $\mathbf{P}^3$ with isotropy group $H^+ = H \cap K_0^+ = I^* \times 1$. Thus

$$I^* \times 1 \subseteq H \subseteq K^- \subseteq N_{S^3 \times S^1}(K_0^-),$$

since $L \subseteq N(L_0)$ for every closed subgroup L of G. Since I^* cannot normalize any circle in S^3, it follows that $p = 0$. Hence $K_0^- = 1 \times S^1$ and $H^- = H \cap K_0^- = 1 \times \mathbb{Z}_k$, for some $k \geq 1$. By Lemma 4.5, $H = \langle H^-, H^+ \rangle = I^* \times \mathbb{Z}_k$. Finally, by Proposition 4.10, $K^+ = K_0^+ H = S^3 \times \mathbb{Z}_k$, $K^- = K_0^- H = I^* \times S^1$, and we have the following diagram:

$$(S^3 \times S^1, I^* \times \mathbb{Z}_k, I^* \times S^1, S^3 \times \mathbb{Z}_k).$$

The action is the one described in Example 2.4. Therefore, by Proposition 4.1, M is equivalent to $\mathbf{P}^3 * \mathbf{S}^1$, which is homeomorphic to the double suspension of $\mathbf{P}^3$. By the Double Suspension Theorem, M is then homeomorphic to $\mathbf{S}^5$.

Dimension 6. Proceeding as in the 5-dimensional case, we find that $5 \leq \dim G \leq 15$ and $\dim H = \dim G - 5$. It follows from Propositions 4.9 and 4.8 that G must be one of $S^3 \times S^3$, $S^3 \times S^3 \times S^1$, $\mathrm{SU}(3)$, $\mathrm{SU}(3) \times S^1$, $\mathrm{Sp}(2)$, $\mathrm{Sp}(2) \times S^1$ or $\mathrm{Spin}(6)$. On the other hand, since $K^+/H = \mathbf{P}^3$, we must have that $\dim K^+ = \dim G - 2$. This dimension restriction rules out all possible G except $S^3 \times S^3$.

Now we determine the possible diagrams for $G = S^3 \times S^3$. In this case, $\dim H_0 = 1$, so H_0 is a circle subgroup of $S^3 \times S^3$, i.e. $H_0 = \{(e^{ip\theta}, e^{iq\theta})\}$ after conjugation by an element of G. Since, by assumption, $K^+/H = \mathbf{P}^3$, we must have that $\dim K^+ = \dim G - 2 = 4$. Thus, $K_0^+ = S^3 \times S^1$ or $K_0^+ = S^1 \times S^3$ by Table 3. These cases differ only by an automorphism of $S^3 \times S^3$ that exchanges the factors of the group and yield the same diagrams up to switching the order of the factors of the isotropy groups. Hence we can assume that $K_0^+ = S^3 \times S^1$, which acts transitively on $\mathbf{P}^3$. According to Lemma 4.11, S^1 acts trivially on $\mathbf{P}^3$ and $H^+ = H \cap K_0^+ = I^* \times S^1$. Now we consider the different possibilities for K^-/H, namely $\mathbf{S}^l$, for $l \geq 2$, $\mathbf{S}^1$, and $\mathbf{P}^3$, and determine the group diagrams corresponding to each case.

First, suppose $K^-/H = \mathbf{S}^l$, $l \geq 2$. By Corollary 4.4, K^+ is connected. Hence $K^+ = K_0^+ = S^3 \times S^1$, and $H = H^+ = I^* \times S^1$. For $l = 2$, $\dim K^- = 3$. Thus K_0^- has to be one of the subgroups $1 \times S^3$, $S^3 \times 1$, or $\Delta S^3 = \{(g,g) \mid g \in S^3\}$ of G. However, since $I^* \times S^1 = H \subset K^-$, we must have $K_0^- = 1 \times S^3$ and hence $K^- = I^* \times S^3$. Therefore, we obtain the following diagram:

$$(S^3 \times S^3, I^* \times S^1, I^* \times S^3, S^3 \times S^1).$$

By Proposition 4.1, M is equivalent to $\mathbf{P}^3 * \mathbf{S}^2$ with the action described in Example 2.4. For $l \geq 3$ there is no subgroup K^- of $S^3 \times S^3$ containing $H = I^* \times S^1$ such that $K^-/H = \mathbf{S}^3$.

Now assume that $K^-/H = \mathbf{S}^1$. In this case, $\dim K^- = 2$. By Table 3, K_0^- is isomorphic to T^2, i.e. K_0^- is a maximal torus of G. Since K^- should contain $I^* \times S^1 = H^+ \subseteq H$, and no finite extension of a maximal torus of $S^3 \times S^3$ contains $I^* \times S^1$, this case cannot happen.

The only remaining case to be considered is when K^-/H is also the Poincaré sphere, i.e. $K^-/H = \mathbf{P}^3$. We now show that the only possible diagram that can occur is

$$(S^3 \times S^3, I^* \times S^1, S^3 \times S^1, S^3 \times S^1). \tag{5.1}$$

Since K_0^- is a 4-dimensional subgroup of G containing H, we must have $K_0^- = S^3 \times S^1$, with the same S^1-factor as in $K_0^+ = S^3 \times S^1$ (considered as subgroups of G). Therefore, $H^+ = H \cap K_0^+ = H \cap K_0^- = H^-$. Because M is simply-connected, H must be generated by H^+ and H^- by Lemma 4.5, so $H = H^- = H^+ = I^* \times S^1$. The last part of Proposition 4.10 then implies that $K^- = S^3 \times S^1 = K^+$. Diagram (5.1) is given by the natural product action of $S^3 \times S^3$ on $\Sigma(\mathbf{P}^3) \times \mathbf{S}^2$, defined by

$$\begin{aligned}(S^3 \times S^3) \times (\Sigma(\mathbf{P}^3) \times \mathbf{S}^2) &\to \Sigma(\mathbf{P}^3) \times \mathbf{S}^2\\ ((g,h),([x,t],y)) &\mapsto ([gx,t], hyh^{-1}).\end{aligned}$$

Here the S^3-action on $\mathbf{S}^2$ is the restriction to the unit sphere of the adjoint action of S^3 on its Lie algebra $\mathfrak{su}(2) \cong \mathbb{R}^3$. Therefore, by Proposition 4.1, M is equivalent to $\Sigma(\mathbf{P}^3) \times \mathbf{S}^2$ with the product action we just described above. By Theorem G, M is in turn homeomorphic to $\mathbf{S}^4 \times \mathbf{S}^2$.

Dimension 7. By Proposition 4.7 we know that $6 \leq \dim G \leq 21$ and $\dim H = \dim G - 6$. As before, Propositions 4.9 and 4.8 give us the possible acting groups: $S^3 \times S^3$, $S^3 \times S^3 \times S^1$, $\mathrm{SU}(3)$, $S^3 \times S^3 \times S^3$, $\mathrm{SU}(3) \times S^1$, $\mathrm{Sp}(2)$, $\mathrm{SU}(3) \times S^3$, $\mathrm{Sp}(2) \times S^3$, G_2, $\mathrm{SU}(4)$, $\mathrm{SU}(4) \times S^1$ and $\mathrm{Spin}(7)$. Since $\dim K^+ = \dim G - 3$, we easily rule out most of the groups and the only possible groups remaining are $S^3 \times S^3$, $S^3 \times S^3 \times S^1$ and $S^3 \times S^3 \times S^3$. We analyze each case separately.

$G = S^3 \times S^3$. Here $\dim H = 0$, so H_0 is trivial. Since $K^+/H = \mathbf{P}^3$ by assumption, $\dim K^+ = 3$. Therefore, K_0^+ is one of $1 \times S^3$, $S^3 \times 1$ or $\Delta_g S^3 = \{(a, gag^{-1}) \mid a \in S^3\}$, for some $g \in G$. The cases where $K_0^+ = 1 \times S^3$ or $S^3 \times 1$ only differ by an automorphism of $S^3 \times S^3$ which exchanges the factors of the product group. Since both cases yield the same diagram up to a reordering of the factors of the isotropy groups, we will only consider the cases where $K_0^+ = S^3 \times 1$ or $K_0^+ = \Delta_g S^3$.

First assume that $K_0^+ = S^3 \times 1$, so $H^+ = K_0^+ \cap H = I^* \times 1$. We now continue by considering the different possibilities for K^-/H, i.e. $\mathbf{S}^l$, for $l \geq 2$, $\mathbf{S}^1$, and $\mathbf{P}^3$.

Suppose first that $K^-/H = \mathbf{S}^l$, $l \geq 2$. Then, by Corollary 4.4, K^+ is connected. Thus, we have $K^+ = S^3 \times 1$ and $H = I^* \times 1$. Since H is discrete and K^-/H is simply-connected, the long exact sequence of homotopy groups corresponding to the fiber bundle

$$H \to K^- \to K^-/H$$

implies that $\pi_1(K_0^-) = 0$. A glance at the subgroups of $S^3 \times S^3$ shows that only $S^3 \times S^3$ and its 3-dimensional subgroups are simply-connected. Since $(S^3 \times S^3)/H$ is not a sphere, K^- is necessarily 3-dimensional. Therefore, K_0^- is one of the subgroups $S^3 \times 1$, $1 \times S^3$ or $\Delta_g S^3$, for some $g \in G$. We will now show that only the case $K_0^- = 1 \times S^3$ occurs. Clearly, $\Delta_g S^3$ does not contain $I^* \times 1 = H$, so this case cannot happen. Assume now that $K_0^- = S^3 \times 1$. Then $H \subseteq K_0^-$ and it follows from Proposition 4.10 that $K^- = K_0^-$. Since $H = I^* \times 1$, K^-/H is not a sphere, which contradicts our assumption that $K^-/H = \mathbf{S}^l$, $l \geq 2$. This rules out the case $K_0^- = S^3 \times 1$. Therefore, we are left with the case $K_0^- = 1 \times S^3$. It follows from Proposition 4.10 that $K^- = K_0^- H = I^* \times S^3$. Hence we have the following diagram:

$$(S^3 \times S^3, I^* \times 1, I^* \times S^3, S^3 \times 1),$$

which corresponds to the join action of $S^3 \times S^3$ on $\mathbf{P}^3 * \mathbf{S}^2$.

Now let $K^-/H = \mathbf{S}^1$. Since $H_0 = 1$, we can write $K_0^- = \{(e^{ip\theta}, e^{iq\theta})\} \subseteq S^3 \times S^3$. As

$$I^* \times 1 \subseteq K^- \subseteq N_{S^3 \times S^3}(K_0^-),$$

and I^* cannot normalize any circle in S^3, we must have $p = 0$, i.e. $K_0^- = 1 \times S^1$. Let $H^- = H \cap K_0^-$. Since H is a finite group, $H^- \subseteq H$ has to be a finite cyclic subgroup of $K_0^- = 1 \times S^1$, say $\mathbb{Z}_k$, $k \geq 1$. By Lemma 4.5 and Proposition 4.10, we have the following diagram:

$$(S^3 \times S^3, I^* \times \mathbb{Z}_k, I^* \times S^1, S^3 \times \mathbb{Z}_k).$$

In this case, the action is non-primitive. In fact, for $L = S^3 \times S^1 \subseteq S^3 \times S^3$, we have the following diagram, which is the diagram of the cohomogeneity one action of $S^3 \times S^1$ on $\mathbf{P}^3 * \mathbf{S}^1$ already described in dimension 5:

$$(S^3 \times S^1, I^* \times \mathbb{Z}_k, I^* \times S^1, S^3 \times \mathbb{Z}_k).$$

Therefore, M is an $\mathbf{S}^5$-bundle over $\mathbf{S}^2$.

Finally, suppose that $K^-/H = \mathbf{P}^3$. Then K_0^- is one of the subgroups $S^3 \times 1$, $1 \times S^3$ or $\Delta_g S^3$ of G. As above, $H^+ = H \cap K_0^+ = I^* \times 1 \nsubseteq \pm\Delta_g S^3 = N_{S^3 \times S^3}(\Delta_g S^3)$, where $-\Delta_g S^3 = \{(a, -gag^{-1}) \mid a \in S^3\}$ and $-gag^{-1}$ denotes the additive inverse of the unit quaternion $gag^{-1} \in S^3$. Thus we have two possibilities: either $K_0^- = S^3 \times 1$ or $K_0^- = 1 \times S^3$. Let $K_0^- = S^3 \times 1$. Then $H^- = H \cap K_0^- = I^* \times 1$, and, since M is simply-connected, Lemma 4.5 implies that $H = \langle H^-, H^+ \rangle = I^* \times 1$. Therefore, we have the following group diagram:

$$(S^3 \times S^3, I^* \times 1, S^3 \times 1, S^3 \times 1), \tag{5.2}$$

which corresponds to the cohomogeneity one product action of $S^3 \times S^3$ on $\Sigma(\mathbf{P}^3) \times \mathbf{S}^3$. Thus M is equivariantly homeomorphic to $\Sigma(\mathbf{P}^3) \times \mathbf{S}^3$. Now assume that $K_0^- = 1 \times S^3$. Hence $H^- = H \cap K_0^- = 1 \times I^*$ and the following diagram appears:

$$(S^3 \times S^3, I^* \times I^*, I^* \times S^3, S^3 \times I^*).$$

It follows that M is equivalent to the join action of $S^3 \times S^3$ on $\mathbf{P}^3 * \mathbf{P}^3$.

Now let $K_0^+ = \Delta_g S^3$. Thus $H^+ = H \cap K_0^+ = \Delta_g I^* = \{(a, gag^{-1}) \mid a \in I^*\}$. As before, we consider the possible cases for K^-/H, i.e. $\mathbf{S}^l$, for $l \geq 2$, $\mathbf{S}^1$, and $\mathbf{P}^3$.

First assume $K^-/H = \mathbf{S}^l$, for $l \geq 2$. By Corollary 4.4, K^+ is connected and we have $H = \Delta_g I^*$. Also, K_0^- is a 3-dimensional subgroup of G. Therefore, K_0^- is one of $1 \times S^3$, $S^3 \times 1$ or $\Delta_h S^3 = \{(a, hah^{-1}) \mid a \in S^3\}$, for some $h \in G$. As before, the cases where $K_0^- = 1 \times S^3$ or $S^3 \times 1$ only differ by an automorphism of $S^3 \times S^3$, so we will only consider the cases where $K_0^- = 1 \times S^3$ or $K_0^- = \Delta_h S^3$.

For $K_0^- = 1 \times S^3$ we can first conjugate all subgroups by $(1, g^{-1})$. Then, by Proposition 4.1, the corresponding diagrams are equivalent to the following:

$$(S^3 \times S^3, \Delta I^*, I^* \times S^3, \Delta S^3).$$

The actions are equivalent to the following action:

$$\begin{aligned}(S^3 \times S^3) \times (\mathbf{P}^3 * \mathbf{S}^3) &\to \mathbf{P}^3 * \mathbf{S}^3\\ ((g,h),([x,y,t],y)) &\mapsto [gx, gyh^{-1}, t].\end{aligned}$$

Now let K_0^- be a diagonal subgroup of $S^3 \times S^3$. Since K^-/H is simply-connected, $\pi_0(K^-) = \pi_0(H)$. However, $\pi_0(H) = I^*$ whereas K^- has at most two components. Thus this case cannot occur.

Now we suppose that $K^-/H = \mathbf{S}^1$. Thus K_0^- is a circle subgroup. Notice that the normalizer of a circle subgroup of G does not contain $H_0 = \Delta_g I^*$. Therefore this case cannot happen either.

Finally, let $K^-/H = \mathbf{P}^3$. In this case K_0^- must be one of the 3-dimensional subgroups of $S^3 \times S^3$. The same argument we used in the paragraph preceding diagram 5.2 rules out the cases $K_0^- = 1 \times S^3$ and $K_0^- = S^3 \times 1$. Indeed, if K_0^- were $1 \times S^3$, then, since $K^-/H = P^3$, we would have $H^- = 1 \times I^*$. However, $K_0^+ = \Delta_g S^3$, which gives $K^+ \subseteq \pm\Delta_g S^3$. Therefore, K^+ cannot contain $H^- = 1 \times I^*$. The case where $K_0^- = S^3 \times 1$ is ruled out similarly. Therefore, there is only one possibility for K_0^-, namely $\Delta_h S^3 = \{(a, hah^{-1}) \mid a \in S^3\}$ for some $h \in S^3$. Now we show that the isotropy groups $K^\pm$ are connected. Notice that $K^+ \subseteq N(\Delta_g S^3) = \pm\Delta_g S^3$ and $K^- \subseteq N(\Delta_h S^3) = \pm\Delta_h S^3$. Hence $K^\pm$ has at most two connected components, so $\pi_1(G/K^\pm)$ is either trivial or $\mathbb{Z}_2$ according to the long exact sequence of homotopy groups of the fiber bundle

$$K^\pm \to G \to G/K^\pm.$$

It also follows from the sequence that $\pi_2(G/K^\pm) = 0$.

Since M is simply-connected, Proposition 4.3 implies that $\pi_1(G/H) = N^+N^-$, where $N^\pm$ are as in the statement of the proposition. From the long exact sequences

$$0 \to \pi_1(K^\pm/H) \xrightarrow{i_*^\pm} \pi_1(G/H) \xrightarrow{\rho_*^\pm} \pi_1(G/K^\pm) \to 0$$

we conclude that $N^{\pm} \cong I^*$, for $\pi_1(K^{\pm}/H) = I^*$. Furthermore,

$$\pi_1(G/K^{\pm}) = \frac{N^+N^-}{N^{\pm}} = \frac{N^{\mp}}{N^+ \cap N^-}.$$

Since the only proper normal subgroup of I^* is $\mathbb{Z}_2$, $\pi_1(G/K^{\pm})$ has to be trivial by the above argument. Therefore, $K^{\pm}$ are connected. Consequently, we have $\Delta_g I^* = H = \Delta_h I^*$. This implies that $hg^{-1} \in C_{S^3}(I^*) = \mathbb{Z}_2$, where $C_{S^3}(I^*)$ is the centralizer of I^* in S^3. Therefore, $g = h$ or $g = -h$. In either case, $K^+ = \Delta_g S^3 = \Delta_h S^3 = K^-$. Hence, after conjugating all subgroups by $(1, g^{-1})$, we get the following equivalent diagram, by Proposition 4.1:

$$(S^3 \times S^3, \Delta I^*, \Delta S^3, \Delta S^3).$$

Thus M is equivariantly homeomorphic to $\Sigma(\mathbf{P}^3) \times \mathbf{S}^3$ with the action of $S^3 \times S^3$ given by

$$\begin{aligned} (S^3 \times S^3) \times (\Sigma(\mathbf{P}^3) \times \mathbf{S}^3) &\to \Sigma(\mathbf{P}^3) \times \mathbf{S}^3 \\ ((g,h),([x,t],y)) &\mapsto ([gx,t], gyh^{-1}). \end{aligned}$$

By Corollary H, M is homeomorphic to $\mathbf{S}^4 \times \mathbf{S}^3$.

$G = S^3 \times S^3 \times S^1$. In this case $\dim H = 1$ and $\dim K^+ = 4$ since, by assumption, $K^+/H = \mathbf{P}^3$. According to Proposition 4.8, $K^-/H = \mathbf{S}^1$ and $K_0^+ \subseteq S^3 \times S^3 \times 1$. Therefore, K_0^- is a 2-torus subgroup of G, and (up to an automorphism of G) we can assume that $K_0^+ = S^3 \times S^1 \times 1$. As a result $H^+ = H \cap K_0^+ = I^* \times S^1 \times 1$ by Lemma 4.11. Since $I^* \times S^1 \times 1 \subseteq K^- \subseteq N(K_0^-) = N_G(T^2)$, K_0^- has to be $1 \times T^2 = 1 \times S^1 \times S^1$. Therefore, we have the following diagram:

$$(S^3 \times S^3 \times S^1, I^* \times S^1 \times \mathbb{Z}_k, I^* \times T^2, S^3 \times S^1 \times \mathbb{Z}_k).$$

This action is a non-primitive action. Indeed, for $L = S^3 \times T^2 \subseteq S^3 \times S^3 \times S^1$, we have the diagram

$$(S^3 \times T^2, I^* \times S^1 \times \mathbb{Z}_k, I^* \times T^2, S^3 \times S^1 \times \mathbb{Z}_k),$$

which is a normal extension of the cohomogeneity one action of $S^3 \times S^1$ on $\mathbf{P}^3 * \mathbf{S}^1$ described in dimension 5. Thus M is a $\mathbf{P}^3 * \mathbf{S}^1$-fiber bundle over $(S^3 \times S^3 \times S^1)/(S^3 \times T^2)$ and is therefore homeomorphic to an $\mathbf{S}^5$-bundle over $\mathbf{S}^2$.

$G = S^3 \times S^3 \times S^3$. We show that no non-reducible diagram for this case occurs. In fact, we will show that all possible diagrams in this case reduce to the diagrams of the case $G = S^3 \times S^3$. First, note that $\dim H = 3$ and $\dim K^+ = 6$. Recall that Proj_k, $k = 1, 2, 3$, denotes the projection onto the k-th factor of $S^3 \times S^3 \times S^3$. Since we assume that the action is non-reducible, it follows from Proposition 4.6 that $\mathrm{Proj}_k(H_0)$, $k = 1, 2, 3$, is not S^3. On the other hand, $\mathrm{Proj}_k(H_0)$ cannot be trivial. Otherwise, H would be a 3-dimensional subgroup of $S^3 \times S^3$ and H_0 must project onto one of the factors. This would yield a reducible action, which would contradict the assumption that the action is non-reducible. Thus, $\mathrm{Proj}_k(H_0) = S^1$, for $k = 1, 2, 3$. An inspection of the subgroups of $S^3 \times S^3 \times S^3$ shows that none of the 6-dimensional subgroups of $S^3 \times S^3 \times S^3$ contains H. Therefore, no non-reducible action can occur. □

6. Proof of Theorem G

To prove that $\Sigma(\mathbf{P}^3) \times \mathbf{S}^2$ is homeomorphic to $\mathbf{S}^4 \times \mathbf{S}^2$ we will use a special instance of the classification of closed, oriented, simply-connected topological 6-manifolds with torsion-free homology. This classification follows from work of Wall [54], Jupp [31], and Zhubr [60]. We first recall Jupp's classification of closed, oriented, simply-connected topological 6-manifolds with torsion-free homology (see [31] or [46, Section 1]). We follow Okonek and Van de Ven's presentation in [46, Section 1].

Definition 6.1. (1) A *system of invariants* is a sextuple $(r, H, \mu, w, p, \Delta)$, where
- r is a nonnegative integer;
- H is a finitely generated free abelian group;
- $\mu\colon H \oplus H \oplus H \to \mathbb{Z}$ is a symmetric trilinear form;
- $w \in H \otimes \mathbb{Z}_2$;
- $p \in \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z})$; and
- $\Delta \in \mathrm{Hom}(H, \mathbb{Z}_2)$.

(2) A system of invariants $(r, H, \mu, w, p, \Delta)$ is *admissible* if, for every $W \in H$ and $T \in \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z})$ such that $\overline{W} \equiv w \pmod 2$ and $\overline{T} \equiv \Delta \pmod 2$, the following relation holds:

$$\mu(W, W, W) \equiv (p + 24T)(W) \pmod{48}.$$

(3) Two systems of invariants $(r, H, \mu, w, p, \Delta)$ and $(r', H', \mu', w', p', \Delta')$ are *equivalent* if $r = r'$ and there exists an isomorphism $\alpha\colon H \to H'$ such that

$$\alpha(w) = w', \quad \alpha^*(\Delta') = \Delta, \quad \alpha^*(\mu') = \mu, \quad \alpha^*(p') = p.$$

Let M be a closed, oriented, simply-connected topological 6-manifold with torsion-free homology. We now recall how to extract from M the system of invariants $(r, H, \mu, w, p, \Delta)$ in Definition 6.1, as explained in [31, Section 1] and [46, Section 1.1, Definition 1 and Theorem 1]. The invariants r, H and μ correspond to invariants of the cohomology ring of M. As pointed out in [46, p. 300, item ii)] the rank of $H^3(M, \mathbb{Z})$ is an even number. We let $b_3(M)$ denote the rank of $H^3(M, \mathbb{Z})$. One obtains r, H and μ as follows:

- $r = b_3(M)/2$, a nonnegative integer;
- $H = H^2(M, \mathbb{Z})$, a finitely generated free abelian group;
- μ corresponds to the symmetric trilinear form $\mu_M\colon H \oplus H \oplus H \to \mathbb{Z}$ given by the cup product evaluated on the orientation class.

The invariants w, p, and Δ correspond to characteristic classes of M and are obtained in the following way:

- $w \in H \otimes \mathbb{Z}_2$ corresponds to the second Stiefel–Whitney class $w_2(M) \in H^2(M, \mathbb{Z}_2) = H \otimes \mathbb{Z}_2$;
- $p \in \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z})$ represents the first Pontryagin class $p_1(M) \in H^4(M, \mathbb{Z})$. By Poincaré duality, $H^4(M, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z})$.
- $\Delta \in \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z}_2)$ is the Kirby–Siebenmann class $\Delta(M) \in H^4(M, \mathbb{Z}_2)$. By Poincaré duality, $H^4(M, \mathbb{Z}_2) = \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z}_2)$.

The invariants of M listed above satisfy the relation

$$\mu_M(W, W, W) \equiv (p_1(M) + 24T)(W) \pmod{48}.$$

for all integral classes $W \in H^2(M, \mathbb{Z})$, $T \in H^4(M, \mathbb{Z})$ with $\overline{W} \equiv w_2(M) \pmod 2$, $\overline{T} \equiv \Delta(M) \pmod 2$. Therefore, any system of invariants coming from a closed, oriented, simply-connected topological 6-manifold with torsion-free homology is admissible.

We now state Jupp's classification theorem (see [31, Theorem 1] and [46, Theorem 1]).

Theorem 6.2 (Jupp)**.** *The assignment*

$$M \mapsto \big(b_3(M)/2, H^2(M, \mathbb{Z}), \mu_M, w_2(M), p_1(M), \Delta(M)\big)$$

induces a bijection between oriented homeomorphism classes of closed, oriented, simply-connected topological 6*-manifolds with torsion-free homology and equivalence classes of admissible systems of invariants* $(r, H, \mu, w, p, \Delta)$*. Such a manifold has a smooth (or PL) structure if and only if* $\Delta(M) = 0$*, and the smooth structure is unique.*

We point out that for closed, oriented, simply-connected topological 6-manifolds the first Pontryagin class is always integral (see [31]). Okonek and Van de Ven [46, pp. 302–303] have summarized the classification in the special case where $r = 0$ and $H = \mathbb{Z}$. This is the case that is relevant to us, since for $\Sigma(\mathbf{P}^3) \times \mathbf{S}^2$ we have $r = 0$ and $H = \mathbb{Z}$. We now recall these results (cf. [46, Example 2]).

Definition 6.3 ([46])**.** (1) A quadruple $(\bar{W}, \bar{T}, d, p) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}$ is *admissible* if

$$d(2x + W)^3 \equiv (p + 24T)(2x + W) \pmod{48}, \tag{6.1}$$

for every integer x.

(2) Two admissible quadruples $(\bar{W}, \bar{T}, d, p)$ and $(\bar{W}', \bar{T}', d', p')$ are *equivalent* if $\bar{W} = \bar{W}'$, $\bar{T} = \bar{T}'$ and $(d', p') = \pm(d, p)$.

(3) A quadruple $(\bar{W}, \bar{T}, d, p) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}$ is *normalized* if $d \geq 0$, and $p \geq 0$ if $d = 0$.

(4) Two normalized quadruples $(\bar{W}, \bar{T}, d, p)$, $(\bar{W}', \bar{T}', d', p')$ are *weakly equivalent* if $d' = d$, $\bar{W}' = \bar{W}$ and

$$\begin{aligned} p + 24T \equiv p' + 24T' &\pmod{48} \quad \text{if } d \equiv 0 \pmod 2, \\ p \equiv p' &\pmod{24} \quad \text{if } d \equiv 1 \pmod 2. \end{aligned}$$

Theorem 6.4 ([46])**.** (1) *Let M be as in Theorem 6.2 with $r = 0$ and $H = \mathbb{Z}$. The system of invariants introduced in Definition 6.1 can be identified with admissible quadruples $(\bar{W}, \bar{T}, d, p) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}$, where the "degree" d corresponds to the cubic form μ (i.e. d is the unique integer such that $\mu(x, x, x) = dx^3$ for every integer x).*

(2) *The assignment*
$$X \to (\bar{W}, \bar{T}, d, p)$$
induces a 1-1 correspondence between oriented homeomorphism classes of closed, oriented, simply-connected topological 6-manifolds with torsion-free homology and equivalence classes of normalized admissible quadruples $(\bar{W}, \bar{T}, d, p)$.

(3) *The assignment*
$$X \to (\bar{W}, \bar{T}, d, p)$$
induces a 1-1 correspondence between homotopy classes of closed, oriented, simply-connected topological 6-manifolds with torsion-free homology and weak equivalence classes of admissible systems of invariants.

Now we use the above results to prove that $\Sigma(\mathbf{P}^3) \times \mathbf{S}^2$ is homeomorphic to $\mathbf{S}^4 \times \mathbf{S}^2$. Let $(\bar{W}, \bar{T}, d, p)$ be the admissible quadruple of $M = \Sigma(\mathbf{P}^3) \times \mathbf{S}^2$ as in Theorem 6.4–(1). Since $(0, 0, 0, 0)$ is the admissible quadruple of $\mathbf{S}^4 \times \mathbf{S}^2$, and M is homotopy equivalent to $\mathbf{S}^4 \times \mathbf{S}^2$, Theorem 6.4–(3) implies that $d = 0$, $\bar{W} = 0$, and $p + 24T \equiv 0 \pmod{48}$. On the other hand, in this case, equation (6.1) is equivalent to $p \equiv 0 \pmod{24}$, i.e. $p = 24k$, for $k = 0, \pm 1, \pm 2, \ldots$. Hence, $T \equiv p/24 \pmod 2$; therefore, it suffices to compute the first Pontryagin class of M. If $p = 0$, then M is homeomorphic to $\mathbf{S}^4 \times \mathbf{S}^2$ by Theorem 6.4–(2); if $p \equiv 0 \pmod{48}$, M is smoothable and, if $p \equiv 24 \pmod{48}$, M is non-smoothable by Theorem 6.2.

In the remainder of this section, we compute the first Pontryagin class p_1 of $\Sigma(\mathbf{P}^3) \times \mathbf{S}^2$ and show that $p_1 = 0$. Note that rational Pontryagin classes are defined for topological manifolds (see [32]) and for polyhedral rational homology manifolds (see [39, Ch. 20] or [58, Ch. 1]; also [25, 53]). In both cases, one can also define Hirzebruch l-classes so that $l_1 = \frac{1}{3}p_1$ (see [39, p. 231] and [32, Section 2.1]). Typically, the l-classes are not integral. When a polyhedral rational homology manifold is a topological manifold, both definitions of Pontryagin classes (and of Hirzebruch l-classes) agree (see [32, p. 244]).

The l-classes are multiplicative, i.e. given any two topological or polyhedral rational homology manifolds X and Y, one has

$$l_i(X \times Y) = \sum_{p+q=i} l_p(X) l_q(Y).$$

Thus in our case, since $\Sigma(\mathbf{P}^3)$ is a polyhedral rational homology manifold, we have

$$\begin{aligned} l_1(\Sigma(\mathbf{P}^3) \times \mathbf{S}^2) &= l_1(\Sigma(\mathbf{P}^3)) + l_1(\mathbf{S}^2) \\ &= l_1(\Sigma(\mathbf{P}^3)). \end{aligned}$$

Therefore, to find $p_1(\Sigma(\mathbf{P}^3) \times \mathbf{S}^2)$, it suffices to find $l_1(\Sigma(\mathbf{P}^3))$.

First observe that $\Sigma(\mathbf{P}^3) \approx \mathbf{S}^4/I^*$, where I^* acts in the obvious way on a round $\mathbf{S}^4$ by orientation-preserving diffeomorphisms. This observation allows us to use a formula of Zagier [59] to compute the Hirzebruch l-class of the quotient space X/G of a closed, oriented smooth manifold X by the orientation-preserving diffeomorphism action of a finite group G. Since $\Sigma(\mathbf{P}^3)$ is 4-dimensional, the top-dimensional component of l is l_1 and, by [59, Observation (ii) on p. 256], l_1 equals the signature. Thus, to compute $l_1(\Sigma(\mathbf{P}^3))$, we need only compute $\mathrm{Sign}(\Sigma(\mathbf{P}^3))$. Recall now that $\Sigma(P^3)$ is homotopy equivalent to $\mathbf{S}^4$. Therefore, their cohomology rings are isomorphic and it follows that $\mathrm{Sign}(\Sigma(\mathbf{P}^3)) = 0$. Hence the top component of $l(\Sigma(\mathbf{P}^3))$ is zero. Since the top component of $l(\Sigma(\mathbf{P}^3))$ is $\mathrm{Sign}(\Sigma(\mathbf{P}^3)) = \frac{1}{3}p_1(\Sigma(\mathbf{P}^3)) = \frac{1}{3}p_1(\Sigma(\mathbf{P}^3) \times \mathbf{S}^2)$, we conclude that the first Pontryagin class of $\Sigma(\mathbf{P}^3) \times \mathbf{S}^2$ is zero. Therefore, $\Sigma(\mathbf{P}^3) \times \mathbf{S}^2$ is homeomorphic to $\mathbf{S}^4 \times \mathbf{S}^2$. □

Remark 6.5. One can also see that $\mathrm{Sign}(\Sigma(\mathbf{P}^3)) = 0$ using results of Atiyah and Singer [2] and of Atiyah and Bott [1], which we briefly outline in the following paragraphs.

Let X be a closed, oriented smooth manifold and G a finite group acting on X by orientation-preserving diffeomorphisms. Let

$$\pi\colon X \to X/G$$

be the projection map of X onto the orbit space X/G. As mentioned above, the top-dimensional component of $l(X/G)$ is $\mathrm{Sign}(X/G)$. By the Atiyah-Singer G-equivariant signature theorem (see [2, Section 6] or [26]), the signature of X/G is given by

$$\mathrm{Sign}(X/G) = \frac{1}{|G|}\sum_{g\in G}\mathrm{Sign}(g, X). \tag{6.2}$$

In our particular case, where $G = I^*$ acts on $X = \mathbf{S}^4$ with only two isolated fixed points, $\mathrm{Sign}(g, X)$ is given by a signature formula of Atiyah and Bott [1, Theorem 6.27], which we state as Theorem 6.6 below. Before quoting the theorem, we recall some notation.

Let $f\colon X \to X$ be an isometry of a compact, oriented, even-dimensional Riemannian manifold X and p be a fixed point of f. Consider the differential

$$df_p\colon T_pX \to T_pX.$$

Because f is an isometry of X, df_p will be an isometry of T_pX. Hence, one may decompose T_pX into a direct sum of orthogonal 2-planes

$$T_pX = E_1 \oplus E_2 \oplus \ldots \oplus E_n,$$

which are stable under df_p. Let (e_k, e'_k) be an orthogonal basis of E_k. We may choose (e_k, e'_k) so that

$$v_p(e_1 \wedge e'_1 \wedge \ldots \wedge e_n \wedge e'_n) = 1,$$

where v is the volume form of X. Relative to such a basis df_p is then given by rotations by angles θ_k in E_k. That is,

$$df_p e_k = \cos\theta_k e_k + \sin\theta_k e'_k,$$

$$df_p e'_k = -\sin\theta_k e_k + \cos\theta_k e'_k.$$

The resulting set of angles $\{\theta_k\}$ is called a *coherent system* for df_p.

Theorem 6.6 (Atiyah and Bott). *Let $f\colon X^{2n} \to X^{2n}$ be an isometry of the compact, oriented, even dimensional Riemannian manifold X. Assume further that f has only isolated fixed points $\{p\}$, and let θ^p_k be a system of coherent angles for df_p. Then the signature of f is given by*

$$\mathrm{Sign}(f, X) = i^{-n}\sum_p\prod_k \cot(\theta^p_k/2).$$

We now use Theorem 6.6 to compute $\mathrm{Sign}(g, X)$ for each $g \in I^*$ and recover $\mathrm{Sign}(\mathbf{S}^4/I^*)$ via equation (6.2). For non-trivial $g \in G$, the fixed point set X^g has two elements, say $\{p, q\}$, corresponding to the cone points of the suspended I^*-action on $\mathbf{S}^4$. Let $\{\alpha, \beta\}$ be the coherent system for p. Then the coherent system for q will be $\{-\alpha, \beta\}$, so $\mathrm{Sign}(g, \mathbf{S}^4) = 0$. On the other hand, $\mathrm{Sign}(e, \mathbf{S}^4) = \mathrm{Sign}(\mathbf{S}^4) = 0$ and $\mathrm{Sign}(-e, \mathbf{S}^4) = 0$ by Theorem 6.6. As a result $\mathrm{Sign}(\mathbf{S}^4/I^*) = 0$.

References

[1] Atiyah, M.F. and Bott, R., *A Lefschetz fixed point formula for elliptic complexes: II. Applications*, Ann. of Math. (2) **88** 1968 451–491.

[2] Atiyah, M.F. and Singer, I.M., *The index of elliptic operators. III*, Ann. of Math. (2) **87** 1968 546–604.

[3] Böhm, C., *Non-compact cohomogeneity one Einstein manifolds*, Bull. Soc. Math. France **127** (1999), no. 1, 135–177.

[4] Bredon, G.E., *On homogeneous cohomology spheres*, Ann. of Math. (2) **73** (1961) 556–565.

[5] Bredon, G.E., *Introduction to Compact Transformation Groups*. Acad. Press, New York, 1972.

[6] Burago, D., Burago, Y. and Ivanov, S., *A course in metric geometry*, Graduate Studies in Mathematics, 33. American Mathematical Society, Providence, RI, 2001.

[7] Burago, Y., Gromov, M. and Perel'man, G., *A.D. Alexandrov spaces with curvatures bounded below*, (Russian) Uspekhi Mat. Nauk 47 (1992), no. 2(284), 3–51, 222; translation in Russian Math. Surveys 47 (1992), no. 2, 1–58

[8] Cannon, J.W., *Shrinking cell-like decompositions of manifolds. Codimension three.* Ann. of Math. (2) **110** (1979), no. 1, 83–112.

[9] Dancer, A. and Swann, A., *Quaternionic Kahler manifolds of cohomogeneity one*, Internat. J. Math. 10 (1999), no. 5, 541–570.

[10] Dantzig, D. van and Waerden, B. L. van der, *Über metrisch homogene räume*, Abh. Math. Sem. Univ. Hamburg **6** (1928), no. 1, 367–376.

[11] Dearricott, O., *A 7-manifold with positive curvature*, Duke Math. J. **158** (2011), no. 2, 307–346.

[12] Dynkin, E.B., *Maximal subgroups of the classical groups* (Russian) Trudy Moskov. Mat. Obšč. **1**, (1952). 39–166.

[13] Edwards, R.D., *Suspensions of homology spheres*, preprint (2006), arXiv:math/0610573 [math.GT].

[14] Edwards, R.D., *The topology of manifolds and cell-like maps*, Proceedings of the International Congress of Mathematicians (Helsinki, 1978), pp. 111–127, Acad. Sci. Fennica, Helsinki, 1980.

[15] Escher, C.M. and Ultman, S.K., *Topological structure of candidates for positive curvature*, Topology Appl. **158** (2011), no. 1, 38–51.

[16] Frank, P., *Cohomogeneity one manifolds with positive Euler characteristic*, Transform. Groups **18** (2013), no. 3, 639–684.

[17] Fukaya, K. and Yamaguchi, T., *Isometry groups of singular spaces*, Math. Z. **216** (1994), no. 1, 31–44.

[18] Galaz-Garcia, F. and Searle, C., *Cohomogeneity one Alexandrov spaces*, Transform. Groups, **16** (2011), no. 1, 91–107.

[19] Goertsches, O. and Mare, A., *Equivariant cohomology of cohomogeneity one actions*, Topology Appl. **167** (2014), 36–52.

[20] Grove, K., Verdiani, L. and Ziller, W., *An exotic* $T_1\mathbb{S}^4$ *with positive curvature*, Geom. Funct. Anal. **21** (2011), no. 3, 499–524.

[21] Grove, K., Wilking, B. and Ziller, W., *Positively curved cohomogeneity one manifolds and 3-Sasakian geometry*, J. Differential Geom. **78** (2008), no. 1, 33–111.

[22] Grove, K. and Ziller, W., *Curvature and symmetry of Milnor spheres*, Ann. of Math. (2) **152** (2000), no. 1, 331–367.

[23] Grove, K. and Ziller, W., *Cohomogeneity one manifolds with positive Ricci curvature*, Invent. Math. **149** (2002), no. 3, 619–646.

[24] Hirsch, M.W. and Mazur, B., *Smoothings of piecewise linear manifolds*, Princeton University Press, Princeton, NJ, 1974, Annals of Mathematics Studies, No. 80.

[25] Hirzebruch, F., *Pontrjagin classes of rational homology manifolds and the signature of some affine hypersurfaces*, Proceedings of Liverpool Singularities Symposium, II (1970), 207–212.

[26] Hirzebruch, F., *The signature theorem: reminiscences and recreation*, Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970), pp. 3–31. Ann. of Math. Studies, No. 70, Princeton Univ. Press, Princeton, N.J. 1971.

[27] Hoelscher, C.A., *Classification of cohomogeneity one manifolds in low dimensions*, Ph.D. Thesis, University of Pennsylvania. 2007.

[28] Hoelscher, C.A., *Diffeomorphism type of six-dimensional cohomogeneity one manifolds*, Ann. Global Anal. Geom. **38** (2010), no. 1, 1–9.

[29] Hoelscher, C.A., *Classification of cohomogeneity one manifolds in low dimensions*, Pacific J. Math. **246** (2010), no. 1, 129–185.

[30] Hoelscher, C.A., *On the homology of low-dimensional cohomogeneity one manifolds*, Transform. Groups **15** (2010), no. 1, 115–133.

[31] Jupp, P., *Classification of certain 6-manifolds*, Proc. Cambridge Philos. Soc. **73** (1973), 293–300

[32] Kahn, P.J., *A note on topological Pontrjagin classes and the Hirzebruch index formula*, Illinois J. Math. **16** (1972), 243–256.

[33] Kervaire, M.A. and . Milnor, J. W, *Groups of homotopy spheres. I*, Ann. of Math. (2) **77** (1963), 504–537.

[34] Kirby, R.C. and Scharlemann, M.G., *Eight faces of the Poincaré homology sphere*, Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977), pp. 113–146, Academic Press, New York-London, 1979.

[35] Kirby, R.C. and Siebenmann, L.C., *Foundational essays on topological manifolds, smoothings, and triangulations*, Princeton Univ. Press, Princeton, 1977.

[36] Kobayashi, S., *Transformation groups in differential geometry*, Reprint of the 1972 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[37] Kobayashi, S. and Nomizu, K., *Foundations of differential geometry*, Vol. I. Reprint of the 1963 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1996.

[38] Maunder, C.R.F., *On the Pontrjagin classes of homology manifolds*, Topology, **10** (1971) 111–118.

[39] Milnor, J.W. and Stasheff, J.D., *Characteristic classes*, Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974.

[40] Montgomery, D. and Yang, C.T., *Orbits of highest dimension*, Trans. Amer. Math. Soc. **87** (1958), 284–293.

[41] Montgomery, D. and Zippin, L., *A theorem on Lie groups*, Bull. Amer. Math. Soc. **48** (1942), 448–452.

[42] Mostert, P.S., *On a compact Lie group acting on a manifold*, Ann. of Math. (2) **65** (1957), 447–455. Errata, "On a compact Lie group acting on a manifold". Ann. of Math. (2) **66** 1957 589.

[43] Munkres, J. R., *Obstructions to the smoothing of piecewise-differentiable homeomorphisms*, Ann. of Math. (2) **72** (1960), 521–554.

[44] Myers, S.B. and Steenrod, N.E., *The group of isometries of a Riemannian manifold*, Ann. of Math. (2) **40** (1939), no. 2, 400–416.

[45] Neumann, W.D., 3-*Dimensional G-manifolds with* 2-*dimensional orbits*. In: Mostert, P.S. (ed.) Proceedings of Conference on Transformation Groups, pp. 220–222. Springer Verlag, Berlin (1968).

[46] Okonek, C. and Van de Ven, A., *Cubic forms and complex* 3-*folds*, Enseign. Math. **41** (1995), no.3-4, 297–333.

[47] Onishchik, A.L., *Topology of transitive transformation groups*, Johann Ambrosius Barth Verlag GmbH, Leipzig, 1994.

[48] Parker J., 4-*Dimensional G-manifolds with* 3-*dimensional orbits*, Pacific J. Math. **129** (1986), no. 1, 187–204.

[49] Perelman, G., *Elements of Morse theory on Aleksandrov spaces*, St. Petersburg Math. J. **5** (1994), 205–213.

[50] Rourke, C.P. and Sanderson, B.J., *Introduction to piecewise-linear topology*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 69. Springer-Verlag, New York-Heidelberg, 1972.

[51] Schwachhöfer, L.J. and Tuschmann, W., *Metrics of positive Ricci curvature on quotient spaces*, Math. Ann. **330** (2004), no. 1, 59–91.

[52] Searle, C., *Cohomogeneity and positive curvature in low dimensions*, Math. Z. **214** (1993), no. 3, 491–498. Corrigendum: Math. Z. 226 (1997), no. 1, **165–167**.

[53] Thom, R., *Les classes caractéristiques de Pontrjagin des variétés triangulées*, 1958 Symposium internacional de topología algebraica International symposium on algebraic topology 54–67 Universidad Nacional Autónoma de México and UNESCO, Mexico City.

[54] Wall, C.T.C., *Classification problems in differential topology. V. On certain 6-manifolds*, Invent. Math. **1** (1966), 355–374.

[55] Weinberger, S., *The topological classification of stratified spaces*, Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1994.

[56] Wolf, J.A., *Spaces of Constant Curvature*, McGraw-Hill Book Co., New York-London-Sydney 1967.

[57] Wu, J.Y., *Topological regularity theorems for Alexandrov spaces*, J. Math. Soc, Japan, **49** (1997), no. 4. 741–757.

[58] Zagier, D., *Equivariant Pontrjagin classes and applications to orbit spaces. Applications of the G-signature theorem to transformation groups, symmetric products and number theory*, Lecture Notes in Mathematics, Vol. 290. Springer-Verlag, Berlin-New York, 1972.

[59] Zagier, D., *The Pontrjagin class of an orbit space*, Topology, **11** (1972), 253–264.

[60] Zhubr, A.V. *Closed simply connected six-dimensional manifolds: proofs of classification theorems*, Algebra i Analiz **12** (2000), no.4, 126–230.

[61] Ziller, W., *On the geometry of cohomogeneity one manifolds with positive curvature*, Riemannian topology and geometric structures on manifolds, 233–262, Progr. Math., 271, Birkhäuser Boston, Boston, MA, 2009.

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