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The challenge and opportunities of quantum literacy for future education and transdisciplinary problem-solving

Laurentiu Nita [a], Laura Mazzoli Smith [b], Nicholas Chancellor [a] and Helen Cramman [b]

[a]Department of Physics, Durham University, Durham, UK; [b]School of Education, Durham University, Durham, UK

ABSTRACT

Background: Knowledge of quantum computing is arguably inaccessible to many, with knowledge of the complex mathematics involving a particular barrier to entry, creating difficulty in terms of teaching and inclusive learning for those without a high level of mathematics. Meanwhile, it is increasingly important that the knowledge of quantum technologies is accessible to those who work with real-world applications and is taught to the younger generation.

Purpose: Resulting from collaborative dialogue between physicists, computer scientists, educationalists, and industrial end users, we propose the concept of quantum literacy as one means of addressing the need for transdisciplinary research in response to the complex problems that we see at the heart of issues around global sustainability. In this way, quantum literacy can contribute to UN Sustainable Development Goal 4, Quality Education.

Methods: We introduce a specific puzzle visualization learning tool through which to achieve the pedagogic ends we set out with respect to quantum literacy. Visualization through puzzles can enable non-specialists to develop an intuitive, but still rigorous, understanding of universal quantum computation and provide a facility for non-specialists to discover increasingly complex and new quantum algorithms. Using the Hong–Ou–Mandel optical effect from quantum mechanics, we demonstrate how visual methods such as those made possible through the puzzle visualization tool can be very useful for understanding underlying complex processes in quantum physics and beyond and therefore support the aims of quantum literacy.

Conclusion: We argue that quantum literacy, as defined here, addresses the challenges of learning within a highly bounded discipline and of access to the kind of powerful knowledge that should be more accessible to a wide group of learners. We therefore argue for the importance of addressing pedagogic issues when powerful knowledge consists of dense concepts, as well as complex and hierarchical relations between concepts, in addition to presenting a strong barrier to entry in the form of mathematics.

KEYWORDS
Quantum literacy; quantum computation; powerful knowledge; mathematics; games

CONTACT Laura Mazzoli Smith laura.d.mazzolismith@durham.ac.uk School of Education, Leazes Road, Durham, DH1 1TA

Introduction

In this paper, we propose the concept of quantum literacy as an innovative route to improved knowledge and understanding, with an associated learning tool designed to circumnavigate the specialist knowledge that currently prevents wider understanding of, and hence applications for, quantum computation. We argue that quantum literacy can contribute to the United Nations Sustainable Development Goal (SDG) 4, Quality Education. We take this SDG as foundational, not only in terms of its inclusive aim; 'access to inclusive education can help equip locals with the tools required to develop innovative solutions to the world's greatest problems' but also as being capable of addressing the access to powerful knowledge and understanding that will be needed at societal level in order to make greater progress in relation to other SDGs. We have developed this approach through collaborative dialogue across disciplines, which is essential when considering the transdisciplinary nature of the complex problems that we see at the heart of issues around global sustainability. We suggest that the highly bounded, specialist knowledge that characterises disciplines such as physics need not prevent innovative educational approaches to widening the base of understanding in such fields, ensuring access for as many in society as possible.

Quantum literacy

Although quantum technologies are at an early stage, impacts they are having on our culture are already observable, with far-reaching benefits identified for a wide range of industries in the coming years, 'Quantum science promises to have a major impact on the finance, defence, aerospace, energy and telecommunications sectors . . . These technologies promise to change our lives profoundly' (QT SAB 2015, 4–5). The UK is part of a global race to industrialise quantum technology through the National Quantum Technology Programme (Knight and Walmsley 2019). Industry managers and government will increasingly be expected to make decisions related to quantum computing technology and, we argue, they will have to become 'quantum literate' to avoid falling victim to misconceptions and hype. Yet at present, there is little understanding of, or expertise in, the skills required for effective quantum computational reasoning outside specialists in physics and mathematics. While quantum computing technologies provide an exciting new paradigm to approach some of the most difficult problems faced by humanity, how these technologies work is currently only understood by a small segment of the population, with strong perceived barriers to entry from outside of the field. It is recognised that investment in education and training is needed (QT SAB 2015). This lack of understanding from outside of the field is problematic because many of the people who know the end use cases and the current state of the art classical computation techniques (application domain experts), which are currently used to solve them, will find it difficult to contribute. The only way quantum computing can succeed in the near term is to find the right niche areas to apply it and leverage the expertise of the people who currently solve these problems on how quantum technologies can make their methods better.

Such a goal is fundamentally transdisciplinary and cannot be achieved otherwise.

Transdisciplinarity, as concerned with complexity, multidimensionality and problem-focused research (Klein 2013), underscores the growing need for science to contribute to persistent, complex problems (Hoffmann-Riem et al. 2008). Hoffman-Reim et al. also state that increase in the availability of scientific knowledge is not reflected in decisive action and they define transdisciplinary orientations in research, education and institutions as tyring 'to overcome the mismatch between knowledge production in academia, and knowledge requests for solving societal problems' (2008, 4). Integration is the core methodological aim underpinning transdisciplinary research and education and we propose that the wider application of quantum computation to complex real-world problems, which depends on understanding that is situated more broadly than within a single discipline, is facilitated by the ideas presented in this paper. Klein (2013) refers to the transgressive imperative of transdisciplinarity, which can challenge disciplinary conventions and hierarchies of expertise through more participatory modes of knowledge across sectors. We demonstrate the concept of quantum literacy as an educational aim that is aligned with this systematic knowledge integration aim and a specific learning model through which it can be achieved.

Funding for quantum computation, however, is largely directed towards highly specialist centres, for instance in the UK (QT SAB 2015), yet approaches must include training, to some extent, far greater numbers of the current workforce as well as educating the next generation of application domain experts to be quantum literate. We suggest that the concept of quantum literacy can help to structure the educational initiatives needed to support these approaches and support the overarching aim through providing clarity of purpose. We advance a definition of quantum literacy as: *learning of the minimal body of fundamental knowledge of quantum mechanics that allows understanding of how quantum computation could be used in diverse application domains and the capability to assess claims related to quantum technology*. Disciplinary-based approaches to dialogue across quantum computing and application domain experts can only go so far. We will need people with an understanding of both fields. We argue that quantum literacy should aim to increase understanding rather than simply awareness, important as quantum computing emerges as a technology and many non-experts are faced with decisions related to it. This is the knowledge physicists and computer scientists do not have, i.e. knowledge not just about quantum computing, but the wider understanding of the potential applications of quantum computing in diverse areas of society, which currently do not benefit from this technological revolution.

There are many concrete examples of areas in which quantum technologies could be game changing. The number of domains for which proof-of-concept quantum computing experiments have been conducted is too many to list here, but includes subjects as diverse as computational chemistry (Kandala et al. 2017), flight gate assignments at airports (Stollenwerk, Lobe, and Jung 2019), decoding of error correction codes (Chancellor et al. 2016), and hydrology (O'Malley 2018). While these experiments are too early to show quantum advantage directly, there are areas in which provable (at least up to standard assumptions about computational complexity) quantum speedups are possible, for instance, the famous Grover search (Grover 1996), or Shor factoring (Shor 1999), algorithms. Building on these insights and others, many more algorithms with provable advantages have been developed, with applications in areas such as optimisation and cryptography. A review of such algorithms is provided by Montanaro (2016) and a running list of quantum algorithms by Jordan (2020). An equation-free review focusing on continuous time quantum information processing is provided by Kendon (2020).

While quantum computing promises the broadest transformations of any quantum technologies, there are other quantum technologies which promise more near-term applications. These include the use of atomic systems to see through objects by detecting terahertz radiation (Downes et al. 2020), more accurate atomic clocks, which would allow the detection of small strains in the earth's crust for earthquake detection (Ludlow et al. 2015), and key distribution protocols for cryptography, which can detect spying by construction (Minder et al. 2019). Since quantum mechanics equates to a different way of understanding reality, quantum literacy therefore potentially offers a different way of conceiving of problems in a range of diverse fields. Classical computer programs are not effective for quantum computing. Instead, quantum algorithms must be used, which employ quantum phenomena, such as interference of states. The problems for which quantum computation can offer solutions are themselves in need of constructing in light of the understanding that would be gained through quantum literacy.

A social realist approach to scientific knowledge and understanding

In this paper, we focus on the educational challenge of quantum literacy and position the concept in social realist debates in the epistemology of knowledge. A social realist framing is useful in that it advances an argument which 'rehabilitates specialised knowledge and binds it back into a social framework on which it depends' (Young and Muller 2013, 247), important given the transdisciplinary nature of the problems that quantum literacy could help to address. We too often lack a theory of knowledge (Maton 2013), in terms of both the sociological structure of knowledge with properties, powers and tendencies, but also in terms of its intrinsic features. We argue that quantum literacy, as we define it here, addresses both the challenges of learning and knowing in a bounded discipline, as well as the relational structures of knowledge practices that create barriers to accessing areas of knowledge. We draw on an important debate in the field of education about 'powerful knowledge' (Moore et al. 2006; Young 2013; Young and Muller 2013). As stated at the outset, we relate this to the UN SDG Quality Education in arguing the case for wider access to such powerful knowledge, which can be defined as

> (1) access to more reliable facts or truths; (2) access to higher level conceptual perspectives of the specialist field; (3) being able to see the specialist, structured form of a knowledge that differs from everyday experience; and (4) working with objective rather than learner-centred or social-interests-centred orientations to curriculum. (Yates and Millar 2016)

If we are concerned to widen access to a body of systematic knowledge built over time, then we must address the issue of disciplinary specialisation. An educational critique concerns elitism, in that, by definition, specialised knowledge will not be distributed equally and those who tend to have access to it are the already powerful (White 2012). However, Young and Muller (2013) note the category mistake here, pointing out that knowledge of the powerful does not necessarily equate to powerful knowledge. Indeed, in our conceptualisation of quantum literacy, we would argue precisely that this powerful knowledge does not readily accrue to the knowledge of those with power to effect change. Young and Muller also answer the critique that education should be about the flourishing of society and the fostering of well-being, not knowledge acquisition *per se* (e.g. White 2012) by saying that this poses a false dualism. We would concur and promote the idea that access to, and understanding of, a specialised body of knowledge, in this case quantum computing, can be linked directly to societal well-being through support of the UN SDG Quality Education. We advance the case of quantum literacy as axiomatic with respect to these and associated arguments.

We assert that quantum mechanics – and quantum computation – represent a body of knowledge that cannot be thought of other than through the conceptualisation of powerful knowledge (Young and Muller 2013; Wheelahan 2007) if we attend fully to the nature of this knowledge as specialised and its likely impact on human endeavour. Powerful knowledge takes account of how we differentiate knowledge in many ways; epistemologically, aesthetically, morally. Yates and Millar (2016) suggest that science, and physics in particular, would seem to be paradigmatic examples of powerful knowledge because of their strong disciplinary boundedness and vertical knowledge structure, in Bernstein's terminology, that: "…takes the form of a coherent, explicit, and systematically principled structure, hierarchically organised, as in the sciences' (Bernstein 1999, 159). Bernstein developed the idea of a hierarchical knowledge structure, which characterizes the natural sciences, to refer to how different knowledge structures build cumulatively and progressively, newer knowledge subsuming earlier knowledge and differing bodies of knowledge then differing in their degrees of verticality. The powerful knowledge we are focused on in relation to quantum literacy can be understood in this way, the verticality pointing to abstraction of real-world knowledge to decontextualized principles, often utilising dense nominalisations, where one word comes to stand for a complex concept (Conana, Marshall, and Case 2016).

The challenge for education is therefore considerable. We demonstrate this idea of dense nominalisation by considering the concept of entanglement. Entanglement emerged through a thought experiment (Einstein, Podolsky, and Rosen 1935) on the mathematics behind a predicted phenomenon in nature by the formulae of quantum physics. This thought experiment predicted a phenomenon so counterintuitive, that the paper presenting the thought experiment concludes, 'We are thus forced to conclude that the quantum-mechanical description of physical reality given by wave functions is not complete' (1935, 1). Entanglement as a word was later coined by Schrödinger, as 'the probability relations [when the quantum state of two corelated physical systems][…] is known by a representative in common' (Schrödinger 1935, 1), and later observed in the laboratory to be correct. Einstein (1947 in, 2001) referred to this as, 'spooky action at a distance'. The difficulty of explaining the concept of entanglement to others lies in the lack of analogy in human experience, with the concept itself being the result of applied mathematics and the coined word being a translation from the German word *verschränkung.*

Morrow (1993) suggests that 'epistemological access' to the discourse is important, with discourse referring to how a discipline presents itself in the many symbolic modes it employs in addition to language. Students need to develop their ability to shift between these. The difficulty of physics lies not in the number of concepts that need to be learnt, but 'in learning the myriad of relations among concepts … However, this structure is rarely (if ever) explicitly taught in physics' (Lindstrøm 2010). It is also suggested that students do not have access to the qualitative representational aspects that expert physicists do, approaching quantitative problems without this kind of additional understanding (Rosengrant, Etkina, and Van Heuvelen 2007; Conana, Marshall, and Case 2016). Rather, they adopt a formula-related approach, grappling with the mathematics straight away, without any recourse to some representation of the physical concreteness of the issue.

> . . . many of the representational aspects of Physics tend to be taken for granted in teaching: although problem-solving is demonstrated in lectures, often the modelling and qualitative representational aspects are glossed over, and what students see written down by the lecturer is merely the mathematical representation of the problem situation . . . (Conana, Marshall, and Case 2016, 32)

In Conana et al.'s study with university physics students, qualitative representations used to understand physical processes, rather than just mathematical representations, supported students' problem-solving practices, which were more congruent with how expert physicists would work. If we approach quantum literacy as an educational challenge, we then consider how to facilitate learning when powerful knowledge consists of such dense concepts, as well as complex and hierarchical relations between concepts and presents a strong barrier to entry in the form of mathematics. We do this by focusing on pedagogic goals to manage these challenges through discussing the benefits of a puzzle visualization tool. In the puzzle visualization tool that we present, boundary crossing is facilitated through the removal of mathematics as a point of entry and through the provision of a learning tool that engages lay people in not only learning, but also the potential creation of real quantum algorithms.

We would propose Nowotny's (1993) 'protoexpert' as a tangible outcome of a more quantum literate society, someone that is able to operate effectively in the kind of transdisciplinary space that we describe. These scientific protoexperts would possess some knowledge of quantum computation by virtue of understanding in addition to knowledge, to differing degrees, but sufficient so that this understanding and knowledge could then be applied in different disciplinary contexts to address varied domain-based problems. The rise of the protoexpert in other areas of science is noted, but quantum computation has not, as yet, been accessible to any extent that there could be said to be protoexperts in this field: 'men and women who possess scientific and technological knowledge of different kinds and degrees and know how to apply them in different contexts, thus contributing to the production of novel configurations of knowledge and knowledge claims' (Nowotny 1993, 308). Probably the best example of protoexperts in classical computing are programmers, who often do not have formal training in computer science, but rather know how to write efficient code based on experience and incomplete understanding of low-level processes and then utilise programming skills in a range of domain areas. There are specific instances of lay communities shaping scientific research, such as Epstein's (1996) analysis of how AIDS activists transformed biomedical research practices in the field of AIDS research.

Mathematics as a barrier to entry

Yates and Millar state that 'One long-standing problem in the physics curriculum is the constitutive role that mathematics plays in physics' (2016, 305).

> ... the relations between specialised and non-specialised knowledge differ in different disciplines. The boundaries between the two are for all practical purposes unbridgeable in physics ... not the least as a result of the lack of ambiguity of the mathematics they use and the abilities they have developed to express the relationships between their concepts in precise mathematical form. (Young and Muller 2013, 244)

The argument about powerful knowledge, introduced above, along with the entry barrier of mathematical knowledge needed in quantum mechanics, can be conceived of, and dealt with, as a pedagogical issue. Indeed Young (2013) suggests that pedagogy is still under-developed as a specialist field of knowledge and suggests that

> ... although knowledge can be experienced as oppressive and alienating, this is not a property of knowledge itself. An appropriate pedagogy, which engages the commitment of the learner to a relationship to knowledge ... can have the opposite consequences – it can free the learner to have new thoughts and even think the 'not yet thought'. (2013, 107)

It is relevant to note here suggestions of common misconceptions of foundational concepts, precisely because of an over-reliance on mathematics. Yates and Millar state that 'A reduction in mathematics is seen to provide room for a closer and more detailed conceptual understanding of such areas' (2016, 305), while at the same time,a high level of mathematics is seen as being necessary for cutting edge topics like quantum mechanics. University physicists who were interviewed by Yates and Millar expressed concern, 'that students who spent all their time mastering the mathematics would not have the sense of the field or the creativity and initiative needed to take it forward' (Yates and Millar 2016, 306).

Access to a particularly difficult notation language in mathematics effectively functions as a powerful and exclusive form of knowledge that is only accessible to those who have achieved well in it. Mathematical Dirac notation equations relay quantum states and concepts precisely, yet this is beyond the reach of most people. We can class this as a valued form of knowledge, which enables access to other learning and one in which issues of inclusion are therefore pertinent (Young and Muller 2010). However, we can also class this as a form of knowledge wherein multiple barriers to learning pertain, resting primarily on 'the gatekeeping function of achievement in school mathematics' (Straehler-Pohl and Gellert 2013, 314) and a form of strong classification, consisting of a highly specialized discourse, with its own specialized set of internal rules (Bernstein 1996). We suggest that circumnavigating conventional forms of mathematics in order to work with the representations of quantum matter directly is a promising way to proceed towards quantum literacy and that doing so can be positioned as a direct challenge to a particular configuration of hierarchical knowledge.

Added to the difficulty of accessing the mathematics is also the fact that these foundational concepts of quantum mechanics are very difficult to communicate verbally, through linguistic description, because they are counter-intuitive, challenging the underlying common core of knowledge that non-specialists would access, derived from classical Newtonian physics. Here specialised knowledge also conflicts with lay, or common-sense knowledge. The issue therefore is not only one of knowledge acquisition, if knowledge is to be defined as knowledge-that (Ryle 1946) but also understanding, if by understanding we mean something holistic, incorporating a creative act that links together knowledge of parts, imposes order, compares and contrasts (Cooper 1995), incorporating knowledge-how. This epistemology of understanding is therefore critical to the concept of quantum literacy:

> ... to compare and contrast, to amplify, abridge and paraphrase, to generalize and to instantiate, to emphasize, and so on, are all capacities which fall under understanding. Anybody who has at least some of these capacities is able not only to answer questions about the subject-matter of inquiry but also to raise new questions and so enlarge understanding. (1995, 209)

Quantum puzzle visualisation tool

Learning through some activity, for instance,playing a game in a socially scaffolded environment, draws on Vygotsky's social-constructivist theory of learning (1997), where social interaction plays a fundamental role in the development of cognition. By engaging in investigative puzzle-solving activities, students acquire new understanding by actively constructing their own knowledge through experience. Using puzzles relies on problem-solving as a means through which concepts that are otherwise challenging, can be learned. 'Game-based learning' uses gamified content to meet instructional goals (Zainuddin et al. 2020) with three positive themes relating to the use of gamification in education evidenced in the literature: learning achievement, motivation and engagement, and interaction and social connection. When designing games for the education context, it is essential to consider the learning or behavioural outcomes of the gamified task (Schöbel et al. 2020). Studies on teaching various aspects of quantum mechanics have shown that learning through gamified means can boost motivation for learning and improve learning outcomes (Eggers Bjælde, Kock Pedersen, and Sherson 2015). It is also shown that such methods can foster collaboration across disciplines (Magnussen 2012).

Making use of puzzle games to describe physical phenomena has been shown to be very good at explaining certain observations in physics, such as remote optimization of ultracold atoms in an experiment by experts and citizen scientists (Heck et al. 2018), quantum speed limit (Sørensen et al. 2016), and quantum simulations (Lieberoth et al. 2015). These games are specific to solving certain problems; however, the puzzle tool discussed in this paper differs in scope, in being complete, to integrate and describe visually the body of quantum physics. It therefore stands as an alternative method to create any types of puzzles based on quantum physics. Solving the puzzles allows the player to observe the dynamics and learn the methods to resolve the puzzle without having to also understand the mathematical framework behind it. The puzzle visualization tool we introduce is not the first work to express the mathematics of quantum mechanics in a visual way. For example, there is significant work being done on graphical calculi such as the ZX graphical calculus (Coecke and Duncan 2011) and related graphical calculi (Backens and Kissinger 2018). In fact, matrix operations have even been represented in a similar fashion to our tool within the graphical calculi community (Zanazi 2015; Bonchi, Sobocinski, and Zanazi 2017). While the goal of our techniques is to educate quantum non-experts on quantum computing, the primary goal of previous work is to provide more powerful tools to those who are already mathematical experts. One exception to this pattern is the graphical tool developed in Roffe et al. (2019), for the design of quantum error correction codes.

We suggest that the entire body of knowledge needed to learn and work with quantum computation can be acquired through a blend of intuition derived from the process of engaging with puzzle playing, alongside scaffolded transmission of conceptual knowledge at particular stages of play. We take a multifaceted conceptualisation of what and how learning may unfold, drawing on understandings of the importance of active learning in science (Wieman and Perkins 2005; Van Heuvelen 1991). The ultimate aim of creating this new learning approach for quantum computation, pioneered by *Quarks Interactive*,[1] is to enable non-specialists to develop an intuitive, but still rigorous, understanding of universal quantum computing and to provide a facility for non-specialists to discover increasingly complex and new quantum algorithms. The visuals are not an approximation of reality but represent what actually happens in the quantum world. The game engine is based on a visualisation of the matrix mathematics which sits behind quantum state representations. This representation is complete in the sense that it can represent all isolated quantum states. Quantum computing is difficult to understand because, while classically, the possibility of reaching an outcome in an additional way can only increase probability of that outcome happening, quantum mechanically, an additional way to reach an outcome can lead to destructive interference and cause the probability to decrease, possibly even to zero. These new principles can be difficult to describe in words and traditional equations and a visualization tool circumvents both these issues, enabling learners to gain an intuitive grasp of concepts.

Quantum physicists use matrix-vector multiplication to calculate the effect of a change (a matrix of complex numbers) on a quantum state (a vector of complex numbers). Multiple such changes executed in a specific order comprise a quantum computation. The matrices representing universal quantum computation are represented as edges on a bipartite graph (a visual map). The colours and the sizes of the balls used in the puzzle tool to represent the quantum states encode the complex numbers used to define quantum states, with colour representing phases, and size representing amplitude. The graphs contain the same information as the matrices but in a way which is more intuitive to use (as any visual puzzle would be), without requiring knowledge in mathematics to understand, process and make use of it.

This aspect is furthermore defined in the puzzle tool as an interaction between the balls representing quantum states and the graphs representing the matrices. The order of such applied changes to the quantum states (the order of the matrix-vector multiplications) is represented in the puzzle tool in real time through the evolution of the balls and graphs. The balls always pass through the graphs from the beginning to the end, representing the order and the dynamic of the quantum circuit as it processes information. The player can perform changes to the graphs by adding puzzle pieces (matrices in visual form representing state changes) at any point in the circuit, equivalent to a quantum physicist designing a quantum circuit to achieve a goal. In the puzzle tool, the goal itself is described as the number, position, colour and size of the balls that should arrive at the end of graphs. Using this tool, problems known by physicists as state compilation or decomposition problems are performed visually, without the need for prior knowledge in the field, allowing non-experts to create quantum algorithms once they are familiar with the visual representation.

We propose that from this visualization tool the players will be able to learn about fundamental principles behind quantum mechanics such as superposition and interference, through using trial and error, as they attempt to solve puzzles that make use of such phenomena in physics. Entanglement and superposition are phenomena at the core of the theoretical framework of quantum mechanics, hence because the puzzle tool is representing this framework, such phenomena are also present through visual outcomes of the dynamics that the player sets, and makes use of. Because high level tools for quantum computing are not yet fully developed, understanding the underlying building blocks is crucial.

The dynamics of classically counter-intuitive processes such as phase amplification can be understood intuitively by engaging with the visual tool and solving puzzles. The visualization tool represents quantum circuits, which include non-Clifford[2] gates and are therefore universal for quantum computing. The fact that the game is a full, exact representation of quantum mechanics, necessarily limits the systems to small sizes (if the game could exactly simulate large quantum computers, we would not need large quantum computers),; however, small-sized examples can build intuition for larger systems. This is therefore a gateway to further learning because it presents complex numbers and linear algebra in a more accessible format.

The difficulty in understanding quantum mechanics processes in language and mathematics

Since quantum mechanics underlies much of our understanding of reality, it is of mass appeal, often attracting interest from those not versed in the level of mathematics and physics required to fully understand it, and misconceptions can easily be generated. We show how to circumvent this issue by describing the Hong–Ou–Mandel effect (Hong, Ou, and Mandel 1987). We do this first by analogy with a classical example – throwing balls on a splitter – followed by a simplified mathematical description of the actual experiment, which involves photons (light particles) travelling through a beam splitter.[3] We then show how the same body of information can be conveyed using the puzzle visualisation tool.

One of the most common misconceptions is that quantum computation works by simply trying all possibilities at once and taking the right solution. While this statement has a kernel of truth in it, the reality is not this simple. While quantum systems can exist in superpositions of different classical states, reading out a solution requires a phenomenon called interference. Classically, probabilities can only add, an additional route to an outcome makes that outcome more likely, and never less. Let us take as an example a situation in which we are dropping two bouncing balls right on the top of a perfectly positioned splitter that is hitting the centre of each ball. The balls can bounce either left or right after hitting the splitter. The drawing below depicts the physical process of dropping the two balls (the upper arrow) on a splinter (the purple symbol), with the lower arrows representing the direction of the bounce that follows. We expect the probability outcome of repeating the experiment multiple times of dropping the balls and measuring the direction of how the balls will bounce, to follow one of the three scenarios:

If the balls behave quantum mechanically and are indistinguishable photons, upon measuring, the result will no longer follow the example above. This is known as the Hong–Ou–Mandel effect and cannot be understood without quantum interference. Its mathematical simplicity makes this a natural example. The outcome for (bosonic) quantum balls is a 50% probability of either of the following two events:

A) Both balls go left.
B) Both balls go right.

The Hong–Ou–Mandel effect explained mathematically

Our understanding of quantum mechanics shows that probabilities can also decrease, or cancel even to reach zero, because of the effects of quantum interference, a phenomenon required to explain the Hong–Ou–Mandel effect. To understand this effect, first we must understand the behaviour of a single photon. We send a single photon through the beam splitter (Equation (1)) that has a 50% chance to allow the photon to pass through. We use the quantum mechanics formalism called Bra-Ket notation to encode our events and their probability for it to not pass through as $|0\rangle$and for it to pass as $|1\rangle$.

$$H_{bsp} = = \frac{1}{\sqrt{2}}(111-1) \tag{1}$$

In this case,the $|0\rangle$state corresponds to the photon travelling vertically, and the $|1\rangle$state corresponds to it travelling horizontally. Note that the factor of −1 is necessary for the beam splitter to be unitary. Our starting state before we send the photon, in Bra-Ket $|\psi_{in}\rangle$is a sum between the events that it did not pass through with 100% probability and that it did with 0% probability, as seen in Equation (2).

$$|\psi_{in}\rangle = 1|0\rangle + 0|1\rangle \tag{2}$$

Now multiplying the matrix from Equation (1) with Equation (2) we can get the probability distribution for the photon to be in both states, $|\psi_{out}\rangle$, which normalized, gives us the 50% split of events in Equation (3). A beam splitter that performs such an operation is called a Hadamard operator in quantum computation and its effect is to place a quantum particle in a superposition state, pre-measurement.

$$H|\psi_{in}\rangle = |\psi_{out}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{3}$$

The Hong–Ou–Mandel effect occurs in the case when we have two identical photons that are going through a beam splitter at the same time. For this quantum effect to happen, we must have identical photons. The condition for a pair of identical photons is that their collective wavefunction is an eigenstate of the swap operator with a + 1 eigenvalue,[4]

$U_{swap}|\psi_{init}\rangle = |\psi_{init}\rangle$, where $U_{swap} = \frac{1}{\sqrt{2}}(\,)$ and our initial state is described by Equation (4)

using the same Bra-ket method as for a single photon, with the difference that now we are representing the behaviour of two photons in the same formula. If the description of the initial state of the photon before passing through the experiment is Equation (2), for two photons, we define the initial state in Equation (4). The states that $|01\rangle$, $|10\rangle$ form the initial state, with one photon travelling vertically and one travelling horizontally. The state $|00\rangle$represents the case where both photons are travelling vertically and $|11\rangle$ represents both travelling horizontally.[5] We, however, consider an initial condition where the photons are travelling perpendicular to each other. Our quantum state is now the sum of two ways of labelling the photons, labelling the horizontally (vertically) travelling one as the first photon and therefore the vertically (horizontally) travelling one as the second photon, shown in Equation (4). Note that the relative phase between the photons must be positive so that the eigenvalue with respect to swapping is +1.

$$|\psi_{init}\rangle = \frac{1}{\sqrt{2}}(0|00\rangle + 1|01\rangle + 1|10\rangle + 0|11\rangle) = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \quad (4)$$

In the case of these two identical photons, the effect of our quantum beam splitter is described by the tensor product U_{bsp} in Equation (5) of two Hadamard operators shown in Equation (1).

$$U_{bsp=}H_{bsp} \otimes H_{bsp} \quad (5)$$

When the beam splitter comes into effect, our $|\psi_{out}\rangle$ defines the outcome as in Equation (3), $\psi_{out}\rangle \; = \; U_{bsp} \, | \, \psi_{init}\rangle$. Our final state is again the sum of all probabilities of all events happening. Here is where we can see the effects of quantum interference effects mathematically, because some of the probability amplitudes of the events have negative signs in Equation (6), and the result shown in Equation (7) gives us the outcome of the experiment in mathematical form.

$$|\psi_{out}\rangle = \frac{1}{2\sqrt{2}} \; |00\rangle + |00\rangle - |01\rangle + |01\rangle + |10\rangle - |10\rangle - |11\rangle - |11\rangle) \quad (6)$$

$$|\psi_{out}\rangle \; = \; \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \quad (7)$$

We get a phase with each reflection that leads to quantum interference between probability amplitudes of different events in particular, the two different ways for one photon to travel in each direction after the output cancels, and the photons therefore must either both be travelling vertically or both be travelling horizontally after the beam splitter. This core concept of interference, explained in mathematics and witnessed in nature, shows how counterintuitive quantum mechanics can be. In the next ssection,we explain this with what we argue is a more intuitive, visual means.

The Hong–Ou–Mandel effect explained through the puzzle visualisation tool

Here we discard the mathematical Dirac formalism and encode this experiment in the quantum puzzle visualization tool.

Figure 2, series (1) to (3), shows a series of snapshots from a dynamic, visual representation of the same topic as described in the previous section. Our two photons are represented as two blue balls, defining the $|\psi_{init}\rangle$ quantum state from Equation (4), shown visually in Figure 1. The position of the photons is encoded in bitstring format, as explained in the previous section.

In the snapshot in Figure 1, we chose to represent a photon falling vertically with bitstring 01, starting to traverse the graph, while a photon traveling horizontally starts at bitstring 10, both identical in size and colour. By using the bitstring encoding and representing the change effect as a graph, we can represent all possible combinations of photons as they travel as a graph, starting with 00 (if both would enter the beam splitter vertically), and ending with 11 (if both would enter horizontally). We show three snapshots of a dynamic animation generated by the tool. The bitstrings at the top, in Figure 1, define their starting positions: in our case, 01 and 10, because the photons are fired from opposite directions at the start.

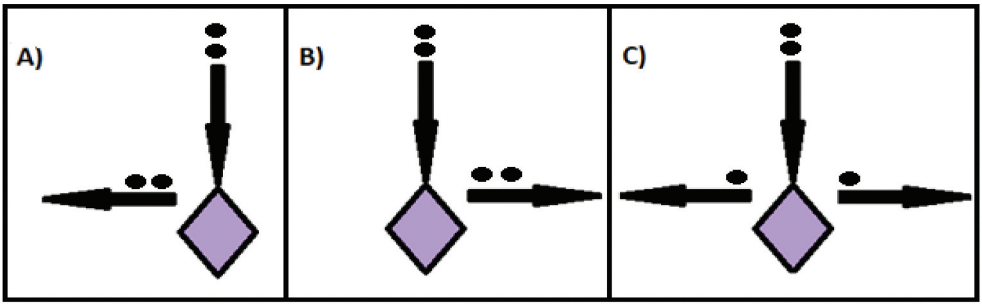

Figure 1. Balls are represented with the black dots and arrows show direction of bounce. (a) 25% Chance for both balls to bounce left. (b) 25% Chance for both balls to bounce right. (c) 50% Chance that each ball will bounce in an opposite direction.

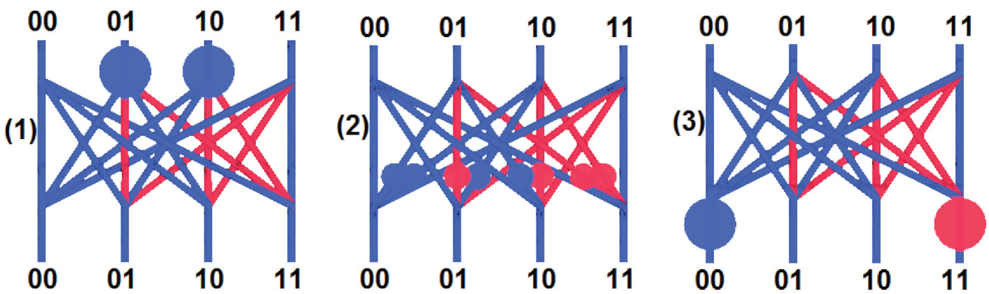

Figure 2. The Hong–Ou–Mandel experiment realised in the puzzle visualization tool.

Figure 2 shows the dynamics of the change effect (the matrix-vector multiplication) performed by the beam splitter on the quantum state. With this method the player can see the actual effect in a visual, instead of calculus. Each of the blue balls split to smaller, equal sizes. The horizontal photon (from 01) splits and becomes two blue balls travelling towards bitstrings 00 and 10 and two red balls towards 01 and 11, while the horizontal photon (from 10) splits in two blue balls towards 00 and 10 and two red balls towards 10 and 11. This is the exact visualisation of the effect mathematically described by Equation (6).

The final quantum state, $|\psi_{out}\rangle$is depicted in Figure 2 and it represents the outcome put forward in Equation (7), after the quantum states of different phases annihilate/amplify, as described in Equation (6). Quantum interference can be seen in the cancellation of blue and red balls in Figure 2. On the other hand, the pair of blue balls and pairs of red balls are reinforced within bitstrings 00 and 11, both travelling either horizontally or vertically. The visual gives us a full representation of the behaviour before the beam splitter (Figure 1), the quantum effect of the beam splitter (Figure 2), and of the outcome after the beam splitter (Figure 2), in accordance with the physical experiment.

The puzzle tool allows the creation of any quantum circuit with the ability to witness in real time the effects of any change in its configuration. We suggest this as a more accessible method to create intuition and understanding of the effect of the beam splitter proposed in the Hong–Ou–Mandel experiment, with considerable gains for learning in this field as a result therefore. Demonstrating the difference in using mathematics, versus visual representation, as the means of explaining the same effect is designed to convey how different barriers to entry are in each method.

Concluding comments

The visual tool we have discussed here is also likely to have benefits beyond just quantum information science. For one, the concept of quantum phase lies at the heart of many important quantum phenomena in a variety of subjects including chemistry, material science and biology. Moreover, as in the example we have given here, these methods teach the player to think in abstractions, which allow phenomena in physically very different systems to be connected by the underlying dynamics. In the example here we abstracted the 1 and 0 to indicate the direction in which particles of light were travelling, but they could have just as easily been spin states of electrons, or energy levels within a chemical system. This abstraction allows the reduction of a system to the fundamental underlying behaviours and allows intuition to be built in this picture. While we have chosen to focus on quantum mechanics, in relation to the concept of quantum literacy and its role in transdisciplinary problem-solving, it is worth noting that our representation is general to all linear systems through the underlying use of matrices. This is a class of systems that includes many classical systems which are often encountered in learning, such as electrical circuits, rotating objects, 'ball-and-spring' systems and many others. While a full exploration of how these systems could be represented with our tool is beyond the scope of our work, these are also likely to be fruitful suggestions.

Through detailed demonstration of how this puzzle visualization tool can relay foundational concepts in quantum computation, we demonstrate a tentative pedagogic answer to the challenge of hierarchical bounded knowledge and complex mathematics as a barrier to entry in the field of quantum computation, which will be subject to further trialling. We suggest that pedagogy can develop from such a visual approach to learning in this field and as such, support the aims of quantum literacy, as we propose them in this paper. We suggest that quantum literacy is a useful concept through which to take account of the normative theory of expertise in the increasingly important domain of quantum mechanics and its associated technologies. By problematising, and foregrounding, pedagogy, as we have done in this paper, and the epistemological features that pertain, we demonstrate how to develop more inclusive teaching and learning in this field and hence wider access to the powerful knowledge that underpins quantum technologies.

Notes

1. *Quarks Interactive* has developed an innovative puzzle visualization tool to explore the process by which non-specialists in quantum mechanics develop their understanding of quantum computational thinking.
2. Clifford gates are a subset of quantum gates (complex matrices that impose a change to a quantum state), which are efficiently classically simulable; the inclusion of at least one non-Clifford gate makes the circuits hard to simulate classically. In the puzzle tool, these are the puzzle pieces the player can place in any order.
3. A beam splitter is the device that forces the photon to either bounce off or pass through it, by analogy going either left and right with the classical example.

4. This is the physics knowledge required *a priori* to understand the experiment, set out here to make our point.
5. An important aspect of working with quantum computation is to be able to encode physical phenomena (or world problems of any kind) in quantum states represented by bitstrings of 0s and 1s, to which other bitstrings of 0s and 1s are the solution to the problem. The quantum algorithm is the process of finding that solution.

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ORCID

Laurentiu Nita http://orcid.org/0000-0002-8884-3701
Laura Mazzoli Smith http://orcid.org/0000-0001-9391-6613
Nicholas Chancellor http://orcid.org/0000-0002-1293-0761
Helen Cramman http://orcid.org/0000-0002-8684-4882

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