STABILIZATION DISTANCE BETWEEN SURFACES

ALLISON N. MILLER AND MARK POWELL

Abstract. Define the *1-handle stabilization distance* between two surfaces properly embedded in a fixed 4-dimensional manifold to be the minimal number of 1-handle stabilizations necessary for the surfaces to become ambiently isotopic. For every nonnegative integer m we find a pair of 2-knots in the 4-sphere whose stabilization distance equals m.

Next, using a *generalized stabilization distance* that counts connected sum with arbitrary 2-knots as distance zero, for every nonnegative integer m we exhibit a knot J_m in the 3-sphere with two slice discs in the 4-ball whose generalized stabilization distance equals m. We show this using homology of cyclic covers.

Finally, we use metabelian twisted homology to show that for each m there exists a knot and pair of slice discs with generalized stabilization distance at least m, with the additional property that abelian invariants associated to cyclic covering spaces coincide. This detects different choices of slicing discs corresponding to a fixed metabolising link on a Seifert surface.

1. Introduction

A 2-knot is an oriented smooth embedding of S^2 in S^4. Our first result shows that 2-knots can be arbitrarily far apart in our 1-handle stabilization distance. For the definitions of our stabilization distances see Section 2 below. Roughly, the 1-handle stabilization distance between two surfaces of the same genera is the minimal number of 1-handle stabilizations, that is ambient surgeries $\Sigma \rightsquigarrow \Sigma \backslash (S^0 \times D^2) \cup D^1 \times S^1$, that must be performed to both surfaces in order to make them ambiently isotopic rel. boundary.

Theorem A. *For every nonnegative integer m, there exists a pair of 2-knots K and J in the 4-sphere with 1-handle stabilization distance m.*

We prove this using homology of cyclic covers, in particular the first Alexander modules $H_1(S^4 \backslash \nu K; \mathbb{Q}[t^{\pm 1}])$ and $H_1(S^4 \backslash \nu J; \mathbb{Q}[t^{\pm 1}])$.

A *slice disc* for a 1-knot $S^1 \subset S^3$ is a smoothly embedded disc $D^2 \subset D^4$ with boundary the 1-knot. The next theorem uses a more general notion of stabilization distance, where addition of arbitrary 2-knots is also permitted and counts as distance zero. Clearly the previous theorem would not hold with the generalized stabilization distance.

Theorem B. *For every nonnegative integer m, there exists a knot $J \subset S^3$ and a pair of slice discs D_1 and D_2 for J with generalized stabilization distance m.*

To prove Theorem B we investigate the kernels

$$\ker \left(H_1(S^3 \backslash \nu J; \mathbb{Q}[t^{\pm 1}]) \to H_1(D^4 \backslash \nu D_i; \mathbb{Q}[t^{\pm 1}]) \right),$$

for $i = 1, 2$. For our last main result we use metabelian twisted homology to detect second order differences between slice discs.

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Theorem C. *For every nonnegative integer m, there exists a knot $J \subset S^3$ and a pair of slice discs D_1 and D_2 for J with generalized stabilization distance at least m, such that the kernels*
$$\ker\left(H_1(S^3\backslash\nu J;\mathbb{Z}[t^{\pm 1}]) \to H_1(D^4\backslash\nu D_i;\mathbb{Z}[t^{\pm 1}])\right)$$
coincide for $i = 1, 2$.

Theorem B is not a corollary of Theorem C, since the former gives us distance exactly m. Theorem B is also somewhat easier to prove, and the method extends easily to distinguish choices of slice discs for many knots beyond the explicit examples we give, while Theorem C requires more involved arguments and more specialized constructions.

Organization of the paper. In Section 2 we introduce our notions of stabilization distance precisely. Section 3 constructs a cobordism corresponding to a stabilization. Our results will follow from analyzing the effects on homology of these cobordisms. Section 4 recalls the notion of generating rank of a module over a commutative PID, as well as recording the facts about generating rank that we shall use. Then Section 5 proves Theorem A, Section 6 proves Theorem B, and Section 7 proves Theorem C.

Conventions. When N is a properly embedded submanifold of M, we write $X_N := M \smallsetminus \nu(N)$. In our context, we will frequently have a canonical isomorphism $\varepsilon\colon H_1(X_N) \to \mathbb{Z}$ and in this case we let X_N^n denote the corresponding n-fold cyclic cover, for $n \in \mathbb{N} \cup \{\infty\}$. We will use $\mathbb{Z}_n$ to denote the finite cyclic group $\mathbb{Z}/n\mathbb{Z}$. Let $g(F)$ be the genus of a surface F.

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2. Stabilization distances

Fix a compact, oriented, smooth 4-manifold W. The following definition is motivated by that of Juhasz and Zemke [JZ18b].

Definition 2.1. Let Σ be an oriented surface with boundary, smoothly and properly embedded in W. Let B be an embedding of D^4 into W such that ∂B intersects Σ transversely in a 2-component unlink L and B intersects Σ in two discs Δ_0 and Δ_1, which can be simultaneously isotoped within B to lie in ∂B. Suppose that a 3-dimensional 1-handle $D^2 \times I$ is embedded into the interior of W such that $D^2 \times \{i\} = \Delta_i$ for $i = 0, 1$. Then $\Sigma' := (\Sigma \cap (W \smallsetminus B)) \cup_L (S^1 \times I)$ is a *1-handle stabilization* of Σ. If $S^1 \times I$ can be isotoped into ∂B relative to L, we call the stabilization *trivial.*

A trivial 1-handle stabilization does not change the fundamental group of the complement of the surface, so frequently there will be no sequence of trivial stabilizations relating two given surfaces. On the other hand, any two homologous surfaces become isotopic after adding finitely many 1-handles [BS15].

Definition 2.2. Define the *1-handle stabilization distance* in $\mathbb{N} \cup \{0, \infty\}$ between smoothly and properly embedded surfaces $(F, \partial F) \subset (W, \partial W)$ and $(F', \partial F') \subset (W, \partial W)$ with $\partial F = \partial F'$, homologous in $H_2(W, \partial W; \mathbb{Z})$, to be the minimal $k \in \mathbb{N}$ such that F and F' become ambiently isotopic rel. boundary after each has been stabilized at most k times. We denote this by $d_1(F, F')$. If F and F' are not homologous or have different boundaries then we say that $d_1(F, F') = \infty$.

In particular for any two 2-knots K and J, $d_1(K,J) < \infty$. For distances between slice discs, we obtain stronger results by defining a coarser notion that permits connected sum with locally knotted 2-spheres. By *adding a locally knotted 2-sphere* to a properly embedded surface $(\Sigma, \partial\Sigma) \subset (W, \partial)$ we mean taking a 2-knot S in S^4 and forming the connected sum of pairs

$$(W,\Sigma)\#(S^4,S) = (W,\Sigma\#S).$$

Definition 2.3. Let $(F,\partial F) \subset (W,\partial W)$ and $(F',\partial F') \subset (W,\partial W)$ be smoothly and properly embedded surfaces. If $\partial F = \partial F'$ and $[F] = [F'] \in H_2(W,\partial W;\mathbb{Z})$, we define the *generalized stabilization distance* $d_2(F,F')$ in $\mathbb{N} \cup \{0,\infty\}$ to be the minimal $k \in \mathbb{N}$ such that F and F' become ambiently isotopic rel. boundary after each has been stabilized at most k times and had arbitrarily many locally knotted 2-spheres added. If F and F' are not homologous or have different boundaries then we say that $d_2(F,F') = \infty$.

Note that for any two slice discs D_1, D_2 in D^4 for a fixed knot in S^3, we have that $d_2(D_1,D_2) < \infty$. It is immediate from the definitions that

$$d_2(F,F') \leqslant d_1(F,F').$$

We also remark that $d_{JZ}(F,F') \leqslant d_2(F,F')$, where d_{JZ} denotes the Juhasz-Zemke stabilization distance [JZ18b] between surfaces.

We pause to advertise the following problem. For a slice knot R, let $n_s(R)$ denote the number of equivalence classes of slice discs for R, where the equivalence relation is generated by connected sum with knotted 2-spheres and ambient isotopy rel. boundary. Note that $n_s(U) = 1$.

Our examples from Theorem B show that for every integer k there is a knot R_k with $n_s(R_k) \geqslant k$. In fact, the knot $\#^k 9_{46}$ has 2^k natural slice discs obtained by choosing 'left band' or 'right band' slice discs for each $i = 1,\dots,k$. By considering the kernels of induced maps on Alexander modules, one can see they are all mutually not ambiently isotopic rel. boundary and so $n_s(\#^k 9_{46}) \geqslant 2^k$.

Problem 2.4. *Determine the value of $n_s(R)$ for some nontrivial knot R, or at least whether $n_s(R) < \infty$.*

Recent related work includes [JZ18a] and [CP19].

3. Cobordisms corresponding to handle additions

Now we construct cobordisms corresponding to handle additions. The following construction will be used in our proofs of all three main theorems.

Construction 3.1. [A cobordism between surface exteriors.] Let W be a compact, oriented, smooth 4-manifold. Suppose that F_1 is a smoothly and properly embedded surface in W with $\partial F_1 = K \subset \partial W$ and that F_2 has been obtained from F_1 by a 1-handle addition such that $g(F_2) = g(F_1) + 1$. We define an ambient cobordism $T \subset W \times I$ as follows:

$$T := (F_1 \times [0,1/2]) \cup ((D^1 \times D^2) \times \{1/2\}) \cup (F_2 \times [1/2,1]),$$

where $D^1 \times D^2 \hookrightarrow W$ is an embedding with $\partial D^1 \times D^2 \subset F_1$ and $D^1 \times \partial D^2 \subset F_2$. (That is, $D^1 \times D^2$ is the 3-dimensional 1-handle h in the definition of 1-handle stabilization.) Observe that

$$\partial T = (F_1 \times \{0\}) \cup_{K\times\{0\}} (K \times [0,1]) \cup_{K\times\{1\}} F_2 \times \{1\}$$

and so $X_T \colon = (W \times I) \smallsetminus \nu(T)$ is a cobordism rel. X_K from X_{F_1} to X_{F_2}.

Since T is obtained from $F_1 \times [0, 1/2]$ by attaching a single 3-dimensional 1-handle to $F_1 \times \{1/2\}$ (and then flowing upwards), it follows from the rising water principle [GS99, Section 6.2] that X_T has a handle decomposition relative to X_{F_1} obtained by attaching a single 5-dimensional 2-handle to $X_{F_1} \times I$. Notice that the attaching sphere of this 2-handle determines a fundamental group element of the form $\gamma = \mu_1 \beta \mu_2^{-1} \beta^{-1}$, where μ_1 and μ_2 are meridians to F_1 near the attaching spheres of h and β is a parallel push-off of the core of h. In particular, γ is null-homologous in $H_1(X_{F_1})$. Taking the dual decomposition, we see that X_T also has a handle decomposition relative to X_{F_2} obtained by attaching a single 5-dimensional 3-handle. By excision, we therefore have that

$$H_*(X_T, X_{F_1}) = \begin{cases} \mathbb{Z} & * = 2 \\ 0 & \text{else} \end{cases} \text{ and } H_*(X_T, X_{F_2}) = \begin{cases} \mathbb{Z} & * = 3 \\ 0 & \text{else.} \end{cases}$$

In particular, the inclusion maps $X_{F_i} \to X_T$ induce isomorphisms on first homology. It will be useful for us later on to know that the inclusion induced map $\pi_1(X_{F_1}) \to \pi_1(X_T)$ is surjective, as follows immediately from applying the Seifert-van Kampen theorem to $X_T = (X_{F_1} \times I) \cup (2\text{-handle})$.

We now comment on basepoints for the fundamental group in this context. Let $x_0 \in X_K \subseteq X_T \times \{0\}$, let $\alpha = \{x_0\} \times I \subseteq X_T \times I$, and let $x_1 = \{x_0\} \times 1$. We will always let $\pi_1(X_K) = \pi_1(X_K, x_0)$, $\pi_1(X_{F_1}) = \pi_1(X_{F_1}, x_0)$, $\pi_1(X_T) = \pi_1(X_T, x_0)$, and $\pi_1(X_{F_2}) = \pi_1(X_{F_2}, x_1)$. There are natural inclusion induced maps $\iota\colon \pi_1(X_K, x_0) \to \pi_1(X_T, x_0)$ and $\iota_1\colon \pi_1(X_{F_1}, x_0) \to \pi_1(X_T, x_0)$. Moreover, we use the arc α to define

$$\iota_2\colon \pi_1(X_{F_2}, x_1) \to \pi_1(X_T, x_1) \to \pi_1(X_T, x_0).$$

Later on, we will often omit basepoints from our notation, always using the above arcs and corresponding inclusion maps. This completes Construction 3.1.

Proposition 3.2. *Fix a compact, oriented, smooth 4-manifold W, a (possibly empty) link L in ∂W, a nonnegative number g, and a homology class $S \in H_2(W, \partial W; \mathbb{Z})$ with $\partial S = [L]$. The distance function d_1 defines a metric on the set of ambient isotopy classes rel. boundary of embedded oriented surfaces of genus g in W with boundary L that represent the class $S \in H_2(W, \partial W; \mathbb{Z})$.*

Proof. We use that the distance is finite within the sets considered [BS15]. If $d_1(\Sigma, \Sigma') = 0$, then Σ and Σ' are ambiently isotopic. The distance function is flagrantly symmetric.

To see the triangle inequality, suppose F and F' are homologous rel. boundary surfaces which stabilize via k 1-handle additions to S and F' and F'' are homologous rel. boundary surfaces which stabilize via h 1-handle additions to S'. Now consider the sequence of stabilizations and destabilizations from F to S to F' to S' to F'' as a 3-dimensional cobordism T embedded in $W \times I$. We may perturb the embedding of T so that $F\colon W \times I \to I$ restricts to a Morse function on T, where stabilizations correspond to index one critical points, and destabilizations correspond to index two critical points. First we argue that we can rearrange this sequence of stabilizations and destabilizations so that all the stabilizations come first, followed by destabilizations. Our desired result will then follow immediately from letting S'' be the preimage of a regular value taken after all index one critical points and before all index two critical points, and observing that both F and F'' stabilize via $(k + h)$ 1-handle additions to S''.

In codimension at least two, critical points of an embedded cobordism can be arranged, by ambient isotopy, to appear in order of increasing index [Per75], [BP16, Theorem 4.1], by the following standard argument, which we include for completeness. Choose a gradient-like embedded vector field subordinate to F [BP16, Definition 3.1]. Rearrangement of critical points is possible in general if the ascending manifold of the lower critical point is disjoint from the descending manifold of the higher critical point. Suppose that an index one critical point of T has critical value t_1 higher than critical value t_2 of an index two critical point, and suppose that there are no critical values between t_2 and t_1. The descending manifold of the index 1 critical point of a 3-dimensional cobordism intersects a generic level set $W \times \{t\}$, with $t_2 < t < t_1$ in a 1-dimensional disc. The descending manifold of the index 2 critical point intersects $W \times \{t\}$ also in a 1-dimensional disc. By general position, we can perturb the gradient-like vector field to make the ascending and descending manifolds disjoint, and we may do so simultaneously for all such t. It follows that the critical points can be rearranged by an ambient isotopy, as desired. ☐

We remark that we do not claim d_2 gives rise to a metric. The next proposition tells us that 2-spheres can be reordered so they come before 1-handle additions.

Proposition 3.3. *Suppose that an embedded surface Σ_2 is obtained from a connected surface Σ_1 by some number m of 1-handle additions, followed by connect summing with a local 2-knot. Then there is an embedded surface Σ' that is obtained from Σ_1 by adding a local 2-knot, and such that Σ_2 is obtained from Σ' by m 1-handle additions.*

Proof. Let Σ_1' denote Σ_1 with the 1-handles attached, so Σ_2 is obtained from Σ_1' by connected sum with a local 2-knot S. The isotopy class of $\Sigma_1' \# S$ is unchanged by where on Σ_1' we take the connected sum, so we can assume that our connected sum takes place far away from the attached 1-handles. But then it is clear that we can attach S first and our 1-handles second. ☐

4. Generating rank of modules over a commutative PID

We recall some facts about generating ranks of finitely generated modules over commutative PIDs.

Let A be a finitely generated module over a commutative PID S. We say that A has *generating rank k over S* if A is generated as an S-module by k elements but not by $k-1$ elements and write $\operatorname{g-rk}_S A = k$. When S is clear from context, we often abbreviate $\operatorname{g-rk}_S A$ by $\operatorname{g-rk} A$.

Lemma 4.1. *Let A, B, and C be finitely generated modules over a commutative PID S.*

(1) If A surjects onto B then $\operatorname{g-rk}_S B \leqslant \operatorname{g-rk}_S A$.
(2) If $B \leqslant A$ then $\operatorname{g-rk}_S B \leqslant \operatorname{g-rk}_S A$.
(3) Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of S-modules. Then $\operatorname{g-rk}_s C \geqslant \operatorname{g-rk}_S(B) - \operatorname{g-rk}_S(A)$.

Proof. The first part follows immediately from the definition of generating rank. The second part is easy to check using the classification of finitely generated modules over a commutative PID. The third property follows from taking minimal S-generating sets $\{a_1, \ldots, a_n\}$ and $\{c_1, \ldots, c_m\}$ for A and C respectively, picking $b_i \in g^{-1}(c_i)$ for each $1 \leqslant i \leqslant m$, and observing that $\{f(a_1), \ldots, f(a_n), b_1, \ldots, b_m\}$ is an S-generating set for B. ☐

We will also make arguments involving the *order* of a finitely generated module A over a commutative PID S. The classification of finitely generated modules over a PID states that there exist $j, k \in \mathbb{N}$ and elements $s_1, \dots, s_k \in S$ such that there is a (non-canonical) isomorphism

$$A \cong S^j \oplus TA \cong S^j \oplus \bigoplus_{i=1}^{k} S/\langle s_i \rangle.$$

When $j > 0$ we say that the *order* of A is $|A| = 0$ and when $j = 0$ we say that the order of A is $|A| = \prod_{i=1}^k s_i$. This is well-defined up to multiplication by units in S. The key property of order we use is that if $f\colon A \to B$ is a map of S-modules with $\ker(f)$ torsion, then $|\operatorname{Im}(f)| = |A|/|\ker(f)|$.

5. Pairs of 2-knots with arbitrary 1-handle distance

In this section, we prove that for every nonnegative integer m, there exists a pair of 2-knots K and J in the 4-sphere with 1-handle stabilization distance m, which is an immediate consequence of the following proposition.

Proposition 5.1. *For each $m \in \mathbb{N}$, there exists a knotted 2-sphere K in S^4 such that the minimal number of 1-handle stabilizations needed to make K an unknotted surface is exactly m.*

Proof of Theorem A. Let $m \in \mathbb{N}$, let K be as in Proposition 5.1, and let J be an unknotted 2-sphere. Since every stabilization of an unknotted 2-sphere is an unknotted surface, we obtain immediately that $d_1(K, J) = m$. □

The next proposition is the key algebraic input into the proof of Proposition 5.1.

Proposition 5.2. *Let $F_1 \subset S^4$ be a smoothly embedded oriented surface and suppose that F_2 is obtained from F_1 by a 1-handle stabilization. Then there is a polynomial $p \in \mathbb{Q}[t^{\pm 1}]$ and a short exact sequence*

$$0 \to \mathbb{Q}[t^{\pm 1}]/\langle p \rangle \to H_1(S^4 \smallsetminus \nu F_1; \mathbb{Q}[t^{\pm 1}]) \to H_1(S^4 \smallsetminus \nu F_2; \mathbb{Q}[t^{\pm 1}]) \to 0.$$

Proof. We consider the relative cobordism X_T between X_{F_1} and X_{F_2} from Construction 3.1, with $W = S^4$. We will consider the infinite cyclic cover $\widetilde{X}_T$. Recall that X_T is obtained from $X_{F_1} \times I$ by attaching a single 5-dimensional 2-handle along $\gamma \times \{1\}$ for $\gamma = \mu_1 \beta \mu_2^{-1} \beta^{-1}$, where μ_1 and μ_2 are meridians of F_1 in S^4 near the attaching spheres of the 1-handle and β is a parallel push-off of the core of this 1-handle. Since $H_1(F_1; \mathbb{Z}) \cong \mathbb{Z}$, and the attaching sphere of the 2-handle is null homologous, the abelianization homomorphism $\pi_1(X_{F_1}) \to \mathbb{Z}$ extends to a homomorphism $\pi_1(X_T) \to \mathbb{Z}$. From now on in this proof we consider homology with $\mathbb{Q}[t^{\pm 1}]$-coefficients induced by this homomorphism. We also note that the handle decomposition lifts to a relative handle decomposition of $\widetilde{X}_T$ with one orbit of 2-handles under the deck transformation action of $\mathbb{Z}$.

Using this relative handle decomposition we obtain that $H_*(X_T, X_{F_1}; \mathbb{Q}[t^{\pm 1}]) = 0$ for $* \neq 2$ and $H_2(X_T, X_{F_1}; \mathbb{Q}[t^{\pm 1}]) \cong \mathbb{Q}[t^{\pm 1}]$. Since dually X_T is obtained from $X_{F_2} \times I$ by attaching a single 5-dimensional 3-handle, we have that $H_*(X_T, X_{F_2}\, \mathbb{Q}[t^{\pm 1}]) = 0$ for $* \neq 3$. Now consider the long exact sequence of the pair (X_T, X_{F_1}) with $\mathbb{Q}[t^{\pm 1}]$-coefficients.

$$\cdots \to H_2(X_T) \to H_2(X_T, X_{F_1}) \to H_1(X_{F_1}) \to H_1(X_T) \to H_1(X_T, X_{F_1}).$$

Since $H_1(X_T, X_{F_1}) = 0$ and $H_2(X_T, X_{F_1}) \cong \mathbb{Q}[t^{\pm 1}]$, and since $\mathbb{Q}[t^{\pm 1}]$ is a PID, this yields a short exact sequence

$$0 \to \mathbb{Q}[t^{\pm 1}]/\langle p \rangle \to H_1(X_{F_1}) \to H_1(X_T) \to 0$$

for some $p \in \mathbb{Q}[t^{\pm 1}]$. Now the long exact sequence of the pair (X_T, X_{F_2}) yields

$$0 = H_2(X_T, X_{F_2}) \to H_1(X_{F_2}) \to H_1(X_T) \to H_1(X_T, X_{F_2}) = 0,$$

from which it follows that the inclusion induced map $H_1(X_{F_2}) \to H_1(X_T)$ is an isomorphism, and so we obtain the short exact sequence

$$0 \to \mathbb{Q}[t^{\pm 1}]/\langle p\rangle \to H_1(X_{F_1}) \to H_1(X_{F_2}) \to 0$$

as desired. □

Example 5.3 (The knot 9_{46} and its two standard slice discs.)**.** Let $R := 9_{46}$ and let D_j for $j = 1, 2$ be the slice discs indicated by the left and right bands, respectively, of the left part of Figure 1. Observe that R has a genus 1 Seifert surface F (illustrated on the right

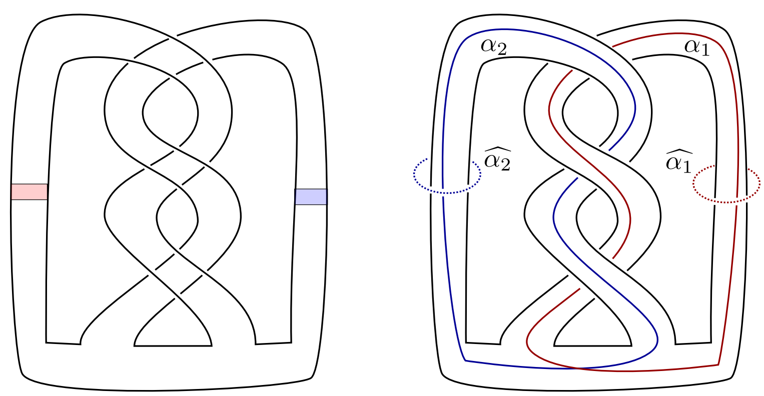

FIGURE 1. The knot $R = 9_{46}$ has slice discs D_1 (left band) and D_2 (right band).

of Figure 1) and that D_j can also be obtained by surgering a pushed in copy of F along the 0-framed unknot α_j. The curves α_1, α_2 also represent a basis for $H_1(F)$ with respect to which the Seifert pairing is given by

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}.$$

The Alexander module is therefore presented by

$$tA - A^T = \begin{bmatrix} 0 & 2t-1 \\ t-2 & 0 \end{bmatrix},$$

and hence is isomorphic to $\mathbb{Z}[t^{\pm 1}]/\langle t-2\rangle \oplus \mathbb{Z}[t^{\pm 1}]/\langle 2t-1\rangle$, where $\widehat{\alpha_1}$ and $\widehat{\alpha_2}$ represent the generators of each summand.

Moreover, the inclusion induced maps $\iota_j \colon \mathcal{A}_{\mathbb{Q}}(R) \to \mathcal{A}_{\mathbb{Q}}(D_j)$ are given by projection onto summands:

$$\begin{gathered}
\mathcal{A}_{\mathbb{Q}}(R) \cong \mathbb{Q}[t^{\pm 1}]/\langle 2t-1\rangle \oplus \mathbb{Q}[t^{\pm 1}]/\langle t-2\rangle \xrightarrow{\iota_1} \mathbb{Q}[t^{\pm 1}]/\langle 2t-1\rangle \cong \mathcal{A}_{\mathbb{Q}}(D_1) \\
(x, y) \mapsto x \\
\mathcal{A}_{\mathbb{Q}}(R) \cong \mathbb{Q}[t^{\pm 1}]/\langle 2t-1\rangle \oplus \mathbb{Q}[t^{\pm 1}]/\langle t-2\rangle \xrightarrow{\iota_2} \mathbb{Q}[t^{\pm 1}]/\langle t-2\rangle \cong \mathcal{A}_{\mathbb{Q}}(D_2) \\
(x, y) \mapsto y.
\end{gathered}$$

Note that $\ker(\iota_1) \cap \ker(\iota_2) = \{0\} \subseteq \mathcal{A}_{\mathbb{Q}}(R)$.

Proof of Proposition 5.1. Let $D := D_2 \subset D^4$ be the "right band" slice disc for the 9_{46} knot shown via a blue band on the left of Figure 1. Let K_0 be the 2-knot obtained from doubling this disc, that is $K_0 = D \cup_{9_{46}} D \subset D^4 \cup D^4 = S^4$. Let $K := \#_{i=1}^m K_0$.

First we use Proposition 5.2 to show that if K stabilizes to an unknotted surface by n 1-handle additions then $n \geqslant m$. We know that

$$H_1(S^3 \smallsetminus \nu(9_{46}); \mathbb{Q}[t^{\pm 1}]) \cong \mathbb{Q}[t^{\pm 1}]/\langle 2t-1\rangle \oplus \mathbb{Q}[t^{\pm 1}]/\langle t-2\rangle$$

where the inclusion induced map to $H_1(D^4 \smallsetminus \nu(D); \mathbb{Q}[t^{\pm 1}]) \cong \mathbb{Q}[t^{\pm 1}]/\langle t-2\rangle$ is given by projection onto the second factor. By using the Mayer-Vietoris sequence corresponding to the decomposition

$$S^4 \smallsetminus \nu K_0 = \big(D^4 \smallsetminus \nu(D)\big) \cup_{S^3 \smallsetminus \nu(9_{46})} \big(D^4 \smallsetminus \nu(D)\big),$$

we can compute that

$$H_1(S^4 \smallsetminus \nu K_0; \mathbb{Q}[t^{\pm 1}]) = \mathbb{Q}[t^{\pm 1}]/(t-2).$$

Since Alexander modules are additive under connected sum of 2-knots we therefore have that

$$H_1(S^4 \smallsetminus \nu K; \mathbb{Q}[t^{\pm 1}]) = \bigoplus_{i=1}^m \big(\mathbb{Q}[t^{\pm 1}]/(t-2)\big).$$

We therefore need to show that one requires at least m stabilizations to trivialize the Alexander module of K. Note that the generating rank of $H_1(S^4 \smallsetminus \nu K; \mathbb{Q}[t^{\pm 1}])$ is m. We claim that the result of stabilizing an embedded surface whose Alexander module has generating rank k is an embedded surface with generating rank at least $k-1$. To see the claim, we use Proposition 5.2 and the fact that if a $\mathbb{Q}[t^{\pm 1}]$-module M has generating rank k and a submodule N has generating rank 1, then the quotient M/N has generating rank at least $k-1$, by Lemma 4.1 (3). Here we use that $\mathbb{Q}[t^{\pm 1}]$ is a PID. By the claim and the fact that the generating rank of $H_1(S^4 \smallsetminus \nu K; \mathbb{Q}[t^{\pm 1}])$ is m, it follows by induction that $d_1(K, J) \geqslant m$.

It remains to show that we can make K unknotted via m 1-handle attachments. Recall that the slice disc D is constructed by a band move "cutting" one of the bands of the obvious Seifert surface Σ for 9_{46} in Figure 1, and then capping off the resulting 2-component unlink with disjoint discs. A single stabilization, tubing these two discs together, results in an embedded genus one surface. This surface could also be obtained by capping off the 2-component unlink with an annulus instead of two discs, and hence is isotopic to the result of pushing the aforementioned Seifert surface into D^4. We assert that $D \cup \Sigma \subset S^4$ is an unknotted genus one surface, and prove this by direct manipulation of handle diagrams for the embedding of the surface in D^4, using the banded knot diagram moves of Swenton [Swe01].[1]

The data of an unlink and bands attached to it with the property that the result of performing the corresponding band moves is also an unlink provides instructions for embedding a surface in S^4: the unlink's components correspond to 0-handles, the bands to 1-handles, and the unlink obtained by banding can be capped off with 2-handles in an essentially unique way, in the sense that any two choices of discs in S^3 capping off the unlink yield isotopic surfaces in S^4. This uses the main result of [Liv82], that any two sets of embedded discs in S^3 are isotopic rel. boundary in D^4.

[1]The reader who is familiar with doubly slice knots may instead observe that $D \cup \Sigma$ is a stabilization of the unknotted 2-knot obtained by gluing the 'left band' and 'right band' discs together, and hence is itself unknotted. We give the longer argument here to be self-contained.

The banded diagram on the far left of Figure 2 gives $D\cup\Sigma$. The top two bands correspond to the Seifert surface, and the green band is the band of the disc D. The center left of

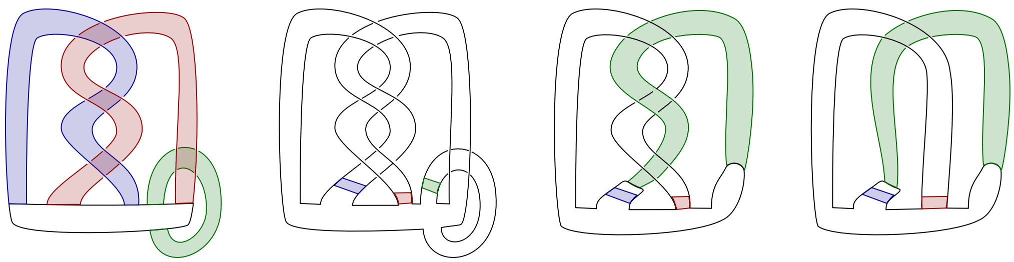

FIGURE 2. Simplifying a banded knot diagram for $D \cup \Sigma$.

Figure 2 gives the 'dual' band description, corresponding to turning our handle diagram upside down. The center right figure is obtained by an isotopy of the banded diagram in S^3, and we perform a 'band-swim' move of the green band through the red band to obtain the diagram on the far right of Figure 2. Now obtain the diagram on the left of Figure 3 by an isotopy of the diagram in S^3, before sliding the green band across the red band to obtain the central diagram. We can then cancel the right-hand unknot with the red

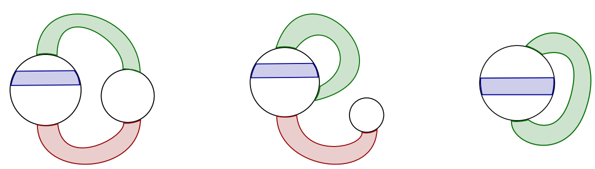

FIGURE 3. Further simplifications of the banded knot diagram for $D \cup \Sigma$, resulting in the standard diagram for an unknotted torus (right).

band, corresponding to canceling a pair of 0- and 1-handles, in order to obtain the standard diagram for an unknotted torus seen on the right of Figure 3. □

6. Pairs of slice discs with large generalized stabilization distance

In this section we prove Theorem B. We use the classical Alexander module of a knot to show that for every nonnegative integer m there is a knot K with slice discs D and D' such that $d_2(D, D')$ equals m. To do this, we investigate the kernel of the induced map on fundamental groups from the knot exterior to the slice disc exteriors by using the homology of cyclic covering spaces.

First, we note that connected sum with a knotted 2-sphere has no effect on the kernel of the map on fundamental groups.

Proposition 6.1. *Suppose that F_2 has been obtained from F_1 by connected sum with a knotted 2-sphere S. Then*

$$\ker(i_1\colon \pi_1(X_K) \to \pi_1(X_{F_1})) = \ker(i_2\colon \pi_1(X_K) \to \pi_1(X_{F_2})).$$

Proof. Let $X_S := S^4 \backslash \nu S$ be the exterior of S in S^4. Construct X_{F_2} from X_{F_1} and X_S by identifying thickened meridians $S^1 \times D^2 \subset \partial X_{F_1}$ and $S^1 \times D^2 \subset \partial X_S$ in the boundaries and smoothing corners. By the Seifert-van Kampen theorem we have that

$$\pi_1(X_{F_2}) \cong \pi_1(X_{F_1}) *_{\mathbb{Z}} \pi_1(X_S).$$

So $\pi_1(X_{F_1})$ is isomorphic to a subgroup of $\pi_1(X_{F_2})$ in such a way that the inclusion-induced maps factor as

$$\pi_1(X_{F_1}) \hookrightarrow \pi_1(X_{F_1}) *_{\mathbb{Z}} \pi_1(X_S) \xrightarrow{\cong} \pi_1(X_{F_2}).$$

It follows that $\ker(i_1) = \ker(i_2)$. □

The following proposition is central to the rest of the paper, and so we state it in some generality. In particular, in later sections we will want to apply this result with twisted coefficients, so in the name of efficiency we state and prove the full version here.

Proposition 6.2. *Let F_1 and F_2 be properly embedded surfaces in D^4 with $\partial F_j = K$, where F_2 has been obtained from F_1 by g 1-handle additions such that $g(F_2) = g(F_1) + g$. Let $T \subseteq D^4 \times I$ be the 3-manifold built as in Construction 3.1. Suppose that $\phi\colon \pi_1(X_K) \to \mathrm{GL}_m(R)$ extends over $\pi_1(X_T)$ to a map Φ (i.e. $\phi = \Phi \circ \iota$). For $j = 1, 2$ define*

$$P_j := \ker\left(H_1^{\phi}(X_K; R) \to H_1^{\Phi\circ\iota_j}(X_{F_j}; R)\right).$$

Then $P_1 \subseteq P_2$ and, assuming in addition that R is a PID, P_2 is generated as an R-module by $P_1 \cup \{x_i\}_{i=1}^{gm}$ for some choice of $x_i \in P_2$.

Proof. The case of general g follows immediately from repeated application of the $g = 1$ case, which we now prove. The map Φ induces a regular cover $\widetilde{X}_T \to X_T$. We wish to lift the relative handle decomposition of Construction 3.1 to $\widetilde{X}_T$. Recall that X_T is obtained from $X_{F_1} \times I$ by attaching a single 5-dimensional 2-handle along $\gamma \times \{1\}$ for $\gamma = \mu_1 \beta \mu_2^{-1} \beta^{-1}$, where μ_1 and μ_2 are meridians of F_1 in D^4 near the attaching spheres of the 1-handle and β is a parallel push-off of the core of this 1-handle. Our assumption that ϕ extends to a map $\Phi\colon \pi_1(X_T) \to \mathrm{GL}_m(R)$ implies that $\phi(\gamma) = \mathrm{Id}_{R^m}$.

We therefore obtain that $H_*^{\Phi}(X_T, X_{F_1}; R) = 0$ for $* \neq 2$ and, since dually X_T is obtained from $X_{F_2} \times I$ by attaching a single 5-dimensional 3-handle, we have that $H_*^{\Phi}(X_T, X_{F_2}; R) = 0$ for $* \neq 3$. For $j = 1, 2$ the long exact sequence in twisted homology with R-coefficients corresponding to the triple (X_T, X_{F_j}, X_K) is

$$\cdots \to H_3^{\Phi}(X_T, X_{F_j}) \to H_2^{\Phi}(X_{F_j}, X_K) \xrightarrow{g_j} H_2^{\Phi}(X_T, X_K) \xrightarrow{h_j} H_2^{\Phi}(X_T, X_{F_j}) \to \ldots \quad (1)$$

and so we see that g_2 is surjective and that g_1 is injective.

Now consider the following diagram, which is commutative since all maps are induced by various inclusions and natural long exact sequences. Note that the horizontal sequences are coming from long exact sequences of various pairs and that all homology is appropriately twisted with coefficients in R.

$$\begin{tikzcd}
H_2^{\Phi}(X_{F_1}) \arrow[r] \arrow[d] & H_2^{\Phi}(X_{F_1}, X_K) \arrow[d, "g_1"] \arrow[dr, "\partial_1"] & & H_1^{\Phi}(X_{F_1}) \arrow[d] \\
H_2^{\Phi}(X_T) \arrow[r] & H_2^{\Phi}(X_T, X_K) \arrow[r, "\partial_T"] & H_1^{\phi}(X_K) \arrow[r, "j_T"] \arrow[ur, "j_1"] \arrow[dr, "j_2"] & H_1^{\Phi}(X_T) \\
H_2^{\Phi}(X_{F_2}) \arrow[r] \arrow[u] & H_2^{\Phi}(X_{F_2}, X_K) \arrow[u, "g_2"] \arrow[ur, "\partial_2"] & & H_1^{\Phi}(X_{F_2}) \arrow[u]
\end{tikzcd}$$

Since g_2 is surjective, we have that $P_2 = \ker(j_2) = \mathrm{Im}(\partial_2) = \mathrm{Im}(\partial_T)$. Also,

$$P_1 = \ker(j_1) = \mathrm{Im}(\partial_1) = \mathrm{Im}(\partial_T \circ g_1) \subseteq \mathrm{Im}(\partial_T) = P_2.$$

So we have established the first conclusion of this proposition.

To establish the second conclusion, we note that $C_k^{\Phi}(X_T, X_{F_1}; R)$ is isomorphic to R^m if $k = 2$ and is 0 for $k \neq 2$, so $H_2^{\Phi}(X_T, X_{F_1}; R) \cong R^m$ has R-generating rank m. Considering the long exact sequence of Equation (1), we see that

$$\operatorname{coker}(g_1) = H_2(X_T, X_K)/\operatorname{Im}(g_1) = H_2(X_T, X_K)/\ker(h_1) \cong \operatorname{Im}(h_1) \subseteq H_2(X_T, X_{F_1})$$

and so $\operatorname{coker}(g_1)$ has generating rank no more than m as an R-module, by Lemma 4.1. We can therefore let $\{a_i\}_{i=1}^m$ be elements of $H_2(X_T, X_K)$ which together with $\operatorname{Im}(g_1)$ generate $H_2(X_T, X_K)$ as an R-module. Note that

$$\begin{aligned} P_1 \cup \{\partial_T(a_i)\}_{i=1}^m &= \operatorname{Im}(\partial_1) \cup \{\partial_T(a_i)\}_{i=1}^m = \operatorname{Im}(\partial_T \circ g_1) \cup \{\partial_T(a_i)\}_{i=1}^m \\ &= \partial_T(\operatorname{Im}(g_1) \cup \{a_i\}_{i=1}^m) \end{aligned}$$

generates $\operatorname{Im}(\partial_T) = P_2$ as an R-module, and so we can let $x_i = \partial_T(a_i)$ for $i = 1, \dots m$. $\square$

For any knot or slice disc L, let $\mathcal{A}_{\mathbb{Q}}(L)$ denote the Alexander module of L with rational coefficients. That is, if we as usual let $\varepsilon \colon \pi_1(X_L) \to \mathbb{Z}$ denote the abelianization map, we have that $\mathcal{A}_{\mathbb{Q}}(L) = H_*^{\phi}(X_L)$ where $\phi \colon \pi_1(X_L) \to \mathbb{Q}[t^{\pm 1}]^\times$ is defined by $\gamma \mapsto t^{\varepsilon(\gamma)}$.

Proposition 6.3. *Let Δ_1 and Δ_2 be slice discs for a knot K. Let $P_j := \ker(\mathcal{A}_{\mathbb{Q}}(K) \to \mathcal{A}_{\mathbb{Q}}(\Delta_j))$ for $j = 1, 2$. Suppose that $\operatorname{g-rk}(P_1) = \operatorname{g-rk}(P_2) = n$ and that $\operatorname{g-rk}(P_1 \cap P_2) = k$. Then $d_2(\Delta_1, \Delta_2) \geqslant n - k$.*

Proof. Suppose that F is a genus g surface to which both Δ_1 and Δ_2 stabilize by g 1-handle additions and some number of 2-knot additions. We will show that $g \geqslant n - k$. By Proposition 3.3, for $j = 1, 2$ there exist a disc Δ_j' obtained from Δ_j by connected sum with some number of knotted 2-spheres such that F is obtained from Δ_j' by g 1-handle additions. It follows from Proposition 6.1 that for $j = 1, 2$ we have

$$P_j' := \ker(\mathcal{A}_{\mathbb{Q}}(K) \to \mathcal{A}_{\mathbb{Q}}(\Delta_i')) = P_j.$$

Let $P := \ker(\mathcal{A}_{\mathbb{Q}}(K) \to \mathcal{A}_{\mathbb{Q}}(F))$. By Proposition 6.2, we see that both P_1' and P_2' are submodules of P. We now argue that the generating rank of P, considered as a $\mathbb{Q}[t^{\pm 1}]$-module, is at least $2n - k$. To see this we show that $\operatorname{Im}(P_1' \oplus P_2' \to P)$ has generating rank at least $2n - k$ and apply Lemma 4.1 (2). Let $i_1 \colon P_1' \to P$ and $i_2 \colon P_2' \to P$ be the inclusion maps. Both P_1' and P_2' are submodules of P, so

$$\ker(i_1 \oplus -i_2 \colon P_1' \oplus P_2' \to P) = \{(p_1, p_2) \in P_1' \oplus P_2' \mid i_1(p_1) = i_2(p_2) \in P\} \cong P_1' \cap P_2'.$$

We obtain a short exact sequence

$$0 \to P_1' \cap P_2' \to P_1' \oplus P_2' \to \operatorname{Im}(i_1 \oplus -i_2) \to 0,$$

and conclude by Lemma 4.1 (3) that $\operatorname{g-rk}(\operatorname{Im}(i_1 \oplus -i_2)) \geqslant 2n - k$.

However, Proposition 6.2 applied with $m = 1$ also tells us that there exist some $x_1, \dots, x_g$ in P such that P is generated by $P_1' \cup \{x_1, \dots, x_g\}$. Therefore the generating rank of P is at most $n + g$, and so we have $n + g \geqslant \operatorname{g-rk}(P) \geqslant 2n - k$, from which it follows as desired that $g \geqslant n - k$. $\square$

The next proposition completes the proof of Theorem B.

Proposition 6.4. *Let K_0 be the knot 9_{46} and let $K = \#_{i=1}^n K_0$. Let $\Delta_1 = \natural_{i=1}^n D_1$ and let $\Delta_2 := \natural_{i=1}^n D_2$ be the 'left band only' and 'right band only' slice discs. Then*

$$d_2(\Delta_1, \Delta_2) = n.$$

Proof. First, note that we can obtain both Δ_1 and Δ_2 from surgery on a genus n Seifert surface for K and so $d_2(\Delta_1, \Delta_2) \leqslant n$.

There is an identification

$$\mathcal{A}_{\mathbb{Q}}(K) \cong \bigoplus_{i=1}^{n} \mathcal{A}_{\mathbb{Q}}(K_0) \cong \bigoplus_{i=1}^{n} \left(\mathbb{Q}[t^{\pm 1}]/\langle 2t-1\rangle \oplus \mathbb{Q}[t^{\pm 1}]/\langle t-2\rangle\right)$$

such that

$$P_1 := \ker(\mathcal{A}_{\mathbb{Q}}(K) \to \mathcal{A}_{\mathbb{Q}}(\Delta_1)) = \bigoplus_{i=1}^{n} \mathbb{Q}[t^{\pm 1}]/\langle t-2\rangle$$
$$\text{and } P_2 := \ker(\mathcal{A}_{\mathbb{Q}}(K) \to \mathcal{A}_{\mathbb{Q}}(\Delta_2)) = \bigoplus_{i=1}^{n} \mathbb{Q}[t^{\pm 1}]/\langle 2t-1\rangle.$$

In particular, $P_1 \cap P_2 = \{0\}$. Now, $\text{g-rk}(P_1) = \text{g-rk}(P_2) = n$, and $\text{g-rk}(P_1 \cap P_2) = 0$. It follows from Proposition 6.3 that $d_2(\Delta_1, \Delta_2) \geqslant n$ as required. □

7. Secondary lower bounds using twisted homology

Now we shall construct subtler examples of pairs of slice discs with high stabilization distance.

7.1. **Satellite knots and satellite slice discs.** Our examples come from the satellite construction. Let R and J be knots and let $\eta \subset S^3 \smallsetminus R$ be an unknotted simple closed curve in the complement of R. Recall that $S^3 \smallsetminus \nu(\eta) \cup X_J \cong S^3$, where the meridian of η is identified with the longitude of J, and vice versa. The image of $R \subset S^3 \smallsetminus \nu(\eta)$ under this homeomorphism is by definition the satellite knot $R_\eta(J)$.

It is a well known fact that if R and J are slice knots and η is any unknot in the complement of R, then the satellite knot $R_\eta(J)$ is also slice. It will be useful to have an explicit construction of a slice disc Δ_D for $R_\eta(J)$ coming from a choice of slice discs Δ_0 for R and D for J, together with compatible degree 1 maps $f\colon X_{R_\eta(J)} \to X_R$ and $g\colon X_{\Delta_D} \to X_{\Delta_0}$.

Construction 7.1 (Satellite slice discs and degree 1 maps)**.** Let R be a knot with slice disc Δ_0 and let η be an unknotted curve in $S^3 \smallsetminus \nu(R)$. Identify $D^4 \supset \Delta_0$ as $D^2 \times D^2$ in such a way that when we consider $\partial(D^2 \times D^2) = (S^1 \times D^2) \cup (D^2 \times S^1)$ we have $D^2 \times S^1 = \nu(\eta)$ and so $R = \partial\Delta_0 \subseteq S^1 \times D^2$.

Now let J be a knot with slice disc D. We obtain a slice disc denoted Δ_D for $R_\eta(J)$ by considering

$$\Delta_0 \subseteq D \times D^2 = \nu(D) \subset D^4.$$

Note that $X_{\Delta_D} = X_{\Delta_0} \cup_{S^1 \times D^2} X_D$, where $S^1 \times D^2$ is identified with $\nu(\eta) \subseteq X_R \subset \partial X_{\Delta_0}$ and with $S^1 \times D \subset \partial X_D$, and that this identification is evidently compatible with the decomposition $X_{R_\eta(J)} = (X_R \smallsetminus \nu(\eta)) \cup_{T^2} X_J$.

For every knot J there is a standard degree 1 map $f_0\colon X_J \to X_U$ which sends μ_J to μ_U and λ_J to λ_U, and for any slice disc D there is a similar degree one map $g_0\colon X_D \to X_E$, where E denotes the standard slice disc for the unknot. For the sake of completeness, we give this construction, emphasizing that one can choose g_0 to be an extension of f_0.

Parametrize

$$\nu\left(\partial X_J\right) = \partial X_J \times [0,\delta] = \{(p,s,t) \in S^1 \times ([0,2\pi]/\sim) \times [0,\delta]\},$$

where $\{(p,0,0)\} = \lambda_J$ and $\{(1,s,0)\} = \mu_J$. Now let $F \subset X_J$ be a (truncated) Seifert surface for J with tubular neighborhood $\nu(F) = F \times [0,\varepsilon]$. We can assume that

$$\nu(F) \cap \nu(\partial X_J) = \{(p,s,t) \in S^1 \times [0,\varepsilon] \times [0,\delta]\},$$

as illustrated below.

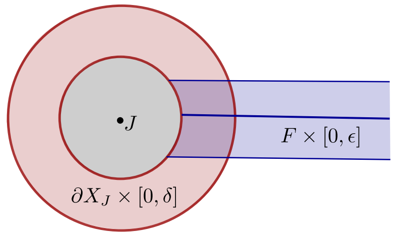

FIGURE 4. A cross section of X_J near its boundary. Note that the grey region represents $\nu(J)$ and is therefore not part of X_J.

We write $X_U = S \times D$ for $S = ([0,\varepsilon]/\sim) \cong S^1$ and $D = (S^1 \times [0,\delta])/(S^1 \times \delta) \cong D^2$. Define f_0 on $\nu(\partial X_J)$ by

$$f_0(p,s,t) = \begin{cases} (s,(p,t)) & \text{if } 0 \leqslant s \leqslant \varepsilon \\ (\varepsilon,(p,t)) & \text{if } \varepsilon < s, \end{cases}$$

and then extend over the rest of $\nu(F) = F \times [0,\varepsilon]$ by $f_0(y,s) = (s,(0,\delta))$. Finally, for any x in neither $\nu(F)$ nor $\nu(\partial X_K)$, we define $f_0(x) = (\varepsilon,(0,\delta))$.

The construction of g_0 is very similar, only with a compact orientable 3-manifold G with boundary $\partial G = F \cup_J D$ playing the rôle of the Seifert surface. We extend f_0 as defined above on X_J over $X_J \times I$, then over the rest of $\nu(\partial X_D)$, then over $\nu(G) \cong G \times I$ and then send the entirety of $X_D \backslash (\nu(\partial X_D) \cup \nu(G))$ to a single point in X_E.

Here are the details, which closely parallel the construction of f_0, but with extra care to ensure that $g_0|_{X_J} = f_0$.

First parametrize a neighborhood of the slice disc D as $D^2 \times D^2$, naturally a manifold with corners, such that $S^1 \times D^2$ is a tubular neighborhood of J and $S^1 \times S^1 = \partial X_J$. Consider a collar on this part of ∂X_D as follows. We think of X_D as a manifold with corners, with ∂X_J the corner set, dividing ∂X_D as $X_J \cup_{\partial X_J} D^2 \times S^1$. Then we consider a collar on the $D^2 \times S^1$ part of the boundary that restricts on X_J to a collar for ∂X_J in X_J. Parametrize this collar as

$$\nu\left(D^2 \times S^1\right) = D^2 \times S^1 \times [0,\delta] = \{(p,s,t) \in D^2 \times ([0,2\pi]/\sim) \times [0,\delta]\},$$

where $\{(p,0,0)\}$ is a push-off of the slice disc with boundary λ_J and $\{(1,s,0)\} = \mu_J$.

Now let $G \subset X_D$ be a (truncated) 3-manifold with $\partial G = F \cup \{(p,0,0)\}$, with tubular neighborhood $\nu(G) = G \times [0,\varepsilon]$. Assume this restricts to the tubular neighborhood of F used above in the definition of f_0. Note that we can assume that

$$\nu(G) \cap \nu(D^2 \times S^1) = \{(p,s,t) \in D^2 \times [0,\varepsilon] \times [0,\delta]\}.$$

We write $X_E = S \times B$ for $S = ([0,\varepsilon]/\sim) \cong S^1$ and $B = (D^2 \times [0,\delta])/(D^2 \times \delta) \cong D^3$. Note that we have a natural inclusion $D \subset B$ corresponding to $X_U = S \times D \subset S \times B = X_E$.

Define g_0 on $\nu(D^2 \times S^1)$ by

$$g_0(p,s,t) = \begin{cases} (s,(p,t)) & \text{if } 0 \leqslant s \leqslant \varepsilon \\ (\varepsilon,(p,t)) & \text{if } \varepsilon < s, \end{cases}$$

and then extend over the rest of $\nu(G) = G \times [0,\varepsilon]$ by $g_0(y,s) = (s,(0,\delta))$. Finally, for any x in neither $\nu(G)$ nor $\nu(D^2 \times S^1)$, we define $g_0(x) = (\varepsilon,(0,\delta))$.

By using the above decompositions $X_{R_\eta(J)} = (X_R \smallsetminus \nu(\eta)) \cup_{T^2} X_J$ and $X_{\Delta_D} = X_{\Delta_0} \cup_{S^1\times D^2} X_D$, we obtain compatible degree 1 maps

$$f = \operatorname{Id} \cup f_0 \colon X_{R_\eta(J)} \to X_R \text{ and } g = \operatorname{Id} \cup g_0 \colon X_{\Delta_D} \to X_{\Delta_0}.$$

This completes Construction 7.1.

For a connected space X equipped with a surjective map $\varepsilon\colon \pi_1(X) \to \mathbb{Z}$, we let $\mathcal{A}(X)$ denote the induced $\mathbb{Z}[t^{\pm 1}]$-twisted first homology. For a knot K or disc D we often let $\mathcal{A}(K)$ denote $\mathcal{A}(X_K)$ and correspondingly let $\mathcal{A}(D)$ denote $\mathcal{A}(X_D)$.

Proposition 7.2. *Let R, Δ_0, η, J, and D be as above. Suppose that the linking number of η and R in S^3 is 0. Letting f and g be the degree 1 maps discussed above, the following diagram commutes, where the horizontal maps are the usual inclusion induced maps:*

$$\begin{tikzcd} \mathcal{A}(R_\eta(J)) \arrow[r] \arrow[d, "f_*"] & \mathcal{A}(\Delta_D) \arrow[d, "g_*"] \\ \mathcal{A}(R) \arrow[r] & \mathcal{A}(\Delta_0). \end{tikzcd}$$

Moreover, f_* and g_* are isomorphisms and so

$$\ker(\mathcal{A}(R_\eta(J)) \to \mathcal{A}(\Delta_D)) = f_*^{-1}(\ker(\mathcal{A}(R) \to \mathcal{A}(\Delta_0))) \cong \ker(\mathcal{A}(R) \to \mathcal{A}(\Delta_0))$$

is independent of the choice of slice disc D for J.

Proof. The fact that the diagram commutes follows immediately from the compatibility of f and g as defined in Construction 7.1. The fact that f_* is an isomorphism is a standard fact (one can also imitate the proof of Proposition 7.7 in a simpler setting). To see that g_* induces an isomorphism consider the following diagram, where the rows are the Mayer-Vietoris sequences in $\mathbb{Z}[t^{\pm 1}]$-coefficients corresponding to the decompositions $X_{\Delta_D} = X_{\Delta_0} \cup_{S^1\times D^2} X_D$ and $X_{\Delta_0} = X_{\Delta_0} \cup_{S^1\times D^2} X_E$. We have replaced the H_0 terms with zeroes, since the maps from $H_0(S^1 \times D^2; \mathbb{Z}[t^{\pm 1}])$ are injective.

$$\begin{tikzcd} H_1(S^1\times D^2;\mathbb{Z}[t^{\pm1}]) \arrow[r] \arrow[d] & H_1(X_{\Delta_0};\mathbb{Z}[t^{\pm1}]) \oplus H_1(X_D;\mathbb{Z}[t^{\pm1}]) \arrow[r] \arrow[d, "\operatorname{Id}\oplus(g_0)_*"] & H_1(X_{\Delta_D};\mathbb{Z}[t^{\pm1}]) \arrow[r] \arrow[d, "g_*"] & 0 \\ H_1(S^1\times D^2;\mathbb{Z}[t^{\pm1}]) \arrow[r] & H_1(X_{\Delta_0};\mathbb{Z}[t^{\pm1}]) \oplus H_1(X_E;\mathbb{Z}[t^{\pm1}]) \arrow[r] & H_1(X_{\Delta_0};\mathbb{Z}[t^{\pm1}]) \arrow[r] & 0 \end{tikzcd}$$

Since the linking number of η and R is 0, the cores of the copies of $S^1 \times D^2$ along which the spaces are glued, when thought of as fundamental group elements, map trivially to $\mathbb{Z}$. Therefore $H_1(S^1 \times D^2; \mathbb{Z}[t^{\pm 1}]) \cong H_1(S^1 \times D^2; \mathbb{Z}) \otimes \mathbb{Z}[t^{\pm 1}] \cong \mathbb{Z}[t^{\pm 1}]$. Similarly, since $S^1 \times D^2 \to X_D$ and $S^1 \times D^2 \to X_E$ are $\mathbb{Z}$-homology equivalences, the maps $\pi_1(X_D) \to \mathbb{Z}$ and $\pi_1(X_E) \to \mathbb{Z}$ are likewise trivial, and so the maps $H_1(S^1 \times D^2; \mathbb{Z}[t^{\pm 1}]) \to H_1(X_D; \mathbb{Z}[t^{\pm 1}])$

and $H_1(S^1 \times D^2; \mathbb{Z}[t^{\pm 1}]) \to H_1(X_E; \mathbb{Z}[t^{\pm 1}])$ are isomorphisms. It follows that the diagram above reduces to the diagram:

$$\begin{tikzcd} H_1(X_{\Delta_0}; \mathbb{Z}[t^{\pm 1}]) \arrow[r, "\cong"] \arrow[d, "\mathrm{Id}"] & H_1(X_{\Delta_D}; \mathbb{Z}[t^{\pm 1}]) = \mathcal{A}(\Delta_D) \arrow[d, "g_*"] \\ H_1(X_{\Delta_0}; \mathbb{Z}[t^{\pm 1}]) \arrow[r, "\cong"] & H_1(X_{\Delta_0}; \mathbb{Z}[t^{\pm 1}]) = \mathcal{A}(\Delta_0). \end{tikzcd}$$

Therefore the right hand vertical map is an isomorphism induced by g, as required. ☐

Example 7.3. Let R be the slice knot 6_1, with unknotted curve $\eta \in S^3 \smallsetminus \nu(R)$ as shown on the left of Figure 5. We will be interested in the satellite knot $R_\eta(J)$, depicted on the right of Figure 5, for certain choices of J. Note that η does not intersect F and so $R_\eta(J)$ has a

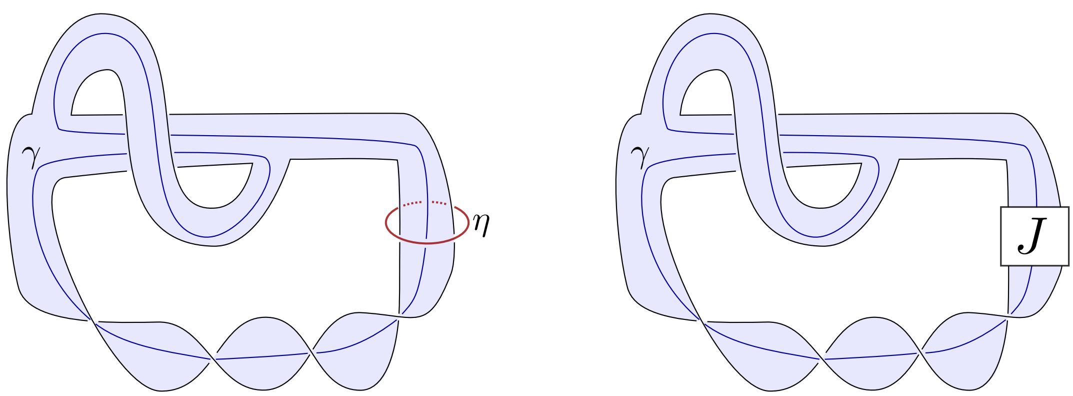

FIGURE 5. The knot $R = 6_1$ with a genus 1 Seifert surface F, a 0-framed curve γ on F, and an infection curve η (left) and the satellite knot $R_\eta(J)$ (right).

genus 1 Seifert surface F_J as shown on the right of Figure 5. The illustrated homologically essential 0-framed curve on F_J that in a mild abuse of notation we also call γ is isotopic to the knot J when thought as a curve in S^3.

Let Δ_0 denote the standard slice disc for R, obtained by surgering F along γ. Given a slice disc D for J, in Construction 7.1 we built a slice disc Δ_D for $R_\eta(J)$. In this context, one can interpret this construction as follows. Push the interior of F_J into the interior of D^4, then remove a small neighborhood of γ in F_J. This creates two new boundary components, which may be capped off with parallel copies of D to yield Δ_D. We note that a single 1-handle attachment to Δ_D that connects the two parallel copies of D returns the (pushed in) Seifert surface F_J, and so if D and D' are two different slice discs for J we always have that $d_2(\Delta_D, \Delta_{D'}) \leqslant 1$, even if $d_2(D, D')$ is large.

As in Example 5.3, we can pick a basis for the first homology of the Seifert surface F for which the Seifert matrix is given by

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$$

and manipulate $tA - A^T$ to see that $\mathcal{A}(R) \cong \mathbb{Z}[t^{\pm 1}]/\langle (2t-1)(t-2) \rangle$. We have that $\mathcal{A}(\Delta_0) \cong \mathbb{Z}[t^{\pm 1}]/\langle 2t-1 \rangle$, and that the kernel of the inclusion induced map $\mathcal{A}(R) \to \mathcal{A}(\Delta_0)$ is exactly $(t-2)\,\mathcal{A}(R)$. Additionally, by substituting $t = -1$ into the above computations we discover the homology of the 2-fold branched covers: $H_1(\Sigma_2(R)) \cong \mathbb{Z}_9$ and $\ker(H_1(\Sigma_2(R);\mathbb{Z}) \to H_1(\Sigma_2(D^4,\Delta_0);\mathbb{Z})) = 3\,\mathbb{Z}_9$.

7.2. **Metabelian twisted homology.** We will use twisted homology coming from metabelian representations that factor through the dihedral group $D_{2n} \cong \mathbb{Z}_2 \ltimes \mathbb{Z}_n$.

Construction 7.4. Given a knot K with preferred meridian μ_0, abelianization map $\varepsilon\colon \pi_1(X_K) \to \mathbb{Z}$, and a map $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_n$ for some prime n, define

$$\phi_\chi\colon \pi_1(X_K) \to \mathbb{Z}_2 \ltimes \mathbb{Z}_n \text{ by } \phi_\chi(\gamma) = ([\varepsilon(\gamma)], \chi(\mu_0^{\varepsilon(\gamma)}\gamma)).$$

Also, letting $\xi_n = e^{2\pi i/n}$, we have a standard map

$$\begin{aligned} \alpha\colon\ \mathbb{Z}_2 \ltimes \mathbb{Z}_n &\to \mathrm{GL}_2(\mathbb{Z}[\xi_n]) \\ (a,b) &\mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^a \begin{bmatrix} \xi_n^b & 0 \\ 0 & \xi_n^{-b} \end{bmatrix} \end{aligned}$$

In particular, we obtain a representation $\alpha_\chi = \alpha \circ \phi_\chi$ of $\pi_1(X_K)$ into $\mathrm{GL}_2(\mathbb{Z}[\xi_n])$. We will be interested in the corresponding twisted homology $H_*^{\alpha_\chi}(X_K, \mathbb{Z}[\xi_n])$, especially when $\mathbb{Z}[\xi_n]$ is a PID, e.g. when $n = 3$ and $\mathbb{Z}[\xi_3]$ is the ring of *Eisenstein integers*. For a connected space X together with a map $\phi\colon \pi_1(X) \to \mathbb{Z}_2 \ltimes \mathbb{Z}_n$, we will let $H_*^\phi(X)$ be shorthand for $H_*^{\alpha\circ\phi}(X;\mathbb{Z}[\xi_n])$.

We note for later use that for a connected space X with a surjection $\varepsilon\colon \pi_1(X) \to \mathbb{Z}$, a preferred element $\mu_0 \in \varepsilon^{-1}(1)$ and a map $\chi\colon \pi_1(X^2) \to \mathbb{Z}_n$, we have that

$$H_0^{\phi_\chi}(X, \mathbb{Z}[\xi_n]) \cong \begin{cases} \mathbb{Z}[\xi_n]^2/\langle(1,-1)\rangle \cong \mathbb{Z}[\xi_n] & \text{if } \chi = 0 \\ \mathbb{Z}[\xi_n]^2/\langle(1,-1),(\xi_n-1,-\xi_n+1\rangle \cong \mathbb{Z}[\xi_n]/\langle \xi_n - 1\rangle & \text{if } \chi \neq 0. \end{cases}$$

We will need a computation of the twisted homology of a knot complement with respect to certain abelian representations into $\mathrm{GL}_2(\mathbb{Z}[\xi_n])$. It will be convenient to have the following notation.

Notation 7.5. Let X be a connected space equipped with a surjection $\varepsilon\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}$, and let ξ be a root of unity. Define $\mathcal{A}_\xi(X) := \mathcal{A}(X) \otimes_{\mathbb{Z}[t^{\pm1}]} \mathbb{Z}[\xi]$, where $\mathbb{Z}[\xi]$ has the $\mathbb{Z}[t^{\pm1}]$-module structure induced by $t \cdot a := \xi a$.

Also, for any $\mathbb{Z}[\xi]$-module M, let $\overline{M}$ denote the module with conjugate $\mathbb{Z}[\xi]$-structure and let $M^{1\oplus\bar{1}} := M \oplus \overline{M}$.

Lemma 7.6. *Let X be a connected space with a surjection $\varepsilon\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}$, and define $\phi\colon \pi_1(X) \to \mathrm{GL}_2(\mathbb{Z}[\xi_n])$ by*

$$\gamma \mapsto \begin{bmatrix} \xi_n^{\varepsilon(\gamma)} & 0 \\ 0 & \xi_n^{-\varepsilon(\gamma)} \end{bmatrix}.$$

Then $H_1^\phi(X;\mathbb{Z}[\xi_n]) \cong \mathcal{A}_\xi(X) \oplus \mathcal{A}_{\bar{\xi}}(X)$.

Proof. First, note that $H_1^\phi(X;\mathbb{Z}[\xi_n]) \cong H_1^\theta(X;\mathbb{Z}[\xi_n])^{1\oplus\bar{1}}$, where $\theta\colon \pi_1(X) \to \mathbb{Z}[\xi_n]^\times$ is given by $\theta(\gamma) = \xi_n^{\varepsilon(\gamma)}$. So it suffices to show that $H_1^\theta(X;\mathbb{Z}[\xi_n]) \cong \mathcal{A}_\xi(X)$.

Let $X^\infty \to X$ be the ε-induced $\mathbb{Z}$-cover of X. Note that the ϕ-induced cover of X is the n-fold cyclic cover X^n, but we can compute $H_1^\theta(X;\mathbb{Z}[\xi_n])$ as

$$H_1^\theta(X;\mathbb{Z}[\xi_n]) = H_1\left(C_*(X^n) \otimes_{\mathbb{Z}[\mathbb{Z}_n]} \mathbb{Z}[\xi_n]\right) = H_1\left(C_*(X^\infty) \otimes_{\mathbb{Z}[t^{\pm1}]} \mathbb{Z}[\xi_n]\right).$$

The Künneth spectral sequence [Wei94, Theorem 5.6.4, p. 143] tells us that since $C_*(X_\infty)$ is a bounded below complex of flat (in fact free) $\mathbb{Z}[t^{\pm 1}]$-modules, there is a boundedly converging upper right quadrant spectral sequence:

$$E^2_{p,q} = \operatorname{Tor}_p^{\mathbb{Z}[t^{\pm 1}]}(H_q(X^\infty), \mathbb{Z}[\xi_n]) \Rightarrow H_{p+q}(C_*(X^\infty) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[\xi_n]).$$

By considering the possible differential maps, we see that the only $E^2_{p,q}$ which could potentially contribute to $H_1(C_*(X^\infty) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[\xi_n])$ are $(p,q) \in \{(1,0),(0,1)\}$. The only relevant differential could be $d^2_{2,0}\colon E^2_{2,0} \to E^2_{0,1}$. However.

$$\begin{aligned} E^2_{2,0} &= \operatorname{Tor}_2^{\mathbb{Z}[t^{\pm 1}]}(H_0(X^\infty), \mathbb{Z}[\xi_n]) = \operatorname{Tor}_2^{\mathbb{Z}[t^{\pm 1}]}(\mathbb{Z}[t^{\pm 1}]/\langle t-1\rangle, \mathbb{Z}[\xi_n]) \\ &= \operatorname{Tor}_2^{\mathbb{Z}[t^{\pm 1}]}(\mathbb{Z}, \mathbb{Z}[\xi_n]) = 0, \end{aligned}$$

since as a $\mathbb{Z}[t^{\pm 1}]$-module $\mathbb{Z}$ has a length 1 projective resolution. Therefore the spectral sequence collapses on the 1-line at the E^2 page, and it suffices to compute $E^2_{0,1}$ and $E^2_{1,0}$. We have that

$$\begin{aligned} E^2_{1,0} &= \operatorname{Tor}_1^{\mathbb{Z}[t^{\pm 1}]}(H_0(X^\infty), \mathbb{Z}[\xi_n]) \\ &= \operatorname{Tor}_1^{\mathbb{Z}[t^{\pm 1}]}(\mathbb{Z}[t^{\pm 1}]/\langle t-1\rangle, \mathbb{Z}[\xi_n]) \\ &\cong \{x \in \mathbb{Z}[\xi_n] \text{ such that } (t-1)\cdot x = 0\} \\ &\cong \{x \in \mathbb{Z}[\xi_n] \text{ such that } (\xi_n - 1)x = 0\} = 0. \end{aligned}$$

Finally, since

$$E^2_{0,1} = \operatorname{Tor}_0^{\mathbb{Z}[t^{\pm 1}]}(H_1(X^\infty), \mathbb{Z}[\xi_n]) \cong H_1(X^\infty) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[\xi_n] = \mathcal{A}_\xi(X)$$

we obtain our desired result. □

Proposition 7.7. *Let R be a slice knot with slice disc Δ_0 and J be a slice knot with slice disc D. Let η be an unknot in the complement of R which generates $\mathcal{A}(R)$. Suppose that n is prime and $\chi\colon H_1(\Sigma_2(R)) \to \mathbb{Z}_n$ is a nontrivial map such that ϕ_χ extends to $\Phi\colon \pi_1(X_{\Delta_0}) \to \mathbb{Z}_2 \ltimes \mathbb{Z}_n$. There are identifications*

$$\begin{aligned} H_1^{f_*\circ\phi_\chi}(X_{R_\eta(J)}, \mathbb{Z}[\xi_n]) &\cong H_1^{\phi_\chi}(X_R, \mathbb{Z}[\xi_n]) \oplus \mathcal{A}_{\xi_n}(J)^{1\oplus\bar{1}} \\ \text{and } H_1^{g_*\circ\Phi}(X_{\Delta_D}, \mathbb{Z}[\xi_n]) &\cong H_1^{\Phi}(X_{\Delta_0}, \mathbb{Z}[\xi_n]) \oplus \mathcal{A}_{\xi_n}(D)^{1\oplus\bar{1}} \end{aligned}$$

Moreover, these are natural with respect to inclusion maps; in particular

$$P := \ker\left(H_1^{f_*\circ\phi_\chi}(X_{R_\eta(J)}, \mathbb{Z}[\xi_n]) \to H_1^{g_*\circ\Phi}(X_{\Delta_D}, \mathbb{Z}[\xi_n])\right)$$

splits as the direct sum of the corresponding kernels $P_R \oplus P_J^{1\oplus\bar{1}}$, where

$$\begin{aligned} P_R &:= \ker\left((H_1^{\phi_\chi}(X_R, \mathbb{Z}[\xi_n]) \to H_1^{g_*\circ\Phi}(X_{\Delta_D}, \mathbb{Z}[\xi_n])\right) \\ P_J^{1\oplus\bar{1}} &:= \ker\left(\mathcal{A}_{\xi_n}(J)^{1\oplus\bar{1}} \to \mathcal{A}_{\xi_n}(D)^{1\oplus\bar{1}}\right) = \ker\left(\mathcal{A}_{\xi_n}(J) \to \mathcal{A}_{\xi_n}(D)\right)^{1\oplus\bar{1}}. \end{aligned}$$

We warn the reader that in the following proof we are a little too enthusiastic in our use of $\oplus$-notation, using $f_1 \oplus f_2$ variously to refer to any of:

- the map $A \to B_1 \oplus B_2$ induced by $\{A \xrightarrow{f_i} B_i\}_{i=1,2}$;
- the map $A_1 \oplus A_2 \to B$ induced by $\{A_i \xrightarrow{f_i} B\}_{i=1,2}$;
- the map $A_1 \oplus A_2 \to B_1 \oplus B_2$ induced by $\{A_i \xrightarrow{f_i} B_i\}_{i=1,2}$.

Since the domain and codomain of all maps are explicitly stated, we trust that this will not be too confusing.

Proof. We abbreviate $X_R \smallsetminus \nu(\eta)$ by $X_R \smallsetminus \eta$ and let $\xi = \xi_n = e^{2\pi i/n}$. In addition, unless otherwise specified, all homology in this proof is taken to be twisted with $\mathbb{Z}[\xi]$-coefficients induced by (restrictions of) the maps ϕ_χ and Φ.

Since $\eta \in \pi_1(X_R)^{(1)}$, when we restrict $\alpha \circ \phi_\chi$to $\pi_1(X_J)$ we see that every element of $\pi_1(X_J)$ is sent to a matrix of the form $\begin{bmatrix} \xi^b & 0 \\ 0 & \xi^{-b} \end{bmatrix}$ for some $b \in \mathbb{Z}_n$. In particular, this restriction factors through $H_1(X_J;\mathbb{Z}) \cong \mathbb{Z}$. The fact that η generates $\mathcal{A}(R)$ implies that the lifts of η to X_R^2 generate $TH_1(X_R^2)$, since $TH_1(X_R^2) \cong \mathcal{A}(R)/\langle t^2-1\rangle$ [Fri04, Lemma 2.2]. However, the longitudes of η are identified with the meridians of J in $X_{R_\eta(J)}$, and so since χ is a nontrivial (hence surjective) character, the map $\pi_1(X_J) \to \mathbb{Z}_n$ given by $\gamma \mapsto b(\gamma) \in \mathbb{Z}_n$ is surjective.

So we are in the setting of Lemma 7.6 and therefore $H_1(X_J) \cong \mathcal{A}_\xi(J)^{1\oplus\bar{1}}$ and $H_1(X_D) \cong \mathcal{A}_\xi(D)^{1\oplus\bar{1}}$. The decompositions outlined in Construction 7.1 are related by inclusion and degree one maps in such a way that, when we take homology with twisted $\mathbb{Z}[\xi]$-coefficients, we obtain the following commutative diagram. All horizontal sequences are exact, since they arise from Mayer-Vietoris sequences.

Note that the twisted homology $H_1(X_U) = H_1(X_E) = H_1(S^1 \times D^2) = 0$, by Lemma 7.6, since each of these spaces have trivial Alexander module. Also, the maps $H_0(T^2) \to H_0(X_*)$ for $* = U, J$ and $H_0(S^1 \times D^2) \to H_0(X_*)$ for $* = E, D$ are isomorphisms. We have simplified the diagram using these observations.

$$\begin{tikzcd}
 & 0 \arrow[rr] \arrow[dd] & & {\begin{matrix} H_1(X_{\Delta_0}) \\ \oplus \\ H_1(X_D) \end{matrix}} \arrow[rr, "\pi_\Delta \oplus \pi_D"] \arrow[dd, "Id \oplus 0"] & & H_1(X_{\Delta_D}) \arrow[r] \arrow[dd, "g_*"] & 0 \\
H_1(T^2) \arrow[ur] \arrow[rr, "j_\eta \oplus j_J"] \arrow[dd, "\mathrm{Id}"] & & {\begin{matrix} H_1(X_R \smallsetminus \eta) \\ \oplus \\ H_1(X_J) \end{matrix}} \arrow[ur, "i_\eta \oplus i_J"] \arrow[rr, "\pi_\eta \oplus \pi_J"] \arrow[dd, "\mathrm{Id} \oplus 0"] & & H_1(X_{R_\eta(J)}) \arrow[ur, "i"] \arrow[rr] \arrow[dd, "f_*"] & & 0 \\
 & 0 \arrow[rr] & & H_1(X_\Delta) \arrow[rr, "\pi"] & & H_1(X_\Delta) \arrow[r] & 0 \\
H_1(T^2) \arrow[ur] \arrow[rr, "j_R = j_\eta"] & & H_1(X_R \smallsetminus \eta) \arrow[ur, "i_\eta"] \arrow[rr, "\pi_R"] & & H_1(X_R) \arrow[ur, "i_R"] \arrow[r] & 0 &
\end{tikzcd}$$

We immediately obtain that

$$\pi_\Delta \oplus \pi_D \colon H_1(X_{\Delta_0}) \oplus H_1(X_D) \to H_1(X_{\Delta_D})$$

is an isomorphism, which is the second identification of the proposition. We also see that

$$H_1(X_R) = \mathrm{Im}(\pi_R) \cong H_1(X_R \smallsetminus \eta)/\ker(\pi_R) = H_1(X_R \smallsetminus \eta)/\mathrm{Im}(j_R)$$

and similarly that

$$H_1(X_{R_\eta(J)}) = \mathrm{Im}(\pi_\eta \oplus \pi_J) \cong \big(H_1(X_R \smallsetminus \eta) \oplus H_1(X_J)\big)/\,\mathrm{Im}(j_R \oplus j_J).$$

We can directly compute that

$$H_1(T^2) = H_1(C_*(\widetilde{T^2}) \otimes_{\mathbb{Z}[\pi_1(T^2)]} \mathbb{Z}[\xi]^2) \cong (\mathbb{Z}[\xi]/(\xi-1))^{1\oplus\bar{1}}$$

is generated as a $\mathbb{Z}[\xi]$-module by $\alpha \otimes [0,1]$ and $\alpha \otimes [1,0]$, where α is the curve on T^2 identified with μ_η in $X_R \smallsetminus \eta$ and λ_J in X_J. Since $[\lambda_J] = 0 \in H_1(X_J^\infty)$, we see that

$$j_J(\alpha \otimes [0,1]) = j_J(\alpha \otimes [1,0]) = 0 \text{ in } H_1(X_J)$$

and hence that $j_J = 0$.

It follows that the map induced by $\pi_\eta \oplus \pi_J$ from $H_1(X_R \smallsetminus \eta)/\,\mathrm{Im}(j_\eta) \oplus H_1(X_J)$ to $H_1(X_{R_\eta(J)})$ is an isomorphism, and that our desired isomorphism for the first identification of the proposition is given by[2]

$$\Phi\colon H_1(X_R) \oplus H_1(X_J) \xrightarrow{\pi_R^{-1} \oplus \mathrm{Id}} H_1(X_R \smallsetminus \eta)/\,\mathrm{Im}(j_\eta) \oplus H_1(X_J) \xrightarrow{\pi_\eta \oplus \pi_J} H_1(X_{R_\eta(J)}). \quad (2)$$

It remains to show that $\Phi^{-1}(\ker(i)) = \ker(i_R) \oplus \ker(i_J)$, which will follow from some diagram chasing,

First we show that $\Phi^{-1}(\ker(i)) \subseteq \ker(i_R) \oplus \ker(i_J)$. Let $x \in \ker(i)$. Since $(\pi_\eta \oplus \pi_J)$ is onto, there exists $a \in H_1(X_R \smallsetminus \eta)$ and $b \in H_1(X_J)$ such that $(\pi_\eta \oplus \pi_J)(a,b) = x$. Moreover, $(\pi_R(a), b) = \Phi^{-1}(x)$, so it suffices to show that

$$i_R(\pi_R(a)) = 0 \in H_1(X_{\Delta_0}) \text{ and } i_J(b) = 0 \in H_1(X_D).$$

Observe that by the commutativity of our large diagram,

$$\pi_R(a) = (\pi_R \circ (\mathrm{Id} \oplus 0))(a,b) = (f_* \circ (\pi_\eta \oplus \pi_J))(a,b) = f_*(x).$$

Therefore

$$(i_R \circ \pi_R)(a) = (i_R \circ f_*)(x) = (g_* \circ i)(x) = g_*(0) = 0.$$

In order to show that $i_J(b) = 0$, observe that

$$((\pi_\Delta \oplus \pi_D) \circ (i_\eta \oplus i_J))(a,b) = (i \circ (\pi_\eta \oplus \pi_J))(a,b) = i(x) = 0.$$

But $\pi_\Delta \oplus \pi_D$ is an isomorphism, and so we know that

$$(i_\eta \oplus i_J)(a,b) = (i_\eta(a), i_J(b)) = 0.$$

So $i_J(b) = 0$ as desired. This completes the proof that $\Phi^{-1}(\ker(i)) \subseteq \ker(i_R) \oplus \ker(i_J)$.

Now we show that $\Phi^{-1}(\ker(i)) \supseteq \ker(i_R) \oplus \ker(i_J)$. It suffices to show that both $\ker(i_R)$ and $\ker(i_J)$ are contained in $\Phi^{-1}(\ker(i))$. Observe that if $b \in \ker(i_J)$ then

$$i(\Phi(b)) = i(\pi_J(b)) = \pi_D(i_J(b)) = \pi_D(0) = 0,$$

[2]The labels of the maps in Equation (2) are mild abuses of notation. In particular, $\pi_R\colon H_1(X_R \smallsetminus \eta) \to H_1(X_R)$ is not itself an isomorphism and hence does not have an inverse until we mod out by $\mathrm{Im}(j_\eta)$, and $\pi_\eta \oplus \pi_J$ actually has domain $H_1(X_R \smallsetminus \eta) \oplus H_1(X_J)$, though it of course induces a well-defined map on $H_1(X_R \smallsetminus \eta)/\,\mathrm{Im}(j_\eta) \oplus H_1(X_J)$. Nevertheless, we hope the reader finds the reminder of how these maps are induced sufficiently helpful so as to outweigh the indignity of slightly misleading labels.

so $b \in \Phi^{-1}(\ker(i))$. Now let $\alpha \in \ker(i_R)$ to show that $\Phi(\alpha) \in \ker(i)$. Let $a \in H_1(X_R \smallsetminus \eta)$ be such that $\pi_R(a) = \alpha$, and observe that $\Phi(\alpha) = \pi_\eta(a)$. We have that

$$(\pi \circ i_\eta)(a) = (i_R \circ \pi_R)(a) = i_R(\alpha) = 0.$$

Since π is an isomorphism, this implies that $i_\eta(a) = 0$ and hence that

$$i\left(\Phi(\alpha)\right) = i\left(\pi_\eta(a)\right) = \pi_\Delta\left(i_\eta(a)\right) = \pi_\Delta(0) = 0,$$

as desired. This completes the proof that $\Phi^{-1}(\ker(i)) = \ker(i_R) \oplus \ker(i_J)$, which completes the proof of Proposition 7.7. □

We also need an analogue of Proposition 6.1 in the context of twisted homology.

Proposition 7.8. *Let D be a properly embedded disc in D^4 with boundary K, and let S be a knotted 2-sphere in S^4. Then given $\chi\colon H_1(\Sigma_2(K))) \to \mathbb{Z}_n$ which extends to $\chi_D\colon H_1(\Sigma_2(D^4, D)) \to \mathbb{Z}_n$, define*

$$\chi_{D\#S}\colon H_1(\Sigma_2(D^4, D\#S)) \cong H_1(\Sigma_2(D^4, D)) \oplus H_1(\Sigma_2(S^4, S)) \xrightarrow{\chi_D \oplus 0} \mathbb{Z}_n\,.$$

Then

$$\ker\left(H_1^{\phi_\chi}(X_K) \to H_1^{\phi_{\chi_D}}(X_D)\right) = \ker\left(H_1^{\phi_\chi}(X_K) \to H_1^{\phi_{\chi_{D\#S}}}(X_{D\#S})\right).$$

Proof. For a submanifold $Y \subset X_{D\#S}$ we can restrict $\phi_{\chi_{D\#S}}$ to $\pi_1(Y)$ and, by a mild abuse of notation we let $H_*^{\phi_{\chi_{D\#S}}}(Y)$ denote the resulting twisted homology with $\mathbb{Z}[\xi_n]$-coefficients. We use the decomposition $X_{D\#S} = X_D \cup_{S^1\times D^2} X_S$. We can directly compute $H_1^{\phi_{\chi_{D\#S}}}(S^1 \times D^2) \cong \mathbb{Z}[\xi_n]$. We can also deduce that

$$H_1^{\phi_{\chi_{D\#S}}}(X_S) \cong \mathbb{Z}[\xi_n] \oplus \left(\mathcal{A}(S) \otimes_{\mathbb{Z}[t^{\pm1}]} \mathbb{Z}[\xi_n]^2\right),$$

where on the right we have the action of $\mathbb{Z}[t^{\pm1}]$ on $\mathbb{Z}[\xi_n]^2$ given by $t \cdot [x, y] = [y, x]$. To see this, use the Künneth spectral sequence [Wei94, Theorem 5.6.4] as in the proof of Lemma 7.6. Obtain

$$\begin{aligned}
E^2_{0,1} &= \mathcal{A}(S) \otimes_{\mathbb{Z}[t^{\pm1}]} \mathbb{Z}[\xi_n]^2 \\
E^2_{1,0} &= \mathrm{Tor}_1^{\mathbb{Z}[t^{\pm1}]}(H_0(X_S^\infty), \mathbb{Z}[\xi_n]^2) = \mathbb{Z}[\xi_n] \\
E^2_{2,0} &= \mathrm{Tor}_2^{\mathbb{Z}[t^{\pm1}]}(H_0(X_S^\infty), \mathbb{Z}[\xi_n]^2) = 0
\end{aligned}$$

This gives rise to a short exact sequence of $\mathbb{Z}[\xi_n]$-modules

$$0 \to \mathcal{A}(S) \otimes_{\mathbb{Z}[t^{\pm1}]} \mathbb{Z}[\xi_n]^2 \to H_1^{\phi_{\chi_{D\#S}}}(X_S) \to \mathbb{Z}[\xi_n] \to 0,$$

which splits since the last module is free. In particular, it follows from naturality of the spectral sequence, comparing the sequences for $S^1 \times D^2$ and X_S, that the map $\mathbb{Z}[\xi_n] \cong H_1^{\phi_{\chi_{D\#S}}}(S^1 \times D^2) \to H_1^{\phi_{\chi_{D\#S}}}(X_S)$ is injective.

Since the restriction of $\phi_{\chi_{D\#S}}\colon \pi_1(X_{D\#S}) \to \mathbb{Z}_2 \ltimes \mathbb{Z}_n$ to $\pi_1(X_S)$ is the map $\gamma \mapsto ([\varepsilon_S(\gamma)], 0)$ we have that $H_0^{\phi_{\chi_{D\#S}}}(S^1 \times D^2) \to H_0^{\phi_{\chi_{D\#S}}}(X_S)$ is an isomorphism. The Mayer-Vietoris sequence for $X_{D\#S} = X_D \cup_{S^1\times D^2} X_S$ with $\mathbb{Z}[\xi_n]$-coefficients therefore gives us that

$$H_1^{\phi_{\chi_{D\#S}}}(X_{D\#S}) \cong H_1^{\phi_{\chi_D}}(X_D) \oplus \left(\mathcal{A}(S) \otimes_{\mathbb{Z}[t^{\pm1}]} \mathbb{Z}[\xi_n]^2\right)$$

and, since $X_K \subset X_D$, we have as desired that

$$\ker(H_1^{\phi_\chi}(X_K) \to H_1^{\phi_{\chi_{D\#S}}}(X_{D\#S})) \cong \ker(H_1^{\phi_\chi}(X_K) \to H_1^{\phi_{\chi_D}}(X_D)). \qquad \square$$

7.3. **Construction of examples and proof of Theorem C.** Let J_0 be a ribbon knot with preferred ribbon disc D_0 such that

$$\mathcal{A}_{\xi_3}(J_0)/\ker\left(\mathcal{A}_{\xi_3}(J_0) \to \mathcal{A}_{\xi_3}(D_0)\right)$$

is nonzero.

One example of such a knot is $J_0 = 6_1$. As noted in Example 7.3, $\mathcal{A}(J_0) = \mathbb{Z}[t^{\pm 1}]/\langle (2t-1)(t-2)\rangle$, $\mathcal{A}(D_0) = \mathbb{Z}[t^{\pm 1}]/\langle t-2\rangle$ and the map $i_0\colon \mathcal{A}(J_0) \to \mathcal{A}(D_0)$ is given by multiplication by $2t-1$. In particular, we have that

$$\begin{aligned}\mathcal{A}_{\xi_3}(J_0)/\ker\left(\mathcal{A}_{\xi_3}(J_0) \to \mathcal{A}_{\xi_3}(D_0)\right) &\cong \mathbb{Z}[\xi_3]/\langle (2\xi_3 - 1)(\xi_3 - 2), \xi_3 - 2\rangle \\ &\cong \mathbb{Z}_7[x]/\langle x-2\rangle \neq 0\end{aligned}$$

Here the $\mathbb{Z}_7$ comes from $\xi_3^2 + \xi_3 + 1 = 0$, combined with $\xi_3 - 2 = 0$.

The knot $J := J_0\# - J_0$ has two simple slice (in fact ribbon) discs: D_1 consists of $D_0 \natural - D_0$ and D_2 is the standard ribbon disc for any knot of the form $K\# - K$ obtained by spinning. Note that $\mathcal{A}(J) \cong \mathcal{A}(J_0)^2$, $\mathcal{A}(D_1) \cong \mathcal{A}(D_0)^2$, and $\mathcal{A}(D_2) \cong \mathcal{A}(J_0)$. Moreover, the map $i_1\colon \mathcal{A}(J) \to \mathcal{A}(D_1)$ is given by $(x, y) \mapsto (i_0(x), i_0(y))$ and the map $i_2\colon \mathcal{A}(J) \to \mathcal{A}(D_2)$ is given by $(x, y) \mapsto x + y$.

Now we prove the following more explicit version of Theorem C.

Theorem 7.9. *Let (R, η, Δ_0) be as in Example 7.3 and let J_0 be a ribbon knot with preferred ribbon disc D_0 such that $\mathcal{A}_{\xi_3}(J_0)/\ker\left(\mathcal{A}_{\xi_3}(J_0) \to \mathcal{A}_{\xi_3}(D_0)\right)$ is nonzero. Let $J = J_0\# - J_0$, D_1, and D_2 be defined as above. Then for any $g \geqslant 0$, the knot $K := \#_{i=1}^{4g} R_\eta(J)$ has ribbon discs Δ_1, the boundary connected sum of $4g$ copies of Δ_{D_1}, and Δ_2, the boundary connected sum of $4g$ copies of Δ_{D_2}, such that*

$$d_2(\Delta_1, \Delta_2) \geqslant g.$$

As discussed above, we also know that $d_2(\Delta_1, \Delta_2) \leqslant 4g$, but we are not able determine $d_2(\Delta_1, \Delta_2)$ precisely in this case.

Proof of Theorem 7.9. Fix $g \in \mathbb{N}$, and let K, Δ_1, and Δ_2 be as above. Define $N = 4g$, $\xi := \xi_3$, and recall that for any knot or slice disc L we have $\mathcal{A}_\xi(L) := \mathcal{A}(L) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[\xi]$. By Proposition 7.2 we have identifications

$$\mathcal{A}(K) \cong \bigoplus_{i=1}^{N} \mathcal{A}(R_\eta(J)) \cong \bigoplus_{i=1}^{N} \mathcal{A}(R)$$
$$\text{and } \mathcal{A}(\Delta_j) \cong \bigoplus_{i=1}^{N} \mathcal{A}(\Delta_{D_j}) \cong \bigoplus_{i=1}^{N} \mathcal{A}(\Delta_0) \text{ for } j = 1, 2$$

in such a way that $\ker(\mathcal{A}(K) \to \mathcal{A}(\Delta_1))$ and $\ker(\mathcal{A}(K) \to \mathcal{A}(\Delta_2))$ are both identified with a sum $\bigoplus_{i=1}^{N} \ker(\mathcal{A}(R) \to \mathcal{A}(\Delta_0))$, and in particular are equal.

Now suppose that F is a genus $h \leqslant g$ surface to which both Δ_1 and Δ_2 stabilize by addition of h 1-handles and some number of local 2-knots. We shall show under these assumptions that $h \geqslant g$. As in the proof of Proposition 6.3, for $j = 1, 2$ there exist discs Δ_j' which are obtained from Δ_j by connected sum with local 2-knots such that F is obtained from Δ_j' by h 1-handle additions. In particular, we can write $\Delta_j' = \Delta_j \# S_j$ for some local 2-knot S_j for $j = 1, 2$.

Let T_1 and T_2 be appropriate unions of the simple cobordisms built in Construction 3.1, such that X_{T_1} is a cobordism from $X_{\Delta_1'}$ to X_F rel. X_K and X_{T_2} is a cobordism from $X_{\Delta_2'}$ to X_F rel. X_K. We let $X_T := X_{T_1} \cup_{X_F} -X_{T_2}$.

Claim 7.10. *There exists a map $\chi = (\chi_i)_{i=1}^N \colon H_1(\Sigma_2(K)) \to \mathbb{Z}_3$ with at least $2g$ of the χ_i nonzero such that $\phi_\chi \colon \pi_1(X_K) \to \mathbb{Z}_2 \ltimes \mathbb{Z}_3$ extends over $\pi_1(X_T)$ to a map $\Phi \colon \pi_1(X_T) \to \mathbb{Z}_2 \ltimes \mathbb{Z}_3$ and for $j = 1, 2$ the composition*

$$\pi_1(X_{S_j}) \to \pi_1(X_{\Delta_j}) *_{\mathbb{Z}} \pi_1(X_{S_j}) \cong \pi_1(X_{\Delta_j'}) \to \pi_1(X_T) \xrightarrow{\Phi} \mathbb{Z}_2 \ltimes \mathbb{Z}_3$$

is given by $\gamma \mapsto ([\varepsilon(\gamma)], 0)$.

We will always construct our extensions in stages, first extending over

$$Y = X_{\Delta_1'} \cup (X_K \times I) \cup X_{\Delta_2'}$$

and then extending over the rest of X_T. Note that

$$H_1(\Sigma_2(K)) \cong \bigoplus_{i=1}^N H_1(\Sigma_2(R_\eta(J))) \cong \bigoplus_{i=1}^N H_1(\Sigma_2(R)) \cong \bigoplus_{i=1}^N \mathbb{Z}_9 \,.$$

Moreover, we have that

$$\begin{aligned} \ker\big(H_1(\Sigma_2(K)) \to H_1(\Sigma_2(D^4, \Delta_j))\big) &\cong \ker\Big(\bigoplus_{i=1}^N H_1(\Sigma_2(R)) \to \bigoplus_{i=1}^N H_1(\Sigma_2(D^4, \Delta_0))\Big) \\ &\cong \bigoplus_{i=1}^N 3\,\mathbb{Z}_9 \,. \end{aligned}$$

It follows that for $j = 1, 2$ any character $\chi \colon H_1(\Sigma_2(K)) \to \mathbb{Z}_3$ extends to a map χ_j on $H_1(\Sigma_2(D^4, \Delta_j))$, up to a priori extending its range to $\mathbb{Z}_{3^a}$ for some $a \geqslant 1$. However, since our slice discs Δ_j are in fact ribbon discs, the inclusion induced map $\pi_1(X_K) \to \pi_1(X_{\Delta_j})$ is surjective for $j = 1, 2$. So we can take $a = 1$.

Note that any map $\chi = (\chi_i)_{i=1}^N \colon H_1(\Sigma_2(K)) \to \mathbb{Z}_3$ induces $\bar{\chi} \colon H_1(X_K^2) \to \mathbb{Z}_3$ by precomposition with the natural inclusion induced map $H_1(X_K^2) \to H_1(\Sigma_2(K))$. Since inclusion induces isomorphisms of $H_1(X_K)$ with $H_1(X_T)$, in order to show that a given ϕ_χ extends over $\pi_1(X_T)$ it suffices to extend the corresponding $\bar{\chi}$ first over $\pi_1(X_{\Delta_1'} \cup (X_K \times I) \cup X_{\Delta_2'})$ and then over $\pi_1(X_T^2)$.

Now, consider the Mayer-Vietoris sequence for $X^2_{\Delta_1'} \cup (X_K^2 \times I) \cup X^2_{\Delta_2'}$, which we note is diffeomorphic to $X^2_{\Delta_1'} \cup_{X_K^2} X^2_{\Delta_2'}$:

$$H_1(X_K^2) \xrightarrow{i_1' \oplus i_2'} H_1(X^2_{\Delta_1'}) \oplus H_1(X^2_{\Delta_2'}) \xrightarrow{j_1 \oplus j_2} H_1(X^2_{\Delta_1'} \cup_{X_K^2} X^2_{\Delta_2'}) \to 0.$$

For $j = 1, 2$ we have that $H_1(X^2_{\Delta_j'}) \cong H_1(X^2_{\Delta_j}) \oplus H_1(\Sigma_2(S^4, S_j))$ in such a way that $i_j' \colon H_1(X_K^2) \to H_1(X^2_{\Delta_j'})$ is given by $i_j \oplus 0$, where $i_j \colon H_1(X_K^2) \to H_1(X^2_{\Delta_j})$ is the inclusion-induced map. We therefore obtain, recalling that the map $H_1(X_K^2) \to H_1(X^2_{\Delta_j})$ is surjective since Δ_j is a ribbon disc, that

$$H_1(X^2_{\Delta_1'} \cup_{X_K^2} X^2_{\Delta_2'}) \cong H_1(X^2_{\Delta_1}) \oplus H_1(\Sigma_2(S^4, S_1)) \oplus H_1(\Sigma_2(S^4, S_2)).$$

Therefore any $\bar{\chi}$ can be extended over

$$X^2_{\Delta_1'} \cup (X_K^2 \times I) \cup X^2_{\Delta_2'} = (X^2_{\Delta_1} \cup X^2_{S_1}) \cup (X_K^2 \times I) \cup (X^2_{\Delta_2} \cup X^2_{S_2}) \subset \partial X_T^2$$

so that the extension is trivial on the $H_1(\Sigma_2(S^4,S_1))\oplus H_1(\Sigma_2(S^4,S_2))$-summand. Moreover, such a map extends over $H_1(X_T^2)$ if and only if it vanishes on

$$H := \ker\left(H_1(X^2_{\Delta_1'} \cup (X^2_K \times I) \cup X^2_{\Delta_2'}) \to H_1(X^2_T)\right).$$

Note that our maps $\bar{\chi}$ have been chosen to vanish on $H_1(\Sigma_2(S^4,S_1))\oplus H_1(\Sigma_2(S^4,S_2))$, and hence vanish on H if and only if they vanish on

$$H \cap H_1(X^2_{\Delta_1}) = \ker\left(H_1(X^2_{\Delta_1}) \to H_1(X^2_T)\right).$$

Moreover, $\ker\left(H_1(X^2_{\Delta_1}) \to H_1(X^2_T)\right)$ is isomorphic to a quotient of $\ker(H_1(X_K) \to H_1(X^2_T))$.

For a space X with surjection $\varepsilon\colon H_1(X) \to \mathbb{Z}$, we consider the map

$$\begin{aligned} e = e_X\colon \pi_1(X) &\to \mathrm{GL}_2(\mathbb{Z}) \\ \gamma &\mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\varepsilon(\gamma)}. \end{aligned}$$

Note that the e_X maps for $X = X_K, X_{\Delta_j'}, X_F, X_T$ are compatible, since inclusion $X_K \hookrightarrow X_*$ induces an isomorphism on first homology. By application of Proposition 6.2, we therefore have that

$$\ker(H_1^e(X_K) \to H_1^e(X_{T_1})) \cong \ker(H_1^e(X_K) \to H_1^e(X_F)) \cong \ker(H_1^e(X_K) \to H_1^e(X_{T_2})).$$

Moreover, this kernel is generated by $\ker(H_1^e(X_K) \to H_1^e(X_{\Delta_1'}))$ along with some $2h$ elements $\{x_k\}_{k=1}^{2h} \subseteq H_1^e(X_K)$.

By the topologists' Shapiro lemma [DK01, p. 100], there is a canonical identification $H_1^e(X) \cong H_1(X^2)$ for all X, and so

$$\ker(H_1(X_K^2) \to H_1(X^2_{T_1})) \cong \ker(H_1(X_K^2) \to H_1(X^2_F)) \cong \ker(H_1(X_K^2) \to H_1(X^2_{T_2}))$$

and this kernel is generated by $\ker(H_1(X_K^2) \to H_1(X^2_{\Delta_1'}))$ along with some $2h$ elements $\{x_k\}_{k=1}^{2h} \subseteq H_1(X_K^2)$.

Therefore, since every map $H_1(X_K^2) \to \mathbb{Z}_3$ extends over $H_1(X^2_{\Delta_1'} \cup_{X_K^2} X^2_{\Delta_2'})$ in our prescribed fashion, in order to ensure that $\bar{\chi}$ extends over $H_1(X_T^2)$ it is enough to have $\bar{\chi}(x_k) = 0$ for all $k = 1, \ldots, 2h$.

Since $\mathrm{Hom}(H_1(\Sigma_2(K)), \mathbb{Z}_3) \cong \mathbb{Z}_3^N$ and using our assumption that $h \leqslant g$, we have

$$N - 2h = (4g) - 2h \geqslant (4g) - 2g \geqslant 2g,$$

so there exists some $\chi = (\chi_i)_{i=1}^N$ vanishing on $\{x_1, \ldots, x_{2h}\}$ with at least $2g$ of the χ_i nonzero (see the proof of [KL05, Theorem 6.1]). This completes the proof of Claim 7.10.

Let $\chi = (\chi_i)_{i=1}^N$ be such a map. By reordering the summands, without loss of generality we may assume that $\chi_1, \ldots, \chi_m$ are nonzero for some $m \geqslant 2g$ and that $\chi_{m+1}, \ldots, \chi_N$ are zero. Let $\Phi\colon \pi_1(X_T) \to \mathbb{Z}_2 \ltimes \mathbb{Z}_3$ be the corresponding extension of ϕ_χ over $\pi_1(X_T)$. By a mild abuse of notation we will refer to the restriction of Φ to $\pi_1(Y)$ for various subsets $Y \subset X_T$ by Φ as well. (Recall that there are some choices of basepoints and paths implicit here as well – see the note at the end of Construction 3.1.)

Observe that X_K is the union of N copies of $X_{R_\eta(J)}$, glued along $(N-1)$ copies of $S^1 \times I$, and that, for $j = 1, 2$, $X_{\Delta_j'}$ is the union of N copies of $X_{\Delta_{D_j}}$, glued along $(N-1)$ copies of $S^1 \times I \times I$, along with a single copy of X_{S_j} glued along $S^1 \times D^2$ away from all the other identifications. These decompositions are compatible. Moreover, for $1 \leqslant i \leqslant m$, the map χ_i is nontrivial and so Proposition 7.7 implies that

$$H_1^{\phi\chi_i}(X_{R_\eta(J)}) \cong H_1^{\phi\chi_i}(X_R) \oplus \mathcal{A}_\xi(J)^{1\oplus\bar{1}} \text{ and } H_1^{\phi\chi_i}(X_{\Delta_{D_j}}) \cong H_1^{\phi\chi_i}(X_{\Delta_0}) \oplus \mathcal{A}_\xi(D_j)^{1\oplus\bar{1}}$$

in such a way that $\ker(H_1^{\phi\chi_i}(X_{R_\eta(J)}) \to H_1^{\phi\chi_i}(X_{\Delta_{D_j}}))$ is identified with

$$\ker(H_1^{\phi\chi_i}(X_R) \to H_1^{\phi\chi_i}(X_{\Delta_0})) \oplus \ker(\mathcal{A}_\xi(J) \to \mathcal{A}_\xi(D_j))^{1\oplus\bar{1}}.$$

Now consider a portion of the Mayer-Vietoris sequences in twisted homology for $X_K = \cup_{i=1}^N X_{R_\eta(J)}$ and $X_{\Delta_j} = \cup_{i=1}^N X_{\Delta_{D_j}}$ for $j = 1, 2$:

$$\begin{tikzcd}
\oplus_{i=1}^{N-1} H_1^{\phi\chi_i}(S^1) \arrow[r, "u"] \arrow[d, "\mathrm{Id}"] & \oplus_{i=1}^{N} H_1^{\phi\chi_i}(X_{R_\eta(J)}) \arrow[r, "v"] \arrow[d, "\oplus_{i=1}^n \iota_j^i"] & H_1^{\phi\chi}(X_K) \arrow[d, "\iota_j"] \\
\oplus_{i=1}^{N-1} H_1^{\Phi}(S^1) \arrow[r, "U_j"] & \oplus_{i=1}^{N} H_1^{\Phi}(X_{\Delta_{D_j}}) \arrow[r, "V_j"] & H_1^{\Phi}(X_{\Delta_j}).
\end{tikzcd}$$

Claim 7.11. *The module $Q := \oplus_{i=1}^m \mathcal{A}_\xi(J)^{1\oplus\bar{1}} \subset \oplus_{i=1}^N H_1^{\phi\chi_i}(X_{R_\eta(J)})$ is carried isomorphically by v to a subgroup of $H_1^{\phi\chi}(X_K)$ such that for $q \in Q$ we have that $v(q) \in \ker(\iota_j)$ if and only if $q \in \ker\left(\oplus_{i=1}^N \iota_j^i\right)$, for $j = 1, 2$.*

The 'carried isomorphically' part holds, since when we decompose

$$\bigoplus_{i=1}^{N} H_1^{\phi\chi_i}(X_{R_\eta(J)}) \cong \bigoplus_{i=1}^{m} \left(H_1^{\phi\chi_i}(X_R) \oplus \mathcal{A}_\xi(J)^{1\oplus\bar{1}}\right) \oplus \bigoplus_{i=m+1}^{N} H_1^{\phi\chi_i}(X_{R_\eta(J)})$$

we can observe that

$$\ker(v) = \operatorname{Im}(u) \subseteq \bigoplus_{i=1}^{m} H_1^{\phi\chi_i}(X_R) \oplus \bigoplus_{i=m+1}^{N} H_1^{\phi\chi_i}(X_{R_\eta(J)}).$$

Similarly, we have that

$$\ker(V_j) = \operatorname{Im}(U_j) \subseteq \bigoplus_{i=1}^{m} H_1^{\Phi}(X_{\Delta_0}) \oplus \bigoplus_{i=m+1}^{N} H_1^{\Phi}(X_{\Delta_{D_j}}).$$

That is, $\ker(v)$ and $\ker(V_j)$ respectively intersect the $\mathcal{A}_\xi(J)^{1\oplus\bar{1}}$ and $\mathcal{A}_\xi(\Delta_0)^{1\oplus\bar{1}}$ summands trivially.

In order to show that $\iota_j^i(x) = 0$ if and only if $\iota_j(v(x)) = 0$, suppose that x is an element of the ith copy of $\mathcal{A}_\xi(J)^{1\oplus\bar{1}}$ for some $1 \leqslant i \leqslant m$. One direction follows immediately from the commutativity of our diagram: if $\iota_j^i(x) = 0$, then $\iota_j(v(x)) = V_j(\iota_j^i(x)) = V_j(0) = 0$. So suppose now that $\iota_j(v(x)) = 0$. It follows that $\iota_j^i(x) \in \ker(V_j) = \operatorname{Im}(U_j)$, and so there exists $y \in \oplus_{i=1}^{n-1} H_1(S^1)$ such that $U_j(y) = \iota_j^i(x)$. Observe that $\iota_j^i(x - u(y)) = \iota_j^i(x) - U_j(y) = 0$, so $x - u(y) \in \ker(\iota_j^i)$. However, since

$$\iota_j^i(x) \in \bigoplus_{i=1}^{m} \mathcal{A}_\xi(D_j)^{1\oplus\bar{1}}$$

and

$$\iota_j^i(u(y)) = U_j(y) \in \operatorname{Im}(U_j) \subseteq \bigoplus_{i=1}^{m} H_1^{\Phi}(X_{\Delta_0}) \oplus \bigoplus_{i=m+1}^{N} H_1^{\Phi}(X_{\Delta_{D_j}})$$

we must have $\iota_j^i(x) = 0 = U_j(y)$, as desired. This completes the proof of Claim 7.11.

Now we finish the proof that $h \geqslant g$, i.e. that if Δ_1' and Δ_2' have a common stabilization F of genus $h \leqslant g$ then in fact we must have $h = g$. For $j = 1, 2$ we have by Claim 7.11 that

$$P_j := v(Q) \cap \ker(\iota_j) \cong Q \cap v^{-1}(\ker(\iota_j)) = Q \cap \bigoplus_{i=1}^{m} \ker(\iota_j^i). \tag{3}$$

Moreover, by the splitting of the kernel from Proposition 7.7 we have that

$$Q \cap \bigoplus_{i=1}^{m} \ker(\iota_j^i) = \bigoplus_{i=1}^{m} \mathcal{A}_\xi(J)^{1\oplus\bar{1}} \cap \bigoplus_{i=1}^{m} \ker(\iota_j^i) = \bigoplus_{i=1}^{m} \ker\left(\mathcal{A}_\xi(J)^{1\oplus\bar{1}} \to \mathcal{A}_\xi(D_j)^{1\oplus\bar{1}}\right). \tag{4}$$

From our computations of the maps $\mathcal{A}_\xi(J) \to \mathcal{A}_\xi(D_j)$, we also have

$$\ker\left(\mathcal{A}_\xi(J)^{1\oplus\bar{1}} \to \mathcal{A}_\xi(D_j)^{1\oplus\bar{1}}\right) = \begin{cases} \ker(\iota_0^\xi\colon \mathcal{A}_\xi(J_0) \to \mathcal{A}_\xi(D_0))^{1\oplus\bar{1}} & j = 1 \\ \{(x, -x) \mid x \in \mathcal{A}_\xi(J_0)\} & j = 2. \end{cases} \tag{5}$$

Let $P_F := \ker(H_1^{\phi_\chi}(X_K) \to H_1^{\phi_\chi}(X_F))$. By Proposition 6.2 applied to Δ_1' and F, we have that P_F is generated as a $\mathbb{Z}[\xi]$-module by $\ker(H_1^{\phi_\chi}(X_K) \to H_1^{\Phi}(X_{\Delta_1'}))$ together with some $2h$ elements $x_1, \ldots, x_{2h}$. Here we use that the ring of Eisenstein integers $\mathbb{Z}[\xi_3]$ is a Euclidean domain and is therefore a PID. However, by Proposition 7.8 we have that

$$\ker\left(H_1^{\phi_\chi}(X_K) \to H_1^{\Phi}(X_{\Delta_1'})\right) = \ker\left(H_1^{\phi_\chi}(X_K) \to H_1^{\Phi}(X_{\Delta_1})\right) = \ker(\iota_1),$$

Similarly, Proposition 6.2 applied to Δ_2' and F together with the fact that by Proposition 7.8

$$\ker\left(H_1^{\phi_\chi}(X_K) \to H_1^{\Phi}(X_{\Delta_2'})\right) = \ker\left(H_1^{\phi_\chi}(X_K) \to H_1^{\Phi}(X_{\Delta_2})\right) = \ker(\iota_2)$$

implies that $\ker(\iota_2) \supseteq P_2$ is contained in P_F. So the generating rank of $P_2/(\ker(\iota_1) \cap P_2))$ is at most $2h$. We will now show that the generating rank of $P_2/(\ker(\iota_1) \cap P_2)$ is at least $2g$, thereby completing our proof. Observe that by Claim 7.11 together with Equations (3) and (5) we have

$$\begin{aligned} P_2/(\ker(\iota_1) \cap P_2) &= P_2/(\ker(\iota_1) \cap v(Q) \cap \ker(\iota_2)) \\ &= P_2/(P_2 \cap P_1) \\ &\cong \bigoplus_{i=1}^{m} \{(x, -x) \mid x \in \mathcal{A}_\xi(J_0)\} / \bigoplus_{i=1}^{m} \left\{(x, -x) \mid x \in \ker(\iota_0^\xi)\right\} \\ &\cong \bigoplus_{i=1}^{m} \mathcal{A}_\xi(J_0)/\ker(\iota_0^\xi). \end{aligned}$$

Since $\mathcal{A}_\xi(J_0)/\ker(\iota_0^\xi)$ is nonzero, this implies that the generating rank of $P_2/(\ker(\iota_1) \cap P_2)$ is $m \geqslant n = 2g$. It follows that $2h \geqslant \operatorname{g-rk} P_2/(\ker(\iota_1) \cap P_2)) \geqslant 2g$, so $h \geqslant g$ as desired. □

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Department of Mathematics, Rice University, Houston, TX, United States
E-mail address: `allison.miller@rice.edu`

Department of Mathematical Sciences, Durham University, United Kingdom
E-mail address: `mark.a.powell@durham.ac.uk`