Received: 30 January 2020 | Revised: 28 February 2021 | Accepted: 17 April 2021

DOI: 10.1111/jpet.12515

ORIGINAL ARTICLE

JOURNAL OF PUBLIC ECONOMIC THEORY WILEY

Optimal taxation in an endogenous growth model with variable population and public expenditure

Thomas I. Renström[1] | **Luca Spataro**[2]

[1]Department of Economics and Finance, Durham University Business School, Durham, UK

[2]Dipartimento di Economia e Management, University of Pisa, Pisa, Italy

Correspondence
Luca Spataro, Dipartimento di Economia e Management, University of Pisa, via C. Ridolfi, 10, 56124 Pisa, Italy.
Email: luca.spataro@unipi.it

Abstract

In this paper we analyze second-best optimal taxation in an endogenous-growth model driven by public expenditure, in presence of endogenous fertility and labor supply. Normative analysis shows positive taxes on the number of children, which are necessary to correct for congestion in the publicly provided input (such as education and healthcare), negative public debt. Results on capital and labor income taxation depend on whether the public input is optimally provided.

1 | INTRODUCTION

In this paper we analyze second-best optimal taxation in an endogenous-growth model driven by public expenditure, in which we allow for endogenous fertility and labor supply.

An extensive literature on optimal taxation of factor incomes in a general-equilibrium dynamic framework has been flourishing in the last three decades, both from a normative and positive point of view.

Under the normative approach, a well-established finding is that in dynastic economies, in the long run, capital income should not be taxed, thus shifting the burden from capital income taxation toward labor (Chamley, 1986; Judd, 1985). Although the result is robust with respect to several extensions, some exceptions may arise, such as in the case of borrowing constraints (Aiyagari, 1995), market imperfections (Judd, 1997), incomplete set of taxes (Correia, 1996), social discounting and disconnected economies (De Bonis & Spataro, 2010), government time-inconsistency and lack of commitment (Reis, 2012), externalities from suboptimal policy rules (Turnovsky, 1996).

In endogenous growth models, Lucas (1990) and Turnovsky (1992) compare the effects of a tax on capital versus a tax on labor and find the former to be inferior to the latter from a welfare point of view[1].

Under the positive approach, several papers have analyzed the impact of fiscal policies on economic growth, such as Barro (1990), Jones and Manuelli (1990), and Rebelo (1991). Turnovsky (1996) analyses the issue of first-best optimal taxation and expenditure policies in an endogenous growth model with externalities stemming from public goods both in the utility and in the production function and Turnovsky (2000) extends the analysis to the case of endogenous labor supply[2]. In this type of models, direct taxation brings about a natural trade-off: on one hand, it distorts incentives to save and work, hence reducing growth; on the other, it increases the marginal productivity of private inputs, thus increasing growth and possibly welfare. This is also the key contribution of Barro (1990) and has been extended in several subsequent studies.

However, in all these papers (with normative or positive analysis) population growth is either absent or exogenous. In fact, the observed large variations in fertility rates both across countries and across times, have led an increasing number of scholars to work on the re-formulation of economic theory and fiscal policy in the presence of endogenous fertility (see, among others, Balestrino et al., 2002; Barro & Becker, 1989; Becker & Barro, 1988; Cigno, 2001; de la Croix et al., 2012; Renström & Spataro, 2011). Moreover, most of the endogenous growth models mentioned above suffer from the "scale effect," meaning that the steady-state growth rate increases with the size (scale) of the economy, as indexed, for example, by population.[3]

To breach this gap, the contribution of our work is twofold: first, we extend the Barro (1990) model, in which the engine of growth is productive public expenditure, by allowing for both endogenous labor supply (as in Turnovsky, 2000) and endogenous population (as in Barro & Becker, 1989; Becker & Barro, 1988 and Spataro & Renström, 2012; Renström & Spataro, 2019) in one encompassing model. Second, in this scenario, we analyze the second-best optimal tax structure. To the best of our knowledge, this has not been done so far.

In fact, the process of economic development and in particular the so-called Demographic Transition (i.e., the transition from a positive to a negative relationship between per-capita income and fertility) have been intensively studied in recent years. Galor (2011) argues that the positive effect of technical progress on the return to education and the feedback effect of higher education on technical progress bring upon a rapid decline in fertility accompanied by accelerated output growth.[4] Although producing relevant results, such works have not focused on the role of fiscal policies. Therefore, in this paper we provide another possible explanation on population changes which can add to the existing ones so far emerged in the literature, by focusing on the role of fiscal policies.

We retain the Barro (1990) approach since there is consolidated evidence that public expenditure in favor of productive services has a sizable impact on growth (for major insights, see,

[1] As pointed out by Lucas (1990), the result holds for the steady state, although along the transition the welfare gains of an efficient tax shift are lower. See also Sinclair and Slater (1990) and Laitner (1995).

[2] A discussion of the effects of taxation in models with endogenous labor supply is also provided by Rebelo (1991). See also Basu et al. (2004) and Basu and Renström (2007) for indivisible labor economies.

[3] To overcome this problem, non-scale models have been provided by Jones (1995) and subsequent works, although still hinging on exogenous population (see, e.g., Strulik et al., 2013; Bucci & Raurich, 2017).

[4] See also Elgin (2012). For recent empirical evidence on development patterns and demograhic dynamics, see Cervellati et al. (2017).

among others, García Peñalosa & Turnovsky, 2005; Turnovsky, 1996). We also note that our model can avoid two shortcomings of the aforementioned nonscale growth models: first, the direct positive link between economic and demographic growth, which is not supported by post-war data (see Acemoglu, 2009, p. 448)[5] and, second, the fact that the long-run equilibrium growth rate is determined by technological parameters and is independent of macro policy instruments.

The assumption of endogenous population, may pose major issues related to welfare analysis. In fact, given that under these circumstances welfare evaluations typically imply the comparisons between states of the world in which the size of population is different, the Pareto criterion cannot be used. To deal with this issue, we adopt a Classical Utilitarianism approach, which allows for social orderings that are based on desirable welfarist axioms in presence of variable population and, under our assumptions, can also avoid unpleasant outcomes as for population (e.g., the so-called Repugnant Conclusion[6]).

In this framework, normative analysis reveals positive taxation on children, which is necessary to correct for congestion in the publicly provided input, negative public debt and constant population. Results on capital and labor income taxation depend on whether the public input is optimally provided.

The work is organized as follows: after laying out the model (Section 2), we characterize the decentralized equilibrium (Section 3), the second-best optimal public debt, tax rules and population dynamics (Section 4). Section 5 concludes.

2 | THE MODEL

In this section we lay out the benchmark model. We denote individual quantities by lower case letters, and aggregate quantities by corresponding upper case letters, so that $V = Nv$, with N population size. We will assume that public expenditure is fixed as a fraction of GDP and we will discuss the cases in which the share of public expenditure is set arbitrarily or it is set optimally. This approach will enhance our understanding of the optimal capital income tax rate, which, as it will be shown, depends upon both the socially optimal level of government expenditure and the deviation of actual expenditure from its social optimum.

We assume that the representative agent is endowed with a unit of time that can be allocated either to leisure or to work or to child rearing. Individuals live for one period and we also assume intergenerational altruism, with life-time utility function $u(c_t, l_t)$, where c_t is life-time consumption for that individual, l_t is time allocated to work and child rearing. This utility is assumed to be increasing in c_t, decreasing in l_t and strictly concave. An individual family chooses consumption, labor supply, savings, and the number of children (i.e., the change in the cohort size N).

[5]See Becker et al. (1990) and Renström and Spataro (2015) for endogenous growth models driven by human capital with variable population. In the present work we take a different approach, in that we focus on taxation and the engine of growth is public expenditure.

[6]According to the Repugnant conclusion, any state in which each member of the population enjoys a life above neutrality is declared inferior to a state in which each member of a larger population lives a life with lower utility (see Parfit 1976, 1984). In particular, in a growth model with endogenous population, the Repugnant Conclusion takes the form of upper-corner solution for the population rate of growth (society reproduces at its physical maximum rate). On this issue, see Marsiglio (2014), Renström and Spataro (2019), and the literature therein. See Spataro and Renström (2012) for an optimal taxation and policy change analysis in the presence of Critical Level Utilitarianism a là Blackorby et al. (1995), although without endogenous growth.

The literature has provided different explanations for parents' demand for children. For example, the latter could be motivated supposing individuals: (a) derive utility from the consumption of their children (e.g., Elon Kohlberg, 1976); (ii) aim at receiving old-age support from their children (e.g., Willis, 1979). Here, we assume that parents derive utility from the utility of their children as in Becker and Barro (1988) and Barro and Becker (1989), Jones and Schoonbroodt (2016) and, in particular, a purely welfarist version of the latter (i.e., social utility as the sum of utilities of all generations) as in Blackorby et al. (1995), Michel and Pestieau (1998), Renström and Spataro (2015, 2019), Spataro and Renström (2012).

As for firms, we assume perfectly competitive markets and constant return to scale technology in private inputs. The consequence of the assumptions on the production side is that we retain the "standard" second-best framework, in the sense that there are no profits.

Finally, we assume the government finances an exogenous stream of per-capita expenditure x, that enters as an input in private sector production function, by issuing debt and levying taxes. To retain the second-best, we levy taxes on the choices made by the families, that is, savings, labor supply, and children. Consequently, we introduce the capital-income tax, labor income tax, and a tax/subsidy on the number of children.

2.1 | Preferences

Following previous literature (i.e., Barro & Becker, 1989), we focus on a single dynasty (household) or a policymaker choosing consumption and population growth over time, so as to maximize:

$$U = \int_0^\infty e^{-\rho t} N_t^{1-\varepsilon} u(c_t, l_t) dt, \tag{1}$$

where N_t is the population (family) size of generation t, $1 - \varepsilon > 0$ represents the preference on the population size, $u_t = u(c_t, l_t) \geq 0$ is the instantaneous utility function of an individual of generation t, such that $u(0, \bullet) = 0$, $u_c > 0$, $u_l < 0$, $u_{cc} < 0$, $u_{ll} < 0$ and $\rho > 0$ the intergenerational discount rate. More precisely, we will assume the following form of the instantaneous utility function:

$$u(c, l) = c^{\mu(1-\sigma)}(1 - l)^{\mu(1-\eta)}, \tag{2}$$

with $1 > \mu(1 - \sigma) > 0, 1 > \mu(1 - \eta) > 0$, and $1 - \mu(2 - \sigma - \eta) > 0$, which are the assumptions which can generate sustained long-run per-capita income growth[7]. To retain the welfarist approach, in the rest of the paper we assume $\varepsilon = 0$, that is, Classical Utilitarianism. Moreover, the population dynamics is described by the following law:

$$\frac{\dot{N}_t}{N_t} = n_t. \tag{3}$$

Hence, the problem of the household is to maximize (1) under (3) and the constraint:

[7]Notice that the value of utility must be strictly positive, otherwise the value of population would be negative, implying a corner solution for n. Hence, the case $\sigma > 1$ (with the general form of utility $u(c, l) = \frac{c^{\mu(1-\sigma)}}{1-\sigma}(1 - l)^{\mu(1-\eta)}$) discussed by Straub and Werning (2015) as a potentially source of violation of Chamley-Judd zero tax result cannot apply to this model.

$$\dot{A}_t = \bar{r}_t A_t + \bar{w}_t[l_t - \theta(n_t)_t]N_t - c_t N_t - \tau_t^n n_t N_t, \tag{4}$$

where A_t is household wealth, $\bar{r}_t \equiv r_t(1 - \tau_t^k)$ is net-of-tax interest rate, $\bar{w}_t \equiv w_t(1 - \tau_t^l)$ is the net-of-tax wage, τ_t^k, τ_t^l are the tax rate on capital income and on labor income, respectively, and τ_t^n is the tax on the number of children. Without loss of generality, a consumption tax has been normalized to zero and we have a complete set of second-best tax instruments.

We assume that $\theta(n_t)$ is a time cost and is specified over the number of children each parent has, so that it is a function of the population growth rate. In fact, in equilibrium each parent has the same number of children, so the per family member population growth rate becomes the economy wide one. More precisely, for the sake of simplicity, we assume that the cost for raising children is increasing in the number of children and linear, so that $\theta(n_t) = \theta \bullet n_t$.[8]

As for the cost function, we follow Strulik (1999, 2003) and Boldrin and Jones (2002), Jones and Schoonbroodt (2016) according to which children's cost is a fraction of the wage. In economic growth models this assumption is usually made to avoid that the child cost vanishes as the economy grows. However, in our model with endogenous labor supply, the deep economic reason for the children cost function is that there is a time requirement for child-rearing and the parent chooses the allocation of time between work and child-rearing activity.

Finally, the individual time constraint reads as

$$\begin{cases} (1 - l_t) \text{ is time allocated to leisure;} \\ \theta \bullet n_t \text{ is time allocated to child rearing;} \\ l_t - \theta \bullet n_t \text{ is time allocated to work.} \end{cases}$$

2.2 | Firms

We assume constant-returns-to-scale production technology with labor-augmenting productive public expenditure. More precisely, the production function is

$$F_t = F(K_t, x_t L_t) = F(K_t, x_t N_t(l_t - \theta n_t)) = T K_t{}^{\gamma}(x_t N_t(l_t - \theta n_t))^{1-\gamma} \tag{5}$$

with T the parameter representing total factor productivity, K_t is capital stock, assumed infinitely durable, x_t the labor-augmenting flow of services from government spending on the economy's infrastructure. $L_t = (l_t - \theta n_t)N_t$ is hired labor, with $(l_t - \theta n_t)$the fraction of time dedicated to work by each household.

Assuming perfect competition, firms hire capital, K, and labor services, L, on the spot market and remunerate them according to their marginal productivity, such that

$$F_{Kt} = r_t, \tag{6}$$

$$F_{Lt} = w_t. \tag{7}$$

Moreover, the economy resource constraint is

$$\dot{K}_t = F(K_t, x_t L_t) - c_t N_t - x_t N_t - \nu K_t. \tag{8}$$

[8]Notice that convex childrearing costs, although questionable in terms of realism, are commonly used in population literature (see, among others Tertilt, 2005), in that convexity is necessary for avoiding a corner solution for n. In our work, the time nature of the childbearing costs ensures interiority of the solution for n.

with $v > 0$ the depreciation rate of capital. Notice that, following Barro (1990), public expenditure is labor-augmenting and, given our assumptions on the neoclassical production technology, is necessary to production and, thus, to obtain sustained long-run growth of per-capita income.

2.3 | The government

We allow the government to finance an exogenous stream of public expenditure x_t by levying taxes, both on capital and labor income and issuing debt, B, which law of motion is

$$\dot{B}_t = r_t B_t - \tau_t^k r_t A_t - \tau_t^l w_t (l_t - \theta n_t) N_t - \tau_t^n n_t N_t + x_t N_t. \tag{9}$$

We assume that public expenditure is a constant fraction of total output:

$$xN = \delta F. \tag{10}$$

We thus assume there is full congestion in the publicly provided expenditure, in the sense that, as it emerges from Equations (5) and (10), only per-capita expenditure x, and not total expenditure, augments labor productivity.[9]

Hence, we can summarize the economy's resource constraint as follows

$$\dot{K}_t = \widetilde{F}_t - c_t N_t - vK_t, \tag{11}$$

where

$$\widetilde{F} \equiv F - xN = (1-\delta)\delta^{\frac{1-\gamma}{\gamma}} T^{\frac{1}{\gamma}} (l - \theta n)^{\frac{1-\gamma}{\gamma}} K. \tag{12}$$

Defining $k \equiv \frac{K}{N}$ as the capital intensity and $f \equiv \frac{F}{N}$ as per capita output, Equations (5)–(8) read as[10]

$$f = \delta^{\frac{1-\gamma}{\gamma}} T^{\frac{1}{\gamma}} (l - \theta n)^{\frac{1-\gamma}{\gamma}} k, \tag{5'}$$

$$r + v = F_K = \gamma \frac{f}{k} = \gamma \delta^{\frac{1-\gamma}{\gamma}} T^{\frac{1}{\gamma}} (l - \theta n)^{\frac{1-\gamma}{\gamma}}, \tag{6'}$$

$$w = F_L = \frac{F_l}{N} = (1-\gamma)\frac{f}{l - \theta n} = (1-\gamma)\delta^{\frac{1-\gamma}{\gamma}} T^{\frac{1}{\gamma}} (l - \theta n)^{\frac{1-2\gamma}{\gamma}} k, \tag{7'}$$

$$\dot{k}_t = f_t - n_t k_t - c_t - x_t - vk_t. \tag{8'}$$

Notice that by Equation (5′) f increasing in l requires $\gamma < 1$ and, by Equation (7′) concavity in l requires, $\frac{1}{2} < \gamma$, so that $\frac{1}{2} < \gamma < 1$. Finally, in per-capita terms (11) reads as:

$$\dot{k}_t = (1-\delta) f_t - c_t - n_t k_t - vk_t. \tag{11'}$$

[9] With only per-capita public expenditure increasing labor productivity we assume full congestion in the prevision of x. Any partial congestion would imply economies of scale in population size, and there would not be balanced growth in per-capita terms. Per-capita growth rate would explode if population growth were positive. In this sense, x is indeed a non-pure public good, because of rivalry, and the way in which we imagine its financing (through taxes on children) could make x be interpreted as the flow of services produced by children-related public expenditure for local public goods, such as primary schools, hospitals, or, more in general, healthcare.

[10] In fact, by using $xN = \delta F = \delta T K^{\gamma} (xN(l - \theta n))^{1-\gamma}$ we get that $xN = (\delta T)^{\frac{1}{\gamma}} (l - \theta n)^{\frac{1-\gamma}{\gamma}} K$, such that $F = \delta^{\frac{1-\gamma}{\gamma}} T^{\frac{1}{\gamma}} (l - \theta n)^{\frac{1-\gamma}{\gamma}} K$ and $\tilde{F} \equiv F - xN = (1-\delta)\delta^{\frac{1-\gamma}{\gamma}} T^{\frac{1}{\gamma}} (l - \theta n)^{\frac{1-\gamma}{\gamma}} K$.

The production maximizing level of x is when (12) is maximized with respect to δ and it turns out that this δ is time invariant.[11]

3 | THE DECENTRALIZED EQUILIBRIUM

We now characterize the decentralized equilibrium of the economy. The problem of the individual (household) is to maximize (1) subject to (3) and (4), taking A_0 and N_0 as given. The current value Hamiltonian is[12]

$$H_t = N_t u_t + q_t[\bar{r}_t A_t + \bar{w}_t(l_t - \bar{\theta}_t n_t)N_t - c_t N_t] + \lambda_t n_t N_t \tag{13}$$

with $\bar{\theta}_t \equiv \theta + \frac{\tau_t^n}{w_t\left(1-\tau_t^l\right)}$ is the net-of-tax cost of child rearing one child in time units, q_t and λ_t the shadow price of wealth and of population, respectively. Put it differently, according to our specification, $w_t\left(1-\tau_t^l\right)\theta + \tau_t^n$ turns out to be the opportunity cost for giving birth to one child. The first-order conditions are the following[13]:

$$\frac{\partial H}{\partial A} = \rho q - \dot{q} \Rightarrow \dot{q} = (\rho - \bar{r})q, \tag{14}$$

$$\frac{\partial H}{\partial c} = 0 \Rightarrow u_c = q, \tag{15}$$

$$\frac{\partial H}{\partial l} = 0 \Rightarrow -u_l = q\bar{w}, \tag{16}$$

$$\frac{\partial H}{\partial N} = \rho\lambda - \dot{\lambda} \Rightarrow \dot{\lambda} = (\rho - n)\lambda - u - q[\bar{w}(l - \bar{\theta}n) - c], \tag{17}$$

$$\frac{\partial H}{\partial n} = 0 \Rightarrow \lambda = q\bar{w}\bar{\theta}, \tag{18}$$

and the transversality conditions are[14]

$$\lim_{t\to\infty} e^{-\rho t} q_t A_t = 0, \lim_{t\to\infty} e^{-\rho t}\lambda_t N_t = 0. \tag{19}$$

The last condition for the competitive equilibrium is capital market clearing condition:

$$A_t = K_t + B_t, \tag{20}$$

which, in per capita terms, is

$$a_t = k_t + b_t. \tag{21}$$

Note that, given the policy under investigation, equilibrium market price (interest rate) and equilibrium labor price are equal to (private) marginal product of capital and labor, respectively (see Equations 6′ and 7′) and the latter can be different from social marginal product of capital and labor, which are

[11] This is the productively efficient level of x as it yields an allocation on the production possibilities frontier and is desired also in the second best (see Diamond-Mirrlees 1971).

[12] We focus on interior solutions for n, in that the problem is concave.

[13] We omit the subscript referring to time when it causes no ambiguity to the reader.

[14] It can proved that transversality conditions are satisfied along the BGP.

$$r^* + v = \widetilde{F}_K = (1-\delta)\delta^{\frac{1-\gamma}{\gamma}}T^{\frac{1}{\gamma}}(l-\theta n)^{\frac{1-\gamma}{\gamma}}, \tag{22a}$$

$$w^* = \frac{\widetilde{F}_l}{N} = \frac{1-\gamma}{\gamma}(1-\delta)\delta^{\frac{1-\gamma}{\gamma}}T^{\frac{1}{\gamma}}(l-\theta n)^{\frac{1-2\gamma}{\gamma}}k. \tag{22b}$$

This difference is due to the presence of externality brought about by public expenditure. More precisely, we get that

$$r \underset{<}{\overset{>}{-}} r^* \Longleftrightarrow w \underset{<}{\overset{>}{-}} w^* \Longleftrightarrow \gamma \underset{<}{\overset{>}{-}} (1-\delta). \tag{23}$$

In case the policymaker aims to correct for this externality, it can either choose δ optimally (i.e., equal to its productively efficient level $1-\gamma$, as in Turnovsky, 1996, for example), or be forced to raise corrective taxes.

3.1 | Balanced growth path (BGP)

Finally, we characterize the BGP, along which all per-capita variables grow at the same rate.

By using Equation (11′), taking time derivatives of (15), (16), and (18) and using (2), (14), and (17) and recognizing that along the BGP n, l, and c/k are constant, we get the expressions for the per-capita consumption, wage rate and capital intensity growth rates (which, along the BGP are equal):

$$\frac{\dot{c}}{c} = \frac{\bar{r}-\rho}{1-\mu(1-\sigma)}, \tag{24a}$$

$$\frac{\dot{\bar{w}}}{\bar{w}} = \bar{r} - \frac{1-\mu(1-\sigma)}{\bar{\theta}\mu(1-\eta)} + l\frac{1-\mu(2-\sigma-\eta)}{\bar{\theta}\mu(1-\eta)}, \tag{24b}$$

$$\frac{\dot{k}}{k} = G\bar{f} - n - v, \tag{24c}$$

with $\bar{f} \equiv \frac{f}{k}$, $G \equiv (1-\delta) - \frac{(1-\sigma)(1-\gamma)(1-\tau^l)}{(l-\theta n)(1-\eta)}(1-l) > 0$.[15] These equations provide the implicit solutions for $(l,\ n,\ c/k)$ and of the per-capita growth rate of the economy along the BGP.

By looking at Equations (24a) and (24b), we can notice that, as usual, the economy growth rate is proportional to the net-of-tax interest rate (Equation 24a). The latter, in turn, depends on the whole set of the endogenous variables $(l,\ n,\ c/k)$ and, thus, and on the deep parameters of the economy, comprising taxes $(\tau^l,\ \tau^k,\ \tau^n)$. Given that the equilibrium relations are highly complex and nonlinear, their analysis is beyond the scope of the present work. However, we can say that the system behaves as an AK model, although with endogenous population and labor supply: any exogenous and unforeseen perturbation of the equilibrium would imply that the economy jumps on its new steady state growth path, due to the jumps in l, n, c.

In the section that follows we will focus on the characterization of the the optimal tax rules.

[15]Note that we have made use of the relationship $\frac{c}{k} = (1-l)\frac{(1-\sigma)}{(1-\eta)}\frac{\bar{w}}{k}$ stemming from Equations (15), (16), and of Equation (5′) and (7′).

4 | THE RAMSEY TAX PROBLEM

We now turn to finding the second-best optimal tax program for the economy in the previous section by solving the Ramsey tax problem. We focus on the second-best although we will also point out the features of the first-best tax program. In doing so, we adopt the primal approach, consisting of the maximization of a direct social welfare function through the choice of quantities (i.e., allocations; see Atkinson & Stiglitz, 1972). For this purpose, we must restrict the set of allocations among which the government can choose to those that can be decentralized as a competitive equilibrium. We now provide the constraints that must be imposed on the government's problem to comply with this requirement.

In our framework there is an implementability constraint associated with the individual family's intertemporal consumption choice. More precisely this constraint is the individual budget constraint with prices substituted for by using the household's first-order conditions, which yields (see Appendix A):

$$A_0 u_{c_0} = \int_0^\infty e^{-\rho t} u_t N_t dt - N_0 \int_0^\infty e^{-\rho t} [u_t - u_{c_t} c_t - u_{l_t} l_t] dt. \tag{25}$$

Finally, there are two feasibility constraints, one which requires that private and public consumption plus investment be equal to aggregate output (Equation 11); the other one is given by Equation (3).

Suppose that the tax program is chosen in period 0, hence the problem of the policymaker is to maximize (1) subject to Equations (25), (11), and (3) $\forall\, t \geq 0$. The current value Hamiltonian is

$$H_t = N_t \left\{ u_t + \Omega \left[u_t \left(1 - \frac{N_0}{N_t} \right) + \frac{N_0}{N_t} u_{l_t} l_t + \frac{N_0}{N_t} u_{c_t} c_t \right] \right\} + \omega_t (\tilde{F}_t - c_t N_t - v K_t) + \varphi_t n_t N_t, \tag{26}$$

where Ω is the multiplier associated with the implementability constraint[16] and ω_t and φ_t are the co-states associated with the other constraints. The first-order conditions for consumption and labor imply (omitting time subscripts):

$$\frac{\partial H}{\partial c} = 0 \Longrightarrow \left\{ 1 + \Omega \left[\left(1 - \frac{N_0}{N_t} \right) - \frac{N_0}{N_t} \Delta_c \right] \right\} u_c = \omega, \tag{27}$$

$$\frac{\partial H}{\partial l} = 0 \Longrightarrow \left\{ 1 + \Omega \left[\left(1 - \frac{N_0}{N_t} \right) - \frac{N_0}{N_t} \Delta_l \right] \right\} u_l = -\omega \frac{\tilde{F}_l}{N}, \tag{28}$$

with $\Delta_c \equiv -\frac{u_{cc} c}{u_c} - \frac{u_{cl} l}{u_c} > 0$ and $\Delta_l \equiv -\frac{u_{ll} l}{u_l} - \frac{u_{cl} c}{u_l} < 0$ usually referred to as the "general equilibrium elasticity" of consumption and of labor, respectively. By using Equations (15) and (16), (27) and (28) can be written as

[16] It is possible to show that Ω is positive if the constraint binds. For this reason, it is usually interpreted as a measure of the deadweight loss brought about by distortionary taxation.

$$1+\Omega\left[\left(1-\frac{N_0}{N_t}\right)-\frac{N_0}{N_t}\Delta_c\right]=\frac{\omega}{u_c}, \tag{29}$$

$$1+\Omega\left[\left(1-\frac{N_0}{N_t}\right)-\frac{N_0}{N_t}\Delta_l\right]=-\frac{\omega}{u_c}\frac{w^*}{\bar{w}}, \tag{30}$$

with $w^* \equiv \frac{\tilde{F}_l}{N}$. Finally, we get:

$$\frac{\partial H}{\partial K}=\omega r^*=\rho\omega-\dot{\omega}, \tag{31}$$

with $r^* \equiv \tilde{F}_K$.

$$\frac{\partial H}{\partial N}=u(1+\Omega)-\omega c+\varphi n=\rho\varphi-\dot{\varphi}, \tag{32}$$

$$\frac{\partial H}{\partial n}=0 \Longrightarrow \omega\tilde{F}_l\theta=\varphi N. \tag{33}$$

In the next Proposition we characterize the BGP:

Proposition 1. *Along the second-best optimal BGP population is constant.*

Proof. See Appendix B.1. □

Moreover, we can provide the following Proposition concerning second-best taxation for the case when x is chosen optimally (i.e., when $\delta = 1-\gamma$):

Proposition 2. *Along the optimal BGP, under productively efficient δ, second-best optimal taxation implies*

$$\tau^k=0,\ (1-\tau^l)=\frac{1+\Omega\left[\left(1-\frac{N_0}{N_t}\right)-\frac{N_0}{N_t}\Delta_c\right]}{1+\Omega\left[\left(1-\frac{N_0}{N_t}\right)-\frac{N_0}{N_t}\Delta_l\right]}\in(0,1)$$

and

$$\bar{\tau}^n \equiv \frac{\tau_t^n}{w_t}=(1-\tau^l)l\frac{k}{c}\left\{1+\frac{1-\mu(2-\sigma-\eta)}{\mu(1-\eta)}\frac{\Omega\frac{N_0}{N_t}}{\left[1+\Omega-\Omega\frac{N_0}{N_t}\Delta_l\right]}\right\}>0.$$

Proof. See Appendix. B.2. □

The analysis of second-best optimal taxation carried out so far shows that the Chamley-Judd result holds also in our scenario: in fact, along the BGP, the capital income tax should be zero and the labor income tax should be positive. The latter stems from our utility function, which yields normality of leisure.

Furthermore, as in Turnovsky (1996), it can be shown that nonzero capital income tax arises, although in a second-best context, for correcting suboptimal public expenditure. In fact, when the fraction of public expenditure is above (below) the social second-best optimum, the social return to capital is less than its private marginal physical product. Consequently, capital income should be taxed to obtain the social optimum.[17]

Moreover, the tax on children in Proposition 2 is positive. Notice that, also in the first best, when $\Omega = 0$, the tax on children is positive. The reason is that there is a congestion externality in the publicly provided input generated by population size, which must be corrected for.

Notice that the government finds it (second-best) optimal to have constant population. Intuitively, this is due to the fact that the tax instrument that it used ($\bar{\theta}$) entails a tax break at one child per adult, (i.e., $n = 0$). In fact, in case $n > 0$, although the associated tax structure would imply zero taxes on both capital income and labor income (see Proposition 2 for $N \to \infty$), the distortion brought about by a nonzero tax on childbearing would be too high and, consequently, the associated allocation of resources would be suboptimal. Hence, the government will implement a tax structure producing constant population.

As for the quantitative dimension of the tax on children, we notice that along the BGP its absolute value will increase at the per-capita income rate of growth. Hence, by recalling that the opportunity cost for rearing children is equal to $w_t(1-\tau_t^l)\theta + \tau_t^n$, we present the tax on children in two forms: the first one is the tax as a share of the "optimal opportunity cost" that is $\frac{\bar{\theta}-\theta}{\theta} = \frac{\tau^n}{w\theta(1-\tau^l)}$, which is independent of the supplied labor, and it represents the additional marginal cost for an extra child (τ^n) as a share of the second-best optimal opportunity cost, that is, the opportunity cost for a child in the case of no child tax (i.e., $w\theta(1-\tau^l)$). The second measure we present is the tax as a share of total pretax wage income, that is, $\frac{\tau^n}{wl}$. Given that our model (being an AK one) does not allow to adequately replicate the features of an existing economy, we present our numerical results mainly for illustrative purposes. Hence, in Figure 1 we depict the shape of the first measure of the tax on children as a function of the parameters of the model[18]. Table 1 reports the results of the sensitivity analysis for both children tax measures and for the optimal labor income tax.

Finally, the Proposition that follows provides the result concerning the sign of the optimal level of debt:

Proposition 3. *Along BGP, optimal debt is negative.*

Proof. See Appendix B.3. □

This result states that along the second-best optimal BGP public expenditure should entirely be financed by labor income taxes and by the returns of public assets or by also capital income taxes, if $\gamma > (1-\delta)$.

5 | CONCLUSIONS

In the present work we have carried out an analysis of optimal taxation in an endogenous growth model in presence of endogenous fertility and labor supply. As far as the normative analysis is concerned, we show that, at the steady state the second-best policy entails a zero capital-income tax, a positive labor income tax, a positive tax on children, and negative debt.

[17]More precisely, building on Appendix B.2, using (24a) and (31) we have $r* = \bar{r}$. Using (6′) and (22a) we obtain $\tau^k = \frac{\gamma-(1-\delta)}{\gamma}$.

[18]While the purpose is not to seek a full calibration, we have yet chosen the parameters of the model so that it replicates certain quantities of the US economy in the benchmark economy with exogenous taxes. Hence, we set the labor and capital tax rates 23% and 9%, respectively. We then choose the parameters to match a growth rate of 2.2%, labor supply of 26% (of total time), and a population growth rate of 1%. By using these parameters we then compute optimal taxes. The full set of parameters for the benchmark case are listed in Figure 1 and Table 1.

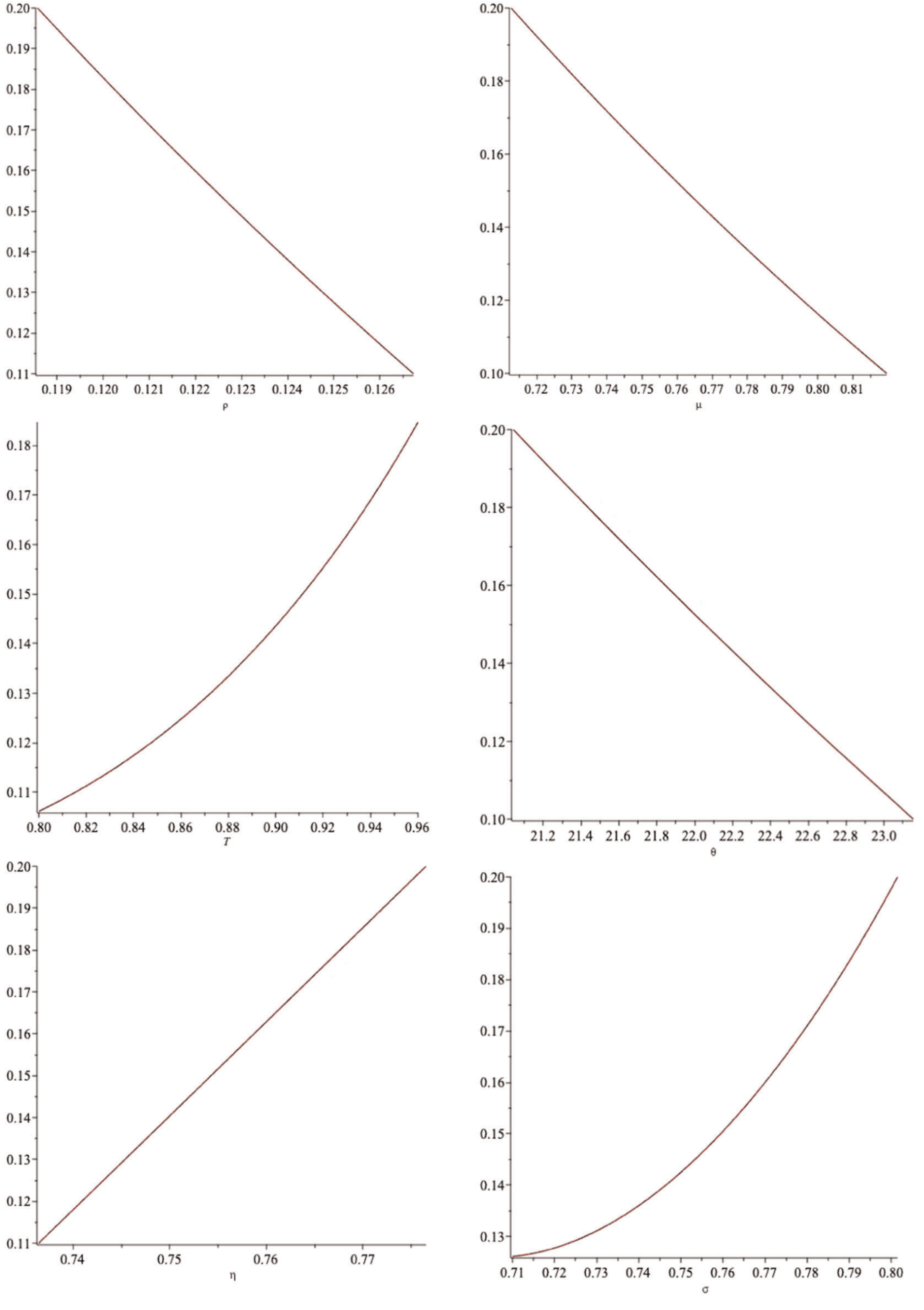

FIGURE 1 Tax on children (as a fraction of children's opportunity cost) as a function of the model parameters

TABLE 1 Sensitivity analysis for optimal (second-best) taxes

Taxes (results)	$\frac{\tau^n}{w\theta(1-\tau^l)}$ (%)	$\frac{\tau^n}{wl}$ (%)	τ^l (%)
Parameters			
$\rho = 0.124$ (Benchmark)	13.36	5.03	14.32
$\rho = 0.112$	28.97	9.61	12.89
$\rho = 0.137$	1.96	0.84	15.28
$\theta = 20.16$	24,7	8.44	10.97
$\theta = 24.64$	3.88	1.60	16.97
$\sigma = 0.71$	12.58	4.25	17.21
$\sigma = 0.80$	19.76	10.22	6.58
$\mu = 0.71$	20.26	7.89	10.21
$\mu = 0.82$	9.94	3.67	16.41
$T = 0.792$	10.44	4.87	11.63
$T = 0.969$	19.21	5.99	17.01
$\eta = 0.694$	1.84	0.82	16.36
$\eta = 0.799$	25.08	8.14	11.40

Parameters for benchmark: $\eta = 0.75$, $\sigma = 0.73$, $\theta = 22.4$, $\mu = 0.78$, $\gamma = \frac{2}{3}$, $\delta = \frac{1}{3}$, $T = 0.88$, $\rho = 0.124$, $\nu = 0$. Capital income tax is equal to zero in all simulations.

Optimal non-zero taxes on capital income result as a corrective device only in the case of suboptimal public spending, as in Turnovsky (1996), although in a second-best analysis. The reason for positive taxes on children is that there is a congestion in the publicly provided input that must be corrected for, even in the first-best: in fact, x is a nonpure public good, because of rivalry, and the way in which we imagine its financing (through taxes on children—i.e., on "household size") could make x be interpreted as the flow of services produced by children-related public expenditure for local public goods, such as schools, hospitals, or, more in general, healthcare.

In this paper we have treated public expenditure as a flow variable (services from current expenditure). A natural extension of our study is to analyze the case of public expenditure as financing a *stock* of public goods (infrastructure): this case is left for future research.

ACKNOWLEDGMENTS

We are grateful to the participants at the European Public Choice Society Conference in Rome, at the PET Conference in Luxemburg, at SIEP Conference in Catania and at the seminars at University of Pisa and Brunell University; we would also like to thank an anonymous Referee for helpful comments. The usual disclaimer applies.

CONFLICT OF INTERESTS

The authors declare that there are no conflict of interests.

ORCID

Thomas I. Renström http://orcid.org/0000-0002-2898-2507
Luca Spataro http://orcid.org/0000-0002-5202-8879

REFERENCES

Acemoglu, D. (2009). *Introduction to modern economic growth*. Princeton Economic Press.

Aiyagari, S. R. (1995). Optimal capital income taxation with incomplete markets, borrowing constraints, and constant discounting. *Journal of Political Economy*, *103*(6), 1158–1175.

Atkinson, A. B., & Stiglitz, J. E. (1972). The structure of indirect taxation and economic efficiency. *Journal of Public Economics*, *1*, 97–119.

Balestrino, A., Cigno, A., & Pettini, A. (2002). Endogenous fertility and the design of family taxation. *International Tax and Public Finance*, *9*, 175–193.

Basu, P., Marsiliani, L., & Renström, T. I. (2004). Optimal dynamic taxation with indivisible labour. *The Manchester School*, *72*(s1), 34–54.

Basu, P., & Renström, T. I. (2007). Optimal dynamic labour taxation. *Macroeconomic Dynamics*, *11*(5), 567–588.

Barro, R. J., & Becker, G. S. (1989). Fertility choice in a model of economic growth. *Econometrica*, *57*, 481–501.

Barro, R. J. (1990). Government spending in a simple model of endogenous growth. *Journal of Political Economy*, *98*, S103–S125.

Becker, G. S., & Barro, R. J. (1988). A reformulation of the economic theory of fertility. *Quarterly Journal of Economics*, *103*, 1–25.

Becker, G. S., Murphy, K. M., & Tamura, R. (1990). Human capital, fertility, and economic growth. *Journal of Political economy*, *98*(5, Part 2), S12–S37.

Blackorby, C., Bossert, W., & Donaldson, D. (1995). Intertemporal population ethics: Critical-level utilitarian principles. *Econometrica*, *63*, 1303–1320.

Boldrin, M., & Jones, L. E. (2002). Mortality, fertility, and saving in a Malthusian economy. *Review of Economic Dynamics*, *5*(4), 775–814.

Bucci, A., & Raurich, X. (2017). Population and economic growth under different growth engines. *German Economic Review*, *18*(2), 182–211.

Cervellati, M., Sunde, U., & Zimmermann, K. F. (2017). Demographic dynamics and long-run development: insights for the secular stagnation debate. *Journal of Population Economics*, *30*(2), 401–432.

Chamley, C. (1986). Optimal taxation of capital income in general equilibrium with infinite lives. *Econometrica*, *54*(3), 607–622.

Cigno, C. (2001). Comparative advantage, observability, and the optimal tax treatment of families with children. *International Tax and Public Finance*, *8*(4), 455–470.

Correia, I. H. (1996). Should capital income be taxed in the steady state? *Journal of Public Economics*, *60*(1), 147–151.

De Bonis, V., & Spataro, L. (2005). Taxing capital income as Pigouvian correction: the role of discounting future. *Macroeconomic Dynamics*, *9*(4), 469–477.

De Bonis, V., & Spataro, L. (2010). Social discounting, migration, and optimal taxation of savings. *Oxford Economic Papers*, *62*(3), 603–623.

de la Croix, D., Pestieau, P., & Ponthiere, G. (2012). How powerful is demography? The Serendipity Theorem revisited. *Journal of Population Economics*, *25*(3), 899–922.

Diamond, P. A., & Mirrlees, J. A. (1971). Optimal taxation and public production I: Production efficiency. *The American Economic Review*, *61*(1), 8–27.

Elgin, C. (2012). A theory of economic development with endogenous fertility. *Macroeconomic Dynamics*, *16*(5), 686–705.

Elon Kohlberg, E. (1976). A model of economic growth with altruism between generations. *Journal of Economic Theory*, *13*(1), 1–13.

Galor, O. (2011). *Unified growth theory*. Princeton University Press.

García Peñalosa, C., & Turnovsky, S. J. (2005). Second-best optimal taxation of capital and labor in a developing economy. *Journal of Public Economics*, *89*, 1045–1074.

Jones, C. I. (1995). R&D-based models of economic growth. *Journal of Political Economics*, *103*, 759–784.

Jones, L., & Manuelli, R. (1990). A Convex Model of Equilibrium Growth: Theory and Policy Implications. *Journal of Political Economy*, *98*(5), 1008–1038.
Jones, L. E., & Schoonbroodt, A. (2016). Baby busts and baby booms: The fertility response to shocks in dynastic models. *Review of Economic Dynamics*, *22*, 157–178.
Judd, K. L. (1985). Redistributive taxation in a simple perfect foresight model. *Journal of Public Economy*, *28*(1), 59–83.
Judd, K. L. (1997). *The optimal tax rate for capital income is negative* (NBER Working Paper, 6004).
Laitner, J. (1995). Quantitative evaluations of efficient tax policies for Lucas' supply side models. *Oxford Economic Papers*, *47*(3), 471–492.
Lucas Jr., R. E. (1990). Supply-side economics: An analytical review. *Oxford economic, papers*, *42*(2), 293–316.
Marsiglio, S. (2014). Reassessing Edgeworth's conjecture when population dynamics is stochastic. *Journal of Macroeconomics*, *42*, 130–140.
Michel, P., & Pestieau, P. (1998). Optimal population without repugnant aspects. *Genus*, *54*, 25–34.
Parfit, D. (1976). On doing the best for our children. In M. Bayles (Ed.), *Ethics and populations*. Schenkman.
Parfit, D. (1984). *Reasons and persons*. Oxford University Press.
Rebelo, S. (1991). Long-run policy analysis and long-run growth. *Journal of Political Economy*, *99*, 500–521.
Reis, C. (2012). Social discounting and incentive compatible fiscal policy. *Journal of Economic Theory*, *147*(6), 2469–2482.
Renström, T. I., & Spataro, L. (2011). The optimum growth rate for population under critical-level utilitarianism. *Journal of Population Economics*, *24*(3), 1181–1201.
Renström, T. I., & Spataro, L. (2015). Population growth and human capital: A welfarist approach. *The Manchester School*, *83*(S2), 110–141.
Renström, T. I., & Spataro, L. (2019). Population growth: A pure welfarist approach. *Journal of Public Economic Theory*, *21*(1), 135–166.
Sinclair, P. J. N., & Slater, M. D. E. (1990). Taxation, private information and capital: An introduction. *Oxford Economic Papers*, *42*, 1–5.
Spataro, L., & Renström, T. I. (2012). Optimal taxation, critical-level utilitarianism and economic growth. *Journal of Public Economics*, *96*, 727–738.
Straub, L., & Werning, I. (2015). Positive long run capital taxation: Chamley-Judd revisited. Mimeo, MIT.
Strulik, H. (1999). Demographic transition, stagnation, and demoeconomic cycles in a model for the less developed economy. *Journal of Macroeconomics*, *21*, 397–413.
Strulik, H. (2003). Mortality, the trade-off between child quality and quantity, and demoeconomic development. *Metroeconomica*, *54*, 499–520.
Strulik, H., Prettner, K., & Prskawetz, A. (2013). The past and future of knowledge-based growth. *Journal of Economic Growth*, *18*, 411–437.
Tertilt, M. (2005). Polygyny, fertility and savings. *Journal of Political Economy*, *113*(6), 1341–1371.
Turnovsky, S. J. (1992). Alternative forms of government expenditure financing: A comparative welfare analysis. *Economica*, *59*, 235–252.
Turnovsky, S. J. (1996). Optimal tax, debt, and expenditure policies in a growing economy. *Journal of Public Economics*, *60*, 21–44.
Turnovsky, S. J. (2000). Fiscal policy, elastic labor supply, and endogenous growth. *Journal of Monetary Economics*, *45*, 185–210.
Willis, R. J. (1979). *The old age security hypothesis and population growth* (National Bureau of Economic Research No. w0372).

How to cite this article: Renström, T. I., Spataro, L. (2021). Optimal taxation in an endogenous growth model with variable population and public expenditure. *J Public Econ Theory*. 23, 639–659. https://doi.org/10.1111/jpet.12515

APPENDIX A: Implementability constraint

First, by integrating Equation (17), having used (18), one gets:

$$\lambda_0 = \int_0^\infty e^{-\rho t}[u_t + q_t(\bar{w}_t - c_t)]dt. \tag{A.1}$$

Second, let us take the following time derivative, $\frac{d(A_t q_t)}{dt} = A_t\dot{q}_t + \dot{A}_t q_t$, which, exploiting Equations (4) and (14) can be written as

$$\frac{d(A_t q_t)}{dt} - \rho q_t A_t = q_t N_t[\bar{w}_t(l_t - \bar{\theta}n_t) - c_t] = q_t N_t[\bar{w}_t l_t - c_t] - \lambda_t \dot{N}_t, \tag{A.2}$$

where the last equality follows from (3) and (18). Hence, pre-multiplying by $e^{-\rho t}$ and integrating both sides of (A.2), making use of transversality conditions we obtain

$$-A_0 q_0 = \int_0^\infty e^{-\rho t} N_t q_t(\bar{w}_t l_t - c_t)dt - \int_0^\infty e^{-\rho t}\lambda_t \dot{N}_t dt. \tag{A.3}$$

Integrating by parts the last integral of (A.3) and using (17) we get

$$\int_0^\infty e^{-\rho t}\lambda_t \dot{N}_t dt = -\lambda_0 N_0 + \int_0^\infty e^{-\rho t} N_t[u_t + q_t(\bar{w}_t l_t - c_t)]dt. \tag{A.4}$$

Substituting (A.4) into (A.3) gives

$$A_0 q_0 + \lambda_0 N_0 = \int_0^\infty N_t e^{-\rho t} u_t dt. \tag{A.5}$$

Finally, using (A.1) in (A.5) and using (15) and (16) gives Equation (25) in the text:

$$A_0 u_{c_0} = \int_0^\infty e^{-\rho t} u_t N_t dt - N_0 \int_0^\infty e^{-\rho t}[u_t - u_{c_t} c_t - u_{l_t} l_t]dt.$$

APPENDIX B

B.1. Proof of Proposition 1

We will write the dynamic system in quantities that are constant at a steady state. Define

$$Q = \frac{\varphi}{\omega k}. \tag{B.1}$$

Then taking the time derivative, and using Equations (31) and (32) as well as $\frac{\dot{k}}{k} = r^* - \frac{c}{k} - n$ gives the first of the dynamic equations for the Jacobian:

$$\dot{Q} = z\left[Q + 1 - \frac{1+\Omega}{\mu(1-\sigma)}\frac{u_c}{\omega}\right], \tag{B.2}$$

where we have defined $z = c/k$ as per capita consumption divided by per capita capital.

Next, by differentiating $u_c = q$ (notice that u is multiplicative in c and l), and using (2) and (14), we have

$$\frac{\dot{c}}{c} = \frac{\bar{r} - \rho}{1 - \mu(1 - \sigma)} - \frac{\mu(1 - \eta)}{1 - \mu(1 - \sigma)} \frac{\dot{l}}{1 - l}. \tag{B.3}$$

Notice that, from (B.3) and (31)

$$\frac{d}{dt}\left(\frac{\omega}{u_c}\right) = (\bar{r} - r^*)\frac{\omega}{u_c}. \tag{B.4}$$

For the second dynamic equation for the Jacobian we have

$$\dot{z} = \left(\frac{\dot{c}}{c} - \frac{\dot{k}}{k}\right)z,$$

or, by using Equations (B.3) and (B.4), as well as the capital accumulation equation

$$\dot{z} = \left[-\frac{\mu(1 - \eta)}{1 - \mu(1 - \sigma)} \frac{\dot{l}}{1 - l} + \frac{\frac{d}{dt}\left(\frac{\omega}{u_c}\right)/\frac{\omega}{u_c} - \rho}{1 - \mu(1 - \sigma)} + \frac{\mu(1 - \sigma)}{1 - \mu(1 - \sigma)}r^* + n + z\right]z. \tag{B.5}$$

Differentiating the log of (27) with respect to time

$$\frac{d}{dt}\left(\frac{\omega}{u_c}\right)/\frac{\omega}{u_c} = \frac{\Omega\frac{N_0}{N}\left[\Delta_c n - \frac{\mu(1-\eta)}{1-l}\frac{\dot{l}}{1-l}\right]}{1 + \Omega - \Omega\frac{N_0}{N}\Delta_c},$$

and substituting into (B.5) we have

$$\begin{aligned}\dot{z} = \Bigg\{&-\frac{\mu\left(1 - \eta\right)}{1 - \mu(1 - \sigma)} \frac{\dot{l}}{1 - l}\left[1 + \frac{\Omega\frac{N_0}{N}\frac{1}{1-l}}{1 + \Omega - \Omega\frac{N_0}{N}\Delta_c}\right] \\ &+ n\left[1 + \frac{1}{1 - \mu(1 - \sigma)} \frac{\Omega\frac{N_0}{N}\Delta_c n}{1 + \Omega - \Omega\frac{N_0}{N}\Delta_c}\right] + \frac{\mu(1 - \sigma)r^* - \rho}{1 - \mu(1 - \sigma)} + z\Bigg\}z.\end{aligned} \tag{B.6}$$

Next, as for the third dynamic equation, from (28), (33), (15), (16) and the relationship $\frac{c}{k} = (1 - l)\frac{(1-\sigma)}{(1-\eta)}\frac{\bar{w}}{k}$, we get:

$$1 + \Omega - \Omega\frac{N_0}{N}\Delta_l = -\frac{\varphi}{u_l\theta} = -\frac{\varphi}{\omega}\frac{\omega}{u_c}\frac{u_c}{u_l}\frac{1}{\theta} = \frac{\varphi}{\omega k}\frac{\omega}{u_c}\frac{1 - \sigma}{1 - \eta}\frac{1 - l}{\theta}\frac{1}{z}.$$

By using (29) and (B.1) we obtain

$$1 + \Omega - \Omega\frac{N_0}{N}\Delta_l = Q\left(1 + \Omega - \Omega\frac{N_0}{N}\Delta_c\right)\frac{1 - \sigma}{1 - \eta}\frac{1 - l}{\theta}\frac{1}{z}. \tag{B.7}$$

Differentiating the log of (B.7) with respect to time gives

$$\frac{\Omega\frac{N_0}{N}\left[\Delta_l n + \frac{1-\mu(1-\eta)}{1-l}\frac{\dot{l}}{1-l}\right]}{1 + \Omega - \Omega\frac{N_0}{N}\Delta_l} = \frac{\dot{Q}}{Q} + \frac{\Omega\frac{N_0}{N}\left[\Delta_c n - \frac{\mu(1-\eta)}{1-l}\frac{\dot{l}}{1-l}\right]}{1 + \Omega - \Omega\frac{N_0}{N}\Delta_c} - \frac{\dot{l}}{1 - l} - \frac{\dot{z}}{z}. \tag{B.8}$$

Rearranging (B.8) we have

$$\frac{\dot{l}}{1-l}D = \Omega\frac{N_0}{N}(1+\Omega)ABn + \frac{\dot{Q}}{Q} - \frac{\dot{z}}{z}, \tag{B.9}$$

where

$$A \equiv (1-l)^{-1}\left[1+\Omega-\Omega\frac{N_0}{N}\Delta_l\right]^{-1} > 0B \equiv \left[1+\Omega-\Omega\frac{N_0}{N}\Delta_c\right]^{-1}$$
$$> 0D \equiv \left\{1+\Omega\frac{N_0}{N}\left[1+\Omega-\Omega\frac{N_0}{N}(1-\mu(2-\sigma-\eta))\right]AB\right\} \geq 1$$

are functions of l. Finally, by using (33) and the production technology

$$Q = \frac{\varphi}{\omega k} = \theta\frac{\tilde{F}_l}{Nk} = \theta\frac{1-\gamma}{\gamma}\frac{\tilde{F}_K}{l-\theta n} \tag{B.10}$$

gives n as a function of l and Q: $n(l,Q)$. In particular (B.10) implies the derivatives

$$\frac{\partial n}{\partial l} = \frac{1}{\theta} > 0, \frac{\partial n}{\partial Q} = \frac{1}{\theta Q}\frac{\gamma}{2\gamma-1} > 0, \frac{d\tilde{F}_K}{dl} = \frac{\partial \tilde{F}_K}{\partial l} + \frac{\partial \tilde{F}_K}{\partial n}\frac{\partial n}{\partial l} = 0.$$

Consequently Equations (B.2), (B.6), and (B.9) form a dynamic system in Q, z, and l (with A, B, and D being functions of l, and n being a function of l and Q). To find the Jacobian we differentiate the system with respect to Q, z, and l, and evaluate the derivatives at a steady state (derivatives of expressions multiplying a time derivative can then be ignored).[19]

We then have (subscripts denoting partial derivatives):

$$\dot{Q}_Q = z, \dot{Q}_z = 0, \dot{Q}_l = -z\frac{1+\Omega}{\mu(1-\sigma)}B_l, \tag{B.11}$$

$$\dot{z}_Q = -z\frac{P}{1-l}\dot{l}_Q, \dot{z}_z = z\left[-\frac{P}{1-l}\dot{l}_z + 1\right], \dot{z}_l = z\left[-\frac{P}{1-l}\dot{l}_l + Sn_l\right], \tag{B.12}$$

$$\dot{l}_Q = \frac{1-l}{D}\left[(D-1)n_Q + \frac{\dot{Q}_Q}{Q} - \frac{\dot{z}_Q}{z}\right], \dot{l}_z = \frac{1-l}{D}\left[\frac{\dot{Q}_z}{Q} - \frac{\dot{z}_z}{z}\right],$$
$$\dot{l}_l = \frac{1-l}{D}\left[(D-1)n_l + \frac{\dot{Q}_l}{Q} - \frac{\dot{z}_l}{z}\right], \tag{B.13}$$

where $P \equiv \frac{\mu(1-\eta)}{1-\mu(1-\sigma)}\left(1+\Omega\frac{N_0}{N}\frac{B}{1-l}\right)$, $S \equiv 1 + \frac{1}{1-\mu(1-\sigma)}\Omega\frac{N_0}{N}\Delta_c B$. Notice that S $n_Q + \frac{\mu(1-\sigma)}{1-\mu(1-\sigma)}\frac{d\tilde{F}_K}{dQ} = 0$ and $\frac{d\tilde{F}_K}{dl} = 0$. Solving (B.11), (B.12), and (B.13) gives the Jacobian. The roots solve:

[19] For example dD/dl can be ignored as it multiples $\frac{\dot{l}}{1-l}$, as well as derivatives of A and B in the expression $\Omega\frac{N_0}{N}(1+\Omega)ABn$, as either n=0 or $N \to +\infty$ at a steady state.

$$\left|J-\nu I\right| = \begin{bmatrix} z-\nu & 0 & -z\frac{1+\Omega}{\mu(1-\sigma)}B_l \\ -z\frac{P}{D-P}\left[(D-1)n_Q+\frac{z}{Q}\right] & z\frac{D}{D-P}-\nu & \left\{\frac{-P}{D-P}\left[(D-1)n_l-Sn_l-\frac{z}{Q}\frac{1+\Omega}{\mu(1-\sigma)}B_l\right] + Sn_l\right\}z \\ \frac{1-l}{D-P}\left[(D-1)n_Q+\frac{z}{Q}\right] & -\frac{1-l}{D-P} & \frac{1-l}{D-P}\left[(D-1)n_l-Sn_l-\frac{z}{Q}\frac{1+\Omega}{\mu(1-\sigma)}B_l\right]-\nu \end{bmatrix} = 0.$$

Multiplying the third row by $zP/(1\text{-}l)$ and adding it to the second, gives

$$\left|J-\nu I\right| = \begin{bmatrix} z-\nu & 0 & -z\frac{1+\Omega}{\mu(1-\sigma)}B_l \\ 0 & z-\nu & Sn_l-\nu\frac{P}{1-l}z \\ \frac{1-l}{D-P}\left[(D-1)n_Q+\frac{z}{Q}\right] & -\frac{1-l}{D-P} & \frac{1-l}{D-P}R-\nu \end{bmatrix} = 0, \tag{B.14}$$

where $R \equiv (D-1)n_l - Sn_l - \frac{z}{Q}\frac{1+\Omega}{\mu(1-\sigma)}B_l$. The characteristic equation is

$$(z-\nu)\nu\left\{\nu^2-\left(\frac{D}{D-P}z+\frac{1-l}{D-P}R\right)\nu+\frac{1-l}{D-P}z(D-1)\left[n_l+n_Q\frac{1+\Omega}{\mu(1-\sigma)}B_l\right]\right\}=0. \tag{B.15}$$

We check the stability of the steady state where $N \to +\infty$. In this case $B_l = 0$, S=1, D=1, $R = -n_l$ and $P = \frac{\mu(1-\eta)}{1-\mu(1-\sigma)}$. Then (B.15) becomes

$$(z-\nu)\nu\left\{\nu-\frac{[1-\mu(1-\sigma)][z-(1-l)n_l]}{1-\mu(1-\sigma)-\mu(1-\eta)}\right\}=0. \tag{B.16}$$

The roots are:

$$\nu_1 = z = \frac{c}{k} > 0, \nu_2 = 0, \nu_3 = \frac{1-\mu(1-\sigma)}{1-\mu(1-\sigma)-\mu(1-\eta)}[z-(1-l)n_l].$$

To see that the last root is positive, we proceed as follows. Equations (B.2) and (B.7) are in this case, respectively, $Q = \frac{1-\mu(1-\sigma)}{\mu(1-\sigma)}$, $z = Q\frac{(1-\sigma)}{(1-\eta)}\frac{1-l}{\theta}$. Then $z = \frac{1-\mu(1-\sigma)}{\mu(1-\eta)}\frac{1-l}{\theta}$. Finally, by differentiating (B.10), $n_l = \frac{1}{\theta}$, then the final root is $\nu_3 = \frac{1-\mu(1-\sigma)}{\mu(1-\eta)}\frac{1-l}{\theta} > 0$.

Consequently, the steady state for $N \to +\infty$ is unstable and cannot be reached asymptotically. Since N cannot jump, this steady state cannot be reached. This leaves only the steady state where n=0. If in this case neither ν_2 nor ν_3 is negative, the system (c/k, Q, l [and consequently n]) jumps to its steady state value (as is usual in AK models). In any case this is the only steady state which can be reached.

B.2. Proof of Proposition 2

As for the capital income tax, from (32) we get $\dot{\omega} = (\rho - r^*)\omega$ and given that $\frac{\omega}{q}$ must be constant along the BGP (by Equation 22), the rate of growth of ω and $_q$ must be equal, so that $\bar{r} = r^*$. If x is productively efficient (i.e., $\delta = 1-\gamma$), $r = r^*$ and then $\tau^k = 0$. As for the labor income tax, from (29) and (31):

$$\frac{\bar{w}}{w^*} = \frac{1 + \Omega\left[\left(1 - \frac{N_0}{N_t}\right) - \frac{N_0}{N_t}\Delta_c\right]}{1 + \Omega\left[\left(1 - \frac{N_0}{N_t}\right) - \frac{N_0}{N_t}\Delta_l\right]}. \tag{B.17}$$

Given that if $\delta = 1 - \gamma$, $w = w^*$, then $(1 - \tau^l) = \frac{1+\Omega\left[\left(1-\frac{N_0}{N_t}\right)-\frac{N_0}{N_t}\Delta_c\right]}{1+\Omega\left[\left(1-\frac{N_0}{N_t}\right)-\frac{N_0}{N_t}\Delta_l\right]} \in (0, 1)$, since $\Delta_c = \left[1 - \mu(2 - \sigma - \eta)\right] + \frac{\mu(1-\eta)}{1-l} > 0$ and $\Delta_l = \left[1 - \mu(2 - \sigma - \eta)\right] - \frac{1-\mu(1-\eta)}{1-l} < 0$.

As for $\bar{\theta}$, along the steady state growth path, after tax wage and consumption grow at the same rate. Combining (24b) and $\frac{\dot{k}}{k} = r^* - n - \frac{c}{k}$, we have

$$\bar{r} - \frac{1 - \mu(1 - \sigma)}{\bar{\theta}\mu(1 - \eta)}(1 - l) - \frac{l}{\bar{\theta}} = r^* - \frac{c}{k} - n. \tag{B.18}$$

Using $n = 0$, $\bar{r} = \tilde{F}_K$, (B.18) becomes

$$\frac{c}{k} = \frac{1 - \mu(1 - \sigma)}{\bar{\theta}\mu(1 - \eta)}(1 - l) + \frac{l}{\bar{\theta}}. \tag{B.19}$$

(B.2) and (B.7) give

$$\frac{c}{k} = \frac{1 - \sigma}{1 - \eta}\frac{1 - l}{\theta}\frac{\left[\frac{1+\Omega}{\mu(1-\sigma)} - \left(1 + \Omega - \Omega\frac{N_0}{N}\Delta_c\right)\right]}{1 + \Omega - \Omega\frac{N_0}{N}\Delta_l}. \tag{B.20}$$

Combining (B.19) and (B.20) we get

$$\bar{\theta} - \theta = l\frac{k}{c} + \frac{1 - \mu(2 - \sigma - \eta)}{\mu(1 - \eta)}l\frac{k}{c}\frac{\Omega\frac{N_0}{N_t}}{\left[1 + \Omega - \Omega\frac{N_0}{N_t}\Delta_l\right]}. \tag{B.21}$$

By definition of $\bar{\theta}$, $\bar{\tau}^n \equiv (1 - \tau^l)(\bar{\theta} - \theta) > 0$.

B.3. Proof of Proposition 3

Let us start from inputs remuneration. Equations (22a) and (22b) give

$$\frac{w^*(l - \theta n)}{r^* k} = \frac{1 - \gamma}{\gamma}\frac{\tilde{F}}{N}\frac{K}{\tilde{F}k} = \frac{1 - \gamma}{\gamma} \tag{B.24}$$

Next, Equation (8′) yields: $\frac{\dot{k}}{k} = \frac{\tilde{F}}{k} - \frac{c}{k} - n = r^* - \frac{c}{k} - n$, so that $\frac{c}{k} = r^* - n - \frac{\dot{k}}{k}$.

Given that along the BGP $\frac{\dot{k}}{k} = \frac{\dot{c}}{c} = \frac{r^* - \rho}{1-\mu(1-\sigma)}$ we get: $\frac{c}{k} = r^* - n - \frac{r^* - \rho}{1-\mu(1-\sigma)} \equiv z$. Next, let us exploit the individuals' budget constraint,

$$\dot{a} = (\bar{r} - n)a + \bar{w}(l - \bar{\theta}n) - c, \tag{B.25}$$

and given that

$$\frac{\dot{a}}{a} = \frac{\dot{c}}{c} = \frac{r^* - \rho}{1 - \mu(1 - \sigma)}, \tag{B.26}$$

it follows:

$$\frac{c}{a} = (\bar{r} - n) - \frac{r^* - \rho}{1 - \mu(1 - \sigma)} + \frac{\bar{w}}{a}(l - \bar{\theta}n). \tag{B.27}$$

Moreover, exploiting (B.24), $\frac{\bar{w}}{a}(l - \bar{\theta}n)$ is $\frac{\bar{w}}{a}(l - \bar{\theta}n) = \frac{\bar{w}}{w^*}\frac{k}{a}\frac{w^*(l - \bar{\theta}n)}{r^* k}r^* = \frac{\bar{w}}{w^*}\frac{k}{a}\frac{1-\gamma}{\gamma}r^*$, so that we can rewrite Equation (B.27) as

$$\frac{c}{a} = z + \frac{\bar{w}}{w^*}\frac{k}{a}\frac{1 - \gamma}{\gamma}r^*. \tag{B.28}$$

Finally,

$$\frac{a}{k} = \frac{\frac{c}{k}}{\frac{c}{a}} = \frac{z}{z + \frac{\bar{w}}{w^*}\frac{k}{a}\frac{1-\gamma}{\gamma}r^*} = \frac{a}{k}\frac{z}{z\frac{a}{k} + \frac{\bar{w}}{w^*}\frac{1-\gamma}{\gamma}r^*},$$

and collecting terms we get:

$$z\left(\frac{a}{k} - 1\right) + \frac{\bar{w}}{w^*}\frac{1 - \gamma}{\gamma}r^* = 0.$$

By recalling that $\left(\frac{a}{k} - 1\right) = \frac{b}{k}$ we get

$$\frac{b}{k} = \frac{\frac{\bar{w}}{w^*}\frac{1-\gamma}{\gamma}r^*}{r^* - n - \frac{r^* - \rho}{1 - \mu\left(1 - \sigma\right)}} < 0. \tag{B.29}$$